Critical indices in a $d = 1$ filled band Fermi system

F.Bonetto
Dipartimento di Matematica, Università di Roma, 00185 Roma, Italia

V.Mastropietro
Dipartimento di Matematica, Università di Tor Vergata, Roma, Italia

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By renormalization group methods we obtain non perturbative results about a $d = 1$ system of interacting spinless fermions in a periodic potential, if the conduction band is filled. Both the strength of the interaction and the amplitude of the periodic potential are assumed small. We obtain that the large distance asymptotic behavior of the two-point Schwinger function is anomalous and described by two critical indices, explicitly computed by convergent series, related to the renormalization of the spectral gap and of the discontinuity at the Fermi surface.

05.30 quantum statistical mechanics

I. INTRODUCTION

The study of $d = 1$ systems of interacting fermions has attracted interest, not only because they can describe suitable strongly anisotropic compounds[1] but also because they have a very rich structure, and their properties may give, hopefully, hints on the intricacies that might be present in higher dimensional models. In Ref. 2 for instance the properties of some $d = 2$ compounds which manifest high T_c-superconductivity are explained by assuming that they have some properties typical of one-dimensional systems ("Luttinger liquid behavior").

The standard Hamiltonian of a system of one dimensional fermions with spin σ which move in a periodic force $-u\partial_{\vec{x}}c(\vec{x})$, $c(\vec{x}+a) = c(\vec{x})$, and interact via a short range pair potential $\lambda v(\vec{x}-\vec{y})$ is:

$$H(\phi) = T(\phi) + uP(\phi) + \lambda\tilde{V}(\phi) + \nu N(\phi) \tag{1}$$

$$T(\phi) = \int_{-L/2}^{L/2} d\vec{x}\, \phi^+_{\vec{x},\sigma} \left[-\frac{\hbar^2\partial^2_{\vec{x}}}{2m} - \mu\right] \phi^-_{\vec{x},\sigma}$$

$$P(\phi) = \int_{-L/2}^{L/2} d\vec{x}\, \phi^+_{\vec{x},\sigma} c(\vec{x}) \phi^-_{\vec{x},\sigma}$$

$$\tilde{V}(\phi) = \int_{-L/2}^{L/2} d\vec{x} d\vec{y}\; v(\vec{x}-\vec{y}) \phi^+_{\vec{x},\sigma}\phi^-_{\vec{x},\sigma}\phi^+_{\vec{y},\sigma'}\phi^-_{\vec{y},\sigma'}$$

$$N(\phi) = \int_{-L/2}^{L/2} d\vec{x}\, \phi^+_{\vec{x},\sigma}\phi^-_{\vec{x},\sigma}$$

where $\phi^\pm_{\vec{x},\sigma}$ are creation or annihilation fermionic field operators with spin σ on the Fock space of fermions confined in a box $\Lambda = [-L/2, L/2]$ with periodic boundary conditions, obeying to the anticommutation rule. If $\sigma = 0$ we say that the fermions are *spinless* while if $\sigma = \pm 1/2$ they are *spinning*, and in this case the sum over the spins is understood. We take $L = Na$, N integer. We assume that u and λ are *dimensionless* and $c(\vec{x}) = \frac{\hbar^2}{2ma^2}\tilde{c}(\vec{x}/a)$, $v(\vec{r}) = \frac{\hbar^2}{2ma^2}\tilde{v}(\vec{r}p_0)$, with $\tilde{c}(\vec{x})$, $\tilde{v}(\vec{r})$ also dimensionless; moreover it is not restrictive to assume $u \geq 0$ and $\int_0^a c(\vec{x}) = 0$ and we choose the sign of λ assuming that $\hat{v}(0) - \hat{v}(2p_F) \geq 0$. Finally $\tilde{c}(\vec{x})$ and $\tilde{v}(\vec{r})$ are assumed "rotationally invariant" *i.e.* even; u is called *amplitude* of the periodic potential, λ *strength* of the interaction and m is the *fermion mass*; ν is a *counterterm* to be fixed to a λ, u dependent value in order to fix the Fermi momentum (see below). Finally we call $V(\phi) = uP(\phi) + \nu N(\phi) + \lambda\tilde{V}(\phi)$, and we denote by E_0^n (*ground state energy*) the minimum value of H over the states $|\psi_n\rangle$ with n particles *i.e.* such that $N|\psi_n\rangle = n|\psi_n\rangle$.

The Grand-canonical state at volume Λ, inverse temperature β and *chemical potential* $\mu + \nu$ can be obtained by the *Schwinger functions* (the imaginary time Green functions[3]):

$$S_n^{L,\beta}(x_1,\sigma_1,\varepsilon_1;\ldots;x_{2n},\sigma_{2n},\varepsilon_{2n}) = \frac{\int\{\mathcal{D}\psi e^{-\int dk \psi^+_{k,\sigma}(-ik_0 - E(\vec{k}))\psi^-_{k,\sigma}}\} e^{-V(\psi)} \prod_{i=1}^{2n} \psi^{\varepsilon_i}_{x_i,\sigma_i}}{\int\{\mathcal{D}\psi e^{-\int dk \psi^+_{k,\sigma}(-ik_0 - E(\vec{k}))\psi^-_{k,\sigma}}\} e^{-V(\psi)}} \tag{2}$$

where $\varepsilon_i = \pm$,$x = (x_0, \vec{x})$, $k = (k_0, \vec{k}) = 2\pi(\frac{n_0 + 2^{-1}}{\beta}, \frac{n_1}{L})$, if n_0, n_1 are integers, $\hbar E(\vec{k}) = \frac{\hbar^2\vec{k}^2}{2m} - \mu$, $\psi^\pm_{x,\sigma}, \psi^\pm_{k,\sigma}$, are *Grassman variables* and from now on we denote by $\int\{\mathcal{D}\psi e^{-\int dk \psi^+_{k,\sigma} h(k_0,\vec{k})^{-1}\psi^-_{k,\sigma}}\}$ the *fermionic integration*, a linear operator defined over the monomials of Grassman variables $\psi^\pm_x$ by the anticommutative Wick rule with propagator $\int dk e^{ik(x-y)} h(k_0, \vec{k})$ with $\int dk = \frac{1}{\beta L}\sum_k$; we set

$$g^{L,\beta}(x-y) = \int dk \frac{e^{ik(x-y)}}{-ik_0 - E(\vec{k})} \tag{3}$$

The fermionic integration is really well defined only if we have a finite number of Grassman variables. Hence we should introduce an ultraviolet cut-off, for example by restricting the sum over k to $|n_0|, |n_1| \leq M$, and we should study the limit $M \to \infty$. However all our bounds will be uniform with respect to M and they easily imply the existence of the limit. Hence, we shall not stress anymore the dependence on M of the various quantities we shall study. Writing $e^{-V} = \sum_{n=1}^{\infty} \frac{(-V)^n}{n!}$ and collecting the contribution of the same order in λ, u, ν one can see, by the *Grahm-Hadamard inequality* (which exploits the anticommutativity of the Fermi variables) that the series resulting by this procedure are absolutely convergent[4,5] in this case we say that the functional integral eq.(2) is well defined. Of course the hard problem is to prove that, for a suitable ν, the thermodynamic limit $L, \beta \to \infty$ is well defined.

In this paper we study the *two-point Schwinger function* defined as:

$$S_2^{L,\beta}(x, \sigma, +; y, \sigma, -) \equiv S^{L,\beta}(x, y) \quad (4)$$

We call $\lim_{L,\beta\to\infty} S_2^{L,\beta}(x, y) = S_2(x, y)$.

If $\lambda = \nu = 0$ the n-particle eigenfunctions of H are the antisymmetrized product of the one-particle wavefunctions, $\phi(\vec{k}, \vec{x}, u)$, solving the Schrödinger equation with periodic potential $uc(\vec{x})$:

$$\left[\frac{-\hbar^2\partial_{\vec{x}}^2}{2m} + uc(\vec{x})\right]\phi(\vec{k}, \vec{x}, u) = \varepsilon(\vec{k}, u)\phi(\vec{k}, \vec{x}, u)$$

The functions $\phi(\vec{k}, \vec{x}, u)$ are called *Bloch waves.* We define

$$\begin{aligned}
&\hat{S}^{L,\beta}(\vec{k}; k_0) = \\
&\quad \frac{1}{L}\int d\vec{x}d\vec{y}d(x_0 - y_0)e^{-ik_0(x_0-y_0)} \\
&\qquad \phi(\vec{k}, -\vec{x}, u)\phi(\vec{k}, \vec{y}, u)S_2^{L,\beta}(x, y) \\
&\bar{S}^{L,\beta}(\vec{k}; t) = \\
&\quad \frac{1}{L}\int d\vec{x}d\vec{y}\phi(\vec{k}, -\vec{x}, u)\phi(\vec{k}, \vec{y}, u)S_2^{L,\beta}(x, y)
\end{aligned}$$

Many important physical properties can be obtained from the two-point Schwinger function. The *occupation number* is defined as

$$n_{\vec{k}}^{L,\beta} = \bar{S}^{L,\beta}(\vec{k}; 0^-)$$

and $n_{\vec{k}} = \lim_{L,\beta\to\infty} n_{\vec{k}}^{L,\beta}$. The *Fermi momentum* $p_F(\lambda, u, \mu + \nu)$ in a $d = 1$ fermionic rotation invariant theory with chemical potential $\mu + \nu$ is defined requiring that $n_{\vec{k}}$ is not regular *i.e.* $n_{\vec{k}}$ or some of its derivatives are singular at $\vec{k} = \pm p_F(\lambda, u, \mu + \nu)$. If there are more than two points where the occupation number is not regular the Fermi momentum is chosen such that it can be continuously parameterized by λ and it reduces for $\lambda = 0$ to the Fermi momentum of the free $\lambda = 0$ theory. Since the early works on the theory of Fermi systems[6] it has been realized that it is more natural to study the properties of weakly interacting fermionic systems when λ is varied at fixed Fermi momentum rather than at fixed chemical potential. Therefore we fix ν so that the Fermi momentum in the interacting $\lambda \neq 0$ system, with chemical potential $\mu + \nu$, is equal to the Fermi momentum of the free $\lambda = 0$ system with chemical potential equal to μ *i.e.* $p_F(\lambda, u, \mu + \nu) = p_F(0, u, \mu) \equiv p_F$. The two points $\pm p_F$ are called *Fermi surface.* The Fermi momentum when L, β is finite can be chosen as $\frac{2\pi}{L}(n_F^L + \frac{1}{2})$, if n_F^L is an integer such that $\frac{2\pi}{L}(n_F^L + \frac{1}{2}) \to_{L\to\infty} p_F$. The *discontinuity at the Fermi surface* is defined as

$$Z^{-1} = n_{p_F^-} - n_{p_F^+} \quad (5)$$

The *spectral gap*

$$\bar{\Delta} = E_0^{n+1} + E_0^{n-1} - 2E_0^n \quad (6)$$

can be computed from the imaginary poles in k_0 of $\hat{S}(\vec{k}; k_0)$; if $i\hbar k_{0,\alpha}, -i\hbar k_{0,\beta}, k_{0,\alpha}, k_{0,\beta} > 0$ are such poles then it is well known that $\bar{\Delta} = \min_\alpha(k_\alpha) + \min_\beta(k_\beta)$.

It is easy to check by an explicit computation that the Schwinger functions of the free $\lambda = \nu = 0$ particle system with Hamiltonian $T + uP$ are given by the Wick's rule in term $S_0^{L,\beta}(x, y)$:

$$S_0^{L,\beta}(x, y) = \int dk \frac{\phi(\vec{k}, \vec{x}, u)\phi(\vec{k}, -\vec{y}, u)e^{ik_0(x_0-y_0)}}{-ik_0 - (\varepsilon(\vec{k}, u) - \mu)\hbar^{-1}} \quad (7)$$

The spectral gap is equal to $\varepsilon((n\pi/a)^+, u) - \varepsilon((n\pi/a)^-, u)$ if $p_F = n\pi/a$ and it is 0 for all the other values of p_F. It is convenient to define the adimensional spectral gap, if $p_F = n\pi/a$, as $\Delta_n = \frac{2ma^2}{\hbar^2}[\varepsilon((n\pi/a)^+, u) - \varepsilon((n\pi/a)^-, u)]$. For small u we have $\Delta_1 \equiv \Delta = c_1 u + O(u^2)$ where c_1, the first Fourier coefficient of $\tilde{c}(\vec{x})$, is assumed from now on equal to 1. If $p_F = \frac{n\pi}{a}$ the system is called *filled band* Fermi system. The occupation number is given by $n_{\vec{k}} = \chi\big(\varepsilon(\vec{k}, u) - \mu \leq 0\big)$ where $\chi(condition)$ is 1 if the condition is verified and 0 otherwise. The Fermi momentum p_F is then defined by the condition $\varepsilon(p_F, u) = \mu$; note that since $\varepsilon(\vec{k}, u)$ is not continuous there is not a one to one correspondence between p_F and μ, *i.e.* to $p_F = n\pi/a$ correspond all the values of μ belonging to $[\varepsilon((n\pi/a)^-, u), \varepsilon((n\pi/a)^+, u)]$. The discontinuity at the Fermi surface is $Z^{-1} = 1$. Contrary to the occupation number, the asymptotic behavior for large values of $|x - y| = a^{-1}\sqrt{(\vec{x} - \vec{y})^2 + v_0^2(x_0 - y_0)^2}$ of the two-point Schwinger function depends critically on the value of the Fermi momentum[7,8]:

1. if $p_F \neq n\pi/a$ the two-point Schwinger function decays as $f(\vec{x}, \vec{y})/|x - y|$, where $f(\vec{x}, \vec{y})$ is a bounded function equal to $\sin(p_F(\vec{x} - \vec{y}))$ if $u = 0$, *i.e.* it behaves for large distances as in the $u = 0$ case;

2. if $p_F = n\pi/a$ for any $N > 1$ one can find constants C_N, C such that

$$|S_0(x, y)| \leq \frac{C_N \Delta_n a^{-1}}{1 + \Delta_n^N |x - y|^N} \quad (8)$$

for $|x-y| > \Delta_n^{-1}$ while

$$|S_0(x,y)| \leq \frac{Ca^{-1}}{|x-y|}$$

for $1 \leq |x-y| \leq \Delta_n^{-1}$, if $\Delta_n < 1$.

In the filled band case the large distance behavior of the two-point Schwinger function is then discriminated by an *intrinsic length* , which is $O(u)$ if the strength of the periodic potential is small.

In the interacting $\lambda \neq 0$ case, the computation of $S(x,y)$ is not an easy task. Some *exactly solvable models* were proposed, which are supposed to be "equivalent" in some sense to the original model eq.(1), as far as the large distance properties of the model are considered. One of these models is the Luttinger model[9] describing two kinds of interacting spinless fermions with linear dispersion relation. In Ref. 10 it was proved that the asymptotic behavior of the two-point Schwinger function for large distances is *anomalous* as $S(x,y) = \frac{g_0(x-y)}{|x-y|^\eta} + \lambda \frac{A(x,y)}{|x-y|^{1+\eta}}$, with $A(x,y)$ bounded by a constant and $g_0(x-y)$ is the free Luttinger model Schwinger function; moreover $\bar{\Delta} = 0$, $Z^{-1} = 0$ and $n_{p_F-\varepsilon} - n_{p_F+\varepsilon} = O(\varepsilon^\eta)$. The explicit expression for the n-particle Schwinger functions were computed, by using different techniques[11,12]. In Ref. 13 some arguments are given for the equivalence of the Luttinger model with the model eq.(1) with spinless fermions and $u = 0$.

Several generalizations[1,14] of the Luttinger model were proposed, conjectured to be equivalent to the model defined by eq.(1) with periodic potential and spinning fermions. They are not exactly solvable and their analysis is essentially equivalent to the study of eq.(1); they are generally studied formally by assuming the convergence of the perturbative series for the Schwinger function, and by truncating it at the second order.

In recent times the techniques developed in constructive quantum field theory for producing examples of non trivial quantum field models (for a review see Ref. 15) were applied to study the model eq.(1). The aim was to obtain the Schwinger functions *non perturbatively* proving the summability of the series expressing them[16]. In Ref. 17, 18, 5 the model eq.(1) in the *spinless* $u = 0$ *case* is considered and the existence of the two-point Schwinger function is proved by showing the uniform in L, β convergence (at small coupling) of the series expressing it. Its behavior for large distances is anomalous and similar to the Luttinger model one, so proving the conjecture in Ref. 13. In Ref. 7 the $u \neq 0$ *spinless case*, if $p_F \neq \frac{n\pi}{a}$ (*not filled band case*[19]) is considered, as well as the $u \neq 0$ *spinning case*, if $p_F \neq \frac{n\pi}{a}, \frac{n\pi}{2a}$ (*not filled nor half filled band case*), and the interaction is *repulsive*. By using the analytic properties of the Bloch waves it was shown that in the above cases the Schwinger function decay is anomalous as in the Luttinger model case. In the spinning case, if the interaction is attractive, there are no available results; it is believed[1] that the system physical behavior is similar to the one of the *Hubbard model*, solved by Ref. 20.

In this paper we continue our analysis by studying the Schwinger function of the Hamiltonian eq.(1) in the spinless case for $p_F = \frac{\pi}{a}$ (similar results of course hold for $p_F = \frac{n\pi}{a}$) *i.e.* for a value of the Fermi momentum corresponding to the *filled band case*.

II. MAIN RESULTS

We consider both λ, u small; the case of small λ and large u (in particular greater than $C|\lambda|$, if C is a proper constant) can be treated by considering as Grassmanian integration the one defined by $S_0^{L,\beta}(x,y)$ eq.(7) instead of $g^{L,\beta}(x-y)$ eq.(3), *i.e.* by considering the periodic potential term in the Hamiltonian not as a perturbation but as a part of the fermionic integration; the large distance behavior of the two-point Schwinger function is substantially identical to the free one eq.(7), so that this "trivial" case is not discussed here, referring for it to Ref. 8.

Our results are for definiteness summarized by the following theorem in which $|k| = \sqrt{v_0^{-2}k_0^2 + (|\vec{k}| - \frac{\pi}{a})^2}$:

Theorem: *There exist an $0 < \varepsilon < 1$ and a $\nu = O(\lambda^2)$ such that, for $|u|, |\lambda| \leq \varepsilon$:*

1. *the two point Schwinger function for spinless fermion $S(x,y)$ is such that, for $|x-y| > \hat{u}(p_F)^{-1}$*

$$|S(x,y)| \leq \hat{Z}(p_F)^{-1} \frac{C_N \hat{u}(p_F) a^{-1}}{1 + (\hat{u}(p_F)|x-y|)^N} \tag{9}$$

for any $N > 1$, if C_N is a suitable constant, while for $1 \leq |x-y| \leq \hat{u}(p_F)^{-1}$, if C is a constant:

$$|S(x,y)| \leq C|x-y|^{-1-\eta_3} \tag{10}$$

where

$$\hat{u}(p_F) = u^{1+\eta_2} \qquad \hat{Z}(p_F)^{-1} = u^{\eta_1} \tag{11}$$

and $\eta_1 = \beta_3\lambda^2 + O(\lambda^3)$ and $\eta_2 = \beta_1\lambda + O(\lambda^2)$, $\beta_1, \beta_3 > 0$, $\eta_3 = \eta_1(1+\eta_2)^{-1}$;

2. *there are two positive constants c_1 and c_2 such that the occupation number discontinuity Z^{-1}, eq.(5) is:*

$$c_1\hat{Z}(p_F)^{-1} \leq Z^{-1} \leq c_2\hat{Z}(p_F)^{-1} \tag{12}$$

3. *the two point Schwinger function can be written as $S(x,y) = S_A(x,y) + \tilde{\varepsilon} S_B(x,y)$ with*

$$S_A(x,y) = \tag{13}$$
$$\int \frac{1}{\hat{Z}(k)} \frac{\phi(\vec{k}, \vec{x}, \hat{u}(k))\phi(\vec{k}, -\vec{y}, \hat{u}(k)) e^{ik_0(x_0-y_0)}}{-ik_0 - (\varepsilon(k, \hat{u}(k)) - \mu)\hbar^{-1}}$$

and $\tilde\varepsilon = \max\{|\lambda|, u, \hat u(p_F)\}$, $\phi(\vec k,\vec x,u), \varepsilon(\vec k,u)$ are the Bloch waves and their correspondent eigenvalue, and $\hat u(k)$ and $\hat Z^{-1}(k)$ are two regular functions such that $|\hat u(k)-u| = O(u\lambda)$, $|\hat Z^{-1}(k)-1| = O(\lambda)$ for $|k| > \pi/2a$, and $\hat u(p_F)$, $\hat Z^{-1}(p_F)$ are given by eq.(11).

Contrary to the not filled band case, we will see in the proof that the choice of ν, which fixes the Fermi momentum of the interacting theory, is not unique; this is not too surprising as also in the free $\lambda = 0$ case to many values of the chemical potential *i.e.* $\mu \in (\varepsilon((\frac{n\pi}{a})^+, u), \varepsilon((\frac{n\pi}{a})^-, u))$ it corresponds a unique value of the Fermi momentum *i.e.* $p_F = \frac{n\pi}{a}$.

Like in the $\lambda = 0$ case, see eq.(8), one can distinguish two regions in the large distance behavior of the two-point Schwinger function, discriminated by an *intrinsic length* produced by the periodic potential, which now is changed by the interaction from $O(u)$ to $O(\hat u(p_F))$. In the first region again the two-point Schwinger function decays faster than any power, with the difference that in the bound the decay rate $\Delta = O(u)$ is replaced by $\hat u(p_F)$ and there is an extra factor $\hat Z(p_F)^{-1}$. In the second region, it is bounded by a power-law bound, but the exponent is not anymore 1 but $1+\eta_3$.

The second statement of theorem says that the occupation number discontinuity is not vanishing but it is $O(u^{\eta_1})$; if $u << e^{-\kappa_1\lambda^{-2}}$, κ_1 is a constant, such discontinuity is $<< 1$ and it is vanishing as $u \to 0$ as $O(u^{\eta_1})$. This is in agreement with the Luttinger behavior of the $u = 0$ case in which $Z^{-1} = 0$. An easy corollary of the theorem derivation is that[21]

$$\bar\Delta \geq c_3 \hbar v_0 a^{-1} \hat u(p_F) \tag{14}$$

This lower bound for the spectral gap is particularly interesting if the interaction is attractive[22], as it states in this case that the spectral gap in the interacting theory, if $u << e^{-\kappa_1|\lambda|}$, is deeply renormalized by the interaction and it is not as in the free case $O(u)$ but surely larger than $O(u^{1+\eta_2})$. We can conclude that also in presence of a strong interaction the occupation number is discontinuous, but the ratio between the "bare discontinuity" *i.e.* the discontinuity of the non interacting fermions, and the "dressed discontinuity", *i.e.* the discontinuity renormalized by the interaction, is $>> 1$, for small u, and diverging as $u \to 0$. There is a non vanishing spectral gap also in presence of an interaction but, at least if the interaction is attractive, it is strongly renormalized by the interaction as the ratio between the "bare gap" and the "dressed gap" is $<< 1$ for small u and vanishing as $u \to 0$.

The last statement of the theorem does not add really new informations about the Schwinger functions, but it allows to formulate a convincing conjecture about the eigenstates of the Hamiltonian eq.(1). In fact, the Schwinger function is written as sum of two terms with $\lim_{\lambda,u\to 0} S_A = g$ and $\lim_{\lambda,u\to 0} S_B = 0$. S_A is formally similar to the $\lambda = 0$ Schwinger function S_0, but the amplitude of the periodic potential u and the wave function normalization are replaced by $\hat u(k)$ and $\hat Z(k)$. If S_A were the "dominant" part of the Schwinger function, at least for large distances (what we are not able to prove), one could interpretate this fact saying that the "interacting one-particle wavefunctions" for $\vec k$ near p_F are approximately ,*i.e.* neglecting corrections, $\frac{\phi(\vec k,\vec x,\hat u(k))}{\sqrt{\hat Z(k)}}$ *i.e. interacting Bloch waves.* This extra momentum dependence is natural as we expect that the interaction changes the one-particle wavefunctions mainly for momenta near the Fermi surface. One can expect then that the spectral gap, which in the non interacting $\lambda = 0$ case is $O(u)$, is deeply renormalized by the interaction between electrons becoming $O(u^{1+\eta_2})$, so becoming much larger or much smaller, if $u << e^{-\kappa_1|\lambda|}$, depending on the attractive or repulsive nature of the interaction. The last statement of the Theorem provides in our opinion a suggestive interpretation of the results we found in the first three statements. It is unclear to us if our results can explain the physical properties of some real compound; they can provide an example of behavior, like the Hubbard or the Luttinger model, that can possibly be useful to understand more realistic cases.

III. THE PERTURBATIVE EXPANSION

We start by integrating the denominator of eq.(2), the *partition function* $\mathcal N$. We set for simplicity $a = \hbar = 2m = 1$. This implies that $p_F = \pi$, $v_0 = 2\pi$.

We perform a decomposition of the propagator:

$$\frac{1}{-ik_0 - \vec k^2 + \pi^2} = \frac{1 - C_0^{-1}(k)}{-ik_0 - \vec k^2 + \pi^2} + \frac{C_0^{-1}(k)}{-ik_0 - \vec k^2 + \pi^2} \tag{15}$$

where $C_0^{-1}(t)$ is a $C^\infty(R^+)$ function which is 0 for $t < \frac{p_F}{2}$ and it is equal to 1 for $t > \gamma\frac{p_F}{2}$, if $\gamma > 1$ is a scaling parameter; moreover $C_0^{-1}(k) \equiv C_0^{-1}(\sqrt{k_0^2 + (|\vec k| - p_F)^2})$, $E(k) = \vec k^2 - p_F^2$, and we call the two addends respectively $g^{(\leq 0)}(k)$ and $g^{(>0)}(k)$. Eq.(15) allows us to represent $\psi^{\pm}_{k,\sigma}$ as sum of two independent Grassmanian variables, $\psi^{\pm(>0)}_{k,\sigma}$, $\psi^{\pm(\leq 0)}_{k,\sigma}$ with Grassmanian integrations $\{\mathcal D\psi^{(>0)} e^{-\int dk \psi^{+(>0)}_{k,\sigma} g^{(>0)}(k)^{-1} \psi^{-(>0)}_{k,\sigma}}\}$ and $\{\mathcal D\psi^{(\leq 0)} e^{-\int dk \psi^{+(\leq 0)}_{k,\sigma} g^{(\leq 0)}(k)^{-1} \psi^{-(\leq 0)}_{k,\sigma}}\}$. The integration on $\psi^{(>0)}$ of $\mathcal N$

$$\int \{\mathcal D\psi^{(>0)} e^{-\int dk \psi^{+(>0)}_{k,\sigma} g^{(>0)}(k)^{-1} \psi^{-(>0)}_{k,\sigma}}\} e^{-V(\psi^{(\leq 0)} + \psi^{(>0)})} \equiv e^{-V^0(\psi^{(\leq 0)})} \tag{16}$$

gives[5,7]:

$$V^0(\psi) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int\prod_{i=1}^{2m} dk_i W_{m,n}(k_1,\ldots,k_{2m};z)\cdot \tag{17}$$
$$\prod_{i=1}^{m}\psi^+_{k_i,\sigma_i}\prod_{i=m+1}^{2m}\psi^-_{k_i,\sigma_i}\delta(\sum_{i=1}^{m}k_i - \sum_{i=m+1}^{2m}k_i + 2n\pi)$$

where $2n\pi$ is a spatial vector and the kernels $W_{m,n}(k_1,\ldots,k_m;z)$ are C^∞ bounded functions such that $W_{m,n} = W_{m,-n}$ and $|W_{m,n}| \le C^m z^{\max(2,m-1)}$ if $z = \max(\lambda,u,\nu)$.

After the integration of the "ultraviolet" field components, *i.e.* of $\psi^{(>0)}$, the problem is reduced to an essentially identical one but with a purely "infrared" propagator $g^{(\le 0)}$ and a new potential with terms of arbitrary degree in the fields. In order to perform the integration $\int\{\mathcal{D}\psi^{(\le 0)}e^{-\int dk\psi^{+(\le 0)}_{k,\sigma}g^{(\le 0)}(k)^{-1}\psi^{-(\le 0)}_{k,\sigma}}\}e^{-V^0(\psi^{\le 0})}$, the propagator $g^{(\le 0)}(k)$ is decomposed as[23]

$$g^{(\le 0)}(k) = \frac{1-C_0^{-1}(k)}{-ik_0 - E(\vec{k})} = \sum_{h=-\infty}^{0}\frac{f_h(k)}{-ik_0 - E(\vec{k})}$$
$$\equiv \sum_{h=-\infty}^{0} g^{(h)}(k) \tag{18}$$

with $f(t) = C_0^{-1}(\gamma t) - C_0^{-1}(t)$, and $f_h(k) \equiv f(\gamma^{-h}\sqrt{k_0^2 + (|\vec{k}|-p_F)^2})$. This allows us to write $\psi^{(\le 0)}_{k,\sigma} = \sum_{h=-\infty}^{0}\psi^{(h)}_{k,\sigma}$. The numbers h will be called *scales.*

Since $g^{(h)}(k)$ does not have good scaling properties (it contains an intrinsic scale length p_F^{-1}) it is convenient to write the propagator $g^{(h)}(k)$ as $g^{(h)}(k) = \sum_{\vec\omega=\pm 1} g^{(h)}_{\vec\omega}(k)$ with:

$$g^{(h)}_{\vec\omega}(k) = \chi(\vec\omega\vec{k})\frac{f_h(k)}{-ik_0 - \vec{k}^2 + p_F^2} \tag{19}$$

where $\chi(\vec{k})$ is the step function, *i.e.* $\chi(\vec{k}) = 0$ if $\vec{k} < 0$ and $\chi(\vec{k}) = 1$ otherwise. The χ-functions select the two "quasi-particles with momenta $\pm p_F$"[15] with propagator $g^{(h)}_{\vec\omega}(k)$; note that $g^{(h)}_1(k)$ and $g^{(h)}_{-1}(k)$ are compact support functions with *disjoint* supports. If $\vec\omega\vec{k} > 0$ we will write in the following $\vec{k} = \vec{k}' + \vec\omega p_F$, where $\vec{k}'$ is the "momentum measured from the Fermi surface" and we shall use the notation $k' = (\vec{k}', k_0)$. We can write $g^{(h)}_{\vec\omega}(k' + \vec\omega\pi) = \gamma^{-h}g_{\vec\omega}(\gamma^{-h}k') + \bar{g}^{(h)}(\gamma^{-h}k')$ with $\bar{g}^{(h)}(k')$ regular and weakly dependent on h and:

$$g_{\vec\omega}(k') = \frac{f(k_0^2 + (\vec{k}')^2)}{-ik_0 - \vec\omega\pi\vec{k}'}$$

This induces a decomposition of $\psi^{(h)}_{k,\sigma} = \sum_{\vec\omega=\pm 1}\psi^{\pm(h)}_{k'+\vec\omega p_F,\vec\omega,\sigma}$ and $\psi^{\pm(h)}_{k'+\vec\omega p_F,\vec\omega,\sigma}$ have a distribution which, "up to scaling", is essentially h independent *i.e.* the distribution of $\psi^{h,\pm}_{k'+\vec\omega p_F,\vec\omega,\sigma}$ is the same of $\gamma^{-h/2}\psi^{(h),\pm}_{\gamma^{-h}k'+\vec\omega p_F,\vec\omega,\sigma}$ up to corrections negligible in the $h\to-\infty$ limit. In other words, from the renormalization group analysis of the model with Hamiltonian eq.(1) "naturally" emerges a description in terms of two kinds of quasi-particles labeled by $\vec\omega = \pm 1$ with linear dispersion relation. The Fourier transform of $g^{(h)}_{\vec\omega}(k)$, called $g^{(h)}_{\vec\omega}(x)$, verifies the bound $|g^{(h)}_{\vec\omega}(x)| \le \frac{\gamma^h C_N}{1+(\gamma^h|x-y|)^N}$ for any $N > 1$, if C_N is a suitable constant.

A naive definition for the *effective potentials* (but, as we will see below, not the most suitable one) could be

$$e^{-V^h(\psi^{\le h})} = \tag{20}$$
$$\int\prod_{l=h+1}^{0}\{\mathcal{D}\psi^{(l)}e^{-\int dk'\psi^{+(l)}_{k',\sigma}f_h^{-1}G(k')^{-1}\psi^{-(l)}_{k',\sigma}}\}e^{-V^0(\psi^{(\le 0)})}$$

where, from now on, we denote by $\psi^{\pm(l)}_{k',\sigma}$ the vector $(\psi^{\pm(l)}_{k'+p_F,1,\sigma},\psi^{\pm(l)}_{k'-p_F,-1,\sigma})$ and

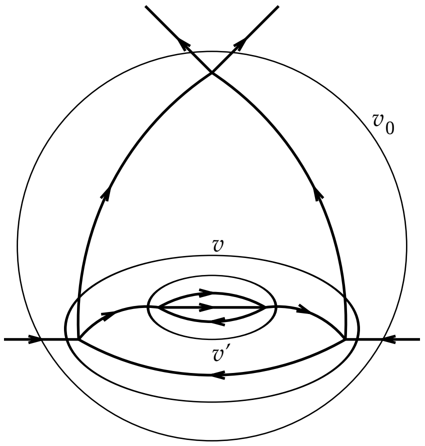
$$G(k')^{-1} = \tag{21}$$
$$\begin{pmatrix} (-ik_0 - \vec{k}'^2) - 2\pi\vec{k}' & 0 \\ 0 & (-ik_0 - \vec{k}'^2) + 2\pi\vec{k}' \end{pmatrix}$$

One can verify that V^h is given by the sum of terms of the form

$$V^h(\psi^{(\le h)}) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int\prod_{i=1}^{2m}dk'_i f^h_{n,m}(k'_1,\ldots,k'_{2m};z)\cdot$$
$$\prod_{i=1}^{m}\psi^{+(\le h)}_{k'_i+\vec\omega_i p_F,\sigma_i}\prod_{i=m+1}^{2m}\psi^{-(\le h)}_{k'_i+\vec\omega_i p_F,\sigma_i} \tag{22}$$
$$\delta(\sum_{i=1}^{m}(k'_i + \vec\omega_i p_F) - \sum_{i=m+1}^{2m}(k'_i + \vec\omega_i p_F) + 2n\pi)$$

and $f^h_{n,m}$, the *kernels* of the effective potential, are expressed by sum of suitable *Feynman diagrams.* A k-th order diagram contributing to $f^h_{n,m}$ can be obtained[15] from k graph-elements representing the addends in eq.(17), formed by vertices with emerging oriented "half-lines" with indices $\vec\omega_i$, k_i, h_i symbolizing the fields $\psi^{\pm(h_i)}_{k'_i+\vec\omega_i p_F,\vec\omega_i}$, by pairing the half lines with consistent orientation and same indices $h_i > h$ (contractions) in such a way that the resulting graph turns out to be connected; $2m$ half-lines with $h_i \le h$ are left not paired. The not paired lines are called *external lines.* To each paired line we associate a propagators $g^{h_i}_{\vec\omega_i}(k'_i + \vec\omega_i p_F)$ and to the vertices are associated the kernels in eq.(17); integrating the product of these factors over all the momenta k_i of the paired lines we obtain the *value* of the graph contributing to $f^h_{n,m}$, if the expression is multiplied by a suitable sign to take into account the Fermi statistic.

A maximal connected subset of lines with scales $\ge h_v$ is called *cluster* with scale h_v, see *e.g.* Ref. 15, and denoted by v. An inclusion relation can be established between the clusters, in such a way that the innermost clusters are the clusters with the higher scale, and so on; see the picture for an example of graphs with its clusters, pictured as boxes including the paired lines.



If $h'_v < h_v$ is the maximal among the scales of the lines entering in or coming out the cluster (called *external lines* of the cluster) one can verify, by a power counting argument that the generic k-order diagram is bounded by $C^k \varepsilon^k \prod_v \gamma^{-D_v(h_v - h'_v)}$ where C is a constant, $\varepsilon = \max(\lambda, u, \nu)$, the product runs over all the clusters v with scale h_v and $D_v > 0$ for each cluster except the clusters with four external lines, for which $D_v = 0$, and the clusters with two external lines, for which $D_v = -1$. Therefore the graphs with clusters with no four or two external lines admit a bound *uniform* in L, β[24] because the damping factor $\gamma^{-D_v(h_v - h'_v)}$, $D_v > 0$, allows to perform the summation over the intermediate scales *i.e.* over $h_v - h'_v$; while the graphs with two or four external lines do not admit a uniform in L, β. More precisely, the clusters v with two or four external lines which, if present in a graph, prevent to obtain a bound uniform in L, β are only the ones whose external lines verify the condition $\sum_i \varepsilon_i \vec{\omega}_i p_F + 2n\pi = 0$, if $\vec{\omega}_i, \varepsilon_i = \pm$ are the indices of the external lines, so that the momenta of the external lines can be *simultaneously* near the Fermi surface; otherwise, for h_v suitably small, the graphs containing clusters not verifying the above condition are vanishing for the compact support properties of the fields associated to the external lines.

We have therefore to set up a different perturbative expansion for computing $\mathcal{N}$ and this is done, in the renormalization group framework[15] by introducing a *localization operator* which is a linear operator acting on the kernels of the effective potential V^h and extracting its *relevant part* $\mathcal{L}V^h$. The localization operator is defined in the following way:

1. if $m > 4$

$$\mathcal{L}f^h_{n,m}(k'_1, \ldots, k'_m) = 0$$

2. if[25] $m = 4$:

$$\mathcal{L}f^h_{n,4}(k'_1, k'_2, k'_3, k'_4) = \delta_{(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega}_3 - \vec{\omega}_4)p_F + n2\pi, 0} f^h_{n,4}(0,0,0,0) \tag{23}$$

3. if $m = 2$

$$\mathcal{L}f^h_{n,2}(k'_1, k'_2) = \delta_{(\vec{\omega}_1 - \vec{\omega}_2)p_F + 2n\pi, 0} [f^h_{n,2}(0,0) + E(\vec{k}' + \vec{\omega}p_F)\partial_{\vec{k}'} f^h_{n,2}(0,0) + k^0 \partial_{k_0} f^h_{n,2}(0,0)] \tag{24}$$

The presence of the Kronecker deltas in the r.h.s. of eq.(23) (24) says that the only relevant processes involve fermions with momenta near the Fermi surface. The relevant part of the effective potential $\mathcal{L}V^h$ depends on the value of p_F and on the value of the spin. In the filled band case we find

$$\mathcal{L}V^h = \gamma^h n_h F^{\leq h}_\nu + \gamma^h s_h F^{\leq h}_\sigma + a_h F^{\leq h}_\alpha + z_h F^{\leq h}_z + + t_h F^{\leq h}_t + i_h F^{\leq h}_\vartheta + \sum_{i=1}^{6} \lambda_{h,i} F^{\leq h}_{\lambda,i} \tag{25}$$

where $v_h = (n_h, s_h, a_h, z_h, i_h, t_h, \lambda_{h,i})$ are called *running coupling constants* and

$$F^{\leq h}_i = \sum_{\vec{\omega}, \sigma} \int dk' \gamma_i \psi^{+(\leq h)}_{k' + \vec{\omega}p_F, \vec{\omega}, \sigma} \psi^{-(\leq h)}_{k' + \varepsilon_i \vec{\omega} p_F, \varepsilon_i \vec{\omega}, \sigma}$$

with $\gamma_\nu = 1$, $\varepsilon_\nu = 1$; $\gamma_\sigma = 1$, $\varepsilon_\sigma = -1$; $\gamma_\alpha = E(\vec{k}' + \vec{\omega}p_F)$, $\varepsilon_\alpha = 1$; $\gamma_\vartheta = E(\vec{k}' + \vec{\omega}p_F)$, $\varepsilon_\vartheta = -1$; $\gamma_z = -ik_0$, $\varepsilon_z = 1$; $\gamma_t = -ik_0$, $\varepsilon_t = -1$; moreover

$$F^{\leq h}_{\lambda,i}(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = \sum_{\sigma, \sigma'} \int \prod_{j=1}^{4} dk'_j \psi^{+(\leq h)}_{k'_1 + \vec{\omega}_1 p_F, \vec{\omega}_1, \sigma} \psi^{+(\leq h)}_{k'_2 + \vec{\omega}_2 p_F, \vec{\omega}_2, \sigma'} \psi^{-(\leq h)}_{k'_3 + \vec{\omega}_3 p_F, \vec{\omega}_3, \sigma'} \psi^{-(\leq h)}_{k'_4 + \vec{\omega}_4 p_F, \vec{\omega}_4, \sigma} \delta(\sum_i \varepsilon_i k'_i)$$

and, if the fermions are spinning,

$$\begin{aligned}
F^{\leq h}_{\lambda,1} &= \sum_{\vec{\omega}} F^{\leq h}(\vec{\omega}, -\vec{\omega}, \vec{\omega}, -\vec{\omega}) \\
F^{\leq h}_{\lambda,2} &= \sum_{\vec{\omega}} F^{\leq 0}(\vec{\omega}, -\vec{\omega}, -\vec{\omega}, \vec{\omega}) \\
F^{\leq h}_{\lambda,3} &= \sum_{\vec{\omega}} F^{\leq h}(\vec{\omega}, \vec{\omega}, -\vec{\omega}, -\vec{\omega}) \\
F^{\leq h}_{\lambda,4} &= \sum_{\vec{\omega}} F(\vec{\omega}, \vec{\omega}, \vec{\omega}, \vec{\omega}) \\
F^{\leq h}_{\lambda,5} &= \sum_{\vec{\omega}} F^{\leq h}(\vec{\omega}, \vec{\omega}, -\vec{\omega}, \vec{\omega}) \\
F^{\leq h}_{\lambda,6} &= \sum_{\vec{\omega}} F^{\leq h}(-\vec{\omega}, \vec{\omega}, \vec{\omega}, \vec{\omega})
\end{aligned} \tag{26}$$

If the fermions are spinless $F^{\leq h}_{\lambda,1} \equiv -F^{\leq h}_{\lambda,2}$, $F^{\leq h}_{\lambda,3} = F^{\leq h}_{\lambda,4} = F^{\leq h}_{\lambda,5} = F^{\leq h}_{\lambda,6} = 0$, and $g_{1,h} = -g_{2,h} = \lambda_h$[26]. In the not filled band case case one finds $s_h = i_h = t_h = 0$ and the relevant part of the effective potential is identical, if the fermions are spinless, to the relevant part of the effective potential of the Luttinger model (we will return on this in the last section).

By means of the localization operator we define a different perturbative expansion for computing $\mathcal{N}$. If $\mathcal{R} = 1 - \mathcal{L}$ and $C_h^{-1} \equiv C_h^{-1}(k') = \sum_{k=-\infty}^{h} f_k(k')$ we write:

$$\int \{\mathcal{D}\psi^{(\leq 0)} e^{-\int dk' \psi^{+(\leq 0)}_{k',\sigma} C_0 G^{-1} \psi^{-(\leq 0)}_{k',\sigma}}\} e^{-\mathcal{L}V^0(\psi^{(\leq 0)}) - \mathcal{R}V^0(\psi^{(\leq 0)})} = \tag{27}$$
$$\int \{\mathcal{D}\psi^{(\leq -1)} e^{-\int dk' \psi^{+(\leq -1)}_{k',\sigma} C_{-1} G^{-1} \psi^{-(\leq -1)}_{k',\sigma}}\} e^{-V^{-1}(\psi^{(\leq -1)})} =$$
$$\int \{\mathcal{D}\psi^{(\leq -1)} e^{-\int dk \psi^{+(\leq -1)}_{k',\sigma} C_{-1} G^{-1} \psi^{-(\leq -1)}_{k',\sigma}}\} e^{-\mathcal{L}V^{-1}(\psi^{(\leq -1)}) - \mathcal{R}V^{-1}(\psi^{(\leq -1)})} = \ldots$$

i.e. we organize each integration by writing the effective potential V^h as $\mathcal{L}V^h + \mathcal{R}V^h$. In this way V^h is a series (*renormalized expansion*) in the running coupling constants $v_k = (n_h, s_h, a_h, z_h, i_h, t_h, \lambda_{i,h})$, $k > h$ and v_h obeys to a recursive relation, called *Beta function*, $v_h = \beta_h(v_{h+1}, \ldots, v_0)$.

The effective potential is still given by a sum of Feynman graphs which differ with respect to others previously introduced because they are obtained contracting the half lines of vertices not only correspondent to the addends in eq.(17) but also to the addends in $\mathcal{L}V^k$, with $k \geq h$; moreover on each cluster v with two or four external lines the $\mathcal{R}$ operation acts so that its value $f_2^{h_v}$ or $f_4^{h_v}$ in the graph is replaced by $\mathcal{R}f_2^{h_v}$ and $\mathcal{R}f_4^{h_v}$; the effect is that the k-th order diagram is bounded by $C^k \varepsilon^k \prod_v \gamma^{-(D_v + d_v)(h_v - h'_v)}$, where everything is again defined as above but $\varepsilon = (\max_{k>h} |v_k|)$ and d_v is 1 for the clusters with four external lines and 2 for the cluster with two external lines, provided that the external lines of the clusters verify $\sum_i \varepsilon_i \vec{\omega}_i p_F + 2n\pi = 0$; in all the other cases $d_v = 0$. The summation over the intermediate scales can be performed and each graph in the renormalized expansion for the effective potential is bounded *uniformly* in h (note however that this is true as we implicitly assumed, in writing eq.(25), that the running coupling constants correspondent to $F^{\leq h}_{\nu}$ and $F^{\leq h}_{\sigma}$ are vanishing as $h \to -\infty$ *i.e.* we have written them as $\gamma^h \nu_h$ and $\gamma^h \sigma_h$).

With respect to the $\mathcal{R} = \text{Id}$ case, we have then in the estimates a factor $\gamma^{-(h_v - h'_v)}$ more corresponding to the clusters with four external lines and an extra factor $\gamma^{-2(h_v - h'_v)}$ for the clusters with two external lines. It is standard to check that the operation $\mathcal{R}$ produces these factors by writing explicitly its effect for instance on a cluster with four external lines:

$$\mathcal{R}f^{h_v}_{n,4}(k'_1, k'_2, k'_3, k'_4) = f^{h_v}_{n,4}(k'_1, k'_2, k'_3, k'_4) - f^{h_v}_{n,4}(0,0,0,0) = \int_0^1 dt \frac{\partial f^{h_v}_{n,4}(tk'_1, tk'_2, tk'_3, tk'_4)}{\partial t}$$

The derivative acting on $f^{h_v}_{n,4}$ has the effect that:

1. some propagator $g^i_{\vec{\omega}}(k')$ in the cluster v with $i \geq h_v$ is derivated and so it is bounded by $A\gamma^{-2i}$, if A is a constant matrix, so there is an extra factor $\gamma^{-i} \leq \gamma^{-h_v}$ in the estimates with respect to the $\mathcal{R} = \text{Id}$ case;
2. a factor k'_i, $i = 1, ..., 4$ is produced which has the effect that in the estimates there is an extra (respect to the $\mathcal{R} = \text{Id}$ case) factor $\leq \gamma^{h'_v}$.

Each graph in the renormalized expansion for the effective potential is then bounded uniformly in L, β[27]. Even more, estimating the fermionic expectations in the renormalized expansion for the effective potential by the *Grahm-Hadamard inequality*, one can prove the analyticity of the effective potential as a function of the running coupling constants in a small domain. This follows from Ref. 4 in which a renormalized expansion for the effective potential essentially identical to our one is discussed for the Gross-Neveu model. Of course the above procedure is a resummation of the previously, apparently divergent series: therefore the difficulties are now hidden in the running coupling constants. To be useful the new expansion for the effective potential requires knowledge of the h-dependence of the running coupling constants, and in particular that $\max_{k>h} |v_k|$ is so small that the series converge. The h-dependence of the running coupling constants can be derived by the study of the Beta function. *However we do not expect that* $\max_{k>h} |v_k|$ *is small* as:

1. we have assumed that $\gamma^h \nu_h$, $\gamma^h \sigma_h$ are vanishing as $h \to -\infty$. One can expect to obtain this for the running coupling constants correspondent to $F^{\leq h}_{\nu}$, by choosing in a suitable way the counterterm ν, see sec.(5), but there are no free parameters in the Hamiltonian for the running coupling constants correspondent to $F^{\leq h}_{\sigma}$.
2. Also the *marginal* couplings give problems. In fact by a second order computation
$$a_{h-1} = a_h + \beta_1 \lambda_h^2 \qquad z_{h-1} = z_h + \beta_1 \lambda_h^2$$
$$\lambda_{h-1} = \lambda_h \tag{28}$$
with $\beta_1 > 0$, so that the vanishing of the second order beta function for λ_h has the effect that λ_h is constant in this approximation and therefore a_h, z_h grows more and more as $h \to -\infty$. One could hope that the third order beta function is not vanishing and negative, but indeed an explicit computation shows that it has the effect of increasing λ_h[28].

IV. ANOMALOUS SCALING

The fact that, at least by a second order computation, the running coupling constants seem to be unbounded (or at least they grow so large to go beyond the possibility of our non-perturbative approach) is an indication that the Schwinger function decay for large distances is different in the free or interacting theory. It is possible to extend the methods followed so far to a more general approach which can take into account a possible modification of the Schwinger function behavior with respect to the free case. Such new behavior is called "anomalous scaling" and it was used for the first time in Ref. 29 for the ultraviolet problem in the ϕ^4_3 model, and among other applications in Ref. 30, 17, 18, 5, 7, 8 for the study of several non relativistic one dimensional Fermi systems and in Ref. 15 for the Bose condensation problem.

The idea is to integrate over the fields on each scale by inserting the terms of the relevant part of the effective potential, that in the previous approach causes the uncontrollable growth of the running coupling constants, in the integration, so taking them into account by a change in the propagator. For simplicity of notations we discuss the spinless case, but everything in this section holds for any value of σ.

We write, setting $Z_0 = 1$:

$$\mathcal{N} = \int \{\mathcal{D}\psi^{(\le 0)} e^{-\int dk' \psi^{(\le 0)}_{k',\sigma} C_0 Z_0 G(k')^{-1} \psi^{(\le 0)}_{k',\sigma}}\} e^{-V^0(\psi^{(\le 0)})} = \tag{29}$$

$$\int \{\mathcal{D}\psi^{(\le 0)} e^{-\int dk' \psi^{(\le 0)}_{k',\sigma} Z_{-1}(k') C_0 \mathcal{G}^{(-1)}(k')^{-1} \psi^{(\le 0)}_{k',\sigma}}\} e^{-\tilde{V}^0(\psi^{(\le 0)})}$$

where, if $\mathcal{L}V^0$ is given by eq.(25)

$$\tilde{V}^0 = \mathcal{L}\tilde{V}^0 + (1-\mathcal{L})V^0$$

and

$$\mathcal{L}\tilde{V}^0 = n_0 F^{\le 0}_\nu + (a_0 - z_0) F^{\le 0}_\alpha + t_0 F^{\le 0}_t + + i_0 F^{\le 0}_\vartheta + \lambda_0 F^{\le 0}_\lambda \tag{30}$$

Moreover

$$\mathcal{G}^{(-1)}(k')^{-1} = \begin{pmatrix} (-ik_0 - \vec{k}'^2 - 2\pi\vec{k}') & \sigma_{-1}(k') \\ \sigma_{-1}(k') & (-ik_0 - \vec{k}'^2 + 2\pi\vec{k}') \end{pmatrix} \tag{31}$$

and $Z_{-1}(k') = Z_0 + C_0^{-1} Z_0 z_0$, $Z_{-1}(k')\sigma_{-1}(k') = C_0^{-1} s_0$. Now we perform the integration with respect to $\psi^{(0)}$ writing

$$\mathcal{N} = \int \{\mathcal{D}\psi^{(\le -1)} e^{-\int dk' \psi^{+(\le -1)}_{k',\sigma} C_{-1} Z_{-1} \mathcal{G}^{(-1)}(k')^{-1} \psi^{-(\le -1)}_{k',\sigma}}\} \int \{\mathcal{D}\psi^{(0)} e^{-\int dk' \psi^{+(0)}_{k',\sigma} \tilde{f}_0^{-1} Z_{-1} \mathcal{G}^{(-1)}(k')^{-1} \psi^{-(0)}_{k',\sigma}}\} e^{-\mathcal{L}\hat{V}^0(\sqrt{Z_{-1}}\psi^{(\le 0)}) - \mathcal{R}\hat{V}^0(\sqrt{Z_{-1}}\psi^{(\le 0)})} \tag{32}$$

where $\tilde{f}_0 \equiv \tilde{f}_0(k') = Z_{-1}[\frac{C_0^{-1}(k')}{Z_{-1}(k')} - \frac{C_{-1}^{-1}(k')}{Z_{-1}}]$, $Z_{-1} \equiv Z_{-1}(0)$, $\mathcal{R}\hat{V}^0(\psi) = \mathcal{R}V^0(\sqrt{\frac{Z_0}{Z_{-1}}}\psi)$ and

$$\mathcal{L}\hat{V}^0(\psi) = \nu_0 F^{\le 0}_\nu + \delta_0 F^{\le 0}_\alpha + \tau_0 F^{\le 0}_t + \vartheta_0 F^{\le 0}_\vartheta + g_0 F^{\le 0}_\lambda \tag{33}$$

with $\nu_0 = \frac{Z_0}{Z_{-1}} n_0$, $\delta_0 = \frac{Z_0}{Z_{-1}}(a_0 - z_0)$, $\tau_0 = \frac{Z_0}{Z_{-1}} t_0$ $\vartheta_0 = \frac{Z_0}{Z_{-1}} i_0$, $g_0 = (\frac{Z_0}{Z_{-1}})^2 \lambda_0$.

In general once the fields $\psi^0, \dots, \psi^{h+1}$ have been integrated we have to evaluate:

$$\mathcal{N} = \int \{\mathcal{D}\psi^{(\le h)} e^{-\int dk' C_h Z_h \psi^{+(\le h)}_{k',\sigma} \mathcal{G}^{(h)}(k')^{-1} \psi^{-(\le h)}_{k',\sigma}}\} e^{-V^h(\sqrt{Z_h}\psi^{(\le h)})} \tag{34}$$

where $\mathcal{L}V^h = n_h F^{\le h}_\nu + s_h F^{\le h}_\sigma + a_h F^{\le h}_\alpha + z_h F^{\le h}_z + i_h F^{\le h}_t + t_h F^{\le h}_\vartheta + \lambda_h F^{\le h}_\lambda$ and

$$\mathcal{G}^{(h)}(k')^{-1} = \begin{pmatrix} (-ik_0 - \vec{k}'^2 - 2\pi\vec{k}') & \sigma_h(k') \\ \sigma_h(k') & (-ik_0 - \vec{k}'^2 + 2\pi\vec{k}') \end{pmatrix} \tag{35}$$

We write eq.(34) as:

$$\int \{\mathcal{D}\psi^{(\le h)} e^{-\int dk' \psi^{+(\le h)}_{k',\sigma} C_h(k') Z_{h-1}(k') \mathcal{G}^{(h-1)}(k')^{-1} \psi^{-(\le h)}_{k',\sigma}}\} e^{-\tilde{V}^h(\sqrt{Z_h}\psi^{(\le h)})} \tag{36}$$

where $\tilde{V}^h = \mathcal{L}\tilde{V}^h + (1-\mathcal{L})V^h$, $\mathcal{L}\tilde{V}^h = n_h F^{\le h}_\nu + (a_h - z_h) F^{\le 0}_\alpha + t_h F^{\le 0}_t + i_h F^{\le h}_\vartheta + \lambda_h F^{\le h}_\lambda$ and

$$Z_{h-1}(k') = Z_h + C_h^{-1}(k') Z_h z_h \tag{37}$$

$$Z_{h-1}(k')\sigma_{h-1}(k') = Z_h \sigma_h(k') + Z_h C_h^{-1}(k') s_h$$

Now one can perform the integration respect to $\psi^{(h)}$ writing eq.(36) as

$$\int \{\mathcal{D}\psi^{(\le h-1)} e^{-\int dk' \psi^{+(\le h-1)}_{k',\sigma} C_{h-1} Z_{h-1} \mathcal{G}^{(h-1)}(k')^{-1} \psi^{-(\le h-1)}_{k',\sigma}}\} \int \{\mathcal{D}\psi^{(h)} e^{-\int dk' \psi^{+(h)}_{k',\sigma} \tilde{f}_h Z_{h-1} \mathcal{G}^{(h-1)}(k')^{-1} \psi^{-(h)}_{k',\sigma}}\} e^{-\mathcal{L}\hat{V}^h(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \mathcal{R}\hat{V}^h(\sqrt{Z_{h-1}}\psi^{(\le h)})} \tag{38}$$

where $\tilde{f}_h \equiv \tilde{f}_h(k') = Z_{h-1}[\frac{C_h^{-1}(k')}{Z_{h-1}(k')} - \frac{C_{h-1}^{-1}(k')}{Z_{h-1}}]$, $Z_{h-1} \equiv Z_{h-1}(0)$, $\mathcal{R}\hat{V}^h(\psi) = \mathcal{R}V^h(\sqrt{\frac{Z_h}{Z_{h-1}}})$ and

$$\mathcal{L}\hat{V}^h = \gamma^h \nu_h F^{\le h}_\nu + \delta_h F^{\le h}_\alpha + \tau_h F^{\le h}_t + \vartheta_h F^{\le h}_\vartheta + g_h F^{\le h}_\lambda \tag{39}$$

with

$$\gamma^h \nu_h = \frac{Z_h}{Z_{h-1}} n_h \quad \delta_h = \frac{Z_h}{Z_{h-1}}(a_h - z_h) \quad \tau_h = \frac{Z_h}{Z_{h-1}} t_h$$

$$\vartheta_h = \frac{Z_h}{Z_{h-1}} i_h \quad g_h = (\frac{Z_h}{Z_{h-1}})^2 \lambda_h \tag{40}$$

and, if:

$$\mathcal{G}^{h-1}(k') = \frac{1}{A_{h-1}(k')} \cdot \begin{pmatrix} (-ik_0 - \vec{k}'^2 + 2\pi\vec{k}') & -\sigma_{h-1}(k') \\ -\sigma_{h-1}(k') & (-ik_0 - \vec{k}'^2 - 2\pi\vec{k}') \end{pmatrix}$$

$$A_{h-1}(k') = \left(\left(-ik_0 + \vec{k}'^2\right)^2 - \left(2\pi\vec{k}'\right)^2 - \sigma_{h-1}(k')^2\right)$$

the *anomalous propagator* corresponding to the second integration in eq.(38) is:

$$g^h(x-y) = \frac{1}{Z_h}\int dk e^{ik(x-y)}\tilde{f}_h(k)\mathcal{G}^{(h-1)}(k) \tag{41}$$

We perform the integration with respect to the field $\psi^{(h)}$ obtaining

$$\int \{\mathcal{D}\psi^{(\leq h-1)} e^{-\int dk'\psi^{+(\leq h-1)}_{k',\sigma}C_{h-1}(k')Z_{h-1}\mathcal{G}^{(h-1)}(k)^{-1}\psi^{+(\leq h-1)}_{k',\sigma}}\} e^{-V^h(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})} \tag{42}$$

and the procedure can be iterated. Note that $\sigma_h(k')$ is "weakly" dependent on k' as

$$\sigma_{h-1}(k') = \frac{\sum_{j=h+1}^{0} s_h + C_h^{-1}(k')s_h}{1 + C_h^{-1}(k')z_h}$$

The effective potential $V^h(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})$ is given by a sum of Feynman graphs similar to the one introduced in the preceding section for the renormalized expansion of the effective potential, with the difference that to the paired lines with scale k joining two vertices with scales k_1 and k_2, the factor $\sqrt{Z_{k_1}Z_{k_2}}g^{(k)}$, if $g^{(k)}$ is the anomalous propagator eq.(41), is associated; moreover in $\mathcal{L}V^k$ there are no vertices correspondent to the running coupling constants which in the previous approach grows, so affecting the convergence of the series.

If $\sigma_h \equiv \sigma_h(0)$, one can check[8] that the anomalous propagator eq.(41) is bounded by, for any $N > 1$:

$$|g^{(h)}(x-y)| \leq A \max_{i=0,1,2}\left|\frac{\sigma_h}{\gamma^h}\right|^i \frac{\gamma^h}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N} \tag{43}$$

if A is a suitable constant matrix and C_N is a constant. In the not filled band case[7] one introduces an anomalous scaling integration *identical* to the one discussed so far with the *only* difference that $\vartheta_h = \tau_h = \sigma_h \equiv 0$, see the considerations after eq.(26). On the other hand, the anomalous propagator obeys to the same bound eq.(43), with $\max_{i=0,1,2}(\left|\frac{\sigma_h}{\gamma^h}\right|^i)$ replaced by a constant not depending on h. In the not filled band case it was proved in Ref. 5, 7 that, if

$$\max_{k\geq h}|\nu_k||\bar{v}_k| = \bar{\varepsilon}_h \qquad \max_{k\geq h}\left|\frac{Z_k}{Z_{k-1}}\right| \leq e^{C\bar{\varepsilon}_h^2} \tag{44}$$

with $\bar{v}_k = (\delta_h, \tau_h, \vartheta_h, g_h)$, the k-order contributions to the kernels of the effective potential V^{h-1} and of the Beta function β^{h-1} is bounded by $C^k\bar{\varepsilon}_h^k$; the same proof holds in this case (up to "trivial" changes) and it allows us to conclude that the k-order contributions to the kernels of the effective potential or to the Beta function is bounded by $C^k\tilde{C}_h^k\bar{\varepsilon_h}^k$, if

$$\tilde{C}_h = \max_{\substack{i=0,1,2\\ k\geq h}}\left|\frac{\sigma_k}{\gamma^k}\right|^i$$

This bound does not depend on L and we can remove the *infrared cut-off i.e.* we can take the limit $L \to -\infty$ on the effective potential.

V. THE FLOW OF THE RENORMALIZATION GROUP

Contrary to the preceding sections, in the following the assumption of *spinless* fermions is crucial. We have seen at the end of sec.(4) that, given a constant $\tilde{C}$[31], *if* $\tilde{C}_h \leq \tilde{C}$ the kernels of the effective potential V^{h-1} and the Beta function β^{h-1} are analytic as functions of their arguments, if $\bar{\varepsilon}_h < \varepsilon < \frac{1}{\tilde{C}C}$ and $\bar{\varepsilon}_h$ is defined in eq.(44). Of course there is no reason for which $\tilde{C}_h \leq \tilde{C}$ for any h so we call $h^* \equiv \tilde{h}(\tilde{C})$ the scale such that $\tilde{C}_{h^*} \leq \tilde{C}$ and $\tilde{C}_{h^*-1} > \tilde{C}$; the scale h^* can be computed once we know the h-dependence of σ_h. Then, if for $h \geq h^*$ it holds that $\bar{\varepsilon}_h < \varepsilon$ we have that the kernels of the effective potential V^{h-1} and the Beta function β^{h-1} are well defined for $h \geq h^*$; to prove this, and compute h^* we need informations on the h-dependence of the running coupling constants, what is provided by the study of the Beta function.

It is possible to choose the counterterm ν so that $|\nu_h| < \varepsilon$ for any $0 \geq h \geq h^*$; in fact, given any sequence of running coupling constants verifying $\max_{i,k\geq h}|\bar{v}_k| \leq \varepsilon$, $\max_{k\geq h}|\frac{Z_k}{Z_{k-1}}| \leq e^{\beta_1\varepsilon}$, ν_h obeys to the equation $\nu_{h-1} = \gamma\nu_h + \beta_\nu^h$, with $|\beta_\nu^h| < K\varepsilon^2$, if K is a constant. One obtains that $\nu_h = \gamma^{-h}(\nu_0 + \sum_{j=h+1}^0 \gamma^{j-1}\beta_\nu^j)$ so that, choosing ν such that ν_0 satisfies $\nu_0 + \sum_{j=h^*}^0 \gamma^{j-1}\beta_\nu^j = 0$ of course $\nu_{h^*} = 0$ and $|\nu_h| \leq \varepsilon$ for any $h \geq h^*$. Of course the value of ν_0 so that $|\nu_h| \leq \varepsilon$ for any $h \geq h^*$ is not unique; from the value of h^* computed in eq.(48) it is clear that, if $\bar{\nu}_0$ is the value such that $\nu_{h^*} = 0$, any $\nu_0 = \bar{\nu}_0 + O(u^2)$ has the effect that $|\nu_h| \leq \varepsilon$ for any $h \geq h^*$.

It is convenient to write the anomalous propagator eq.(41) as[8]:

$$g^h_{\vec{\omega},\vec{\omega}}(x-y) = g^h_{\vec{\omega},L}(x-y) + C^h_{1,\vec{\omega}}(x-y) + C^h_{2,\vec{\omega}}(x-y)$$

$$g^h_{\vec{\omega},L}(x-y) = \int \frac{dk'}{(2\pi)^2}\frac{e^{ik'x}}{Z_h}\frac{f(\gamma^{-2h}(k_0^2+\vec{k}'^2))}{-ik_0 - 2\pi\vec{\omega}\vec{k}'} \tag{45}$$

where $g^{(h)}_{\vec\omega,L}(x-y)$ is just the propagator "at scale h" of the Luttinger model[18] and $|g^{(h)}_{\vec\omega,L}(x-y)| \leq B\frac{\gamma^h}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$, for a suitable constant B. Moreover $|C_1(x-y)| \leq B\frac{\gamma^{2h}}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$, $|C_2(x-y)| \leq B\frac{\gamma^h}{Z_h}\left(\frac{\sigma_h}{\gamma^h}\right)^2\frac{C_N}{1+(\gamma^h|x-y|)^N}$ and $|g^{(h)}_{\vec\omega,-\vec\omega}(x-y)| \leq B\frac{\gamma^h}{Z_h}\frac{\sigma_h}{\gamma^h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$. This decomposition of the propagator will allow us to extract in the Beta function a part coinciding with the Luttinger model Beta function. In fact we can write, from eq.(31)(35):

$$\begin{aligned}
g_{h-1} &= g_h + G^{1,h}_\lambda + G^{2,h}_\lambda + \gamma^h R^h_\lambda \\
\sigma_{h-1} &= \sigma_h + G^{1,h}_\sigma + \gamma^h R^h_\sigma \\
\delta_{h-1} &= \delta_h + G^{1,h}_\delta + G^{2,h}_\delta + \gamma^h R^h_\delta \\
\tau_{h-1} &= \tau_h + G^{2,h}_\tau + \gamma^h R^h_\tau \\
\vartheta_{h-1} &= \vartheta_h + G^{2,h}_\vartheta + \gamma^h R^h_\vartheta \\
\frac{Z_{h-1}}{Z_h} &= 1 + G^{1,h}_z + G^{2,h}_z + \gamma^h R^h_z
\end{aligned} \tag{46}$$

where:

1. $G^{1,h}_\lambda \equiv G^{1,h}_\lambda(g_h,\delta_h;...;g_0,\delta_0)$,
   $G^{1,h}_\delta \equiv G^{1,h}_\delta(g_h,\delta_h;...;g_0,\delta_0)$ and
   $G^{1,h}_z \equiv G^{1,h}_z(g_h,\delta_h;...;g_0,\delta_0)$ are given by series of terms involving only the "Luttinger model" part of the propagator $g^k_{\vec\omega,L}(x-y)$, $k\geq h$, see eq.(45).

2. $G^{1,h}_\sigma$, $G^{2,h}_\lambda, G^{2,h}_\delta, G^{2,h}_z, G^{2,h}_\tau, G^{2,h}_\vartheta$ depend on all the running coupling constants and are given by a series of terms involving at least a propagator $C^k_{2,\vec\omega}(x-y)$ or $g^{(k)}_{\vec\omega,-\vec\omega}(x-y)$, $k\geq h$.

3. R^h_i, $i=\lambda,z,\sigma,\delta,\tau,\vartheta$ depend on all the running coupling constants and are given by a series of terms involving at least a propagator $C^k_{1,\vec\omega}(x-y)$, $k\geq h$.

We have not written the Beta function for ν_h, as we known that with the right choice of ν we have that $|\nu_h|\leq\varepsilon$ for any $h\geq h^*$.

The Beta function generates a recursion that is a *short memory* dynamical system in the sense that is a set of equations of the form $v_{h-1}=\beta_h(v_h,v_{h+1},\ldots,v_0)$ which behaves "essentially" as a system without memory $v_{h-1}=\beta_h(v_h,v_h,\ldots,v_h)$, (this is a consequence of the convergence of the Beta function as function of its arguments[5]). Even more, the convergence of eq.(46) allows to say that the lowest non zero terms determinate the evolution of the running coupling constants.

Let we call $\{G^h\}_2$ the second order contribution to the beta function with equal arguments, $G^h \equiv G^h(v_h,\ldots,v_h)$. Then by an explicit computation[32]

$$\{G^{1,h}_\lambda\}_2 = 0 \qquad \{G^{1,h}_\delta\}_2 = 0$$

$$\{G^{1,h}_z\}_2 = \beta_3 g_h^2, \qquad \{G^{1,h}_\sigma\}_2 = -\beta_1 g_h\sigma_h$$

$$\{G^{2,h}_\tau\}_2 = \beta_4 g_h\frac{\sigma_h}{\gamma^h}, \quad \{G^{2,h}_\tau\}_2 = \beta_5 g_h\frac{\sigma_h}{\gamma^h}$$

with $\beta_1,\beta_2,\beta_3,\beta_4>0$.

The fact that $\{G^{1,h}_\lambda\}_2=0$, $\{G^{1,h}_\delta\}_2=0$ could generate a problem as one cannot exclude *a priori* there there is some non-vanishing term at some large order, and the evolution of the running coupling constants will depend critically on this unknown term. Fortunately it is possible to prove that:

$$\begin{aligned}
\lim_{h\to-\infty} G^{1,h}_\lambda(g,\delta;\ldots;g,\delta) &= 0 \\
\lim_{h\to-\infty} G^{1,h}_\delta(g,\delta;\ldots;g,\delta) &= 0
\end{aligned}$$

i.e. it is vanishing at *all* orders. In fact $G^{1,h}_\lambda$, $G^{1,h}_\delta$ coincide, up to terms vanishing as $h\to-\infty$ as $O(\gamma^h)$, with the correspondent quantities of the Luttinger model , and, by using the properties of the exact solution[10,12] one can prove that it is vanishing[15,18,5,7,33]

Finally, as $|R^h_i| \leq K\varepsilon_h^2$, $i=\lambda,\delta,\sigma,z$ and $|G^{2,h}_\lambda|, |G^{2,h}_\delta|, |G^{2,h}_z|, |G^{2,h}_\tau|, |G^{2,h}_\vartheta| \leq K\varepsilon_h\tilde C_h$ one finds, see (B.M.2):

$$\begin{aligned}
&-c_1\lambda^2 < |g_{h-1}-g_0| < c_2\lambda^2 \\
&\lambda\beta_1 c_4 h \leq \left|\log\left(\frac{\sigma_{h-1}}{\sigma_0}\right)\right| \leq \lambda\beta_1 c_3 h \\
&-\beta_3 c_3\lambda^2 h \leq \log(|Z_{h-1}|) \leq -\beta_3 c_4\lambda^2 h \\
&-c_1|\lambda| < |\tau_{h-1}-\tau_0| < c_2|\lambda| \\
&-c_1|\lambda| < |\vartheta_{h-1}-\vartheta_0| < c_2|\lambda| \\
&-c_1|\lambda| < |\delta_{h-1}-\delta_0| < c_2|\lambda|
\end{aligned} \tag{47}$$

for suitable constants $c_1,c_2>0$ and $c_4>c_3>0$, *i.e.* the flow is essentially described by the second order truncation of the beta function. We use that $\sum_{h=h^*}^0 \tilde C_h \leq K$, if K is a constant.

From eq.(47) it follows that it possible to choose λ small enough so that the series for V^{h-1} and β^{h-1} , if $h\geq h^*$, are convergent; moreover from eq.(47) it is possible to obtain an upper and lower bound for h^*:

$$c_3\frac{\log_\gamma(\tilde C^{-1}u)}{1-\lambda\beta_1} \leq h^* \leq c_4\frac{\log_\gamma(\tilde C^{-1}u)+1}{1-\lambda\beta_1} \tag{48}$$

The above analysis is performed by approximating the anomalous propagator eq.(41) with the Luttinger model propagator by eq.(45). This approximation is of course not reasonable for large $|h|$ corresponding to momenta negligible with respect to the σ_h term due to the periodic potential. But for the scales $h<h^*$ the bound, see (B.M.2):

$$|g^{(<h^*)}(x-y)| \leq A\frac{\gamma^{h^*}}{Z_{h^*}}\left(\frac{\gamma^{h^*}}{\sigma_{h^*}}\right)\frac{C_N}{\sigma^N|x-y|^N} \tag{49}$$

holds and, from eq.(47),(48), $\left(\frac{\gamma^{h^*}}{\sigma_{h^*}}\right) \leq \frac{K}{\tilde{C}}$, if K is a constant. In other words the propagator $g^{(\leq h^*)}_{\vec{\omega},\vec{\omega}'}(x-y)$ obeys to the same bound of $g^{(h)}_{\vec{\omega},\vec{\omega}'}(x-y)$ for $h \geq h^*$; our choice of h^* is made just to obtain this. The integration of the scale between $-\infty$ and h^* is equivalent to the integration of a single scale in a not filled band theory, so that for the same considerations at the end of sec.(4) also the series expressing

$$\int \{\mathcal{D}\psi^{(<h^*)} e^{-\int dk \psi^{+(<h^*)}_{k,\sigma} C_{h-1}(k) Z_{h-1} \mathcal{G}^{(h^*)}(k)^{-1} \psi^{+(<h_*)}_{k,\sigma}}\} \\ e^{-V^{h^*}(\sqrt{Z_{h^*}})\psi^{(\leq h^*)})} \tag{50}$$

is convergent. The n-th order of the series in the running coupling constants for the effective potential for $h > h^*$ is bounded by $(\bar{\varepsilon}_{h^*})^n \tilde{C}^n C_1^n$ while the n-th order term of the series eq.(50) is bounded by $(\bar{\varepsilon}_{h^*-1})^n (\frac{C_2}{\tilde{C}})^n$ so that there is a non ambiguous way to fix $\tilde{C}$ so that $\bar{\varepsilon}_{h^*}$ is the largest possible.

The Schwinger function eq.(2) admits a perturbative expansion similar to the one of the partition function, whose convergence follows from the partition function expansion convergence[5] for a detailed discussion of the ("standard") analysis required; it holds that

$$S(x,y) = S^{u.v.}(x,y) + \sum_{h=h^*}^{0} \sum_{\vec{\omega}_1,\vec{\omega}_2} e^{i\pi(\vec{\omega}_1\vec{x}-\vec{\omega}_2\vec{y})} \\ (g^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y) + \bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)) \tag{51}$$

where we call $g^{(\leq h^*)}(x-y)$ simply $g^{(h^*)}(x-y)$ and the first addend is bounded by $\frac{C_N}{1+|x-y|^N}$; $\bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)$ is given by a sum of Feynman graphs similar to the ones contributing to the effective potential, see sec.3, with the difference that:

1. they have two external lines, to which the propagators $g^{(h_1)}_{\vec{\omega}_1}(x-x')$ or $g^{(h_2)}_{\vec{\omega}_1}(y-y')$ are associated, with $h_1, h_2 \geq h$ and if x', y' are respectively the coordinates of the vertex the external lines is entering in or coming out. Note that $\bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)$ is translation invariant; the reason is that, for the compact support properties (in momentum space) of the propagators, if $\vec{k}_1, \vec{k}_2$ are the momenta associated to the external lines, then $\vec{k}_1 - \vec{k}_2 = 2np_F$, with $n = 0, \pm 1$.

2. h is the smallest scale of the propagators contributing to $\bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)$ (and not, as for the graphs for the effective potential, the scale of the external lines).

3. No $\mathcal{R}$ acts on clusters containing the external lines.

Note that σ_{h^*}, Z_{h^*} depend on $\tilde{C}$, but it is easy to check that they can be written as $\sigma_{h^*} = \bar{\sigma}_{h^*}(1+\lambda f_1)$, $Z_{h^*} = \bar{Z}_{h^*}(1+\lambda f_2)$, with $|f_1|, |f_2|$ bounded by some constant and $\bar{\sigma}_{h^*}$, $\bar{Z}_{h^*}$ *independent* on $\tilde{C}$. Let us define

$$\eta_1 = -\frac{\log(\bar{Z}_{h^*})}{\log u} \qquad 1+\eta_2 = \frac{\log(\bar{\sigma}_{h^*})}{\log u} \tag{52}$$

which are respectively $O(\lambda^2)$ and $O(\lambda)$, with $\mathrm{sign}(\eta_2) = \mathrm{sign}(\lambda)$.

One can check, repeating the analysis in (B.G.P.S.) up to "trivial" changes, that $|\bar{S}^{(h)}_{\vec{\omega},\vec{\omega}}(x-y)| \leq A\tilde{\varepsilon}\frac{\gamma^h}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$ and $|\bar{S}^{(h)}_{\vec{\omega},-\vec{\omega}}(x-y)| \leq A\tilde{\varepsilon}\frac{\sigma_h}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$, if $\tilde{\varepsilon} = \max(u, u^{1+\eta_1}, |\lambda|)$; the extra factor $\frac{\sigma_h}{\gamma^h}$ in the second bound follows from the fact that in the graph contributing to $\bar{S}^{(h)}_{\vec{\omega},-\vec{\omega}}(x-y)$ there is at least a non diagonal propagator, which obeys, see the lines after eq.(45), to a bound similar to the diagonal propagator but with a factor more $\frac{\sigma_k}{\gamma^k} \leq \frac{\sigma_h}{\gamma^h}$, $k \geq h$.

We now prove the first statement of the theorem. The bounds eq.(9)(10) can be obtained by eq.(51), as for $\gamma^{-h_x-1} < |x-y| \leq \gamma^{-h_x}$, $h_x > h^*$, if $|g^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y) + \bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)| \leq \bar{g}^{(h)}(x-y)$:

$$\sum_{h=0}^{h^*} \bar{g}^{(h)}(x-y) \leq A_1\left[\sum_{h=h^*}^{h_x-1}\frac{\gamma^h}{Z_h} + \sum_{h=h_x}^{0}\frac{\gamma^h}{Z_h}\frac{C_N}{\gamma^{Nh}|x-y|^N}\right]$$

$$\leq A_2\gamma^{h_x(1-\eta_3)}$$

while for $|x-y| \geq \sigma^{-1}_{h^*}$:

$$\sum_{h=0}^{h^*} \bar{g}^{(h)}(x-y) \leq \frac{C_N}{|x-y|^N}\sum_{h=h^*+1}^{1}\frac{\gamma^{-(N-1)h}}{Z_h} \\ \leq A_2\frac{\gamma^{h^*}}{Z_{h^*}}\frac{C_N}{(\gamma^{h^*}|x-y|)^N} \tag{53}$$

The second statement of the theorem follows noting that, from the theory of the Bloch waves, $\phi(p_F^+, \vec{x}) = \cos(p_F\vec{x}) + O(u)$ and $\phi(p_F^-, \vec{x}) = i\sin(p_F\vec{x}) + O(u)$ so that the occupation number discontinuity eq.(5) can be written as:

$$Z^{-1} = \frac{1}{L}\int d\vec{x}d\vec{y}[\cos(p_F(\vec{x}+\vec{y}) + r(u)]S^{L,\beta}(x,y)$$

with $r(u) = O(u)$. By inserting in the above expression eq.(51) we obtain several terms that we can bound in the following way, if K_i are positive constants:

$$\frac{r(u)}{L}\int d\vec{x}d\vec{y}\sum_{h=h^*}^{0}\sum_{\vec{\omega}_1,\vec{\omega}_2}|g^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y) + \bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y))| \leq \\ K_1 u\sum_{h=h^*}^{0}\frac{1}{Z_h} \leq K_2 u\log u \leq K_3 u^{\eta_1} \tag{54}$$

$$\frac{1}{L}\int d\vec{x}d\vec{y}\cos(p_F(\vec{x}+\vec{y}))\sum_{h=h^*}^{0}\sum_{\vec{\omega}_1,\vec{\omega}_2}|\bar{S}^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y))|\leq$$

$$K_4\sum_{h=h^*}^{0}\frac{\sigma_h}{Z_h\gamma^h}\leq\tilde{\varepsilon}K_5 u^{\eta_1} \tag{55}$$

in which we take into account that in the sum only the terms $\vec{\omega}_1=-\vec{\omega}_2$ survive;

$$\frac{1}{L}\int d\vec{x}d\vec{y}\cos(p_F(\vec{x}+\vec{y}))g^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)= \tag{56}$$

$$\int dk_0\sum_{h=h^*}^{0}\tilde{f}_h(0,k_0)\frac{1}{Z_h}(\mathcal{G}_h(0,k_0)^{-1})_{\vec{\omega},-\vec{\omega}}$$

verifies the bound eq.(12); finally:

$$\frac{1}{L}\int d\vec{x}d\vec{y}[\cos(p_F(\vec{x}+\vec{y})+r(u)]\int dk S^{u.v.}(x,y)\leq K_7 u \tag{57}$$

Let us prove the last statement of the theorem. By diagonalizing the quadratic form $\hat{g}^{(h)}(x,y)=\sum_{\vec{\omega}_1,\vec{\omega}_2}e^{i\pi(\vec{\omega}_1\vec{x}-\vec{\omega}_2\vec{y})}g^{(h)}_{\vec{\omega}_1,\vec{\omega}_2}(x-y)$ it is possible to see that:

$$\hat{g}^{(h)}(x,y)=\int dk' e^{ik'(x-y)}\frac{\tilde{f}_h(k')}{Z_h}[\frac{F_{xy}(k',\sigma_h)}{A+B}+$$

$$+\frac{F_{xy}(-k',-\sigma_h)}{A-B}]$$

where

$$F_{xy}(k',\sigma_h)=\hat{\phi}(k',\vec{x},\sigma_h)\hat{\phi}(k',-\vec{y},\sigma_h)$$

$$\hat{\phi}(k',\vec{x},\sigma_h)=\frac{1}{\sqrt{2B}}[\sqrt{B-C}e^{ip_F\vec{x}}$$

$$-\frac{e^{-ip_F\vec{x}}}{\sqrt{B-C}}]$$

and $A=-ik_0+\vec{k}'^2$, $B=\sqrt{(}C^2+\sigma_h^2)$ and $C=2\pi\vec{k}'$. We can rewrite the above integral in terms of the k variable. Recall that, if $\vec{\omega}=\mathrm{sign}(\vec{k})$ and $f_h(k')\neq 0$, then $k=\vec{\omega}p_F+\vec{k}'$. Hence

$$\hat{g}^h(x-y)=\sum_{\vec{\omega}_1,\vec{\omega}_2}e^{i\pi(\vec{\omega}_1\vec{x}-\vec{\omega}_2\vec{y})}g^h_{\vec{\omega}_1,\vec{\omega}_2}(x-y)=$$

$$\int dk\frac{\tilde{f}_h(k)}{Z_h(k)}\frac{\hat{\phi}(\vec{k},\vec{x},\sigma_h))\hat{\phi}(\vec{k},-\vec{y},\sigma_h)e^{ik_0(x_0-y_0)}}{-ik_0-(\hat{\varepsilon}(k,\sigma_h)-\pi^2)} \tag{58}$$

where:

$$\hat{\varepsilon}(k,\sigma_h)=\left(|\vec{k}|-\pi\right)^2 \tag{59}$$

$$+2\pi\mathrm{sign}\left(|\vec{k}|-\pi\right)\sqrt{\left(|\vec{k}|-\pi\right)^2+\sigma_h^2}+\pi^2$$

and

$$\hat{\phi}(\vec{k},\vec{x},\sigma_h))=e^{i\vec{k}\vec{x}}u(\vec{k},\vec{x},\sigma_h) \tag{60}$$

$$u(\vec{k},\vec{x},\sigma_h)=e^{-i\mathrm{sign}(\vec{k})\pi\vec{x}}$$

$$\left[\cos(\pi\vec{x})\sqrt{1+\frac{\mathrm{sign}(|\vec{k}|-\pi)\sigma_h}{\sqrt{(|\vec{k}|-\pi)^2+\sigma_h^2}}}+\right. \tag{61}$$

$$\left.+i\mathrm{sign}(\vec{k})\sin(\pi\vec{x})\sqrt{1-\frac{\mathrm{sign}(|\vec{k}|-\pi)\sigma_h}{\sqrt{(|\vec{k}|-\pi)^2+\sigma_h^2}}}\right]$$

In (B.M.2) by a careful analysis of the Bloch waves it is shown that

$$|\hat{\phi}(\vec{k},\vec{x},u)-\phi(\vec{k},\vec{x},u)|=O(u)$$
$$|\hat{\varepsilon}(\vec{k},u)-\varepsilon(\vec{k},u)|=O(u^2) \tag{62}$$

if $\phi(\vec{k},\vec{x},u)$ are the Bloch waves *i.e.* the solutions of eq.(4) and $\varepsilon(\vec{k},u)$ the dispersion relation. The fourth statement of the theorem then follows.

At the end, note that our results are independent on the constant $\tilde{C}$, which is arbitrary; in fact η_1,η_2 as well as the constants entering in the bounds of the theorem do not depend on $\tilde{C}$, which only affects the convergence radius of the series *i.e.* there is an "optimal" way to choose it; analogue considerations can be made for the parameter γ.

VI. CONCLUSIVE REMARKS

As we said in the first section, in literature are often studied model in some sense "equivalent" to eq.(1) for some values of the parameters. A renormalization group treatment allows us to clarify this intuition of "equivalence" between different models in this context. Namely we can consider two models *equivalent* if their Beta functions coincide up to terms vanishing as $h\to-\infty$. In this sense, the analysis in Ref. 17, 18, 5 shows that the"realistic model" defined by eq.(1) with no periodic potential in the spinless case is equivalent to the Luttinger model; in the same sense in Ref. 7 it is shown that the realistic model in the spinning case, if the interaction is repulsive and $p_F\neq\frac{n\pi}{a},\frac{n\pi}{2a}$, is equivalent to the Mattis model[34], which is an exactly solvable generalization of the Luttinger model.

We point out in this paper one more of such "equivalence": the realistic model, in the spinless case and at $p_F=\frac{n\pi}{a}$, is equivalent to a relativistic quantum field model, the Yukawa$_2$ model, describing a relativistic massive fermion in $d=1$. One can easily convince himself of this fact noting that the anomalous propagator eq.(41) can be written as

$$g^{(h)}(x-y)=g^{(h)}_Y(x-y)+C^h_1(x-y) \tag{63}$$

where

$$g_Y^{(h)}(x-y) = \frac{1}{Z_h}\int dk \tilde{f}_h(k) e^{ik(x-y)} \frac{\not{k}+\sigma_h I}{k^2+\sigma_h^2} \qquad (64)$$

with $\not{k} = ik_0\gamma_0 + 2\pi\vec{k}\gamma^1$, if γ_0, γ_1 are the γ-matrices for a relativistic $d=1$ Fermi field with light velocity 2π; $g_Y^{(h)}$ is dominant as $|C_1^{(h)}(x-y)| \le A\frac{\gamma^{2h}}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N}$ with A a suitable constant matrix, *i.e.* it obeys to a similar bound with an extra factor γ^h; g_Y^h is just the infrared propagator at scale h of the Yukawa$_2$ model[35] (σ_h is just the fermion mass at scale h) and it is trivial to check that the two models are equivalent according to the above definition.

The massive Luttinger model[9,36] can be considered an ultraviolet regularization of the Yukawa$_2$ model; it describes two kinds of interacting fermions, described by $\psi_\omega^\pm$, with linear dispersion relation, with a mass term $u\psi_1^+\psi_{-1}^-$ in the Hamiltonian. One can expect that also this model is equivalent to eq.(1) in the spinless filled band case discussed here. In Ref. 37, 14 it was shown by physical arguments that the Schwinger function decays faster than any power for large distances with rate $u^{1+\eta}$, $\eta = O(\lambda) > 0$, in agreement with our results.

As a conclusion let us discuss briefly the spinning case. The only difference with the spinless case treated here is that there are more running coupling constants corresponding to quartic monomials in the fields, and the second order beta function is in this case much more complex involved. One can verify that the dynamical system correspondent to the second order Beta function has a variety of behaviors; it seems that only for special choices of the potential everything might stay unchanged, with respect to the spinless case, while in general new ideas seem necessary.

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16 the same techniques were used for the the $d > 1$ was studied in Ref. 17 and in Ref. 38, 39; in particular in Ref. 39 is started a very ambitious program to construct the Schwinger functions in $d=2$ and $u=0$, which is however still incomplete. Similar ideas are also in Ref. 41

17 G.Benfatto G.Gallavotti: *Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one dimensional systems*, Jour. of Stat. Phis., **59**, 541-664 (1990).

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20 E.Lieb F.Y.Wu *Absence of Mott transition in an exact solution of the short-range, one band model in one dimension* Phys. Rev. Lett. 20, 1445-1448 (1968)

21 By repeating the computations in the following sections using an analytic partition of the unity instead of a C^∞ one, see sec.3, the bound eq.(14) follows as well as an exponential bound for the Schwinger function instead of the faster than any power one of eq.(9) . We prefer a C^∞ decomposition, as it makes the exposition more readable and it allows us to avoid some technicalities, but we stress that it should be no problem to use an analytic decomposition

22 here by attractive we mean $\lambda < 0$

23 As L, β are finite, there exist a scale $h_{L,\beta}$ such that $f_h \equiv 0$ for $h \le h_{L,\beta}$

24 Remember that if L, β are finite than $h_v \ge h_{L,\beta}$

25 eq.(23)(24) are written for the $L = \beta = \infty$ case; more complicated formulae should be written in the case of finite L, β,[7]

26 One can verify by a second order computation that $\lambda_0 = \lambda(\hat{v}(0) - \hat{v}(2p_F)) + O(\lambda^2)$, $\sigma_0 = u + O(u\lambda)$, $t_0, i_0 = O(u\lambda)$, $a_0, z_0 = O(\lambda^2)$, $\nu_0 = \nu + O(\lambda^2)$

27 If there is a chain of k renormalized clusters enclosed

one in the other, one can worry that there is a propagator derived k times, so obtaining in the bound a factor $C_k \geq O(k!)$. However this is not the case, as one can check that there are no propagators derived more then two times[7]. This follows from the fact that $\partial \mathcal{R} f_4^h(k_1', k_2', k_3', k_4') = \partial f_4^h(k_1', k_2', k_3', k_4')$, $\partial^2 \mathcal{R} f_2^h(k_1', k_2') = \partial^2 f_2^h(k_1', k_2')$, $\partial \mathcal{R} f_2^h(k_1', k_2') = \partial f_2^h(k_1', k_2') - \partial f_2^h(0,0)$.

[28] And even if it were $\lambda_{h-1} = \lambda_h - \beta\lambda_h^n$, $\beta \geq 0$,the most favorable case, this would mean that $\lambda_h \geq O(|h|^{-\frac{1}{n-1}})$ and the couplings z_h would still be *unbounded* in this approximation.

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[31] of course our results are independent on the value of $\tilde{C}$

[32] the second order computation for the beta function correspondent to g_h, δ_h and Z_h are identical to the ones in Ref. 17, 18

[33] Of course it should be possible to prove eq.(47) without using the exact solution but, possibly, exploiting some well known symmetries of the Lagrangian, but they are not so immediate to exploit in this approach as they hold only when the regularizations are removed[40].

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