REMARKS ON VIRTUAL BOUND STATES FOR SEMI-BOUNDED OPERATORS

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Abstract. We calculate the number of bound states appearing below the spectrum of a semi-bounded operator in the case of a weak, non-signdefined perturbation. The abstract result generalizes the Birman–Schwinger principle to this case. We discuss a number of examples, in particular higher order differential operators, critical Schrödinger operators, systems of second order differential operators, Schrödinger type operators with magnetic fields and the two-dimensional Pauli operator with a localized magnetic field.

1. Introduction

Let A be a self-adjoint semi-bounded operator on some separable Hilbert space H with the lower bound zero. Let V be a suitable hermitian perturbation, such that the operator

$$A(\alpha) := A - \alpha V$$

is well-defined and self-adjoint for $\alpha \in (0, \alpha_0)$, $\alpha_0 > 0$. Assume moreover, that the negative spectrum of $A(\alpha)$ consists only of a finite number $n_-(A(\alpha), 0)$ of negative eigenvalues for all $\alpha \in (0, \alpha_0)$. Clearly these eigenvalues will tend to zero as $\alpha \to +0$. If the point zero belongs to the essential spectrum of A, some of these eigenvalues might "disappear" at zero. We are interested in the quantity

$$\lim_{\alpha \to +0} n_-(A(\alpha), 0), \tag{1.1}$$

which is the number of negative eigenvalues, that remain present for arbitrary small $\alpha > 0$. This problem is well understood for the case of sign-defined V. Our main purpose is to study the limit (1.1) in the case of *non-signdefined* perturbations.

A typical example of the situation described is the Schrödinger operator in lower dimensions. Indeed, consider the operator

$$Q(\alpha) := -\Delta - \alpha V$$

on $L_2(\mathbb{R}^d, \mathbb{C})$, where V denotes the multiplication by the real-valued function $V(x)$. It is well-known that due to the Hardy inequality $n_-(Q(\alpha), 0) = 0$ as $\alpha \to +0$ for a large

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class of potentials $V \geq 0$ if $d \geq 3$. On the other hand, if $d = 1, 2$ and $V \geq 0, V \not\equiv 0$, the operator $Q(\alpha)$ has negative spectrum for arbitrary $\alpha > 0$. Simon proved [32], that for non-signdefined V with $(1 + |x|)V(x) \in L_1$ for $d = 1$ or $(1 + |x|)^\epsilon V(x) \in L_1$ for $d = 2$, the criteria

$$\lim_{\alpha \to +0} n_-(Q(\alpha), 0) = 1 \quad \text{if} \quad \int V dx \geq 0, \qquad \lim_{\alpha \to +0} n_-(Q(\alpha), 0) = 0 \quad \text{if} \quad \int V dx < 0,$$

holds. Moreover he calculated the asymptotics of the respective bound state in the limit $\alpha \to +0$. Analogous questions for second order operators with variable coefficients have been considered by Pinchover [24] in terms of criticality. A renewed interest in this subject has been shown in [35]. In case of more general operators, for example higher order differential operators, the situation becomes more complicated, since "second order techniques" cannot be applied. In particular we have to deal with several eigenvalues appearing for arbitrary small coupling constants.

If the point zero is an isolated eigenvalue of finite multiplicity for A, the quantity (1.1) can be obtained by analytic perturbation theory. As our main result Theorem 4.1 shows, that the perturbation of the operator A with a possibly complicated spectral structure at zero can be reduced to the perturbation of an isolated eigenvalue of finite multiplicity of a certain auxiliary operator. Standard analytic perturbation theory is then applied to this auxiliary operator. In some sense this construction completes the well-known operator-theoretical approach by Birman [6] and Schwinger [30] to the study of the discrete lower portion of the spectrum of semi-bounded operators. In particular Birman realized that the appearance of spectrum for arbitrary small coupling constants requires a certain attention in the formulation of the Birman-Schwinger principle, see [6] Lemma 1.3a. We modify this scheme to the case of non-signdefined perturbations.

We point out that our approach does not lead to asymptotics on the negative eigenvalues of $A(\alpha)$ as $\alpha \to +0$. However it is applicable under very general conditions and provides a complete answer to (1.1).

At a first glance our abstract constructions might appear involved and difficult to apply. Therefore we attach extensive material on its applications, which is of interest by itself. In this we do not aim for the most general statements but try to highlight the efficiency and features of Theorem 4.1 (see also Remark 5.2).

First we deal with higher order Schrödinger type operators, the coefficients of which might degenerate, see Theorem 6.1. As perturbations we consider localized lower order differential expressions. The material generalizes Simons result on the *existence* of negative eigenvalues in [32].

Next we study second order elliptic systems. Because of the internal degrees of freedom one must be careful about the kernel of the perturbation, see Theorem 7.1.

We also attach a section on critical Schrödinger operators. Although a similar problem has been solved by Pinchover [24], we believe a short deduction of this result from our approach, using techniques entirely different from those in [24], to be of interest.

The final part of the paper is devoted to operators with magnetic fields. We show that in the "generic" case second order Schrödinger type operators with magnetic field do not have negative eigenvalues for weakly coupled potentials. Generic means, that the magnetic field cannot be removed by gauge transformation. In particular this result holds for the two-dimensional operator in the setting of the Aharonov–Bohm effect.

On the other hand the situation for the Pauli operator in the dimension two with a localized magnetic field is quite the opposite. Here the unperturbed operator has a non-trivial spectral structure at zero: The point zero belongs to the essential spectrum and by the Aharonov–Casher Theorem [1], [11] might be an eigenvalue of finite multiplicity. This multiplicity depends only on the magnetic flux. Under perturbation not only these eigenvalues may turn into negative ones, but the essential part of the spectrum contributes also to the appearance of negative eigenvalues.

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Sections 3-4, 7 and 9 are based on material from the authors PhD-thesis [36].

2. Notation

2.1. Let H be a separable Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Consider a bounded or unbounded self-adjoint operator T on H with domain $D(T)$. Let $\sigma(T)$ denote the spectrum of T. A point $\gamma \in \sigma(T)$ is an eigenvalue of T, if there exists a non-trivial $\psi \in H$ such that $(T-\gamma)\psi = 0$. The discrete spectrum $\sigma_d(T)$ is the set of isolated eigenvalues of finite multiplicity of T. The set $\sigma_{\mathrm{ess}}(T) := \sigma(T)\backslash\sigma_d(T)$ forms the essential spectrum of T.

Fix some value $\lambda \in \mathbb{R}$. Assume that the spectrum of T above λ is discrete and finite, and let $n_+(T,\lambda)$ be the total multiplicity of all eigenvalues of T, which are larger than λ. If the spectrum of T above λ is not finite or not purely discrete we set $n_+(T,\lambda) = \infty$. In analogy, if the spectrum of T below λ is discrete and finite, then $n_-(T,\lambda)$ denotes the total multiplicity of all eigenvalues less than λ of T. If the spectrum of T below λ is not finite or not purely discrete we set $n_-(T,\lambda) = \infty$.

2.2. Put $\mathbb{N} = \{0,1,2,\dots\}$ and $\mathbb{N}_+ = \{1,2,\dots\}$. For typographical reasons we use sometimes the shorthand notation

$$\kappa(d,l) := \binom{d+l}{l}, \qquad d,l \in \mathbb{N}.$$

Let $\iota \in \mathbb{N}^d$ be a multiindex. Then $|\iota| = \sum_{j=1}^d \iota_j$ and $\iota! = \prod_{j=1}^d \iota_j!$. We write $\partial_x^\iota = \frac{\partial^\iota}{\partial x^\iota}$, but also $\partial_{x_k} = \frac{\partial}{\partial x_k}$, $k = 1, \ldots, d$, if $|\iota| = 1$. The vector ∇^l consists of the elements $\sqrt{\frac{l!}{\iota!}}\partial_x^\iota$ with $|\iota| = l \in \mathbb{N}_+$. Further $\Omega_{d,l}$ denotes the $\kappa(d,l)$-dimensional lineal of all polynomials over $\mathbb{R}^d$ or $\mathbb{C}^d$, the degree of which does not exceed $l \in \mathbb{N}$.

2.3. Set $\mathcal{B}_R^d(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < R\}$ and $\mathcal{B}_R^d = \mathcal{B}_R^d(0)$. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Assume that $\Xi \subset \mathbb{R}^d$ is a connected, open domain. Then $C^\infty(\Xi, \mathbb{K}^s)$ denotes the set of all smooth functions on Ξ with values in $\mathbb{K}^s$, $s \in \mathbb{N}_+$, such that the function and all its derivatives of arbitrary order are uniformly bounded on Ξ. If Ξ is bounded, then $C_0^\infty(\Xi, \mathbb{K}^s)$ denotes all functions from $C^\infty(\Xi, \mathbb{K}^s)$, vanishing in some ϵ-neighbourhood of $\partial\Xi$, and we put $C_c^\infty(\Xi, \mathbb{K}^s) := C^\infty(\Xi, \mathbb{K}^s)$. If Ξ is unbounded, then $C_c^\infty(\Xi, \mathbb{K}^s)$ is the set of all functions $C^\infty(\Xi, \mathbb{K}^s)$ with bounded support, and $C_0^\infty(\Xi, \mathbb{K}^s)$ is the set of all functions from $C_c^\infty(\Xi, \mathbb{K}^s)$, vanishing in some ϵ-neighbourhood of $\partial\Xi$. The Sobolev space $W_2^l(\Xi, \mathbb{K}^s)$ is given by the norm

$$\|u\|^2_{W_2^l(\Xi,\mathbb{K}^s)} := \sum_{\tau \in \mathbb{N}^d : |\tau| \le l} \int_\Xi |\partial_x^\tau u|^2_{\mathbb{K}^s}\, dx.$$

A measurable function $f : \Xi \to \mathbb{K}^s$ is said to be in $L_p^{\mathrm{loc}}(\Xi, \mathbb{K}^s)$, $s \in \mathbb{N}_+$, $0 < p \le \infty$, if $f \in L_p(\Xi \cap \mathcal{K}, \mathbb{K}^s)$ for all compacts $\mathcal{K} \subset \mathbb{R}^d$.

In contrast to this we equip the space $W_{2,l}^{\mathrm{loc}}(\Xi, \mathbb{K}^s)$ with the Frechet topology, induced by the family of semi-norms $\|\cdot\|_{W_2^l(\mathcal{B}^d_{n^{-1}}(x_k),\mathbb{K}^s)}$, where $\{x_k\}_{k\in\mathbb{N}_+}$ is a dense sequence in Ξ and $n, k \in \mathbb{N}_+$ are such that $\overline{\mathcal{B}^d_{n^{-1}}(x_k)} \subset \Xi$.

2.4. By $< \cdot, \cdot >$ or $< \cdot, \cdot >_{\mathbb{C}^n}$, $\mathbb{O}_n$ and $\mathbb{I}_n$ we denote the standard scalar product, the zero and the identity operator on $\mathbb{C}^n$, $n \in \mathbb{N}_+$. The linear space $GL(n,n)$ consists of all $n \times n$-matrices equipped with the usual operator norm as mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$. For a hermitian matrix $T = \{t_{ij}\}_{i,j=1}^n \in GL(n,n)$ let $\mathfrak{n}_+, \mathfrak{n}_-$ be the total multiplicities of the positive and negative eigenvalues of T and $\mathfrak{n}_0 := \dim\ker T$. The triple $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_0$ is said to be the *signature of* T. Obviously it holds $\mathfrak{n}_+ + \mathfrak{n}_- + \mathfrak{n}_0 = \mathfrak{n}$.

If f is a real-valued function, then set $2f_\pm(x) := |f(x)| \pm f(x)$. If $F : \Xi \to GL(n,n)$ takes its values in the set of the hermitian matrices, then $F_+(x)$ and $F_-(x)$ are the positive and negative part of $F(x)$, respectively.

2.5. For a real number y put

$$\{y\} := \max\{m \in \mathbb{Z} : \ m < y\}, \qquad [y] := \max\{m \in \mathbb{Z} : \ m \le y\}. \tag{2.1}$$

3. Auxiliary results.

3.1. Assume that T is a self-adjoint bounded operator on the separable Hilbert space H and that $t \in \sigma_d(T)$ is an isolated eigenvalue of finite multiplicity n of T. Let H_t denote the respective eigenspace, let P_t be the orthogonal projection on H_t, and set $Q_t = \mathbb{I} - P_t$. Let $\delta > 0$ be such that $\sigma(T) \cap (t - 2\delta, t + 2\delta) = \{t\}$. We consider the perturbation

$$T(\epsilon) := T + \epsilon S$$

of T by another bounded, self-adjoint operator S with a coupling constant $\epsilon \in \mathbb{R}$.

According to analytic perturbation theory [28], II §2 Th.1, there exist n real analytic functions $t_j(\epsilon)$, $t_j = t(0)$, and a family of n orthonormalized vector-functions $\psi_j(\epsilon) \in H$, such that for sufficiently small $|\epsilon| > 0$ the spectrum of $T(\epsilon)$ in $(t - \delta, t + \delta)$ coincides as values and in its multiplicity with the set $\{t_1(\epsilon), \ldots, t_n(\epsilon)\}$, and

$$T(\epsilon)\psi_j(\epsilon) = t_j(\epsilon)\psi_j(\epsilon), \qquad j = 1, \ldots, n. \tag{3.1}$$

Both $t_j(\epsilon)$ and $\psi_j(\epsilon)$ can be represented as power series

$$t_j(\epsilon) = \sum_{k=0}^{\infty} \epsilon^k t_{j,k} \qquad \text{and} \qquad \psi_j(\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \psi_{j,k}\ , \qquad j = 1, \ldots, n,$$

which converge for sufficiently small $|\epsilon| > 0$.

Below we give the expressions for $t_{j,k}$ for $k = 0, 1, 2$ leaving the details to [15] pp. 91-92. Obviously $t_{j,0} = t$ for all $j = 1, \ldots, n$. Extracting the terms being linear in ϵ from (3.1), we find

$$(T - t)\psi_{j,1} = (t_{j,1} - S)\psi_{j,0}, \qquad j = 1, \ldots, n. \tag{3.2}$$

This equation can be solved if and only if $(t_{j,1} - S)\psi_{j,0} \perp H_t$. By normalization we have $\psi_{j,0} \perp \psi_{k,0}$ if $j \neq k$. Hence

$$\{t_{1,1}, \ldots, t_{n,1}\} \quad \text{is the set of eigenvalues of} \quad P_t S P_t \quad \text{on} \quad H_t. \tag{3.3}$$

The vectors $\psi_{j,0}$ are the respective eigenvectors. The vector $\psi_{j,1}$ is of the form

$$\psi_{j,1} = \phi_j + R_t(t_{j,1} - S)\psi_{j,0}, \qquad \phi_j \in H_t,$$

where $R_t = 0 \oplus (Q_t(t - T)Q_t)^{-1}$ is the restricted inverse of $(t - T)$ on $H = H_t \oplus H_t^{\perp}$.

Assume that the eigenvalue $t_{j,1}$ of $P_t S P_t$ has the multiplicity $n_0 \leq n$. Reodering the eigenvalue branches we get $t_{j,1} = \cdots = t_{j,n_0}$. Let $H_{t,0} \subseteq H_t$ be the respective eigenspace of $P_t S P_t$, and $P_{t,0}$ is the projection on it.

Considering the terms multiple to ϵ^2 in (3.4) one obtains the equation

$$(T - t_{j,1})\psi_{2,j} = (t_{j,1} - S)\psi_{j,1} + t_{j,2}\psi_{0,j}, \qquad j = 1, \ldots, n_0. \tag{3.4}$$

This problem has a solution in $\psi_{2,j}$, if and only if $(t_{j,1}-S)\psi_{j,1}+t_{j,2}\psi_{0,j}\perp H_t$. Since the projections P_t and $P_{t,0}$ commute and $P_{t,0}(t_{j,1}-S)P_t=0$, one concludes

$$Z_{t,0}\psi_{j,0}:=P_{t,0}(t_{j,0}-S)R_t(t_{j,0}-S)P_{t,0}\psi_{0,j}=-t_{j,2}\psi_{0,j},\qquad j=1,\ldots,n_0.$$

Hence

$$\{t_{1,2},\ldots,t_{n_0,2}\}\quad\text{are the eigenvalues of}\quad -Z_{t,0}\quad\text{on}\quad H_{t,0}. \tag{3.5}$$

If t is the upper bound for the operator T, the restriction of R_t to $H_t^{\perp}$ is a strictly negative operator and $Z_{t,0}\le 0$. Hence

$$t_{j,2}\ge 0,\qquad j=1,\ldots,n_0\qquad\text{if}\quad t=\max\sigma(T). \tag{3.6}$$

If moreover $t_{j,2}=0$ for some $j\in\{1,\ldots,n_0\}$, then $(t_{j,1}-S)\psi_{j,0}\in\ker R_t=H_t$. Thus

$$(t_{j,1}-S)\psi_{j,0}=P_t(t_{j,1}-S)P_t\psi_{j,0}=0,$$

and $\psi_{j,0}$ being an eigenvector for T at t is also an eigenvector for S at $t_{j,1}$. For such j we have the identity

$$t_j(\epsilon)=t+\epsilon t_{j,1},\qquad t_{j,2}=0,\quad t=\max\sigma(T), \tag{3.7}$$

and the respective eigenvalue branch $t_j(\epsilon)$ is *linear* in ϵ.

3.2. Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Consider a quadratic closable form $a[\cdot,\cdot]$, defined on a dense linear subset $d[a]\subseteq H$. Assume that the form a is hermitian and semi-bounded from below with the lower bound $m(a)>-\infty$. For $\gamma+m(a)>0$ the form $a_\gamma[\cdot,\cdot]:=a[\cdot,\cdot]+\gamma\|\cdot\|^2$ turns $d[a]$ into a pre-Hilbertspace. The respective completions $\mathcal{H}_\gamma[a]$ have equivalent norms and are embedded in H. If $a_{-m(a)}[u,u]$ is strictly positive on $d[a]$ [1], it induces the completion $H[a_{-m(a)}]$ of $d[a]$. This Hilbert space cannot be realized as a subset of H.

Let $\overline{a}$ be the closure of a on $d[\overline{a}]=\mathcal{H}_\gamma[a]$, $\gamma+m(a)>0$. It generates a unique self-adjoint, semi-bounded from below operator A and $\min\sigma(A)=m(a)$.

Consider now a second hermitian quadratic form $v[\cdot,\cdot]$, $d[v]\supseteq d[a]$, satisfying

$$|v[u,u]|\le c_{3.1}a_{-m(a)}[u,u]+c_{3.2}\|u\|^2,\qquad u\in d[a]. \tag{3.8}$$

Then the form

$$a(\alpha)[u,u]:=a[u,u]-\alpha v[u,u],\qquad u\in d[a], \tag{3.9}$$

is semi-bounded from below and closable for all $\alpha\in(0,\alpha_0)$ with $0<\alpha_0:=1/c_{3.1}$. Its closure $\overline{a-\alpha v}$ induces a self-adjoint operator $A(\alpha)$. Formally attaching an operator V to the form v, the operator $A(\alpha)$ shall be considered as the sum $A(\alpha):=A-\alpha V$

[1] $a[u,u]>0$ for all $0\ne u\in d[a]$

in the form sense. If the form v is zero-bounded with respect to (w.r.t.) a, that is for all $\epsilon > 0$ there exists a finite constant $c_{3.3}$ such that

$$|v[u,u]| \le \epsilon a[u,u] + c_{3.3}(\epsilon)\|u\|^2, \qquad u \in d[a],$$

the operator $A(\alpha)$ is well-defined for *all* $\alpha > 0$. In this case we put formally $\alpha_0 = +\infty$.

3.3. Below we give a short account of the Birman–Schwinger principle [6], [30], [8].

Fix some $\lambda \in \mathbb{R}$ and $\alpha \in (0, \alpha_0)$. According to Glazman's Lemma the value of $n_-(A(\alpha), \lambda)$ is given by the upper bound over the dimensions of all lineals $F \subset d[a]$, for which

$$a[u,u] - \alpha v[u,u] < \lambda \|u\|^2 \qquad \text{for all} \quad u \in F: \ u \ne 0. \tag{3.10}$$

Assume that $\gamma + m(a) > 0$. In view of (3.8) v can be extended to a bounded form on $\mathcal{H}_\gamma[a]$. There it induces the bounded self-adjoint operator $\mathcal{V}_\gamma : \mathcal{H}_\gamma[a] \to \mathcal{H}_\gamma[a]$. Again by Glazman's Lemma for $s > 0$ the counting function $n_+(s, \mathcal{V}_\gamma)$ coincides with the upper bound of the dimensions of all lineals $F \subset d[a]$, for which

$$v[u,u] > s a_\gamma[u,u], \qquad \text{for all} \quad u \in F: \ u \ne 0. \tag{3.11}$$

Comparing (3.10) with (3.11) one arrives at

$$n_-(A(\alpha), \lambda) = n_+(\mathcal{V}_{-\lambda}, \alpha^{-1}), \qquad \lambda < m(a), \quad \alpha \in (0, \alpha_0). \tag{3.12}$$

This identity is the key to study the number of eigenvalues of $A(\alpha)$ below a regular point $\lambda < m(a)$. The condition (3.8) implies $n_-(A(\alpha), \lambda) = 0$ for sufficiently small $\alpha > 0$. The spectrum of $A(\alpha)$ below λ is finite and discrete for arbitrary positive α, if and only if the positive part of the auxiliary operator $\mathcal{V}_{-\lambda}$ is compact.

3.4. The case $\lambda = -\gamma = m(a)$ is more subtle. In this subsection we suppose the quadratic form v to be *non-negative* and the auxiliary form

$$g_v[u,u] := a[u,u] - m(a)(u,u) + v[u,u] \tag{3.13}$$

to be strictly positive on its domain $d[g_v] = d[a]$. Let $\mathfrak{H}[g_v]$ denote the completion of $d[a]$ to a Hilbert space w.r.t. the form (3.13). Obviously v is bounded w.r.t. g_v and can be extended to $\mathfrak{H}[g_v]$. There it induces the non-negative operator $\mathfrak{V} : \mathfrak{H}[g_v] \to \mathfrak{H}[g_v]$, the norm of which does not exceed one. By a similar argument as in the previous subsection on obtains

$$n_-(A(\alpha), m(a)) = n_+(\mathfrak{V}, (1+\alpha)^{-1}), \qquad \alpha \in (0, \alpha_0). \tag{3.14}$$

In particular $n_-(A(\alpha), m(a))$ is finite for all positive α, if and only if $\mathfrak{V}$ is compact.

In the analysis of (3.14) we shall distinguish between the following two cases $1 \in \sigma(\mathfrak{V})$ and $1 \notin \sigma(\mathfrak{V})$. Suppose first $1 \notin \sigma(\mathfrak{V})$, or equivalently

$$v[u,u] \le c_{3.4} a_{-m(a)}[u,u], \qquad u \in d[a], \tag{3.15}$$

for some finite constant $c_{3.4}$. This holds if and only if $n_-(A(\alpha), m(a)) = 0$ for sufficiently small positive α. By (3.15) the form $a_{-m(a)}$ is strictly positive and one can consider the space $H[a_{-m(a)}]$. The form v defines a bounded self-adjoint operator $\mathcal{V}$ on $H[a_{-m(a)}]$, and one finds

$$n_-(A(\alpha), m(a)) = n_+(\alpha^{-1}, \mathcal{V}) \ , \qquad \alpha \in (0, \alpha_0) \ . \tag{3.16}$$

This is a more convenient version of the Birman-Schwinger principle for the number of all bound states of $A(\alpha)$ below $m(a)$.

Assume now instead, that (3.15) fails. This is equivalent to $1 \in \sigma(\mathfrak{V})$ and by (3.14) one has $n_-(A(\alpha), m(a)) > 0$ for all $\alpha \in (0, \alpha_0)$. In this case we say, that *the operator A has $\lim_{\alpha \to +0} n_-(A(\alpha), m(a))$ virtual bound states at $m(a)$ for V.* Of course this does not mean that $m(a)$ is an eigenvalue of A.

If $1 \in \sigma_{\text{ess}}(\mathfrak{V})$, then $n_-(A(\alpha), m(a)) = \infty$ for all $\alpha \in (0, \alpha_0)$. We shall stress the case, when $n_-(A(\alpha), m(a))$ is finite for some α. Then 1 is an isolated eigenvalue of finite multiplicity for $\mathfrak{V}$. The corresponding eigenspace is the subspace $\mathfrak{H}_0$ of $\mathfrak{H}[g_v]$, on which the extension of $a_{-m(a)}$ to $\mathfrak{H}[g_v]$ vanishes. Thus

$$n_-(A(\alpha), m(a)) = n_+((1+\alpha)^{-1}, \mathfrak{V}) = \dim \mathfrak{H}_0 \tag{3.17}$$

for $\alpha \in (0, \alpha_1)$ for some sufficiently small $0 < \alpha_1(a, v)$.

Let $\mathfrak{H}_1$ denote the orthogonal complement of $\mathfrak{H}_0$ in $\mathfrak{H}[g_v]$. This subspace reduces the operator $\mathfrak{V}$. The forms $a_{-m(a)}$ and g_v induce equivalent norms on $\mathfrak{H}_1$, and we equip $\mathfrak{H}_1$ with the inner product $a_{-m(a)}$. The form v induces a *bounded* self-adjoint operator $\mathfrak{V}_1$ on $\mathfrak{H}_1$ w.r.t. $a_{-m(a)}$. Thus (3.14) turns into

$$n_-(A(\alpha), m(a)) = \dim \mathfrak{H}_0 + n_+(\alpha^{-1}, \mathfrak{V}_1), \qquad \alpha \in (0, \alpha_0). \tag{3.18}$$

This identity replaces (3.16), which does not make sense if 1 is an eigenvalue of $\mathfrak{V}$. Indeed, fix some non-zero element $u \in \ker a_{-m(a)}$, and consider one of its representative sequences $\{u_n\} \in d[a]$. We have $a_{-m(a)}[u_n, u_n] \to 0$, $v[u_n - u_m, u_n - u_m] \to 0$, but $v[u_n, u_n] \not\to 0$ as $m, n \to \infty$. Hence the form v is not closable in $H[a_{-m(a)}]$, and the operator V is not defined.

By the same argument we see, that for $1 \in \sigma(\mathfrak{V})$ the Hilbert spaces $H[a_{-m(a)}]$ and $\mathfrak{H}[g_v]$ are topologically non-compatible.

4. Virtual bound states for non-signdefined perturbations.

4.1. We shall discuss the phenomenon of virtual bound states in the case of a hermitian *non-signdefined* form v. We represent v as a difference of two non-negative forms

$$v[u, u] = v_+[u, u] - v_-[u, u], \qquad u \in d[v] \subseteq d[v_+], d[v_-], \tag{4.1}$$

$$v_\pm[u, u] \le c^\pm_{4.1} a_{-m(a)}[u, u] + c^\pm_{4.2} \|u\|^2, \qquad u \in d[a] \subset d[v], \quad \gamma + m(a) > 0. \tag{4.2}$$

If (3.8) is satisfied, this can always be done by associating $v_\pm$ with the positive and negative part of the bounded, self-adjoint operator $\mathcal{V}_\gamma$ on $\mathcal{H}_\gamma[a]$, $\gamma + m(a) > 0$. However, such a canonical decomposition is not necessarily the most practical one.

We shall consider the self-adjoint operators

$$
\begin{aligned}
A(\alpha) &= A - \alpha(V_+ - V_-) \qquad \text{induced by the closure of} \quad a - \alpha(v_+ - v_-),\\
\tilde{A}(\alpha) &= A - \alpha(V_+ + V_-) \qquad \text{induced by the closure of} \quad a - \alpha(v_+ + v_-).
\end{aligned}
\tag{4.3}
$$

Both operators are well defined for coupling constants $\alpha \in (0, \alpha_0)$, $\alpha_0 := 1/\max\{2c^\pm_{4.1}\}$.

Condition 4.1. *The inequality* (4.2) *holds and* $n_-(\tilde{A}(\alpha), m(a)) < \infty$ *for some* $\alpha \in (0, \alpha_0)$.

Although v is not sign-defined, the function $n_-(A(\alpha), m(a))$ is monotone increasing in $\alpha > 0$, therefore its limit as $\alpha \to +0$ exists. If $\lim_{\alpha\to+0} n_-(\tilde{A}(\alpha), m(a)) = 0$ then we have also $\lim_{\alpha\to+0} n_-(A(\alpha), m(a)) = 0$ as $\alpha \to +0$. Consequently we concentrate on the case

Condition 4.2. $\lim_{\alpha\to+0} n_-(\tilde{A}(\alpha), m(a)) \geq 1$.

In general the non-negative form

$$g_{v_++v_-}[u,u] := a[u,u] - m(a)(u,u) + v_+[u,u] + v_-[u,u]$$

might vanish on some non-trivial lineal from its domain $d[g_{v_++v_-}] = d[a]$. Therefore we introduce an auxiliary form w, $d[a] \subset d[w]$, such that

$$h[u,u] := g_{v_++v_-+w}[u,u] = a[u,u] - m(a)(u,u) + v_+[u,u] + v_-[u,u] + w[u,u]$$

is strictly positive on $d[h] = d[a]$. Let $\mathfrak{H}[h]$ be the completion of $d[a]$ w.r.t. h. The forms $v, v_\pm, w$ induce the bounded operators $\mathfrak{V}, \mathfrak{V}_\pm, \mathfrak{W}$ on $\mathfrak{H}[h]$. Set $\mathfrak{B}^{(\alpha)} := \mathfrak{W} + \mathfrak{V}_+ + \mathfrak{V}_- + \alpha\mathfrak{V}$.

Let $\mathfrak{H}_0$ be the subspace of $\mathfrak{H}[h]$ where $g_{v_++v_-}$ vanishes. Since all summands in this form are non-negative, the decomposition $\mathfrak{H}[h] = \mathfrak{H}_0 \oplus_h \mathfrak{H}_1$ reduces the operators $\mathfrak{W}, \mathfrak{V}_\pm$ as

$$\mathfrak{W} = \mathbb{I}_0 \oplus \mathfrak{W}_1, \qquad \mathfrak{V}_\pm = \mathbb{O}_0 \oplus \mathfrak{V}_{\pm_1},$$

and hence

$$\mathfrak{V} = \mathbb{O}_0 \oplus \mathfrak{V}_1, \qquad \mathfrak{B}^{(\alpha)} = \mathbb{I}_0 \oplus \mathfrak{B}_1^{(\alpha)}, \qquad \mathfrak{B}_1^{(\alpha)} = \mathfrak{W}_1 + \mathfrak{V}_{+,1} + \mathfrak{V}_{-,1} + \alpha\mathfrak{V}_1.$$

Our third assumption is

Condition 4.3. *The operator* $\mathfrak{W}_1$ *is compact on* $\mathfrak{H}_1$.

4.2. Evaluating the variational expressions according to Glazman's Lemma we find

$$n_-(\tilde{A}(\alpha), m(a)) = n_+(\mathfrak{W} + (1+\alpha)(\mathfrak{V}_+ + \mathfrak{V}_-), 1) = n_+(\mathfrak{B}_1^{(0)} + \alpha(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1}), 1). \tag{4.4}$$

The operator norm of $\mathfrak{B}_1^{(0)} = \mathfrak{W}_1 + \mathfrak{V}_{+,1} + \mathfrak{V}_{-,1}$ does not exceed 1. Hence $1 \notin \sigma(\mathfrak{B}_1^{(0)})$ implies $n_-(\tilde{A}(\alpha), m(a)) = 0$ for sufficiently small positive α .

On the other hand by Condition 4.3 $1 \in \sigma_{\text{ess}}(\mathfrak{B}_1^{(0)})$ implies $1 \in \sigma_{\text{ess}}(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})$, or

$$(1+\alpha) \in \sigma_{\text{ess}}((1+\alpha)(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})) = \sigma_{\text{ess}}(\mathfrak{B}_1^{(0)} + \alpha(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})), \tag{4.5}$$

and thus $n_-(\tilde{A}(\alpha), m(a)) = \infty$ for all $\alpha \in (0, \alpha_0)$.

Finally consider the case $1 \in \sigma_d(\mathfrak{B}_1^{(0)})$. Let $\mathfrak{H}_2 \subseteq \mathfrak{H}_1$ be the $\mathfrak{n}$-dimensional kernel of $\mathfrak{B}_1^{(0)} - \mathbb{I}_1$, that is

$$\mathfrak{H}_2 = \{u \in \mathfrak{H}_1 : a_{-m(a)}[u, u] = 0\}. \tag{4.6}$$

Below $\mathfrak{P}_2$ denotes the orthogonal projection onto $\mathfrak{H}_2$. According to subsection 3.1 we have some $\epsilon > 0$ such that the spectrum of $\mathfrak{B}_1^{(0)} + \alpha(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})$ on $(1-\epsilon, \infty)$ consists of $\mathfrak{n}$ eigenvalue branches

$$\tilde{\lambda}_j(\mathfrak{B}_1^{(0)} + \alpha(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})) = 1 + \alpha\tilde{\lambda}_{1,j} + o(\alpha)$$

for sufficiently small positive α. By (3.3) the coefficients $\tilde{\lambda}_{j,1}$ are the eigenvalues of the finite rank operator $\mathfrak{P}_2(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})\mathfrak{P}_2$ restricted to $\mathfrak{H}_2$. Obviously the form $a_{-m(a)}$ vanishes on $\mathfrak{H}_2$. Since $g_{v_+ + v_-}$ is strictly positive on $\mathfrak{H}_1$, the form $v_+ + v_-$ is strictly positive on $\mathfrak{H}_2$. Hence $\mathfrak{P}_2(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})\mathfrak{P}_2$ is positively defined on $\mathfrak{H}_2$, and

$$\tilde{\lambda}_j(\mathfrak{B}_1^{(0)} + \alpha(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})) > 1 \quad \text{for all} \quad j = 1, \dots, \mathfrak{n} \quad \text{and sufficiently small} \quad \alpha > 0.$$

Thus $n_-(\tilde{A}(\alpha), m(a)) = \mathfrak{n}$ for all sufficiently small positive α.

4.3. In analogy to (4.4) for the non-signdefined perturbation the identity

$$n_-(A(\alpha), m(a)) = n_+(\mathfrak{B}^{(\alpha)}, 1) = n_+(\mathfrak{B}_1^{(\alpha)}, 1) = n_+(\mathfrak{B}_1^{(0)} + \alpha\mathfrak{V}_1, 1) \tag{4.7}$$

holds. From the previous subsection we see, that Conditions 4.1 and 4.2 imply $1 \in \sigma_d(\mathfrak{B}_1^{(0)})$. This eigenvalue has the finite multiplicity

$$\mathfrak{n} = \lim_{\alpha \to +0} n_-(\tilde{A}(\alpha), m(a)).$$

Again according to subsection 3.1 there exists some $\epsilon > 0$, such that the spectrum of $\mathfrak{B}_1^{(0)} + \alpha(\mathfrak{V}_{+,1} + \mathfrak{V}_{-,1})$ on $(1-\epsilon, \infty)$ consists of the eigenvalue branches

$$\lambda_k(\mathfrak{B}_1^{(\alpha)}) = 1 + \alpha\lambda_{k,1} + \alpha^2\lambda_{k,2} + o(\alpha^2)$$

for sufficiently small positive α. By (3.3) the coefficients $\lambda_{k,1}$ are the eigenvalues of the operator $\mathfrak{P}_2\mathfrak{V}_1\mathfrak{P}_2$ on $\mathfrak{H}_2$.

Let $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_0$ be the cardinalities of the positive, negative and zero-valued elements of $\{\lambda_{j,1}\}_{j=1}^{\mathfrak{n}}$,

$$\mathfrak{n}_+ + \mathfrak{n}_- + \mathfrak{n}_0 = \dim \mathfrak{H}_2.$$

These numbers give the signature of the restriction of the form v to $\mathfrak{H}_2$.

Assume that the eigenvalues $\lambda_{j,1}$ are numbered in non-increasing order. Then evidently

$$\begin{aligned} &\lambda_j(\mathfrak{B}_1^{(\alpha)}) > 1 \qquad \text{for} \quad j = 1, \dots, \mathfrak{n}_+, \\ &\lambda_j(\mathfrak{B}_1^{(\alpha)}) < 1 \qquad \text{for} \quad j = \mathfrak{n}_+ + \mathfrak{n}_0 + 1, \dots, \mathfrak{n} \end{aligned} \tag{4.8}$$

if $\alpha > 0$ is sufficiently small.

The point 1 is the upper bound of the unperturbed operator $\mathfrak{B}_1^{(0)}$. Then by (3.6) all second order coefficients $\lambda_{j,2}$ are non-negative. Let $\mathfrak{H}_3 \subseteq \mathfrak{H}_2$ be the $\mathfrak{n}_0$-dimensional kernel of $\mathfrak{P}_2\mathfrak{V}\mathfrak{P}_2$ on $\mathfrak{H}_2$, corresponding to

$$\lambda_{j,1} = 0 \qquad \text{for} \quad j = \mathfrak{n}_+ + 1, \dots, \mathfrak{n}_+ + \mathfrak{n}_0. \tag{4.9}$$

Assume that the coefficients $\lambda_{j,2}$ for $j = \mathfrak{n}_+ + 1, \dots, \mathfrak{n}_+ + \mathfrak{n}_0$ are numbered in increasing order and exactly the first $\mathfrak{n}_{0,0} \leq \mathfrak{n}_0$ coefficients take the value zero. By (3.2) there exists a $\mathfrak{n}_{0,0}$-dimensional subspace $\mathfrak{H}_4 \subseteq \mathfrak{H}_3$ such that

$$\mathfrak{B}_1^{(0)} u = u \quad \text{and} \quad \mathfrak{V}_1 u = 0 \qquad \text{for all} \quad u \in \mathfrak{H}_4.$$

This subspace can be evaluated as

$$\mathfrak{H}_4 = \{u \in \mathfrak{H}_2 : v[u, \psi] = 0 \qquad \text{for all} \quad \psi \in d[a]\}. \tag{4.10}$$

Obviously

$$\lambda_j(\mathfrak{B}^{(\alpha)}) > 1 \qquad \text{for} \quad j = \mathfrak{n}_+ + \mathfrak{n}_{0,0} + 1, \dots, \mathfrak{n}_+ + \mathfrak{n}_0 \tag{4.11}$$

if $\alpha > 0$ is small. By (3.7) the remaining $\mathfrak{n}_{0,0}$ eigenvalue branches $\lambda_j(\mathfrak{B}^{(\alpha)})$, $j = \mathfrak{n}_+ + 1, \dots \mathfrak{n}_+ + \mathfrak{n}_{0,0}$ are linear in α, and from (4.9) we conclude

$$\lambda_j(\mathfrak{B}^{(\alpha)}) = 1 \qquad \text{for} \quad j = \mathfrak{n}_+ + 1, \dots, \mathfrak{n}_+ + \mathfrak{n}_{0,0} \tag{4.12}$$

and all $\alpha > 0$. Combining (4.7), (4.8), (4.11) and (4.12) we arrive at

Theorem 4.1. *Let $A(\alpha)$ be the operator defined in* (4.3) *and assume Conditions* 4.1, 4.2 *and* 4.3 *to be satisfied. Let $\mathfrak{H}_2$ and $\mathfrak{H}_4$ be the finite-dimensional spaces defined in* (4.6), (4.10). *Then*

$$\lim_{\alpha \to +0} n_-(A(\alpha), m(a)) = \mathfrak{n}_+ + \mathfrak{n}_0 - \mathfrak{n}_{0,0}, \tag{4.13}$$

where $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_0$ is the signature of v on $\mathfrak{H}_2$ and $\mathfrak{n}_{0,0} = \dim \mathfrak{H}_4$.

Remark 4.1. The statement of Theorem 4.1 remains formally correct if we drop Condition 4.2. In absence of virtual bound states for $V_+ + V_-$ we have $\mathfrak{n}_\pm = \mathfrak{n}_0 = 0$.

Remark 4.2. If the lower bound of A is an isolated eigenvalue of finite multiplicity, a straightforward application of perturbation theory gives (4.13). The goal of Theorem 4.1 is, that it holds for eigenvalues appearing from the essential spectrum of A.

Remark 4.3. Assume that $\dim \mathfrak{H}_0 < \infty$. Then Theorem 4.1 remains true if we replace in its formulation

(4.14)

$$\mathfrak{H}_2 \quad \text{by} \quad \tilde{\mathfrak{H}}_2 = \mathfrak{H}_2 \oplus \mathfrak{H}_0 = \{u \in \mathfrak{H}[h] : a_{-m(a)}[u,u] = 0\} = \ker(\mathfrak{B}^{(0)} - \mathbb{I}),$$

$$\mathfrak{H}_4 \quad \text{by} \quad \tilde{\mathfrak{H}}_4 = \mathfrak{H}_4 \oplus \mathfrak{H}_0 = \{u \in \mathfrak{H}_2 : v[u,\psi] = 0 \quad \forall \psi \in d[a]\} = \ker \mathfrak{V} \cap \ker(\mathfrak{B}^{(0)} - \mathbb{I}).$$

Remark 4.4. The application of Theorem 4.1 reduces to the following three steps

- the description of $\mathfrak{H}_2$ by (4.6) or $\tilde{\mathfrak{H}}_2$ by (4.14),
- the calculation of the signature $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_0$ of the form v on $\mathfrak{H}_2$ or $\tilde{\mathfrak{H}}_2$,
- the calculation of $\dim \mathfrak{H}_4$ or $\dim \tilde{\mathfrak{H}}_4$ by (4.10), (4.14).

Since the signature of a form on a finite-dimensional Hilbert space does not depend on the underlying scalar product, we can equip $\mathfrak{H}_2$ or $\tilde{\mathfrak{H}}_2$ with an arbitrary suitable inner product what simplifies calculations.

5. Virtual bound states for partial differential operators.

5.1. Theorem 4.1 can be applied to various types of semi-bounded partial differential operators. The analysis of these cases follows the same basic scheme. In this section we demonstrate this approach and some of its variations to outline the various features of Theorem 4.1 and its connections to the theory of functions. There is an extensive literature on related topics. As a general reference we quote the book [19].

Consider an open, connected domain $\Xi \subseteq \mathbb{R}^d, d \geq 1$. Let $a^{(\iota,\tau)}(x)$, $\iota, \tau \in \mathbb{N}^s$, $|\iota|, |\tau| \leq l \in \mathbb{N}_+$, be a family of measurable $s \times s$-matrices with the complex valued entrees $a^{(\iota,\tau)}_{m,n}(x)$, $m, n = 1, \ldots, s$, $x \in \Xi$. Fixing the lexicographic order among the elements $\iota, \tau \in \mathbb{N}^s$ this family forms the matrix function

$$a(x) = \left\{ \left\{ a^{\iota,\tau}_{n,m}(x) \right\}_{n,m=1}^{s} \right\}_{\iota,\tau \in \mathbb{N}^s : |\iota|,|\tau| \leq l} : \mathbb{C}^{s\kappa(d,l)} \to \mathbb{C}^{s\kappa(d,l)}. \tag{5.1}$$

Let $\mathcal{J} = \{\mathcal{J}^{(\iota,\tau)}\}_{\iota,\tau \in \mathbb{N}^s, |\iota|,|\tau| \leq l} : \mathbb{C}^{s\kappa(d,l)} \to \mathbb{C}^{s\kappa(d,l)}$ be the operator, consisting of the entrees $\mathcal{J}^{(\iota,\tau)} = \mathbb{I}_s$ for $\iota = \tau$, $|\iota| = |\tau| = l$ and $\mathcal{J}^{(\iota,\tau)} = \mathbb{O}_s$ otherwise. We shall assume that $a^{(\iota,\tau)}(x) = (a^{(\tau,\iota)}(x))^*$ and that there exist two positive finite functions ρ_0, ρ_1 being continuous in Ξ, which satisfy

$$\rho_0(x)\mathcal{J} \leq a(x) \leq \rho_1(x)\mathbb{I}_{s\kappa(d,l)} \qquad \text{for a.e.} \quad x \in \Xi. \tag{5.2}$$

Thus $a(x)$ is non-negative and locally bounded but might degenerate near the boundary or at infinity. On the subspace corresponding to $|\iota| = l$ we have a local estimate from below.

We introduce the positive quadratic form

$$a[u,u] := \sum_{\iota,\tau\in\mathbb{N}^s:|\iota|,|\tau|\le l} i^{|\iota|-|\tau|} \int_\Xi \left\langle a^{(\iota,\tau)}\partial_x^\tau u, \partial_x^\iota u\right\rangle_{\mathbb{C}^s} dx, \qquad u\in d[a], \tag{5.3}$$

with some domain $d[a]$, such that

$$C_0^\infty(\Xi,\mathbb{C}^s) \subseteq d[a] \subseteq C_c^\infty(a;\Xi,\mathbb{C}^s) := \{u\in C_c^\infty(\Xi,\mathbb{C}^s):\ a[u,u]<\infty\}\ . \tag{5.4}$$

The form (5.2), (5.3) is closable in $L_2(\Xi,\mathbb{C}^s)$. If the bound (5.2) is satisfied we call the form (5.3) *locally non-degenerated* (*l.n.d.*).

A function $\chi\in W_{2,l}^{\mathrm{loc}}(\Xi,\mathbb{C}^s)$ is said to be a *limit element* for a l.n.d. form a, if and only if there exists a *limit sequence* $\{u_n\}\subset d[a]$ for χ, that is

$$u_n\to\chi \quad \text{in } W_{2,l}^{\mathrm{loc}}(\Xi,\mathbb{C}^s) \quad \text{as} \quad n\to\infty \qquad \text{and} \qquad \lim_{n\to\infty} a[u_n,u_n]=0. \tag{5.5}$$

A limit sequence for a non-trivial limit function $\chi\not\equiv 0$ is said to be *singular*. Obviously the set of all limit elements of a forms a linear subset $\Lambda(a)$ of $\Omega_{d,l-1}^s|_\Xi$. We refer to this lineal as the *limit space*.

Let $|\!|\cdot|\!|$ be a non-negative functional defined on $d[a]$, which is positively homogeneous [2]. Then $|\!|\cdot|\!|$ is said to *regularize* $\Lambda(a)$, if and only if

$$\liminf_{n\to\infty} |\!|u_n|\!| > 0$$

for all singular sequences $\{u_n\}$ of a.

Lemma 5.1. *Let a be the l.n.d. form* (5.3) *with the limit space $\Lambda(a)$. Suppose that $\Theta\subset\Xi$ is a compact set of positive Lebesgue measure. Then the inequality*

$$\|u\|^2_{L_2(\Theta,\mathbb{C}^s)} \le c_{5.1}\left(a[u,u]+|\!|u|\!|^2\right), \qquad u\in d[a], \tag{5.6}$$

holds, if and only if $|\!|\cdot|\!|$ regularizes $\Lambda(a)$. In particular $\dim\Lambda(a)=0$ *if and only if*

$$\|u\|^2_{L_2(\Theta,\mathbb{C}^s)} \le c_{5.2}\, a[u,u], \qquad u\in d[a]. \tag{5.7}$$

Proof. Assume that $|\!|\cdot|\!|$ does not regularize $\Lambda(a)$. Then there exists a singular sequence $u_n\to\chi\in\Lambda(a)$, $\chi\not\equiv 0$, such that $a[u_n,u_n]+|\!|u_n|\!|^2\to 0$ as $n\to\infty$. At the same time $\lim_{n\to\infty}\|u_n\|_{L_2(\Theta,\mathbb{C}^s)}\to\|\chi\|_{L_2(\Theta,\mathbb{C}^s)}>0$, what disproves (5.6).

Reversely assume that $|\!|\cdot|\!|$ is regularizing for $\Lambda(a)$ but (5.6) fails. Hence we find a sequence of normalized functions u_n, such that

$$1=\|u_n\|^2_{L_2(\Theta,\mathbb{C}^s)} > n\left(a[u_n,u_n]+|\!|u_n|\!|^2\right), \qquad \{u_n\}_{n\in\mathbb{N}}\subset d[a]. \tag{5.8}$$

[2] $|cu| = c|u| \ge 0 \quad$ for all $\quad c>0, \quad u\in d[a]$

Fix a monotone increasing sequence of open, connected, bounded Lipschitz domains $\mathcal{U}_m$, $\overline{\mathcal{U}_m} \subset \Xi$, such that $\Theta \subset \mathcal{U}_1$ and $\mathcal{K} \subset \mathcal{U}_m$ as $m \geq m_0(\mathcal{K})$ for all compacts $\mathcal{K} \subset \Xi$. The theorem on equivalent norms and (5.8) imply

$$\|u_n\|^2_{W_2^l(\mathcal{U}_m,\mathbb{C}^s)} \leq c_m \left(a[u_n,u_n] + \|u_n\|^2_{L_2(\Theta,\mathbb{C}^s)}\right) \leq c_m(n^{-1}+1) \leq 2c_m .$$

Since the embedding $W_2^l(\mathcal{U}_m,\mathbb{C}^s) \hookrightarrow L_2(\mathcal{U}_m,\mathbb{C}^s)$ is compact, we can extract consecutive $L_2(\mathcal{U}_m,\mathbb{C}^s)$-converging subsequences

$$\{u_{n,m}\}_{n\in\mathbb{N}} \subseteq \{u_{n,m-1}\}_{n\in\mathbb{N}} \subseteq \cdots \subseteq \{u_{n,1}\}_{n\in\mathbb{N}} \subseteq \{u_n\}_{n\in\mathbb{N}},$$

such that

$$u_{n,m}|_{\mathcal{U}_m} \overset{L_2(\mathcal{U}_m,\mathbb{C}^s)}{\longrightarrow} \chi|_{\mathcal{U}_m} \qquad \text{as} \quad n \to \infty.$$

We enumerate these subsequences preserving the initial order. From (5.2) and (5.8) we conclude $a[u_{n,m},u_{n,m}] < n^{-1}$, hence $\partial_x^\tau u_{n,m} \to 0$ on $L_2(\mathcal{U}_m,\mathbb{C}^s)$ for all $|\tau| = l$ as $n \to \infty$. This implies $u_{n,n} \to \chi \in \Omega^s_{d,l-1}|_\Xi$, $\chi \not\equiv 0$ in $W_{2,l}^{\mathrm{loc}}(\Xi,\mathbb{C}^s)$ and $\{u_{n,n}\}$ is a limit sequence for $\chi \in \Lambda(a)$. But then $\liminf_{n\to\infty} \|u_{n,n}\| > 0$ in contradiction to (5.8). □

Remark 5.1. It is easy to see that for each $\chi \in \Lambda(a)$ and each compact $\mathcal{K} \subset \Xi$ there exists a limit sequence $u_{n,K} \to \chi$, such that $u_n(x) = \chi(x)$ for all $x \in K$ and $n \in \mathbb{N}$.

5.2. The closure of the form (5.3), (5.4) induces a non-negative self-adjoint realization of the operator

$$A = \sum_{\iota,\tau\in\mathbb{N}^s:|\iota|,|\tau|\leq l} i^{-|\tau|-|\iota|} \partial_x^\iota a^{(\iota,\tau)}(x) \partial_x^\tau \tag{5.9}$$

on $L_2(\Xi,\mathbb{C}^s)$. If $\dim\Lambda(a) > 0$ then $\min\sigma(A) = m(a) = 0$.

Fix some bounded, non-negative function $W \not\equiv 0$ with compact support $\operatorname{supp} W \subset \Xi$. Let w be the form

$$w[u,u] = \int_\Xi W(x)|u|^2_{\mathbb{C}^s}\,dx, \qquad u \in d[w] = d[a]. \tag{5.10}$$

Then w is relatively compact w.r.t. $g_w := a + w$. The closure $\mathfrak{H}[g_w]$ of $d[a]$ w.r.t. g_w can be realized as a subset in $W_{2,l}^{\mathrm{loc}}(\Xi,\mathbb{C}^s)$ and the integral expression for (5.3) extends to $\mathfrak{H}[g_w]$. Let $\mathfrak{H}_0[g_w]$ be the subspace of $\mathfrak{H}[g_w]$ where a vanishes. Obviously a g_w-fundamental sequence $\{u_n\} \subset d[a]$ is a representative for $0 \not\equiv \chi \in \mathfrak{H}_0[g_w]$, if and only if $\{u_n\}$ is singular with the limit element χ. Hence

$$\mathfrak{H}[g_w] = \Lambda(a). \tag{5.11}$$

Consider the operator $\hat{A}(\alpha) := A - \alpha W$. In accordance with (5.11) and subsection 1.5 it holds

$$\lim_{\alpha\to+0} n_-(\hat{A}(\alpha), 0) = \dim \Lambda(a). \tag{5.12}$$

5.3. Let a be a l.n.d. form (5.3) with the limit space $\Lambda(a)$. Let v be the difference $v = v_+ - v_-$ of two non-negative forms $v_\pm$ for which (4.1), (4.2) hold. The operator $A(\alpha)$ and $\tilde{A}(\alpha)$ are defined in accordance to (4.3). Assume that Condition 4.1 is satisfied.

Let w be the form (5.10) with the same function W. Since w is relatively compact w.r.t. $a + w$, it is relatively compact w.r.t. $h = a + w + v_+ + v_-$.

As in section 4 let $\mathfrak{H}[h]$ be the closure of $d[a]$ in h. The forms w, v, $v_\pm$ induce the bounded, self-adjoint operators $\mathfrak{W}$, $\mathfrak{V}$, $\mathfrak{V}_\pm$ on $\mathfrak{H}[h]$. Set $\mathfrak{B}^{(\alpha)} = \mathfrak{W} + \mathfrak{V}_+ + \mathfrak{V}_- + \alpha\mathfrak{V}$. The operator $\mathfrak{W}$ is compact, and thus Condition 4.3 is fulfilled.

Lemma 5.2. *Let a be a l.n.d. form* (5.3) *with the limit space $\Lambda(a)$ and suppose that Conditions* 4.1, 4.2 *hold. Then the point* 1 *is an eigenvalue of $\mathfrak{B}^{(0)}$ and its multiplicity n is not less than* $\dim \Lambda(a)$. *If the topology induced by h on $d[a]$ is compatible with the topology on $W_{2,l}^{\mathrm{loc}}(\Xi, \mathbb{C}^s)$, then $n = \dim \Lambda(a)$ and the kernel of $\mathfrak{B}^{(0)} - \mathbb{I}$ coincides with $\Lambda(a)$.*

Proof. We use the notations of section 4. Since $\mathfrak{W}$ is compact Condition 4.1 implies $1 \in \sigma_d(\mathfrak{B}^{(0)})$. Put $k := \dim \Lambda(a)$. By (5.12) for arbitrary $\epsilon > 0$ there exists a k-dimensional lineal $\mathcal{F}_\epsilon \subset d[a]$, such that

$$1 - \epsilon < \frac{w[u,u]}{a[u,u] + w[u,u]} \leq \frac{w[u,u] + v_+[u,u] + v_-[u,u]}{a[u,u] + w[u,u] + v_+[u,u] + v_-[u,u]}, \qquad 0 \not\equiv u \in \mathcal{F}_\epsilon.$$

Thus $n_+(\mathfrak{B}^{(0)}, 1 - \epsilon) \geq k$ Since $\mathfrak{B}^{(0)}$ has no spectrum above one we find $n \geq k$.

If the topologies of h and $W_{2,l}^{\mathrm{loc}}(\Xi, \mathbb{C}^s)$ are compatible, so are the topologies of h and g_w. Hence $\mathfrak{H}[h]$ can be realized as a subset of $\mathfrak{H}[g_w] \subset W_{2,l}^{\mathrm{loc}}(\Xi, \mathbb{C}^s)$ and

$$\tilde{\mathfrak{H}}_2 = \mathfrak{H}_2 \oplus \mathfrak{H}_0 = \{u \in \mathfrak{H}[h] : a[u,u] = 0\} = \ker(\mathfrak{B}^{(0)} - \mathbb{I}) \subseteq \mathfrak{H}_0[g_w] = \Lambda(a).$$

Hence $n = \dim \tilde{\mathfrak{H}}_2 \leq k$. We conclude $n = k$ and $\tilde{\mathfrak{H}}_2 = \Lambda(a)$. $\square$

Lemma 5.2 and Theorem 4.1 imply

Corollary 5.1. *Let a be a l.n.d. form* (5.3) *with the limit space $\Lambda(a)$. Assume that the non-negative form v satisfies* (4.1) *and that the topology induced by $a + w + v$ on $d[a]$ is compatible with the topology on $W_{2,l}^{\mathrm{loc}}(\Xi, \mathbb{C}^s)$. Then we have either*

$$\lim_{\alpha\to+0} n_-(A(\alpha), 0) = \infty \qquad \textit{or} \qquad \lim_{\alpha\to+0} n_-(A(\alpha), 0) \leq \dim \Lambda(a).$$

Corollary 5.2. *Let a be a l.n.d. form* (5.3) *with the limit space $\Lambda(a)$. Assume that for $v = v_+ - v_-$ Condition* 4.1 *is satisfied and that the topology induced by $a + w + v_+ + v_-$ is compatible with the topology on $W^{\mathrm{loc}}_{2,l}(\Xi, \mathbb{C}^s)$. Then*

$$\lim_{\alpha \to +0} n_-(A(\alpha), 0) = \mathfrak{n}_+ + \mathfrak{n}_0 - \mathfrak{n}_{0,0},$$

where $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{n}_0$ is the signature of the form v on $\Lambda(a)$ and

$$\mathfrak{n}_{0,0} := \dim\{\chi \in \Lambda(a) : v[\chi, u] = 0 \text{ for all } u \in d[a]\}.$$

5.4. Let us discuss a typical application of these corollaries. Consider a family of $s \times s$-matrix valued functions $v^{(\iota,\tau)}(x)$, $\iota, \tau \in \mathbb{N}^s$, $|\iota|, |\tau| \leq l - 1$, with the entrees $v^{(\iota,\tau)}_{m,n}(\cdot) \in L^{\mathrm{loc}}_1(\Xi, \mathbb{C})$, $m, n = 1, \dots, s$. Assume that $\left(v^{(\iota,\tau)}(x)\right)^* = v^{(\tau,\iota)}(x)$ for a.e. $x \in \Xi$. For each x the hermitian matrix

$$v(x) = \left\{ \left\{ v^{\iota,\tau}_{n,m}(x) \right\}^s_{n,m=1} \right\}_{\iota,\tau \in \mathbb{N}^s : |\iota|, |\tau| \leq l-1} : \mathbb{C}^{s\kappa(d,l-1)} \to \mathbb{C}^{s\kappa(d,l-1)}$$

can be represented as the difference $v_+(x) - v_-(x)$ of two non-negative $s\kappa(d, l-1) \times s\kappa(d, l-1)$-matrices $v_\pm(x)$ with the $s \times s$-matrix block entrees $v^{(\iota,\tau)}_{(\pm)}(x)$. The forms

$$\begin{aligned} v[u, u] &:= \int_\Xi \sum_{\iota,\tau \in \mathbb{N}^s : |\iota|, |\tau| \leq l-1} i^{|\iota| - |\tau|} \left\langle v^{(\iota,\tau)} \partial^\tau_x u, \partial^\iota_x u \right\rangle_{\mathbb{C}^s} dx, \\ v_\pm[u, u] &:= \int_\Xi \sum_{\iota,\tau \in \mathbb{N}^s : |\iota|, |\tau| \leq l-1} i^{|\iota| - |\tau|} \left\langle v^{(\iota,\tau)}_{(\pm)} \partial^\tau_x u, \partial^\iota_x u \right\rangle_{\mathbb{C}^s} dx, \end{aligned} \tag{5.13}$$

are well defined on functions $u \in d[v] = d[a]$. Take w as in subsection 5.3. A simple limit argument shows that the topology on $d[a]$ induced by $a + w + v_+ + v_-$ is compatible with the topology of $W^{\mathrm{loc}}_{2,l}(\Xi, \mathbb{C}^s)$. Condition 4.1 implies, that the form v is well defined on $\Lambda(a)$:

Lemma 5.3. *Let a be a l.n.d. form* (5.3) *with the limit space $\Lambda(a)$. Assume that the forms* (5.13) *satisfy Condition* 4.1. *Then the integral*

$$\int_\Xi \sum_{\iota,\tau \in \mathbb{N}^s : |\iota|, |\tau| \leq l-1} i^{|\iota| - |\tau|} \left\langle v^{(\iota,\tau)} \partial^\tau_x \chi, \partial^\iota_x \chi \right\rangle_{\mathbb{C}^s} dx \tag{5.14}$$

converges absolutely for all $\chi \in \Lambda(a)$ and gives the value of $v[\chi, \chi]$.

Proof. By Lemma 5.2 $\Lambda(a) \subset \mathfrak{H}[h]$ with $h = a + w + v_+ + v_-$. Let $\{u_n\}$ be a representative sequence for $0 \not\equiv \chi \in \Lambda(a)$ in $\mathfrak{H}[h]$. Set

$$\eta^\pm_n(x) := \sum_{\iota,\tau \in \mathbb{N}^s : |\iota|, |\tau| \leq l-1} i^{|\iota| - |\tau|} \left\langle v^{(\iota,\tau)}_{(\pm)} \partial^\tau_x u_n, \partial^\iota_x u_n \right\rangle_{\mathbb{C}^s},$$

while $\eta(x)$ denotes the subintegral function in (5.14). Obviously $\eta_n^{\pm}(x) \geq 0$ and it is not difficult to verify that the sequence $\eta_n^{\pm}$ is fundamental in $L_1(\Xi, \mathbb{R})$. Hence there exists a non-negative function $\tilde{\eta}^{\pm} \in L_1(\Xi, \mathbb{R})$, such that $\eta_n^{\pm} \to \tilde{\eta}$ in $L_1(\Xi, \mathbb{R})$.

On the other hand, since $u_n \to \chi$ in $W_{2,l}^{\mathrm{loc}}(\Xi, \mathbb{C}^s)$ we can extract a suitable subsequence, for which $\partial_x^{\iota} u_n \to \partial_x^{\iota}\chi$ a.e. for all $\iota \in \mathbb{N}^s$, $|\iota| \leq l$, and consequently

$$\eta_n^{\pm} \to \eta^{\pm} := \sum_{\iota,\tau \in \mathbb{N}^s : |\iota|, |\tau| \leq l-1} i^{|\iota|-|\tau|} \left\langle v_{(\pm)}^{(\iota,\tau)} \partial_x^{\tau}\chi, \partial_x^{\iota}\chi \right\rangle_{\mathbb{C}^s} \quad \text{a.e.}$$

The limit functions $\eta^{\pm}$ and $\tilde{\eta}^{\pm}$ must coincide, thus

$$v[u_n, u_n] = \int_{\Xi} (\eta_n^+ - \eta_n^-) dx \to \int_{\Xi} (\eta^+ - \eta^-) dx = \int_{\Xi} \eta dx \qquad \text{as} \quad n \to \infty. \quad \square$$

Remark 5.2. Below we shall always use implicit conditions on $v_{n,m}^{(\iota,\tau)}(x)$ of the type of Condition 4.1. Firstly, this kind of assumptions reflects the true essence of the problem. Secondly, sufficient analytic conditions on $v_{n,m}^{(\iota,\tau)}(x)$ for Condition 4.1 to be true are widely known for concrete operators, among others see [19], [9]. Thirdly, our results are of interest even for the case of bounded $v_{n,m}^{(\iota,\tau)}(x)$ with compact support in Ξ, when Condition 4.1 is always satisfied.

In general Condition 4.1 can be reformulated in terms of capacity type conditions with restrains [19]. We restrict ourself to this remark, since a detailed exposition of this material would go beyond the scope of this paper.

6. Schrödinger type operators with degenerating coefficients

6.1. Put $\Xi = \mathbb{R}^d$ and let $a(x)$ be a measurable $\binom{d-1+l}{l} \times \binom{d-1+l}{l}$ -matrix function on $\mathbb{R}^d$, $l, d \in \mathbb{N}_+$, with hermitian values such that

$$c_0 \mathbb{I}_{\kappa(d-1,l)} (1 + |x|)^{\mu} \leq a(x) \leq c_1 \mathbb{I}_{\kappa(d-1,l)} (1 + |x|)^{\mu} \tag{6.1}$$

for suitable constants $0 < c_0, c_1 < \infty$ and some $\mu \in \mathbb{R}$. Consider the form

$$a[u, u] = \int_{\mathbb{R}^d} \left\langle a(x) \nabla^l u, \nabla^l u \right\rangle_{\mathbb{C}^{\kappa(d-1,l)}} dx, \qquad u \in C_c^{\infty}(\mathbb{R}^d, \mathbb{C}). \tag{6.2}$$

We recall the following

Lemma 6.1. *Fix $\mu \in \mathbb{R}$, $l, d \in \mathbb{N}_+$ and set $m = \left[l - \frac{d+\mu}{2}\right]$. Let a be the form* (6.1), (6.2). *Then*

$$\begin{aligned}
&\Lambda(a) = \{0\} && \text{and} \quad \dim \Lambda(a) = 0 && \text{if} \quad 2l - d < \mu, \\
&\Lambda(a) = \Omega_{d,m} && \text{and} \quad \dim \Lambda(a) = \binom{d+m}{m} && \text{if} \quad 2 - d < \mu \le 2l - d, \\
&\Lambda(a) = \Omega_{d,l-1} && \text{and} \quad \dim \Lambda(a) = \binom{d+l-1}{l-1} && \text{if} \quad \mu \le 2 - d.
\end{aligned}$$

Proof. 1. Fix some function $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$, such that $\psi(t) \equiv 1$ for $t < 1$, $\psi(t) \equiv 0$ for $t > 2$ and $0 \le \psi \le 1$ for $1 \le t \le 2$. Define $\psi_\epsilon \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ by $\psi_\epsilon(x) := \psi(\epsilon \ln |x|)$, $\epsilon > 0$. A short calculation shows, that $u_n(x) = \chi(x)\psi_{n^{-1}}(x)$ is a limit sequence for an element χ from the indicated limit spaces, cf. [33] p 123.

2. In the third case we exhaust the maximal possible space $\Omega_{d,l-1}$, what settles the part $\mu \le 2 - d$.

3. According to Theorem 10.4 in [18] the inequality

$$\int_0^\infty r^\beta |\phi(r)|^2 dr \le c_{6.1}(\beta, l) \int_0^\infty r^{\beta+2l} |\phi^{(l)}(r)|^2 dr, \tag{6.3}$$

holds on all functions $\phi \in C_c(\mathbb{R}_+, \mathbb{C})$ if $\beta > -1$, $\beta \notin 2\mathbb{Z} + 1$, and on all functions $\phi \in C_c(\mathbb{R}_+, \mathbb{C})$ with $\phi^{(k)}(0) = 0$, $k = 0, \ldots, [-(1+\beta)/2]$ if $-1 - 2l < \beta < -1$, $\beta \notin 2\mathbb{Z} + 1$.

Introducing spherical coordinates (r, θ) in $\mathbb{R}^d$, applying (6.3) with $\beta = \mu + d - 2l - 1$ to a function $u(r, \theta)$ for fixed $\theta \in \mathbb{S}^{d-1}$ and integrating over θ one arrives at

$$\int_{\mathbb{R}^d} |x|^{\mu-2l} |u|^2 dx \le c_{6.2}(\mu, l) \int_{\mathbb{R}^d} |x|^\mu |\nabla^l u|^2 dx, \tag{6.4}$$

$$-d < \mu < 2l - d + 2, \qquad 2l - d - \mu \notin 2\mathbb{Z}, \qquad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}),$$
$$\partial^\tau u / \partial x^\tau |_{x=0} = 0 \quad \text{for all} \quad |\tau| \le m \quad \text{if} \quad 2l - d - \mu > 0.$$

4. Consider now the case $2 - d < \mu < 2l - d + 2$, $2l - d - \mu \notin 2\mathbb{Z}$. Suppose there exists a $\chi \in \Lambda(a) \backslash \Omega_{d,m}$. Since $\Lambda(a)$ is a lineal and $\Omega_{d,m} \subseteq \Lambda(a)$ it is sufficient to consider $\chi \in \Omega_{d,l-1}$, which do not contain monomials up to the degree m. By Remark 5.1 let u_n be a limit sequence for χ, such that $u_n(x) = \chi(x)$ for all $|x| < 1$. Then u_n satisfies the conditions of (6.4), and

$$0 < \int_{|x| \le 1} |x|^{\mu-2l} |\chi|^2 dx \le \int_{|x| \ge 1} |x|^\mu |\nabla^l u_n|^2 dx \le c_{6.3} a[u_n, u_n], \tag{6.5}$$

what contradicts to u_n being a singular sequence. Hence $\Lambda(a) = \Omega_{d,m}$. If $2l - d - \mu \in 2\mathbb{Z}$ or $\mu \ge 2l - d + 2$ the same argument holds, if we replace μ in the integrals in (6.5) by $2l - d - \frac{1}{2}$ or $2l - d + \frac{3}{2}$, respectively. $\square$

6.2. Let $V(x)$ be a hermitian $\binom{d-1+k}{k} \times \binom{d-1+k}{k}$-matrix function $V_{\tau,\iota}(x) \in L_1^{\text{loc}}(\mathbb{R}^d, \mathbb{C})$, $\iota, \tau \in \mathbb{N}^d : |\iota|, |\tau| = k$, $0 \le k < l$, $k \in \mathbb{N}$. Then we define

$$v[u,u] := \int_{\mathbb{R}^d} \langle V(x)\nabla^k u, \nabla^k u\rangle_{\mathbb{C}^{\kappa(d,m)}}\, dx, \qquad u \in d[a] = C_c^\infty(\mathbb{R}^d, \mathbb{C}). \tag{6.6}$$

The non-negative forms $v_\pm$ can be obtained, if one replaces $V(x)$ by $V_\pm(x)$ in (6.6), respectively. If (4.2) holds, the operators $A(\alpha)$, $\tilde{A}(\alpha)$ defined in accordance to (4.3) are self-adjoint realizations of

$$\begin{aligned} A(\alpha) &= (-1)^l (\nabla^l)^T a(x) \nabla^l - \alpha(-1)^k (\nabla^k)^T V(x) \nabla^k, \\ \tilde{A}(\alpha) &= (-1)^l (\nabla^l)^T a(x) \nabla^l - \alpha(-1)^k (\nabla^k)^T (V_+(x) + V_-(x)) \nabla^k, \end{aligned} \tag{6.7}$$

on $L_2(\mathbb{R}^d, \mathbb{C})$.

Theorem 6.1. *Assume that* $2l \ge d + \mu$, $0 \le k < l$, $k \in \mathbb{N}$, $d, l \in \mathbb{N}_+$, *and set* $m = \min\left\{l-1, \left[l - \frac{d+\mu}{2}\right]\right\}$, $r = \binom{d+m}{d} - \binom{d+k-1}{d}$ *for* $n = 0, \ldots, d$. *Assume that for the forms* a *in* (6.1), (6.2) *and* v, $v_\pm$ *in* (6.6) *the inequalities* (4.2) *hold and that the operator* $\tilde{A}(\alpha)$ *in* (6.7) *has finite negative spectrum for some* $\alpha > 0$. *Then*

$$\lim_{\alpha\to+0} n_-(A(\alpha), 0) = 0 \qquad \textit{for} \quad k > m, \tag{6.8}$$

$$\lim_{\alpha\to+0} n_-(A(\alpha), 0) = \mathfrak{n}_+ + \mathfrak{n}_0 - \mathfrak{n}_{0,0} = r - \mathfrak{n}_- - \mathfrak{n}_{0,0} \qquad \textit{for} \quad k \le m. \tag{6.9}$$

Herein $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_0$ *denotes the signature of the* $r \times r$ *matrix*

$$\left\{ \int_{\mathbb{R}^d} \sum_{\iota,\tau\in\mathbb{N}^d : |\iota|,|\tau| = k} (\partial_x^\tau x^\nu)\,(\partial_x^\iota x^\varkappa)\, V_{\tau,\iota}(x) dx \right\}_{\varkappa,\nu\in\mathbb{N}^d : k\le|\varkappa|,|\nu|\le m}, \tag{6.10}$$

while $\mathfrak{n}_{0,0}$ *is the dimension of the lineal of all polynomials* $\chi(x) = \sum_{\nu\in\mathbb{N}^d : k\le|\nu|\le m} c_\nu x^\nu$ *such that*

$$(\nabla^k)^T V(x) \nabla^k \chi(x) = 0 \quad \textit{as distrbution in} \quad \mathcal{D}'. \tag{6.11}$$

Proof. We apply Corollary 5.2 with $\Lambda(a) = \Omega_{d,m}$. On $\Omega_{d,m}$ we introduce the Hilbert structure, in which the monomials form an orthonormed base. Evidently $v[\chi, u] = 0$ for all $\chi \in \Omega_{d,k-1}$ and all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Hence we can reduce our considerations to the restriction of the form v to the lineal of polynomials, generated by the monomials of degree $k, \ldots, m$, that is matrix (6.10). The integrals in this expression converge because of Lemma 5.3. The condition $v[\chi, u] = 0$ for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ is equivalent to (6.11). $\square$

6.3. Let us emphasize on the special case $k=0$. For $V \in L_1^{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$ set

$$v_\pm[u,u] = \int V_\pm |u|^2 dx, \qquad v[u,u] = v_+[u,u] - v_-[u,u] \tag{6.12}$$

on functions $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$. The closures of these forms induce the operators of multiplication by V, V_+, V_- on $L_2(\mathbb{R}^d, \mathbb{C})$. Assume that (4.2) is satisfied. Then (4.3) defines

$$A(\alpha) := (-1)^l (\nabla^l)^T a(x) \nabla^l - \alpha V, \qquad \tilde{A}(\alpha) := (-1)^l (\nabla^l)^T a(x) \nabla^l - \alpha (V_+ + V_-) \tag{6.13}$$

on $L_2(\mathbb{R}^d, \mathbb{C})$ in the form sum sense (4.3) for sufficiently small $\alpha > 0$.

Corollary 6.1. *Let a be the form (6.2) with $2l \geq d + \mu$, $d, l \in \mathbb{N}_+$ and let v be the form (6.12). Assume that (4.2) holds and that the operator $\tilde{A}(\alpha)$ in (6.13) has finite negative spectrum for some $\alpha > 0$. Set $m = \min\left\{l-1, \left[l - \frac{d+\mu}{2}\right]\right\}$. Then $(1 + |x|^{2m}) V(x) \in L_1(\mathbb{R}^d, \mathbb{R})$, and for the operators (6.13) it holds*

$$\lim_{\alpha \to +0} n_-(\tilde{A}(\alpha), 0) = \binom{d+m}{m}, \qquad \lim_{\alpha \to +0} n_-(A(\alpha), 0) = \mathfrak{n}_+ + \mathfrak{n}_0 = \binom{d+m}{m} - \mathfrak{n}_-. \tag{6.14}$$

Herein $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{n}_0$ denotes the signature of the matrix of the momenta of V:

$$\left\{ \int_{\mathbb{R}^d} x_1^{\iota_1 + \tau_1} \cdots x_d^{\iota_d + \tau_d} V(x) dx \right\}_{\iota, \tau \in \mathbb{N}^d : |\iota|, |\tau| \leq m}.$$

Proof. The first part of the statement follows from (5.12) and Corollary 5.1. Lemma 5.3 implies $(1 + |x|^{2m}) V(x) \in L_1(\mathbb{R}^d, \mathbb{R})$. In view of Corollary 5.2 it remains to show, that there are no non-trivial functions $\chi \in \Omega_{d,m}$ for which

$$\int_{\mathbb{R}^d} V(x) \chi(x) \overline{u(x)} dx = 0$$

for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Indeed, for such a χ we would have $V(x)\chi(x) \equiv 0$, that is $V(x) = 0$ a.e. what gives the contradiction. $\square$

Remark 6.1. Set for example $\mu = 0$. Then according to [14], Theorems 6 and 7, pp. 130-131 the assumptions

$$\begin{aligned} &(1 + |x|)^{2l-d} V(x) \in L_1(\mathbb{R}^d, \mathbb{R}) \quad &&\text{if } d \text{ is odd}, \quad d < 2l, \\ &(1 + |x|)^{2l-d+\epsilon} V(x) \in L_1(\mathbb{R}^d, \mathbb{R}) \quad &&\text{if } d \text{ is even}, \quad d \leq 2l, \quad 0 < \epsilon, \end{aligned} \tag{6.15}$$

guarantee Condition 4.1 to be true in the setting of Corollary 6.1. Both assumptions can be relaxed significantly, see e.g. [9].

Remark 6.2. If $k = \mu = 0$, $d = 1, 2$ and $l = 1$ the momenta matrix has only the entree $\int V dx$. Then we arrive at Simons result [32], that the Schrödinger operator $-\Delta - \alpha V$ in the dimensions $d = 1, 2$ has negative spectrum for arbitrary $\alpha > 0$, if and only if $\int V dx \geq 0$. Condition 4.1 replaces the analytic assumptions on V in [32]. We point out, that in [32] the asymptotics of the ground state for $\alpha \to +0$ has been calculated.

Remark 6.3. Set $\mu = 0$ and consider the form

$$a[u,u] := \int_{\mathbb{R}^d} \left(|\nabla^{l_0} u|^2 + |\nabla^{l_1} u|^2 \right) dx, \qquad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}), \quad l_0, l_1 \in \mathbb{N}_+, \tag{6.16}$$

instead of (6.2). Following the previous arguments one finds, that $\Lambda(a) = \{0\}$ for $d > 2l$ and $\Lambda(a) = \Omega_{d,m}$, $m = \left[l - \frac{d}{2} \right]$, where $l := \min\{l_0, l_1\}$. Consequently Theorems 6.1 and Corollary 6.1 remain true, if we replace l by $\min\{l_0, l_1\}$.

7. Second order matrix operators

7.1. In this section we study virtual bound states for systems of elliptic equations. For the sake of simplicity we reduce ourself to second order differential operators, perturbed by a potential matrix.

Consider a connected, open domain $\Xi \subset \mathbb{R}^d$. Let the form

$$a[u,u] = \int_\Xi \sum_{j,k=1}^d \left\langle a^{(j,k)} \partial_{x_k} u, \partial_{x_j} u \right\rangle_{\mathbb{C}^s} dx, \tag{7.1}$$

on the domain $d[a]$,

$$C_0^\infty(\Xi, \mathbb{C}^s) \subseteq d[a] \subseteq C_c^\infty(a; \Xi, \mathbb{C}^s) := \{u \in C_c^\infty(\Xi, \mathbb{C}^s) : \; a[u,u] < \infty\},$$

be induced by a hermitian $sd \times sd$-matrix function

$$a(x) = \left\{ a^{(j,k)}(x) \right\}_{j,k=1}^d, \tag{7.2}$$

consisting of measurable $s \times s$-matrices $a^{(j,k)}(x) = \left\{ a_{p,q}^{(j,k)}(x) \right\}_{p,q=1}^s$, such that

$$\rho_0(x) \mathbb{I}_{sd} \leq a(x) \leq \rho_1(x) \mathbb{I}_{sd} \tag{7.3}$$

holds for suitable positive functions ρ_0, ρ_1, which are continuous in Ξ.

The closure of a generates a self-adjoint realization of the differential operator

$$A := - \sum_{j,k=1}^d \partial_{x_j} a^{(j,k)}(x) \partial_{x_k}$$

on $L_2(\Xi, \mathbb{C}^s)$. The limit space $\Lambda(a)$ may consists of constant functions only. These functions can be identified with their values in $\mathbb{C}^s$, and thus $\Lambda(a)$ corresponds to a subspace $\mathbb{L}(a) \subseteq \mathbb{C}^s$. Below $\mathbb{P}(a)$ is the orthogonal projection in $\mathbb{C}^s$ onto $\mathbb{L}(a)$.

7.2. The perturbation form

$$v[u,u] = \int_\Xi \langle V(x)u, u\rangle_{\mathbb{C}^s}\, dx, \qquad u \in d[v] = d[a], \tag{7.4}$$

corresponds to a hermitian $s \times s$-matrix $V(x) = \{v_{m,n}(x)\}_{m,n=1}^s$, $v_{m,n} \in L_1^{\mathrm{loc}}(\Xi, \mathbb{C})$. The subspace $\mathbb{M}(V) \subseteq \mathbb{C}^s$ consists of all vectors $\xi \in \mathbb{C}^s$, such that $\xi \in \ker V(x)$ for a.e. $x \in \Xi$. With $V_\pm(x)$ being the positive and negative part of $V(x)$ set

$$v_\pm[u,u] := \int_\Xi \langle V_\pm u, u\rangle\, dx, \qquad u \in d[a]. \tag{7.5}$$

We assume that for a, v and $v_\pm$ the bounds (3.8), (4.2) hold and introduce the operators

$$\begin{aligned} A(\alpha) &:= -\sum_{j,k=1}^d \partial_{x_j} a^{(j,k)}(x) \partial_{x_k} - \alpha V(x), \\ \tilde{A}(\alpha) &:= -\sum_{j,k=1}^d \partial_{x_j} a^{(j,k)}(x) \partial_{x_k} - \alpha \left(V_+(x) + V_-(x)\right), \end{aligned} \tag{7.6}$$

in accordance to (4.3).

Theorem 7.1. *Let a the form (7.1)–(7.3) with the limit space $\Lambda(a)$. Assume, that for v in (7.5) the inequality (4.2) holds, that the operator $\tilde{A}(\alpha)$ in (7.6) has finite negative spectrum for some $\alpha > 0$, and set $\dim \Lambda(a) = k$. Then $\mathbb{P}(a)V(x)\mathbb{P}(a) \in L_1(\Xi, GL(s,s))$ and*

$$\lim_{\alpha \to +0} n_-(A(\alpha), 0) = \mathfrak{n}_+ + \mathfrak{n}_0 - \dim\left(\mathbb{M}(V) \cap \mathbb{L}(a)\right), \tag{7.7}$$

where $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{n}_0$ is the signature of the $k \times k$-matrix $\tilde{V}$, being the natural restriction of

$$\tilde{V} \oplus_{\mathbb{C}^s} \mathbb{O}_{s-k} = \int_\Xi \mathbb{P}(a)V(x)\mathbb{P}(a)dx$$

to a mapping from $\mathbb{L}(a)$ to $\mathbb{L}(a)$.

Proof. The integrability follows from Lemma 5.3 and (7.7) is an consequence of Theorem 4.1. For this it only remains to show that $\mathfrak{n}_{0,0} = \dim\left(\mathbb{M}(V) \cap \mathbb{L}(a)\right)$. Indeed, if $\xi \in \mathbb{L}(a) \subseteq \mathbb{C}^s$ and

$$\int_\Xi \langle V(x)\xi, u(x)\rangle\, dx = 0 \qquad \text{for all} \quad u \in d[a], \tag{7.8}$$

then $V(x)\xi \equiv 0$ and hence $\xi \in \mathbb{M}(V)$. $\square$

For example set $\Xi = \mathbb{R}^d$, $d[a] = C_c^\infty(\mathbb{R}^d, \mathbb{C}^s)$ and $\rho_1(x) \le C(1+|x|)^\alpha$ with $\alpha \le 2l-d$. As in Lemma 6.1 we conclude $\mathbb{L}(a) = \mathbb{C}^s$. In particular, this holds for the Schrödinger operator with a matrix potential acting on vector functions in the case of spatial dimensions one and two.

8. Critical Schrödinger operators

8.1. Sometimes it is possible to reduce the problem on virtual bound states for more complicated operators to the setting of Corollary 5.2 by a suitable transformation. In particular this is the case for critical Schrödinger operators. The respective problem regarding virtual bound states has been solved by Pinchover [24], using entirely different methods. For the notion of criticality, although formally not used below, we point among others to [12], [22]-[25] and the references therein. There is also an extensive literature on the existence of positive solutions and the so-called principal eigenvalues, see among others [2]-[4], [26], [27], [20]. Our construction of a suitable positive solution of the Schrödinger operator follows [29].

8.2. In this section we set $\Xi = \mathbb{R}^d$ and

$$a[u,u] = \int_{\mathbb{R}^d} |\nabla u|^2 dx, \qquad u \in d[a] = C_c^\infty(\mathbb{R}^d, \mathbb{C}). \tag{8.1}$$

For a function $W \in L_1^{\rm loc}(\mathbb{R}^d, \mathbb{R})$ define

$$w[u,u] = \int_{\mathbb{R}^d} W(x)|u(x)|^2 dx, \qquad u \in d[w] = C_c^\infty(\mathbb{R}^d, \mathbb{C}). \tag{8.2}$$

The forms $w_\pm$, $w = w_+ - w_-$ correspond to the functions $W_\pm$, respectively. We assume, that $w_\pm$ are zero-bounded w.r.t. a. Then $a^{(\beta)} := a - \beta w$ is a closable form for all $\beta \ge 0$. Its closure induces the self-adjoint operator

$$A^{(\beta)} = -\Delta - \beta W, \qquad \beta \ge 0, \tag{8.3}$$

on $L_2(\mathbb{R}^d, \mathbb{C})$. The lower bound $m(a^{(\beta)})$ of $A^{(\beta)}$ is continuous in β and $\lim_{\beta\to\infty} m(a^{(\beta)}) = \infty$ if $W_- \not\equiv 0$. Moreover if $m(a^{(\tilde\beta)}) < 0$ for some $\tilde\beta > 0$, then $m(a^{(\beta)})$ is a monotone decreasing for all $\beta > \tilde\beta$. Set

$$\beta_0 := \max\{\beta \ge 0 : \ m(a^{(\beta)}) \ge 0\}. \tag{8.4}$$

Then it holds

$$a^{(\beta)}[u,u] \ge (\beta_0 - \beta) w[u,u], \qquad 0 \le \beta \le \beta_0, \quad u \in d[a]. \tag{8.5}$$

Below we shall assume that the inequality

$$w_+[u,u] \le \theta w_-[u,u] + c_{8.1}(\theta) a[u,u] \tag{8.6}$$

holds for some $0 \leq \theta < 1$ and a suitable finite constant $c_{8.1}(\theta)$.

Condition 8.1. *The number β_0 is positive and there exists a $\beta_1 > \beta_0$, such that the negative spectrum of the operator $A^{(\beta_1)}$ is finite.*

Fix a $\beta \in (0, 1/c_{8.1}(\theta))$. By (8.6) the form metric $a^{(\beta)} = a^{(\beta_0)} + (\beta_0 - \beta)w$ is equivalent to $a + w_-$ on $C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Hence the completion $\mathfrak{H}[a^{(\beta)}]$ of $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ w.r.t. $a^{(\beta)}$ can be realized as a subset of $W_{2,1}^{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$. Let $\mathfrak{W}$ be the bounded operator induced by w on $\mathfrak{H}[a^{(\beta)}]$. Since $a^{(\beta_0)} \geq 0$, this operator has no spectrum above $\beta_0 - \beta > 0$. Then Glazman's Lemma implies

$$n_-(A^{(\delta)}, 0) = n_+\left(\mathfrak{W}, (\delta - \beta)^{-1}\right) < \infty, \qquad \delta > \beta.$$

By Condition 8.1 and the definition of β_0 the point $(\beta_0 - \beta)^{-1}$ is an isolated eigenvalue of finite multiplicity for $\mathfrak{W}$ with the eigenspace $\mathfrak{H}_0 \subset \mathfrak{H}[h^{(\beta)}] \subset W_{2,1}^{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$. Hence the upper bound $(\beta - \beta_0)^{-1}$ of the operator $\mathfrak{W}$ is achieved on some element $0 \not\equiv \phi \in \mathfrak{H}_0$, what implies

$$a^{(\beta_0)}[\phi, u] = 0 \qquad \text{for all} \quad u \in \mathfrak{H}[a^{(\beta)}] \supset C_c^\infty(\mathbb{R}^d, \mathbb{C}). \tag{8.7}$$

Let $\phi \in \mathfrak{H}_0$, such that $\Re\phi \not\equiv 0$. Then $|\Re\phi| \in \mathfrak{H}_0$. Hence $\mathfrak{H}_0$ contains a non-negative distributional solution $0 \leq \phi \in W_{2,1}^{\mathrm{loc}} \cap \mathfrak{H}_0$ of the equation

$$(-\Delta - \beta_0 W)\,\phi = 0\ . \tag{8.8}$$

In addition assume $W \in K_d^{\mathrm{loc}}$, that is for any compact $\Omega \subset \mathbb{R}^d$ it holds

$$\lim_{t \to +0} \sup_{x \in \Omega} \int_{|y-x| \leq t} |x-y|^{2-d} |W(y)| dy = 0 \qquad \text{if} \quad d \geq 3,$$

$$\lim_{t \to +0} \sup_{x \in \Omega} \int_{|y-x| \leq t} \ln|x-y|^{-1} |W(y)| dy = 0 \qquad \text{if} \quad d = 2,$$

$$\sup_{x \in \Omega} \int_{|x-y| \leq 1} |W(y)| dy < \infty \qquad \text{if} \quad d = 1.$$

Then ϕ is (a.e.) a continuous function, see e.g. Theorem 2.4 in [11], and by Harnacks inequality there exist two positive continuous functions ρ_0 and ρ_1, such that

$$\rho_0(x) \leq \phi(x) \leq \rho_1(x), \qquad x \in \mathbb{R}^d, \tag{8.9}$$

see e.g. [11] Theorem 2.5. But then $u\phi^{-1} \in \mathfrak{H}[a^{(\beta)}]$ for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Applying (8.7) one concludes

$$a^{(\beta_0)}[\phi, u\phi^{-1}] = 0 \qquad \text{for all} \quad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}).$$

This equation implies the well-known identity [13]

(8.10)
$$a^{(\beta_0)}[u,u] = a_\phi\left[u\phi^{-1}, u\phi^{-1}\right] := \int_{\mathbb{R}^d} \phi^2 \left|\nabla(u\phi^{-1})\right|^2 dx, \qquad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}).$$

For applications of (8.10) see among others [17] and the references therein.

8.3. Consider a second potential $V \in L_1^{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$ and introduce the forms

$$\begin{aligned} v[u,u] &:= \int_{\mathbb{R}^d} V|u|^2 dx = \int_{\mathbb{R}^d} \phi^2 V \left|u\phi^{-1}\right|^2 dx =: v_\phi\left[u\phi^{-1}, u\phi^{-1}\right], \\ v_\pm[u,u] &:= \int_{\mathbb{R}^d} V_\pm|u|^2 dx = \int_{\mathbb{R}^d} \phi^2 V_\pm \left|u\phi^{-1}\right|^2 dx =: v_{\pm,\phi}\left[u\phi^{-1}, u\phi^{-1}\right], \end{aligned} \tag{8.11}$$

with $u \in d[v_\pm] = d[v] = C_c^\infty(\mathbb{R}^d, \mathbb{C})$. If the inequalities

$$v_\pm[u,u] \le c_{8.2}^\pm a[u,u] + c_{8.3}^\pm \|u\|^2_{L_2(\mathbb{R}^d,\mathbb{C})}, \qquad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}), \tag{8.12}$$

hold, then the forms $a^{(\beta_0)} - \alpha(v_+ - v_-)$ and $a^{(\beta_0)} - \alpha(v_+ + v_-)$ are semi-bounded form below and closable for sufficiently small $\alpha > 0$. Their closures induce the self-adjoint operators

$$\begin{aligned} A^{(\beta_0)}(\alpha) &= A^{(\beta_0)} - \alpha V &&= -\Delta - \beta_0 W - \alpha V, \\ \tilde{A}^{(\beta_0)}(\alpha) &= A^{(\beta_0)} - \alpha(V_+ + V_-) &&= -\Delta - \beta_0 W - \alpha(V_+ + V_-), \end{aligned} \tag{8.13}$$

on $L_2(\mathbb{R}^d, \mathbb{C})$. By a density argument (8.10) and (8.11) imply

$$\begin{aligned} n_-(A^{(\beta_0)}(\alpha), 0) &= \sup\{\dim \mathcal{F}_1 :\ a_\phi[u,u] - \alpha v_\phi[u,u] < 0 \quad && \forall\, 0 \not\equiv u \in \mathcal{F}_1\}, \\ n_-(\tilde{A}^{(\beta_0)}(\alpha), 0) &= \sup\{\dim \mathcal{F}_2 :\ a_\phi[u,u] - \alpha(v_{+,\phi} + v_{-,\phi})[u,u] < 0 \quad && \forall\, 0 \not\equiv u \in \mathcal{F}_2\}, \end{aligned}$$

where $\mathcal{F}_j$ are lineals in $C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Therefore the study of the negative spectrum of the operator $A^{(\beta_0)}(\alpha)$ is reduced to the study of the form $a_\phi - \alpha v_\phi$, and we are in the setting of section 5. In particular set $V = W$, then

$$0 < \lim_{\alpha \to +0} n_-(A^{(\beta_0+\alpha)}, 0) = \dim \mathfrak{H}_0 = \dim \Lambda(a_\phi) \le \dim \Omega_{d,0} = 1.$$

Hence the function ϕ is uniquely defined up to multiplication by a positive constant and $\Lambda(a_\phi)$ consists of the constant functions. Corollary 5.2 now implies

Theorem 8.1. *Let a be the form* (8.1). *Assume that for the real-valued function $0 \not\equiv W \in K_d^{\mathrm{loc}}$ the bound* (8.6) *and Condition* 8.1 *hold and that the forms $w_\pm$ are zero-bounded w.r.t.* (8.1). *Let for the potential $0 \not\equiv V \in L_1^{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$ the inequality* (8.12) *be satisfied, and assume that the operator $\tilde{A}^{(\beta_0)}(\alpha)$ in* (8.13) *has finite negative* *spectrum for some $\alpha > 0$. Finally let $\phi \in \mathfrak{H}_0$ be the positive distributional solution of* (8.8). *Then*

$$
\begin{aligned}
&\lim_{\alpha\to+0} n_-(A^{(\beta_0)}(\alpha),0) = 1 \qquad \text{if} \quad \int_{\mathbb{R}^d} V(x)\phi^2(x) \geq 0,\\
&\lim_{\alpha\to+0} n_-(A^{(\beta_0)}(\alpha),0) = 0 \qquad \text{if} \quad \int_{\mathbb{R}^d} V(x)\phi^2(x) < 0\ .
\end{aligned}
\tag{8.14}
$$

Remark 8.1. If the operators $-\Delta - \beta W_+$ and $-\Delta - \alpha(V_+ + V_-)$ have finite negative spectrum for all $\alpha, \beta > 0$, then $\tilde{A}^{(\beta_0)}(\alpha)$ has a finite negative spectrum for all $\alpha > 0$.

Remark 8.2. Since $|x|^{-2} \notin K_d^{\mathrm{loc}}$, Theorem 8.1 cannot be applied to Schrödinger operators with subtracted "Hardy terms"

$$-\Delta - 4^{-1}(d-2)^2|x|^{-2} - \alpha V(x)$$

on $L_2(\mathbb{R}^d, \mathbb{C})$, $d \geq 3$. However, for such operators an analogous result holds with $\phi(x) = |x|^{(2-d)/2}$.

9. Schrödinger type operators with magnetic fields.

9.1. Let $\mathfrak{a}(x) = (\mathfrak{a}_1(x), \ldots, \mathfrak{a}_d(x))$ be a real valued vector field on an open, connected domain $\Xi \subseteq \mathbb{R}^d$ and

$$\mathfrak{a} \in L_d^{\mathrm{loc}}(\Xi, \mathbb{R}^d) \quad \text{if} \quad d > 2, \qquad \text{or} \qquad \mathfrak{a} \in L_{2+\epsilon}^{\mathrm{loc}}(\Xi, \mathbb{R}^2) \quad \text{if} \quad d = 2. \tag{9.1}$$

Assume that $a(x)$ is a $d \times d$-matrix function with hermitian values, and

$$\rho_0(x)\mathbb{I}_d \leq a(x) \leq \rho_1(x)\mathbb{I}_d, \tag{9.2}$$

hold for suitable continuous functions $0 < \rho_0, \rho_1 < \infty$ and a.e. $x \in \Xi$. Then

$$a(\mathfrak{a})[u,u] = \int_\Xi \langle a(x)(i\nabla + \mathfrak{a})u, (i\nabla + \mathfrak{a})u\rangle\, dx, \tag{9.3}$$

defines a non-negative closable magnetic form on some domain $d[a(\mathfrak{a})]$,

$$C_0^\infty(\Xi, \mathbb{C}) \subseteq d[a(\mathfrak{a})] \subseteq C_c^\infty(a(\mathfrak{a}); \Xi, \mathbb{C}) := \{u \in C_c^\infty(\Xi, \mathbb{C}) :\ a(\mathfrak{a})[u,u] < \infty\}\ .$$

Its closure generates a self-adjoint realization of the operator

$$A(\mathfrak{a}) = (i\nabla + \mathfrak{a})^T a(x)(i\nabla + \mathfrak{a})\ . \tag{9.4}$$

In case of $\Xi = \mathbb{R}^d$, $a(x) = \mathbb{I}$ this definition gives the "free" magnetic Schrödinger operator

$$Q(\mathfrak{a}) = (i\nabla + \mathfrak{a})^T(i\nabla + \mathfrak{a}), \tag{9.5}$$

corresponding to the closure of the form

$$q(\mathfrak{a})[u,u] := \int_{\mathbb{R}^d} |(i\nabla + \mathfrak{a})u|^2 dx, \qquad d[q(\mathfrak{a})] = C_c^\infty(\mathbb{R}^d, \mathbb{C}) \ . \tag{9.6}$$

For magnetic Schrödinger operators in [5] so-called extreme properties have been obtained. In particular it holds

$$\int_{\mathbb{R}^d} |u|^2 |x|^{-2} dx \le 4(d-2)^{-2} q(\mathfrak{a})[u,u], \qquad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}), \quad d \ge 3.$$

It turns out, that for generic magnetic forms Hardy-type inequalities hold, even if such inequalities fails for the corresponding "non-magnetic" form (5.3). In this generic means, that the vector field $\mathfrak{a}$ cannot be removed by a global gauge transformation. This shall be discussed below.

9.2. In view of the regularity assumption (9.1) the magnetic form is locally equivalent to the W_2^1-norm, see e.g. [7], [37]:

Lemma 9.1. *Let $\mathfrak{a}$ be the vector field* (9.1)*. Assume that $\mathcal{U} \subset \Xi$ is a bounded, connected Lipschitz domain and let $\Theta \subseteq \mathcal{U}$ be an arbitrary set of positive Lebesgue measure. Then the form metric*

$$\|(i\nabla + \mathfrak{a}) \cdot \|^2_{L_2(U, \mathbb{C}^d)} + \| \cdot \|^2_{L_2(\Theta, \mathbb{C})} \tag{9.7}$$

defines an equivalent norm on $W_2^1(\mathcal{U}, \mathbb{C})$.

The function $\phi \in W_{2,1}^{\mathrm{loc}}(\Xi, \mathbb{C})$ is said to be a *limit element* for $a(\mathfrak{a})$ if there exists a *limit sequence* $\{u_n\} \subset d[a(\mathfrak{a})]$ for it:

$$u_n \to \phi \quad \text{in} \quad W_{2,1}^{\mathrm{loc}}(\Xi, \mathbb{C}) \quad \text{and} \quad a(\mathfrak{a})[u_n, u_n] \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\Lambda(a(\mathfrak{a}))$ denote the lineal of all limit elements of $a(\mathfrak{a})$.

Lemma 9.2. *Let $\Theta \subset \Xi$ be a compact set of positive Lebesgue measure and let $a(\mathfrak{a})$ be a form* (9.3) *for which* (9.1), (9.2) *are satisfied. Then* $\dim \Lambda(a(\mathfrak{a})) = 0$ *if and only if*

$$\|u\|^2_{L_2(\Theta, \mathbb{C})} \le c_{9.1} a(\mathfrak{a})[u,u], \qquad u \in d[a(\mathfrak{a})]. \tag{9.8}$$

Any limit element $\phi \in \Lambda(a(\mathfrak{a}))$ is a solution of $(i\nabla + \mathfrak{a})\phi = 0$ on Ξ.

Proof. We adapt the proof of Lemma 5.1. If (9.8) holds then obviously $\dim \Lambda(a(\mathfrak{a})) = 0$. Fix the monotone increasing exhausting sequence of open, connected, bounded Lipschitz domains $\mathcal{U}_m$, $\overline{\mathcal{U}_m} \subset \Xi$, such that $\Theta \subset \mathcal{U}_1$. Assume that there exists a $\{u_n\} \subset d[a(\mathfrak{a})]$ for which

$$\mathrm{meas}(\Theta) = \|u_n\|^2_{L_2(\Theta, \mathbb{C})} > n a(\mathfrak{a})[u_n, u_n]. \tag{9.9}$$

By Lemma 9.1 we consecutively select subsequences $\{u_{n,m}\} \subset \{u_{n,m-1}\}$, for which $u_{n,m} \to \phi$ in $W_2^1(\mathcal{U}_m, \mathbb{C})$ and $\|(i\nabla + \mathfrak{a})u_{n,m}\|^2_{L_2(\mathcal{U}_m, \mathbb{C}^d)} \to 0$ as $n \to \infty$. By (9.1) the operator $i\nabla + \mathfrak{a}$ is closed on $W_2^1(\mathcal{U}_m, \mathbb{C})$. Hence ϕ satisfies

$$(i\nabla + \mathfrak{a})\phi|_{\mathcal{U}_m} = 0, \qquad \phi \in W_2^1(\mathcal{U}_m, \mathbb{C}). \tag{9.10}$$

From (9.10) and its complex conjugate one concludes $\nabla|\phi|^2 = 0$. Hence by (9.9) $|\phi| = 1$ on $\mathcal{U}_m$. The diagonal subsequence $\{u_{n,n}\}$ is singular and its limit ϕ solves the equation $(i\nabla + \mathfrak{a})\phi = 0$ globally. Repeading this procedure for an arbitrary non-trivial limit element ϕ, its limit sequence $\{u_n\}$ and a suitable compact $\Theta \subset \Xi$ on which $\|\phi\|_{L_2(\Theta, \mathbb{C})} > 0$, we conclude the proof. $\square$

9.3. Assume now that ϕ is a non-trivial solution of the equation

$$(i\nabla + \mathfrak{a})\phi = 0, \qquad \phi \in W_{2,1}^{\mathrm{loc}}(\Xi, \mathbb{C}). \tag{9.11}$$

Since $|\phi(x)|$ is a constant, we can normalize such that $|\phi(x)| \equiv 1$. Then the multiplication by ϕ and $\overline{\phi}$ are unitary mappings in $L_2(\Xi, \mathbb{C})$, and

$$a_\phi[u,u] = \int_\Xi \langle a(x)\nabla u, \nabla u\rangle\, dx = a(\mathfrak{a})[\phi u, \phi u], \qquad u \in d[a_\phi] := \overline{\phi}\mathcal{H}_\gamma[a(\mathfrak{a})], \quad \gamma > 0.$$

The form a_ϕ is closed on $d[a_\phi]$ and the inequalities (9.8) for $a(\mathfrak{a})$ and (5.7) for a_ϕ fail simultaneously. That means

Theorem 9.1. *Assume that* (9.1) *and* (9.2) *hold. Then* $\dim \Lambda(a(\mathfrak{a})) \neq 0$ *if and only if there is a solution* $\phi \not\equiv 0$ *of the equation* (9.11) *and* $\dim \Lambda(a_\phi) \neq 0$.

Corollary 9.1. *Assume that* (9.1) *and* (9.2) *hold and that the equation* (9.11) *has no solution* $\phi \not\equiv 0$. *Let* Θ *be a compact subset of* Ξ. *Then the inequality* (9.8) *holds.*

Remark 9.1. In (9.11) we seek for a global solution ϕ. Hence magnetic forms are regular even for topological non-trivial domains Ξ and *locally* removable vector fields $\operatorname{rot}\, \mathfrak{a} = 0$, if only $\mathfrak{a}$ cannot be removed *globally.* This setting is typical for the Aharonov–Bohm effect.

Remark 9.2. Let V be a bounded, real-valued function with compact support $\operatorname{supp} V \subset \Xi$. Assume that (9.1) and (9.2) hold and set

$$A(\mathfrak{a}, \alpha) := A(\mathfrak{a}) - \alpha V, \qquad \alpha > 0. \tag{9.12}$$

If the equation (9.11) has no non-trivial solution, then

$$\lim_{\alpha \to +0} n_-(A(\mathfrak{a}, \alpha), 0) = 0.$$

10. The Pauli operator in dimension two with a localized magnetic field

10.1. In contrast to Schrödinger type operators with magnetic field the Pauli operator with magnetic field in dimension two has virtual bound states at zero. This fact is closely related to the Aharonov-Casher Theorem, see [1] and [11] Theorem 6.5. The author is grateful to A. Sobolev for valuable remarks when writing this section.

10.2. Put $\Xi = \mathbb{R}^d$ with $d = 2$ and fix some $\epsilon > 0$. We consider a real-valued vector field

$$\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2) \in W_{2,1}^{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \cap L_{2+\epsilon}^{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \tag{10.1}$$

and assume, that the magnetic field

$$\mathfrak{b}(x) := \partial_{x_1}\mathfrak{a}_2(x) - \partial_{x_2}\mathfrak{a}_1(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \tag{10.2}$$

is bounded and of compact support. The normalized magnetic flux is given by

$$\Phi := (2\pi)^{-1} \int_{\mathbb{R}^2} \mathfrak{b} dx.$$

The form $q(\mathfrak{a})$ and the respective operator $Q(\mathfrak{a})$ are defined as in (9.6), (9.5).

Let $W \in L_1^{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ be a non-negative function and consider the form

$$w[u, u] = \int_{\mathbb{R}^2} W|u|^2 dx, \qquad u \in C_c^\infty(\mathbb{R}^2, \mathbb{C}). \tag{10.3}$$

If w is zero-bounded w.r.t. $q(0)$, then by Theorem 2.7 in [5] w is zero-bounded w.r.t. $q(\mathfrak{a})$ and the operator

$$Q(\mathfrak{a}, \alpha) := Q(\mathfrak{a}) - \alpha W$$

is well-defined in the form sum sense for all $\alpha > 0$. The proof of the following statement bases on Lemma 9.1 and on the fact, that outside the support of $\mathfrak{b}$ the magnetic field can be locally removed. It uses standard variational arguments and shall be omited.

Lemma 10.1. *Assume that* (10.1) *holds and that* $\mathfrak{b}$ *is bounded and of compact support. If* $n_-(Q(0, \alpha), 0) < \infty$ *for all* $\alpha > 0$ *then* $n_-(Q(\mathfrak{a}, \alpha), 0) < \infty$ *for all* $\alpha > 0$.

10.3. Let

$$\varsigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \varsigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \varsigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices. We consider the non-negative form

$$p(\mathfrak{a})[u, u] := \int_{\mathbb{R}^2} |(D_1\varsigma_1 + D_2\varsigma_2)\, u|^2\, dx, \qquad D_j = i\partial_{x_j} + \mathfrak{a}_j, \quad j = 1, 2, \tag{10.4}$$

on functions $u=(u_1,u_2)\in d[p(\mathfrak{a})]=C_c^\infty(\mathbb{R}^2,\mathbb{C}^2)$. The form (10.4) is closable and in view of the identity

$$p(\mathfrak{a})[u,u]=q(\mathfrak{a})[u_1,u_1]-\int \mathfrak{b}|u_1|^2dx+q(\mathfrak{a})[u_2,u_2]+\int \mathfrak{b}|u_2|^2dx$$

its closure generates the self-adjoint non-negative operator

$$P(\mathfrak{a}):=Q(\mathfrak{a})\mathbb{I}_2-\mathfrak{b}\varsigma_3 \tag{10.5}$$

on $L_2(\mathbb{R}^2,\mathbb{C}^2)$. Since $\mathfrak{b}$ is bounded and of compact support by Theorem 6.1 in [11] we have $0\in\sigma_{\mathrm{ess}}(P(\mathfrak{a}))$. Depending on the flux Φ the point zero might also be an eigenvalue of finite multiplicity. Put

$$\psi(x):=-(2\pi)^{-1}\int_{\mathbb{R}^2}\mathfrak{b}(y)\ln|x-y|dy,\qquad x\in\mathbb{R}^2. \tag{10.6}$$

Then it holds [1], [11]

Proposition 10.1. *If $\Phi>1$ the point zero is an eigenvalue of multiplicity $\{\Phi\}$ for the operator $P(\mathfrak{a})$ with the eigenfunctions $(e^{\psi(x)}\chi_1(x_1+ix_2),0)$, $\chi_1\in\Omega_{1,\{\Phi\}-1}$. If $\Phi<-1$ the point zero is an eigenvalue of multiplicity $\{-\Phi\}$ for the operator $P(\mathfrak{a})$ with the eigenfunctions $(0,e^{-\psi(x)}\chi_2(x_1-ix_2))$, $\chi_2\in\Omega_{1,\{-\Phi\}-1}$. If $-1\le\Phi\le 1$ then 0 is not an eigenvalue of $P(\mathfrak{a})$.*

10.4. We shall study perturbations of the operator $P(\mathfrak{a})$ by a matrix potential. Therefore consider the form

$$v[u,u]=\int_{\mathbb{R}^2}\langle V(x)u,u\rangle\,dx,\qquad u\in C_c^\infty(\mathbb{R}^2,\mathbb{C}^2), \tag{10.7}$$

where $V(x)=\{V_{kl}(x)\}_{k,l=1}^2$ are 2×2 hermitian matrices with locally integrable entrees. The forms $v_\pm$ are defined as in (10.7), replacing V by $V_\pm$, respectively. We assume that

$$v_\pm[u,u]\le c_{10.1}^{\pm}p(\mathfrak{a})[u,u]+c_{10.2}^{\pm}\|u\|^2_{L_2(\mathbb{R}^2,\mathbb{C}^2)},\qquad u\in d[p(\mathfrak{a})], \tag{10.8}$$

holds. The operators

$$P(\mathfrak{a},\alpha):=P(\mathfrak{a})-\alpha V,\qquad \tilde P(\mathfrak{a},\alpha):=P(\mathfrak{a})-\alpha(V_++V_-)$$

are defined in the form sum sense for sufficiently small $\alpha>0$. It is clear that the eigenvalues of $P(\mathfrak{a})$ at zero, described in Proposition 10.1, shall turn into negative eigenvalues under arbitrary small negative perturbations. Additional negative eigenvalues will appear due to the essential spectrum at zero. It holds

Theorem 10.1. *Suppose the vector field* (10.1) *induces in* (10.2) *a bounded function $\mathfrak{b}$ of compact support. Assume that for the forms* (10.4) *and* (10.7) *the bound* (10.8) *holds and that the operator $\tilde P(\mathfrak{a},\alpha)$ has finite spectrum for some $\alpha>0$. Let ψ be the function* (10.6).

- *If* $\Phi = 0$ *let* $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{n}_0$ *be the signature of the* 2×2*-matrix*

$$\tilde{V} = \int_{\mathbb{R}^2} \tilde{V}(x) dx, \qquad \tilde{V}(x) := \begin{pmatrix} e^{2\psi(x)} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & e^{-2\psi(x)} V_{22}(x) \end{pmatrix},$$

and put

$$\mathfrak{n}_{0,0} = \dim\{\xi \in \mathbb{C}^2 : \ \xi \in \ker \tilde{V}(x) \ \forall \ a.e. \ x \in \mathbb{R}^2\} \ .$$

- *If* $\Phi < 0$ *let* $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{n}_0$ *be the signature of the* $([-\Phi] + 1) \times ([-\Phi] + 1)$*-matrix*

$$\hat{V} = \int_{\mathbb{R}^2} \hat{V}(x) dx, \qquad \hat{V}(x) = \left\{ e^{-2\psi(x)} V_{22}(x)(x_1 - ix_2)^k (x_1 + ix_2)^l \right\}_{k,l=0}^{[-\Phi]} \ .$$

Put $\mathfrak{n}_{0,0} = [-\Phi] + 1$ *if* $V_{12} \equiv V_{21} \equiv V_{22} \equiv 0$ *or set* $\mathfrak{n}_{0,0} = 0$ *otherwise.*

- *If* $\Phi > 0$ *let* $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{n}_0$ *be the signature of the* $([\Phi] + 1) \times ([\Phi] + 1)$*-matrix*

$$\check{V} = \int_{\mathbb{R}^2} \check{V}(x) dx, \qquad \check{V}(x) = \left\{ e^{2\psi(x)} V_{11}(x)(x_1 + ix_2)^k (x_1 - ix_2)^l \right\}_{k,l=0}^{[\Phi]} \ .$$

Put $\mathfrak{n}_{0,0} = [\Phi] + 1$ *if* $V_{11} \equiv V_{12} \equiv V_{21} \equiv 0$ *or set* $\mathfrak{n}_{0,0} = 0$ *otherwise.*

Then

$$\lim_{\alpha \to +0} n_-(P(\mathfrak{a}, \alpha), 0) = \mathfrak{n}_+ + \mathfrak{n}_0 - \mathfrak{n}_{0,0}.$$

Remark 10.1. Let $\|V(x)\|$ be the norm of the matrix $V(x)$. If the operator

$$-\Delta - \alpha \|V(x)\|$$

on $L_2(\mathbb{R}^2, \mathbb{C})$ has finite negative spectrum for all $\alpha > 0$, by Lemma 10.1 and Lemma 9.1 the same holds for $\tilde{P}(\mathfrak{a}, \alpha)$.

10.5. We shall need the following version of the Hardy inequality.

Lemma 10.2. *Assume that* $s > 1$, $s + t > 1$ *or* $s < 1$, $s + t < 1$. *Then*

$$\int_0^\infty x^{s-2} (1 + x)^t |f|^2 dx \le c(s,t) \int_0^\infty x^s (1 + x)^t |f'|^2 dx$$

for all functions $f \in C_0^\infty(\mathbb{R}_+, \mathbb{C})$.

Proof. By integration by part and Hölders inequality one finds

$$\int_0^\infty \omega |f|^2 dx \le 4 \int_0^\infty \tilde{\omega}^2 \omega^{-1} |f'|^2 dx, \qquad f \in C_0^\infty(\mathbb{R}_+, \mathbb{C}),$$

for arbitrary $0 < \omega(x) = \tilde{\omega}'(x)$. Put $\omega(x) = x^{s-2}(1 + x)^t$ and choose $\tilde{\omega}(y) = \int_0^y \omega dx$ or $\tilde{\omega}(y) = -\int_\infty^y \omega dx$, respectively. $\square$

10.6. Sketch of the proof of Theorem 10.1.

1. Put

$$\tilde{\mathfrak{a}} := (\tilde{\mathfrak{a}}_1, \tilde{\mathfrak{a}}_2) = (\partial_{x_2}\psi, -\partial_{x_1}\psi)\,.$$

Then

$$\mathfrak{b} = \partial_{x_1}\mathfrak{a}_2 - \partial_{x_2}\mathfrak{a}_1 = \partial_{x_1}\tilde{\mathfrak{a}}_2 - \partial_{x_2}\tilde{\mathfrak{a}}_1$$

holds, and by gauge transformation the operators $P(\mathfrak{a},\alpha)$ and $P(\tilde{\mathfrak{a}},\alpha)$ are unitary equivalent. Therefore we can work in the gauge $\mathfrak{a} = \tilde{\mathfrak{a}}$.

2. The function ψ and its first derivatives are bounded functions. Put

$$y := (y_1, y_2) = (e^{-\psi}u_1, e^{\psi}u_2), \qquad u = (u_1, u_2) \in C_c^\infty(\mathbb{R}^2, \mathbb{C}^2).$$

Then y and its first derivatives are bounded functions of compact support. It holds

(10.9)
$$p_\psi[y,y] := \int e^{2\psi}|\overline{\partial} y_1|^2 dx + \int e^{-2\psi}|\partial y_2|^2 dx = p(\mathfrak{a})[u,u],$$

where $\partial := \partial_{x_1} - i\partial_{x_2}$ and $\overline{\partial} := \partial_{x_1} + i\partial_{x_2}$, while

(10.10)
$$v_\psi[y,y] := \int \left\langle \begin{pmatrix} e^{2\psi}V_{11} & V_{12} \\ V_{21} & e^{-2\psi}V_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle dx = v[u,u].$$

In a similar way we define $v_{\psi,\pm}[y,y] := v_\pm[u,u] \geq 0$, $v_\psi = v_{\psi,+} - v_{\psi,-}$. By a density argument it holds

$$n_-(P(\mathfrak{a},\alpha),0) = \sup\dim\{\mathcal{F}:\ p_\psi[y,y] - \alpha v_\psi[y,y] < 0 \quad \text{for all} \quad 0 \not\equiv y \in \mathcal{F}\},$$

where the $\mathcal{F}$ are lineals in $C_0^\infty(\mathbb{R}\backslash\{0\}, \mathbb{C}^2)$. Therefore our initial problem is reduced to the study of the form $p_\psi - \alpha v_\psi$ on $C_0^\infty(\mathbb{R}\backslash\{0\}, \mathbb{C}^2)$.

3. Since ψ is locally bounded and shows the asymptotical behavior

$$\psi(x) = -\Phi \ln|x| + O(|x|^{-1}),$$

the weight functions $e^{\pm 2\psi}$ permit the global two-sided estimates

$$c_\pm(1+|x|)^{\mp 2\Phi} \leq e^{\pm 2\psi(x)} \leq C_\pm(1+|x|)^{\mp 2\Phi} \qquad \text{for all} \quad x \in \mathbb{R}^2.$$

Put $W(x) := (1+x)^{-2-\delta}$, $\delta > 0$. The form (10.3) is zero-bounded w.r.t. $q(0)$ and $Q(0,\alpha W)$ has finite negative spectrum for arbitrary $\alpha > 0$, see e.g. [14]. By Lemma 10.1 $Q(\mathfrak{a},\alpha W)$ has finite negative spectrum for all $\alpha > 0$. But then the operator

$$Q(\mathfrak{a}) - |\mathfrak{b}| - \alpha W, \qquad \text{and thus} \qquad P(\mathfrak{a}) - \alpha W,$$

have finite negative spectrum for arbitrary $\alpha > 0$. Hence w is relatively compact w.r.t. $p(\mathfrak{a}) + w$, and

$$w_\psi[y, y] := \int (1 + |x|)^{-2-\delta-2\Phi} |y_1|^2 dx + \int (1 + |x|)^{-2-\delta+2\Phi} |y_2|^2 dx$$

is relatively compact w.r.t. $p_\psi + w_\psi$.

4. Inserting smooth cut-off functions and integrating by part one easily obtains

$$\begin{aligned} \int_{\mathcal{B}_R} |\nabla f|^2 dx &\le C_R \int_{\mathcal{B}_{2R}} (|\partial f|^2 + |f|^2) dx, \\ \int_{\mathcal{B}_R} |\nabla f|^2 dx &\le C_R \int_{\mathcal{B}_{2R}} (|\overline{\partial} f|^2 + |f|^2) dx, \end{aligned} \tag{10.11}$$

on functions $f \in C^\infty(\mathcal{B}_{2R}, \mathbb{C})$.

Let $\{y_j\} = \{(y_{1,j}, y_{2,j})\}$ be a fundamental sequence in $p_\psi + w_\psi$. Then by (10.11) $y_j \to \chi = (\chi_1, \chi_2)$ in $W^{\mathrm{loc}}_{2,1}$. If $\Lambda(p_\psi)$ denotes the sublineal in the completion of $C^\infty_c(\mathbb{R}^2, \mathbb{C}^2)$ w.r.t. $p_\psi + w$, on which p_ψ vanishes, then

$$\int (1 + |x|)^{-2\Phi} |\overline{\partial} \chi_1|^2 dx = 0 \qquad \text{and} \qquad \int (1 + |x|)^{+2\Phi} |\partial \chi_2|^2 dx = 0$$

for all $\chi = (\chi_1, \chi_2) \in \Lambda(p_\psi)$. That means χ_1 is analytic in $z = x_1 + ix_2$, while χ_2 is analytic in $\overline{z} = x_1 - ix_2$.

5. Let f be a function in $C^\infty_0(\mathbb{R}^2 \backslash \{0\}, \mathbb{C})$. We pass to the spherical coordinates $x = (r, \theta)$ in $\mathbb{R}^2$ and develop f in a Fourier series in θ

$$f(x) = \sum_{m \in \mathbb{Z}} e^{im\theta} \rho^{[f]}_m(r).$$

The functions $\rho^{[f]}_m(r)$ belong to $C^\infty_0(\mathbb{R}_+, \mathbb{C})$. In view of Lemma 10.2 the inequalities

$$\begin{aligned} \int (1 + r)^{-2\Phi} |\overline{\partial} f|^2 dx &= \sum_{m \in \mathbb{Z}} \int_0^\infty (1 + r)^{-2\Phi} r^{2m+1} \left| \frac{d}{dr} \left(r^{-m} \rho^{[f]}_m \right) \right|^2 dr \\ &\ge \sum_{m < 0} C(m, \Phi) \int_0^\infty (1 + r)^{\min\{-2\Phi, -2m-1\}} r^{-2} |\rho^{[f]}_m|^2 dr \\ &\qquad + \sum_{m : m > \max\{0, \Phi\}} C(m, \Phi) \int_0^\infty (1 + r)^{-2\Phi} r^{-2} |\rho^{[f]}_m|^2 dr, \end{aligned}$$

$$\int (1+r)^{2\Phi}|\partial f|^2 dx = \sum_{m\in\mathbb{Z}} \int_0^\infty (1+r)^{2\Phi} r^{1-2m} \left| \frac{d}{dr}\left(r^m \rho_m^{[f]}\right)\right|^2 dr$$
$$\geq \sum_{m>0} C(m,\Phi) \int_0^\infty (1+r)^{\min\{2\Phi,2m-1\}} r^{-2} |\rho_m^{[f]}|^2 dr$$
$$+ \sum_{m:m<\min\{0,\Phi\}} C(m,\Phi) \int_0^\infty (1+r)^{2\Phi} r^{-2} |\rho_m^{[f]}|^2 dr,$$

hold. Hence for a fundamental sequence $y_j = (y_{1,j}, y_{2,j}) \to \chi = (\chi_1, \chi_2) \in \Lambda(p_\psi)$ we find

$$\begin{array}{rlll}
(1+r)^{-\Phi} r^{-1} \rho_m^{[y_{1,j}]} & \to 0 & \text{in } L_2(\mathbb{R}_+,\mathbb{C}) & \text{if } m > \max\{0,\Phi\}, \\
(1+r)^{\min\{-\Phi,-m-1/2\}} r^{-1} \rho_m^{[y_{1,j}]} & \to 0 & \text{in } L_2(\mathbb{R}_+,\mathbb{C}) & \text{if } m < 0, \\
\rho_m^{[y_{1,j}]} & \to c_m r^m & \text{in } W_{2,1}^{\mathrm{loc}}(\mathbb{R}_+,\mathbb{C}) & \text{if } 0 \leq m \leq \Phi, \\
(1+r)^{\Phi} r^{-1} \rho_m^{[y_{2,j}]} & \to 0 & \text{in } L_2(\mathbb{R}_+,\mathbb{C}) & \text{if } m < \min\{0,\Phi\}, \\
(1+r)^{\min\{\Phi,m-1/2\}} r^{-1} \rho_m^{[y_{2,j}]} & \to 0 & \text{in } L_2(\mathbb{R}_+,\mathbb{C}) & \text{if } m > 0, \\
\rho_m^{[y_{2,j}]} & \to d_m r^{-m} & \text{in } W_{2,1}^{\mathrm{loc}}(\mathbb{R}_+,\mathbb{C}) & \text{if } \Phi \leq m \leq 0,
\end{array}$$

for $m \in \mathbb{Z}$ as $j \to \infty$. Therefore $\chi \in \Lambda(p_\psi)$ implies

$$\chi = \begin{cases} (\chi_1, \chi_2) & , \quad \chi_1,\ \chi_2 \in \mathbb{C} & & \text{if } \Phi = 0 \\ (\chi_1(z), 0) & , \quad \chi_1 \in \Omega_{1,[\Phi]} & \text{and } z = x_1 + ix_2 & \text{if } \Phi > 0 \\ (0, \chi_2(\overline{z})) & , \quad \chi_2 \in \Omega_{1,[-\Phi]} & \text{and } \overline{z} = x_1 - ix_2 & \text{if } \Phi < 0 \end{cases} . \tag{10.12}$$

Test sequences as in the proof of Lemma 6.1 show that $\Lambda(p_\psi)$ coincides with the described sets (10.12).

6. Use the notations of section 3 and put

$$h = p_\psi + w_\psi + v_{+,\psi} + v_{-,\psi}.$$

Let $\mathfrak{H}[h]$ be the completion of $C_c^\infty(\mathbb{R}^2, \mathbb{C}^2)$ in h. In analogy to Lemma 5.3 the integrals in the forms $v_{\pm,\psi}$ converge absolutely on $\chi \in \Lambda(p_\psi)$. Hence $\Lambda(p_\psi) = \tilde{\mathfrak{H}}_2$ is the subspace in $\mathfrak{H}[h]$, where p_ψ vanishes. It remains to apply Theorem 4.1. □

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