HUTMP 95/445
April 25, 1997

Quantized Kronecker flows

and almost periodic quantum field theory

Sławomir Klimek[*] [1] and **Andrzej Leśniewski**[**2]

[*]Department of Mathematics
IUPUI
Indianapolis, IN 46205, USA

[**]Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138, USA

Abstract. We define and study the properties of the infinite dimensional quantized Kronecker flow. This $\mathbb{C}^*$-dynamical system arises as a quantization of the corresponding flow on an infinite dimensional torus. We prove an ergodic theorem for a class of quantized Kronecker flows. We also study the closely related almost periodic quantum field theory of bosonic, fermionic and supersymmetric particles. We prove the existence and uniqueness of KMS and super-KMS states for the $\mathbb{C}^*$-algebras of observables arising in these theories.

[1] Supported in part by the National Science Foundation under grant DMS–9500463
[2] Supported in part by the National Science Foundation under grant DMS–9424344

I. Introduction

In this paper, we introduce and study the properties of quantized infinite dimensional Kronecker flows. Very much like its classical counterpart, a quantized Kronecker flow is defined in terms of an infinite sequence Ω of frequencies satisfying certain genericity assumptions. We define a natural $\mathbb{C}^*$-algebra of observables and show that it can be identified with an infinite tensor product of standard Toeplitz algebras. A quantized Kronecker flow is a one parameter group of automorphisms of this $\mathbb{C}^*$-algebra.

Our work has been inspired by the recent preprint [BC]. The structures studied in [BC] can be interpreted as an example of a quantized Kronecker flow with the set of frequencies Ω equal to $\{\log p : \ p \text{ prime number}\}$.

The infinite tensor product of standard Toeplitz $\mathbb{C}^*$-algebras referred to above arises naturally when one quantizes the infinite dimensional Kronecker flow on a Bohr compactification of $\mathbb{R}$. Even though there is no parameter in this theory which would play the role of Planck's constant, one can introduce a natural concept of the classical limit [C1], [S1], [Z]. We prove that, under additional assumptions on Ω, the classical limit of the quantized Kronecker flow exists. The proof of this theorem relies on an Ingham type tauberian theorem. Furthermore, we study the ergodic properties of that quantum Kronecker flow. We show that whenever the classical limit exists, the quantized Kronecker flow is quantum ergodic in the sense of Zelditch [Z].

Infinite dimensional Kronecker flows lead to models of free quantum field theory in one space dimension. In these field theories, the field operators are almost periodic functions of the space coordinate x. There is a natural notion of a mean in the theory of almost periodic functions, the Bohr mean, which plays the role of the integral. Using it, we carry over much of the formalism of quantum field theory to the almost periodic setup. The construction of an almost periodic field theory requires "doubling" the Hilbert space of the Kronecker flow. On this Hilbert space, we define the field and momentum operators, which are fundamental objects in canonical quantum field theory. It turns out that the dynamics of the quantum Kronecker flow and the almost periodic wave equation are essentially the same. We focus our discussion on the theory of free fields.. Interacting (i.e. nonlinear) field theories exhibit some new striking phenomena and will be discussed in a future publication.

Furthermore, we extend the construction of almost periodic field theory to incorporate fermionic and supersymmetric (i.e. $\mathbb{Z}_2$-graded) fields. The supersymmetric extension leads to a natural construction of a Fredholm module over the algebra of observables associated with the almost periodic quantum field and the corresponding super-KMS functional. Super-KMS functionals [K], [JLW], appear naturally in the $\mathbb{Z}_2$-graded entire cyclic cohomology theory, but have been studied less intensively than their nongraded antecedents, namely KMS states. We prove the uniqueness of the super-KMS functionals for the supersymmetric model of an almost periodic quantum field theory.

The paper is organized as follows. In Section II we introduce the notation and recall basic facts from the theory of almost periodic functions. We then study the ergodic properties of the quantized almost periodic Kronecker flow. The main technical input is a variant of Ingham's tauberian theorem proved in Appendix A. As it turns out, ergodicity of the quantized Kronecker flow depends on the growth properties of the set Ω. Examples of ergodic Kronecker systems are given in Appendix B. We start Section III with a discussion of the canonical formalism of the almost periodic classical field theory and illustrate it with an analysis of the almost periodic wave equation. Then we show how to formulate the quantum version of the almost periodic wave equation. Free fermionic and supersymmetric models are introduced in Section IV. Finally, in Section V, we prove the uniqueness of the super-KMS functionals for the free supersymmetric almost periodic quantum field theory.

II. Kronecker flows

In this section we introduce the central concept of this paper, namely the infinite dimensional Kronecker flow. After a brief summary of the classical theory, we present its quantum version and study the properties of the resulting dynamics.

II.A. First, we review some facts from the theory of almost periodic functions on $\mathbb{R}$ and fix our notation.

Definition II.1. A countable ordered subset Ω of $\mathbb{R}$ is called a *Kronecker system* if it satisfies conditions 1-4 below.

1. $0 \notin \Omega$.
2. Let $\Omega = \Omega_+ \cup \Omega_-$, where Ω_+ and Ω_- are the subsets of positive and negative elements of Ω, respectively. Then Ω is even i.e. $\Omega_- = -\Omega_+$.
3. Let ω_n, $n = 1, \ldots$, be the elements of Ω_+ listed in increasing order. Then $\omega_n \to \infty$, as $n \to \infty$.
4. The elements of Ω_+ are algebraically independent over $\mathbb{Z}$.

Two natural examples of a Kronecker system arise as follows.

Example 1. Let $\mathfrak{K}$ be an algebraic number field and $\mathfrak{P}$ the the set of its prime ideals. Set

$$\Omega = \{\pm \log NP : \ P \in \mathfrak{P}\},$$

where NP denotes the norm of P. Then Ω is a Kronecker system. In particular, the unique factorization property of ideals implies that Ω_+ consists of numbers algebraically independent over $\mathbb{Z}$. This example is taken from [BC], [C2], [J], [S2].

Example 2. Let m be a transcendental real number, and let

$$\Omega = \{\pm\sqrt{n^2 + m^2} : \ n \in \mathbb{N}\}.$$

Then Ω is a Kronecker system. This example is motivated by two dimensional quantum field theory.

Let $\overline{\Omega} \subset \mathbb{R}$ be the set of linear combinations of elements of Ω with coefficients in $\mathbb{Z}$. In other words, $\overline{\Omega}$ is the free abelian group generated by Ω_+:

$$\overline{\Omega} = \mathbb{Z}\left[\Omega_+\right] \ .$$

We will use the notation $\mathcal{AP}(\Omega)$ for the vector space of continuous almost periodic functions on $\mathbb{R}$ with frequencies in Ω. This means that a function $f \in \mathcal{AP}(\Omega)$ has the following uniformly convergent Fourier expansion:

$$f(x) = \sum_{\omega \in \Omega} f_\omega e^{i\omega x}.$$

Likewise, $\mathcal{AP}(\overline{\Omega})$ will denote the space of continuous almost periodic functions on $\mathbb{R}$ with frequencies in $\overline{\Omega}$. Unlike $\mathcal{AP}(\Omega)$, the space $\mathcal{AP}(\overline{\Omega})$ forms a unital commutative $\mathbb{C}^*$-algebra. This $\mathbb{C}^*$-algebra can be identified with the $\mathbb{C}^*$-algebra of continuous functions on the following Bohr compactification of $\mathbb{R}$.

Let $\overline{\mathbb{R}}_\Omega$ denote the infinite cartesian product of unit circles, $\overline{\mathbb{R}}_\Omega = \prod_{\omega\in\Omega_+} S^1$, equipped with the Tikhonov topology. The embedding

$$\mathbb{R} \ni x \longrightarrow \prod_{\omega\in\Omega_+} \exp i\omega x \in \prod_{\omega\in\Omega_+} S^1 \tag{II.1}$$

induces an isomorphism $C(\overline{\mathbb{R}}_\Omega) \simeq \mathcal{AP}(\overline{\Omega})$, see [HR].

The product of Lebesgue measures on S^1 defines an integral $\int_{ap}$ on $C(\overline{\mathbb{R}}_\Omega)$ and, consequently, on $\mathcal{AP}(\overline{\Omega})$. Explicitly, in terms of Fourier series we have

$$\int_{ap} \sum_{\eta\in\overline{\Omega}} f_\eta e^{i\eta x}\, dx = f_0 \ . \tag{II.2}$$

For later reference, we note that the almost periodic Dirac's delta distribution δ_Ω defined by

$$\delta_\Omega(x) = \sum_{\omega\in\Omega} e^{i\omega x} \tag{II.3}$$

is the Schwartz kernel of the projection

$$\mathcal{AP}(\overline{\Omega}) \ni f \longrightarrow \int_{ap} \delta_\Omega(x-y) f(y)\, dy \in \mathcal{AP}(\Omega) \ .$$

This projections "forgets" all terms in the Fourier series whose frequencies are not in Ω.

The embedding (II.1) defines a Kronecker type flow α_t on $\overline{\mathbb{R}}_\Omega$ given by

$$\alpha_t\big(\prod_{\omega\in\Omega_+} e^{ix_\omega}\big) = \prod_{\omega\in\Omega_+} e^{ix_\omega + it\omega} \ . \tag{II.4}$$

As a consequence of our assumptions on Ω, this flow is ergodic.

II.B. We will now construct a quantization of the classical dynamical system $(\mathcal{AP}(\overline{\Omega}), \alpha_t)$, and study the ergodic properties of the resulting quantum Kronecker flow. The quantization will be given in terms of an algebra of operators on a Hilbert space, the "algebra of observables".

Set $\mathcal{H}_+ = l^2(\Omega_+)$, and consider the bosonic Fock space $\mathcal{F}_b\mathcal{H}_+$ defined as the symmetric tensor algebra over $\mathcal{H}_+$,

$$\mathcal{F}_b\mathcal{H}_+ = \bigoplus_{n=0}^{\infty} S^n\mathcal{H}_+ \ ,$$

where $S^n\mathcal{H}_+$ is the n^{th} symmetric tensor power of $\mathcal{H}_+$ (with $S^0\mathcal{H}_+ = \mathbb{C}$). The vector $v_0 = (1, 0, 0, \ldots) \in \mathcal{F}_b\mathcal{H}_+$ is called the *vacuum.* The Hilbert space $\mathcal{F}_b\mathcal{H}_+$ can be naturally identified with $l^2(\mathbb{N}[\Omega_+])$, where $\mathbb{N}[\Omega_+]$ is the nonnegative cone in the lattice $\mathbb{Z}[\Omega_+]$. In the example of an algebraic number field $\mathfrak{K}$, the set $\mathbb{N}[\Omega_+]$ can be identified with the set of all ideals of $\mathfrak{K}$.

Alternatively, there is a natural isomorphism of $\mathcal{F}_b\mathcal{H}_+$ with an infinite tensor product

$$\mathcal{F}_b\mathcal{H}_+ \simeq \bigotimes_{\omega\in\Omega_+} l^2(\mathbb{N}[\omega]) \ . \tag{II.5}$$

In von Neumann's terminology, if $e_n(\omega)$, $n = 0, 1, 2, \ldots$, is the canonical basis in $l^2(\mathbb{N}[\omega])$, then the above tensor product is the incomplete tensor product associated with the sequence of vectors $(e_0(\omega), e_0(\omega), e_0(\omega), \ldots)$. Furthermore, the Fourier transform allows us to identify the space $l^2(\mathbb{Z}[\Omega_+])$ with $L^2(\overline{\mathbb{R}}_\Omega)$, and the Fock space $\mathcal{F}_b\mathcal{H}_+ \simeq l^2(\mathbb{N}[\Omega_+])$ with the closed subspace $L^2_+(\overline{\mathbb{R}}_\Omega)$ of $L^2(\overline{\mathbb{R}}_\Omega)$ consisting of functions with nonnegative frequencies.

Let P be the orthogonal projection onto $L^2_+(\overline{\mathbb{R}}_\Omega) \subset L^2(\overline{\mathbb{R}}_\Omega)$. Every $f \in C(\overline{\mathbb{R}}_\Omega)$ defines a *Toeplitz operator* $T(f)$ on $L^2_+(\overline{\mathbb{R}}_\Omega) \simeq \mathcal{F}_b\mathcal{H}_+$ by

$$T(f) = PM(f)P,$$

where $M(f)$ is the operator on $L^2(\overline{\mathbb{R}}_\Omega)$ of multiplication by f. Recall that $T(f)$ is continuous in f, $\|T(f)\| \leq \|f\|_\infty$, where $\|f\|_\infty$ denotes the sup norm of f.

Let $\mathcal{A}_+$ be the C^*-algebra generated by the Toeplitz operators. It is not difficult to see that $\mathcal{A}_+$ coincides with the (reduced) $\mathbb{C}^*$-algebra of the semigroup $\mathbb{N}[\Omega_+]$. For $\eta \in \mathbb{N}[\Omega_+]$, let $e(\eta)$ denote the canonical basis element in $l^2(\mathbb{N}[\Omega_+]) \simeq \mathcal{F}_b\mathcal{H}_+$. Let H_+ be an unbounded, self-adjoint operator in $\mathcal{F}_b\mathcal{H}_+$ defined by

$$H_+ e(\eta) = \eta e(\eta), \tag{II.6}$$

and let $U(t) = e^{itH_+}$ be the corresponding one parameter group of unitary operators. The pair $(\mathcal{A}_+, U(t))$ is a quantization of $(C(\overline{\mathbb{R}}_\Omega), \alpha_t)$ which we call the *quantum Kronecker flow.*

Recall that the standard Toeplitz $\mathbb{C}^*$-algebra $\mathfrak{T}$ is the $\mathbb{C}^*$-algebra generated by a single generator u satisfying the relation $u^*u = I$. The following proposition can be proved by the method used in the proofs of Propositions 7 and 8 in [BC].

Proposition II.2.

1. *The* $\mathbb{C}^*$*-algebra* $\mathcal{A}_+$ *is an infinite tensor product of Toeplitz* $\mathbb{C}^*$*-algebras* $\mathfrak{T}_\omega$*,*
$$\mathcal{A}_+ = \bigotimes_{\omega\in\Omega_+} \mathfrak{T}_\omega \ ,$$
where $\mathfrak{T}_\omega$ *is generated by the unilateral shift* $u_\omega = T(e^{ix_\omega})$*.*
2. *For every* $\beta > 0$ *there exists a unique KMS*$_\beta$ *state for* $(\mathcal{A}_+, U(t))$*.*

II.C. We will show now that the quantum dynamical system constructed above is in fact a quantization of the classical Kronecker flow. Even though there is no Planck's constant in this theory, one can still introduce a natural concept of its *classical limit.* Such a construction of the classical limit was originally proposed in [C1], [S1], [Z], and consists in the following. If a is an operator on $\mathcal{F}_b\mathcal{H}_+$ and $E > 0$, we set

$$\tau_E(x) := \frac{1}{N(E)} \sum_{\mathbb{N}[\Omega_+]\ni\eta\leq E} (e(\eta), ae(\eta)), \tag{II.7}$$

where $N(E)$ is the number of eigenvalues of H_+ less than or equal to E. The theorem which we are about to formulate describes the classical limit of $(\mathcal{A}_+, U(t))$.

We will need additional assumptions on the set Ω_+ to guarantee the existence of the classical limit. Specifically, assume that for every $s > 0$,

$$\theta(s) := \sum_{n=1}^{\infty} e^{-s\omega_n} < \infty. \tag{II.8}$$

This implies that the following ζ-type function

$$\zeta_\Omega(s) := \prod_{n=1}^{\infty} (1 - e^{-s\omega_n})^{-1} \tag{II.9}$$

converges for all $s > 0$. Expanding each term of $\zeta_\Omega(s)$ in a power series and multiplying out the terms, we can express $\zeta_\Omega(s)$ as the following Lebesque-Stietljes integral:

$$\zeta_\Omega(s) = 1 + \int_0^\infty e^{-sx}\, dN(x),$$

where, as above, $N(x)$ is the counting function for the eigenvalues of H_+. Equivalently, we can write this formula as

$$e^{\phi(s)} = 1 + \int_0^\infty e^{-sx}\, dN(x), \tag{II.10}$$

where $\phi(s) := -\sum_{n=1}^\infty \log(1 - e^{-s\omega_n})$. In Appendix A we study (II.10) recast in the following form:

$$\frac{e^{\phi(s)}}{s} = \int_0^\infty e^{-sx} (N(x) + 1)\, dx. \tag{II.11}$$

Theorem II.3. *In addition to the assumptions above, let $\phi(s)$ satisfy conditions (1)–(3) and (α)–(γ) of Appendix A. Then:*

1. *For every $f \in C(\overline{\mathbb{R}}_\Omega)$,*
$$\lim_{E\to\infty} \tau_E(T(f)) = \int_{ap} f(x)\,dx.$$
2. *For every $f, g \in C(\overline{\mathbb{R}}_\Omega)$,*
$$\lim_{E\to\infty} \tau_E(T(f)T(g) - T(fg)) = 0.$$
3. *For every $a \in \mathcal{A}_+$ the limit*
$$\lim_{E\to\infty} \tau_E(a) =: \tau(a)$$
exists.
4. *Let π_τ be the GNS representation of $\mathcal{A}_+$ with respect to the state τ. Then,*
$$\pi_\tau(\mathcal{A}_+) \simeq C(\overline{\mathbb{R}}_\Omega),$$
and for every $a \in \mathcal{A}_+$ we have
$$\pi_\tau(U(t)aU(-t)) = \alpha_t(\pi_\tau(a)).$$

Proof. We first introduce some notation. As before, $u_\omega = T(e^{ix_\omega})$ denotes the unilateral shift. We set
$$u_\omega(n) := \begin{cases} u_\omega^n, & \text{if } n \geq 0, \\ (u_\omega^*)^{-n}, & \text{if } n < 0. \end{cases}$$
It is easy to verify that
$$U(t)u_\omega(n)U(-t) = e^{in\omega t}u_\omega(n). \tag{II.12}$$
If $f \in C(\overline{\mathbb{R}}_\Omega)$ is a trigonometric polynomial,
$$f\big(\prod e^{ix_\omega}\big) = \sum_{\{n_\omega\}} f_{n_\omega}\, e^{in_\omega x_\omega},$$
then the corresponding Toeplitz operator $T(f)$ is explicitly given by
$$T(f) = \sum_{n_\omega} f_{n_\omega}\, u_\omega(n_\omega).$$
It follows that for any η,
$$(e(\eta), T(f)e(\eta)) = f_0 = \int_{ap} f(x)\,dx.$$

This and the continuity of $T(f)$ in f prove part 1 of Theorem II.3.

For later reference, notice also that as a consequence of (II.12),

$$U(t)T(f)U(-t) = T(\alpha_t(f)). \tag{II.13}$$

The structure of the standard Toeplitz algebra implies that operators $T(f)T(g) - T(fg)$ generate the commutator ideal $\mathcal{I}$ of $\mathcal{A}_+$. The quotient $\mathcal{A}_+/\mathcal{I}$ is isomorphic to $C(\overline{\mathbb{R}}_\Omega)$, and the quotient map $\pi : \mathcal{A}_+ \to C(\overline{\mathbb{R}}_\Omega)$ is called the symbol map. We claim that

$$\tau(a) = \lim_{E\to\infty} \tau_E(a) = \int_{ap} \pi(a)(x)\, dx, . \tag{II.14}$$

for all $a \in \mathcal{A}_+$. In other words, the state τ is trivial on the commutator ideal, and it coincides with the Lebesgue integral on the abelian quotient. Parts 2 and 3 of Theorem II.3 are straightforward consequences of (II.14). Formula (II.14) implies also that $\pi_\tau(\mathcal{A}_+)$ is isomorphic the algebra $C(\overline{\mathbb{R}}_\Omega)$. The last statement of the theorem follows now from (II.13).

To prove (II.14), it is enough, in view of part 1 of Theorem II.3, to show that

$$\lim_{E\to\infty} \tau_E(a) = 0, \quad \text{if} \quad a \in \mathcal{I} .$$

The structure of the standard Toeplitz algebra $\mathfrak{T}$ implies that $\mathcal{I}$ is generated by the operators of the form

$$a = a_{\omega_1} \otimes a_{\omega_2} \otimes \ldots \otimes a_{\omega_N} \otimes I \otimes I \ldots, \tag{II.15}$$

where at least one of the operators $a_{\omega_1} \ldots a_{\omega_N}$, say a_{ω_k}, is compact. By a density argument, it is no loss of generality to assume that a_{ω_k} is a finite rank operator whose range is spanned by finitely many elements of the canonical basis. Let Π be the orthogonal projection onto this subspace. Then

$$\tau_E(a) \le \frac{\|a\|}{N(E)} \sum_{\mathbb{N}[\Omega_+]\ni\eta\le E} (e(\eta), \Pi e(\eta)) .$$

Since the spectrum of H_+ is the set $\{\sum n_\omega \omega : n_\omega \ge 0\}$, we have to show that for any integer M,

$$\frac{\#\{n_\omega \ge 0\ ,\ n_{\omega_k} \le M : \sum n_\omega \omega \le E\}}{\#\{n_\omega \ge 0 : \sum n_\omega \omega \le E\}} \longrightarrow 0, \text{ as } E \to 0 . \tag{II.16}$$

The numerator of the LHS of (II.16) is equal to $N(E) - N(E - \omega_k(M+1))$ and so we have to show that

$$\frac{N(E) - N(E - \omega_k(M+1))}{N(E)} \longrightarrow 0, \text{ as } E \to 0.$$

Formula (II.11) and Corollary A.3 yield $N(E+1) = N(E)\big(1+o(1)\big)$. Consequently, (II.16) follows, and the theorem is proved. □

Remark. If $\Omega_+ = \{\log p : \ p \text{ prime number}\}$, then it is easy to see that $N(E) \sim e^E$ and (II.16) is not true. It would be interesting to determine the classical limit in this case. Note also that the prime number theorem implies that $\phi(s)$ is divergent for $s \leq 1$, and so the Tauberian theorem of Appendix A does not apply.

It follows now from the general results in [Z] that the quantum Kronecker flow $(\mathcal{A}_+, U(t))$ is quantum ergodic in the following sense.

Theorem II.4. *Under the assumptions of Theorem II.3, for every $a \in \mathcal{A}_+$,*

$$\lim_{M\to\infty} \frac{1}{M} \int_0^M U(t)aU(-t)\, dt = \tau(a)I + A,$$

where A is in the weak closure of $\mathcal{A}_+$, and

$$\lim_{E\to\infty} \tau_E(A^*A) = 0 \ .$$

In other words, the time average of a quantum observable is equal to its spatial average plus a correction which vanishes in the classical limit.

III. Almost periodic Bose field

In this section we define the free almost periodic quantized field. It arises as the result of canonical quantization of the classical almost periodic wave equation. Using Bohr's mean, we propose a canonical formulation of the latter, and apply the standard quantization procedure. The resulting quantum dynamical system is a "double" of the Kronecker dynamics studied in previous section.

III.A. We first introduce some notation. For a smooth function $H : \mathcal{AP}(\Omega) \to \mathbb{C}$ we let $\nabla_f H$ denote the Frechet derivative of H in the direction of $f \in \mathcal{AP}(\Omega)$. The functional derivative $\frac{\delta H(\phi)}{\delta\phi(x)}$ is, by definition, the distribution on $\mathcal{AP}(\Omega)$ such that

$$\nabla_f H(\phi) = \int_{ap} \frac{\delta H(\phi)}{\delta\phi(x)} f(x)\, dx \ .$$

The canonical complex structure on $\mathcal{AP}(\Omega)$ defines a symplectic structure on $\mathcal{AP}(\Omega)$ for which positions are real functions and momenta are purely imaginary functions. The respective coordinates will be denoted by $\phi(x)$ and $\pi(x)$ so that $\frac{\delta H(\phi,\pi)}{\delta\phi(x)}$ is the functional derivative in the real direction, and $\frac{\delta H(\phi,\pi)}{\delta\pi(x)}$ is the functional derivative in the imaginary direction. The symplectic space $\mathcal{AP}(\Omega)$ is the phase space for almost periodic field theory.

Every smooth function $H : \mathcal{AP}(\Omega) \to \mathbb{R}$ defines a Hamiltonian flow on $\mathcal{AP}(\Omega)$ by

$$\begin{aligned} \frac{d\phi(x,t)}{dt} &= \frac{\delta H}{\delta \pi(x)}, \\ \frac{d\pi(x,t)}{dt} &= -\frac{\delta H}{\delta \phi(x)} \ . \end{aligned} \tag{III.1}$$

The Poisson bracket of two functions F and G on $\mathcal{AP}(\Omega)$ is defined by

$$\{F, G\} = \int_{ap} \left(\frac{\delta F}{\delta \phi(x)} \frac{\delta G}{\delta \pi(x)} - \frac{\delta F}{\delta \pi(x)} \frac{\delta G}{\delta \phi(x)} \right) dx$$

and so the flow (III.1) can be written as

$$\frac{dF(\phi(t), \pi(t))}{dt} = \{F, H\} \ . \tag{III.2}$$

A straightforward calculation shows that

$$\begin{aligned} \{\phi(x,t), \pi(y,t)\} &= \delta_\Omega(x-y), \\ \{\phi(x,t), \phi(y,t)\} &= 0, \\ \{\pi(x,t), \pi(y,t)\} &= 0 \ . \end{aligned} \tag{III.3}$$

We will now formulate the almost periodic free field theory. The dynamics is given by the wave equation,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \ . \tag{III.4}$$

Equation (III.4) can be written in the form (III.1) with the Hamiltonian

$$H(\phi, \pi) = \frac{1}{2} \int_{ap} \left(\pi(x)^2 + (\partial_x \phi(x))^2 \right) dx.$$

For this Hamiltonian, equations (III.1) read

$$\frac{d\phi}{dt} = \pi, \quad \frac{d\pi}{dt} = \partial_x^2 \phi, \tag{III.5}$$

and lead to (III.4). The most general solution of (III.5) can be written in the following form

$$\begin{aligned} \phi(x,t) &= \sum_{\omega \in \Omega} (\phi_{1,\omega} e^{i\omega(x+t)} + \phi_{2,\omega} e^{i\omega(x-t)}), \\ \pi(x,t) &= \sum_{\omega \in \Omega} i\omega (\phi_{1,\omega} e^{i\omega(x+t)} - \phi_{2,\omega} e^{i\omega(x-t)}) \ . \end{aligned} \tag{III.6}$$

The reality condition implies $\overline{\phi}_{1,\omega} = \phi_{1,-\omega}$ and $\overline{\phi}_{2,\omega} = \phi_{2,-\omega}$. In quantum field theory it is convenient to parameterize the above solutions slightly differently. We set

$$a_\omega = \sqrt{2}\,|\omega|^{1/2} \times \begin{cases} \phi_{1,-\omega} & \text{if } \omega > 0, \\ \phi_{2,-\omega} & \text{if } \omega < 0 . \end{cases}$$

The new variables a_ω have the following Poisson brackets

$$\begin{aligned} \{a_\omega, \overline{a}_{\omega'}\} &= \delta_{\omega,\omega'}, \\ \{a_\omega, a_{\omega'}\} &= \{\overline{a}_\omega, \overline{a}_{\omega'}\} = 0 . \end{aligned} \tag{III.7}$$

Equations (III.6) can then be recast in the following form:

$$\begin{aligned} \phi(x,t) &= \frac{1}{\sqrt{2}} \sum_{\omega\in\Omega} |\omega|^{-1/2} (\overline{a}_\omega e^{it|\omega|} + a_{-\omega} e^{-it|\omega|}) e^{i\omega x}, \\ \pi(x,t) &= \frac{1}{\sqrt{2}} \sum_{\omega\in\Omega} |\omega|^{1/2} (\overline{a}_\omega e^{it|\omega|} - a_{-\omega} e^{-it|\omega|}) e^{i\omega x} . \end{aligned} \tag{III.8}$$

III.B. We shall now describe a quantization of the algebra of functions $\mathcal{AP}(\Omega)$ and of the dynamics (III.5). We will follow the procedure of canonical quantization which is adopted in quantum field theory.

The standard rule of quantization consists in replacing classical observables by operators and Poisson brackets by $\frac{1}{i}\times$commutators. For simplicity we set $\hbar = 1$. More precisely, quantization of the almost periodic wave equation proceeds as follows. We find almost periodic, hermitian, operator valued distributions $\phi(x,t)$ and $\pi(x,t)$ such that (III.5) is satisfied. Furthermore, we require that (see (III.3))

$$\begin{aligned} [\phi(x,t), \pi(y,t)] &= i\delta_\Omega(x-y), \\ [\phi(x,t), \phi(y,t)] &= 0, \\ [\pi(x,t), \pi(y,t)] &= 0. \end{aligned} \tag{III.9}$$

The quantum hamiltonian H_b determines the time evolution of the field operators given by the Heisenberg equations of motion,

$$\frac{d\phi}{dt} = \frac{1}{i}[\phi, H_b],\ \frac{d\pi}{dt} = \frac{1}{i}[\pi, H_b]. \tag{III.10}$$

We construct operators a_ω and a^*_ω as in (III.8), satisfying the commutation relations

$$\begin{aligned} [a_\omega, a^*_{\omega'}] &= \delta_{\omega,\omega'}, \\ [a_\omega, a_{\omega'}] &= [a^*_\omega, a^*_{\omega'}] = 0, \end{aligned} \tag{III.11}$$

and such that a_ω^* is the hermitian conjugate of a_ω. Operators a_ω and a_ω^* are called annihilation and creation operators, respectively.

If one additionally assumes the existence of a cyclic vector v_0 such that $a_\omega v_0 = 0$ and some natural domain restrictions, it is known that the algebra (III.11) has a unique representation in terms of a Fock space which we will describe now.

Let $\mathcal{H} = l^2(\Omega)$ and consider the bosonic Fock space $\mathcal{F}_b\mathcal{H}$. As before, the vector $v_0 = (1, 0, 0, \ldots) \in \mathcal{F}_b\mathcal{H}$ is called the vacuum. The Hilbert space $\mathcal{F}_b\mathcal{H}$ can be naturally identified with $l^2(\mathbb{N}[\Omega])$, where $\mathbb{N}[\Omega]$ is the set of nonnegative, integer, finite combinations of elements of Ω. Alternatively,

$$\mathcal{F}_b\mathcal{H} \simeq \bigotimes_{\omega\in\Omega} l^2(\mathbb{N}[\omega]), \tag{III.12}$$

as in (II.5).

Let $e(\eta)$, $\eta \in \mathbb{N}[\Omega]$ be the canonical orthonormal basis in $l^2(\mathbb{N}[\Omega]) \simeq \mathcal{F}_b\mathcal{H}$. The set $\mathbb{N}[\Omega]$ is, in a natural way, a semigroup with respect to addition. Writing

$$\mathbb{N}[\Omega] \ni \eta = \sum n_\omega \omega, \ \omega \in \Omega,$$

where almost all numbers n_ω are zero, we define the creation operators a_ω^* by

$$a_\omega^* e(\eta) = \sqrt{n_\omega + 1}\, e(\eta + \omega). \tag{III.13}$$

The field operators $\phi(x,t)$ and $\pi(x,t)$ are then defined by means of formula (III.8).

The Hamiltonian of the free almost periodic quantum field theory is given by the familiar expression

$$H_b = \sum_{\omega\in\Omega} |\omega| a_\omega^* a_\omega = \frac{1}{2}\int_{ap} :\big(\pi(x)^2 + (\partial_x\phi(x))^2\big): \, dx \ , \tag{III.14}$$

where : : means Wick ordering. The canonical basis $\{e(\eta)\}$ is the basis of eigenvectors for H_b,

$$H_b e(\eta) = (\sum n_\omega |\omega|) e(\eta). \tag{III.15}$$

Let $\mathcal{D}$ be the dense subspace of $\mathcal{F}_b\mathcal{H}$ consisting of finite linear combinations of the basis elements $e(\eta)$. It is an invariant domain for a_ω and a_ω^*, and is a core for H_b.

Proposition III.1. *With the above definitions, the operator valued distributions $\phi(x,t)$ and $\pi(x,t)$, and the Hamilton operator H_b satisfy equations (III.5), (III.9) and (III.10) on $\mathcal{D}$.*

Proof. The proof is a direct calculation following essentially the similar argument in standard quantum field theory, see e.g. [GJ]. □

The quantum dynamics described in this section is very closely related to the quantum Kronecker flow. Indeed, denoting $\mathcal{H}_- := l^2(\Omega_-)$, we have a natural decomposition

$$\mathcal{F}_b\mathcal{H} \simeq \mathcal{F}_b\mathcal{H}_- \otimes \mathcal{F}_b\mathcal{H}_+.$$

With respect to this decomposition the Hamiltonian H_b can be split into the positive and negative frequency parts:

$$H_b = \sum_{\omega\in\Omega} |\omega| a_\omega^* a_\omega = \sum_{\omega\in\Omega_-} |\omega| a_\omega^* a_\omega + \sum_{\omega\in\Omega_+} \omega a_\omega^* a_\omega = H_- + H_+ \ .$$

Since H_- is unitarily equivalent to H_+, the almost periodic free field theory is a double of the quantum Kronecker flow.

Consider the family of operators $U_\omega(t)$ and $V_\omega(s)$, $s, t \in \mathbb{R}$, $\omega \in \Omega$, defined as

$$U_\omega(t) = e^{it(a_\omega + a_\omega^*)} \ , \ V_\omega(s) = e^{s(a_\omega - a_\omega^*)} \ .$$

The $\mathbb{C}^*$-algebra generated by $U_\omega(t)$ and $V_\omega(s)$ is an example of a CCR algebra [BR]. It is a nonseparable C^*-algebra which is usually studied in quantum field theory. It is, however, not well suited for our purposes and, following [BC], we define the bosonic algebra of observables $\mathcal{A}_b$ to be the C^*-algebra generated by the canonical unilateral shifts in each factor of (III.12). The hamiltonian H_b defines a dynamics σ_t^b on $\mathcal{A}_b$ by

$$\sigma_t^b(A) = e^{itH_b} A e^{-itH_b} \ , \ A \in \mathcal{A}_b \ .$$

We have the following analog of Proposition II.2.

Proposition III.2.

1. *The $\mathbb{C}^*$-algebra $\mathcal{A}_b$ is an infinite tensor product of standard Toeplitz $\mathbb{C}^*$-algebras $\mathfrak{T}_\omega$:*

$$\mathcal{A}_b = \bigotimes_{\omega\in\Omega} \mathfrak{T}_\omega,$$

   where $\mathfrak{T}_\omega$ is generated by the canonical unilateral shift in $l^2(\mathbb{N}[\omega])$.
2. *For every $\beta > 0$, there exists a unique KMS_β state on $(\mathcal{A}_b, \sigma_t^b)$.*

Proof. This follows from Propositions 7 and 8 of [BC]. □

IV. Almost Periodic Fermions

In this section we will define the almost periodic fermionic quantum free field.

IV.A. Let, as before, $\mathcal{H} = l^2(\Omega)$, and consider the fermionic Fock space $\mathcal{F}_f\mathcal{H}$. The Hilbert space $\mathcal{F}_f\mathcal{H}$ is defined as

$$\mathcal{F}_f\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigwedge^n \mathcal{H},$$

where $\bigwedge^n \mathcal{H}$ is the n^{th} exterior power of $\mathcal{H}$ with $\bigwedge^0 \mathcal{H} = \mathbb{C}$. The vector $v_0 = (1, 0, 0, \ldots) \in \mathcal{F}_f\mathcal{H}$ is called the vacuum. The Hilbert space $\mathcal{F}_f\mathcal{H}$ can be naturally identified with $l^2(\mathbb{Z}_2[\Omega])$, where $\mathbb{Z}_2$ is the group $\{0, 1\}$ with addition modulo 2.

Let $f(\eta)$, $\eta \in \mathbb{Z}_2[\Omega]$ be the canonical orthonormal basis in $l^2(\mathbb{Z}_2[\Omega]) \simeq \mathcal{F}_f\mathcal{H}$. The set $\mathbb{Z}_2[\Omega]$ has a natural group structure with respect to addition modulo 2. Writing

$$\mathbb{Z}_2[\Omega] \ni \eta = \sum_{\omega\in\Omega} n_\omega \omega,$$

where almost all numbers n_ω are zero, we define the creation operators b_ω^* and the annihilation operators b_ω by

$$\begin{aligned} b_\omega^* f(\eta) &= \sqrt{(n_\omega + 1) \bmod 2}\ f(\eta + \omega), \\ b_\omega f(\eta) &= \sqrt{n_\omega}\ f(\eta - \omega). \end{aligned}$$

It easy to verify the following anticommutation relations

$$\begin{aligned} [b_\omega, b_{\omega'}^*]_+ &= \delta_{\omega,\omega'}, \\ [b_\omega, b_{\omega'}]_+ &= [b_\omega^*, b_{\omega'}^*]_+ = 0, \end{aligned} \tag{IV.1}$$

where $[x, y]_+ := xy+yx$ is the anticommutator. Unlike in the bosonic case, the operators b_ω are bounded. Let $\mathcal{A}_f$ be the $\mathbb{C}^*$-algebra generated the fermionic creation and annihilation operators. This algebra is called in the literature the CAR algebra [BR].

Fermionic field operators $\psi_1(x)$ and $\psi_2(x)$ at time 0 are then defined by

$$\begin{aligned} \psi_1(x) &= \sum_{\omega\in\Omega} \left(\Theta(\omega) b_\omega^* + \Theta(-\omega) b_{-\omega}\right) e^{-i\omega x}, \\ \psi_2(x) &= i\sum_{\omega\in\Omega} \left(\Theta(-\omega) b_\omega^* + \Theta(\omega) b_{-\omega}\right) e^{-i\omega x}, \end{aligned} \tag{IV.2}$$

where Θ is the Heaviside function. One can directly verify the following anticommutation relations

$$[\psi_i(x), \psi_j(y)]_+ = 2\delta_{ij}\delta_\Omega(x - y)\ .$$

The fermionic almost periodic free hamiltonian H_f is then given by

$$H_f = \sum_{\omega\in\Omega} |\omega| b_\omega^* b_\omega \ ,$$

so that

$$H_f f(\eta) = (\sum_{\omega\in\Omega} n_\omega |\omega|) f(\eta).$$

It defines a dynamics σ_t^f on $\mathcal{A}_f$ by

$$\sigma_t^f(A) = e^{itH_f} A e^{-itH_f} \ , \ A \in \mathcal{A}_f.$$

IV.B. The supersymmetric almost periodic quantum free field theory is defined as the tensor product of bosonic and fermionic field theories. This means that the Hilbert space of that theory is the tensor product $\mathcal{F}_b\mathcal{H}\otimes\mathcal{F}_f\mathcal{H}$ with the natural $\mathbb{Z}_2$ grading $\Gamma = I\otimes(-1)^F$, where

$$Ff(\eta) = (\sum_{\omega\in\Omega} n_\omega) f(\eta).$$

The relevant $\mathbb{C}^*$-algebra $\mathcal{A}$ is then the tensor product $\mathcal{A} = \mathcal{A}_b \otimes \mathcal{A}_f$. The supersymmetric hamiltonian H is defined by

$$H = H_b \otimes I + I \otimes H_f,$$

and the corresponding dynamics on $\mathcal{A}$ is denoted by σ_t. The new feature of the supersymmetric theory is the existence of a supercharge, namely a self-adjoint operator Q which is odd under the $\mathbb{Z}_2$-grading, and has the property that $Q^2 = H$. The operator Q can be defined in the following way:

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}} \int_{ap} \psi_1(x) \left((\pi(x) - \partial_x\phi(x)) + \psi_2(x)(\pi(x) + \partial_x\phi(x))\right) dx \\ &= \sum_{\omega\in\Omega} \sqrt{|\omega|}(a_\omega^* b_\omega + a_\omega b_\omega^*) \ . \end{aligned} \tag{IV.3}$$

The system $(\mathcal{A}, \Gamma, \sigma_t, Q)$ is an example of a quantum algebra to be discussed in the next section.

V. Super KMS States

In this section we construct and prove the uniqueness of super-KMS functionals for the free supersymmetric almost periodic quantum field theory. Super-KMS functionals are $\mathbb{Z}_2$-graded counterparts of KMS states and play an important role in index theory.

V.A. We will recall the definitions of quantum algebras and the super-KMS states on quantum algebras [K], [JLW].

Definition V.1. A *quantum algebra* is a quadruple $(\mathcal{A}, \Gamma, \sigma_t, d)$ satisfying conditions 1-4 below.

1. $\mathcal{A}$ is a $\mathbb{C}^*$-algebra.
2. Γ is a $\mathbb{Z}_2$ grading on $\mathcal{A}$ i.e. a $*$-automorphism of $\mathcal{A}$ such that $\Gamma^2 = I$. For $a \in \mathcal{A}$ we denote $a^\Gamma := \Gamma(a)$.
3. $\sigma_t : \mathcal{A} \to \mathcal{A}$ is a continuous, one-parameter group of even, bounded automorphisms of $\mathcal{A}$. σ_t do not have to be $*$-automorphisms.
4. Let $\mathcal{A}_\alpha$ be the subalgebra of $\mathcal{A}$ such that for every $a \in \mathcal{A}_\alpha$ the function $t \to \sigma_t(a)$ extends to an entire $\mathcal{A}$-valued function. It is known that $\mathcal{A}_\alpha$ is norm dense. On $\mathcal{A}_\alpha$ we set
$$D := -i\frac{d\sigma_t}{dt}\Big|_{t=0}.$$
d is a superderivation on $\mathcal{A}_\alpha$ i.e.
$$d^\Gamma = -d \ , \ d(ab) = da\, b + a^\Gamma\, db,$$
such that $d^2 = D$.

In the theory of the previous section set $da := [Q, a]_s$, and $Da := [H, a]_s$, where $[a, b]_s$ is the supercommutator, i.e. $[a, b]_s = [a, b]$, if at least one of the operators a, b is even, and $[a, b]_s = [a, b]_+$, if both are odd. Then $(\mathcal{A}, \Gamma, \sigma_t, d)$ is a quantum algebra. In the following, this quantum algebra will be referred to as the *almost periodic quantum algebra.*

Definition V.2. Let $(\mathcal{A}, \Gamma, \sigma_t, d)$ be a quantum algebra. A continuous linear functional $\mu : \mathcal{A} \to \mathbb{C}$ is called a super-KMS$_\beta$ functional if for $a, b \in \mathcal{A}_\alpha$,

1. $\mu(da) = 0$,
2. $\mu(ab) = \mu(b^\Gamma \sigma_{i\beta}(a))$.

If, for a $\mathbb{Z}_2$-graded $\mathbb{C}^*$ dynamical system $(\mathcal{A}, \Gamma, \sigma_t)$, a linear continuous functional μ satisfies only the condition 2 above, then it is called a *pre super-KMS$_\beta$ functional.*

Unlike for KMS states, no positivity assumptions are or can be made for super-KMS$_\beta$ functionals. It follows from the definition that a super-KMS$_\beta$ functional $\mu : \mathcal{A} \to \mathbb{C}$ satisfies

$$\begin{aligned}
&\mu(\sigma_t(a)) = \mu(a),\\
&\mu^\Gamma = \mu,\\
&\mu(a\, db) = \mu(da\, b^\Gamma).
\end{aligned}$$

In our example of the almost periodic quantum algebra, assuming additionally that $\mathrm{tr}(e^{-\beta H}) < \infty$, set

$$\mu_\beta(a) := \mathrm{Str}(ae^{-\beta H}) \ , \tag{V.1}$$

where Str is the supertrace, i.e. $\mathrm{Str}(a) = \mathrm{tr}(\Gamma a)$. It is then easy to verify that μ_β is a super-KMS$_\beta$ functional.

V.B. The remainder of this section is devoted to the proof of the uniqueness of the super-KMS functional for the almost periodic quantum algebra. We start with two propositions of independent interest.

Proposition V.3. *Let V be a finite dimensional Hilbert space, Γ a $\mathbb{Z}_2$-grading on V, Q an odd self-adjoint operator on V, $H := Q^2$, and $\mathcal{A} := \mathcal{L}(V)$ the algebra of linear operators on V. For $a \in \mathcal{A}$, we set $da := [Q, a]_s$ and $\sigma_t(a) := e^{itH} ae^{-itH}$. Then, for every $\beta > 0$, there is a unique, up to a multiplicative constant, pre super-KMS$_\beta$ functional μ_β on $(\mathcal{A}, \Gamma, \sigma_t)$ given by*

$$\mu_\beta(a) = \mathrm{Str}(ae^{\beta H}).$$

Moreover, μ_β is automatically a super-KMS$_\beta$ functional on $(\mathcal{A}, \Gamma, \sigma_t, d)$.

Proof. Let μ_β be any pre super-KMS$_\beta$ functional on $(\mathcal{A}, \Gamma, \sigma_t, d)$. Consider $\tilde{\mu}_\beta := \mu_\beta(\Gamma ae^{-\beta H})$. Using condition 2 of Definition V.2, one easily verifies that $\tilde{\mu}_\beta(ab) = \tilde{\mu}_\beta(ba)$, and so $\tilde{\mu}_\beta$ is proportional to the trace and the claim follows. □

Proposition V.4. *Let $(\mathcal{A}^i, \Gamma^i, \sigma^i_t)$, $i = 1, 2$ be two $\mathbb{Z}_2$-graded $\mathbb{C}^*$-dynamical systems which have unique, up to a multiplicative constant, pre super-KMS$_\beta$ functionals μ^i_β. Then $\mu^1_\beta \otimes \mu^2_\beta$ is a unique, up to a multiplicative constant, pre super-KMS$_\beta$ functional on the tensor product $(\mathcal{A}^1 \otimes \mathcal{A}^2, \Gamma^1 \otimes \Gamma^2, \sigma^1_t \otimes \sigma^2_t)$.*

Proof. Let μ_β be any pre super-KMS$_\beta$ functional on the tensor product. The statement follows easily from the fact that for any $b \in \mathcal{A}^2$ the following functional on $\mathcal{A}^1$:

$$\mu_{\beta,b}(a) := \mu_\beta(a \otimes b)$$

is a pre super-KMS$_\beta$ functional. □

The following theorem can now be easily deduced from Proposition V.3, Proposition V.4, and Proposition 8 of [BC].

Theorem V.5. *For every $\beta > 0$, there exists a unique, up to a multiplicative constant, super-KMS$_\beta$ functional on the almost periodic quantum algebra $(\mathcal{A}, \Gamma, \sigma_t, d)$.*

Proof. We are going to prove that there is a unique pre super-KMS$_\beta$ functional on the $\mathbb{Z}_2$-graded $\mathbb{C}^*$-dynamical system $(\mathcal{A}, \Gamma, \sigma_t)$. It will follow from the construction that that functional is, in fact, a super-KMS$_\beta$ functional.

It follows from Proposition III.2 and the structure theorems for $\mathcal{A}_f$, see [BR], that the $\mathbb{C}^*$-algebra $\mathcal{A}$ is isomorphic with the following infinite tensor product:

$$\mathcal{A} = \bigotimes_{\omega\in\Omega} \mathfrak{T}_\omega \otimes \mathfrak{A}_\omega,$$

where $\mathfrak{T}_\omega$ is the Toeplitz algebra and $\mathfrak{A}_\omega$ is generated by the fermionic creation and annihilation operators b_ω^*, b_ω, and is isomorphic with $M_2(\mathbb{C})$, the algebra of 2×2 matrices. Additionally, both the grading Γ and the dynamics σ_t factor with respect to the above decomposition,

$$\Gamma = \bigotimes_{\omega\in\Omega} \Gamma_\omega, \quad \sigma_t = \bigotimes_{\omega\in\Omega} \sigma_{t,\omega}.$$

It is easy to verify that Γ_ω is trivial on $\mathfrak{T}_\omega$, so that $\Gamma_\omega = I \otimes \Gamma_\omega^f$. The generator of $\sigma_{t,\omega}$ is the supersymmetric harmonic oscillator hamiltonian $H_\omega = |\omega|(a_\omega^* a_\omega + b_\omega^* b_\omega)$, and so we have a further decomposition: $\sigma_{t,\omega} = \sigma_{t,\omega}^b \otimes \sigma_{t,\omega}^f$. The system $(\mathfrak{A}_\omega, \Gamma^f, \sigma_{t,\omega}^f)$ is finite dimensional, and thus by Proposition V.3 it has a unique pre super-KMS$_\beta$ functional. The uniqueness of a pre super-KMS$_\beta$ functional on $(\mathfrak{T}_\omega, \Gamma^b = I, \sigma_{t,\omega}^b)$ follows from Proposition 8 of [BC] since the proof of that proposition does not require any positivity assumptions on the functional. Moreover, any pre super-KMS$_\beta$ functional on $\mathfrak{T}_\omega \otimes \mathfrak{A}_\omega$ is proportional to

$$a \to \operatorname{Str}(a e^{-\beta H_\omega}),$$

and consequently is a super-KMS$_\beta$ functional. The theorem now follows from Proposition V.4. □

Appendix A. An Ingham type tauberian theorem

In this appendix we prove a technical result used in Section II to establish the quantum ergodicity of the quantized Kronecker dynamics. This result is a variant of Ingham's tauberian theorem [I], see also [P], and differs from the original theorem in some of the hypotheses.

Let $N(x)$ be a nondecreasing function of bounded variation satisfying the following assumptions:

(1) $N(x) = 0$, for all $x \leq 0$;

(2) for all $\sigma > 0$, $\int_0^\infty e^{-\sigma x} dN(x) < \infty$;
(3) for all $s = \sigma + it$, $\sigma > 0$, $t \in \mathbb{R}$, the function $\phi(s)$ defined by

$$e^{\phi(s)} = \int_0^\infty e^{-sx} dN(x) \tag{A.1}$$

is holomorphic.

The function $\phi(s)$ will play a fundamental role in the following analysis. Ingham's original theorem requires detailed knowledge of the asymptotic of $\phi(s)$ as s approaches 0 within an angle. Such an asymptotic is usually difficult to obtain. Somewhat different assumptions on $\phi(s)$ lead to a result which is well tailored for our purposes. Specifically, we require that:

(α)

$$-\sigma\phi'(\sigma) \nearrow \infty, \quad \text{and } \sigma^2\phi''(\sigma) \nearrow \infty, \text{ as } \sigma \searrow 0; \tag{A.2}$$

(β)

$$\left|\frac{\sigma\phi'''(\sigma)}{\phi''(\sigma)}\right| = O(1), \text{ as } \sigma \searrow 0; \tag{A.3}$$

(γ) for any $\Delta > 0$, there is $\sigma_0 > 0$ such that the triangle

$$T(\Delta, \sigma_0) = \{\sigma + it :\ 0 < \sigma < \sigma_0,\ |t| < \Delta\sigma\}$$

does not contain nonreal roots of $\operatorname{Im} \phi'(s)$.

We can now formulate the main result of this appendix..

Theorem A.1. *Under the above assumptions (1 - 3) and (α - γ),*

$$N(E) = \left(2\pi\sigma_E^2\phi''(\sigma_E)\right)^{-1/2} e^{\sigma_E E + \phi(\sigma_E)}\left(1 + o(1)\right), \text{ as } E \to \infty, \tag{A.4}$$

where σ_E is the unique solution to the equation

$$\phi'(\sigma) + E = 0. \tag{A.5}$$

Proof. The existence and uniqueness of the solution of (A.5) follows from assumption (α). Integrating by parts in the right hand side of (A.1) we obtain the identity

$$\frac{e^{\phi(\sigma+it)}}{\sigma + it} = \int_0^\infty e^{-(\sigma+it)x} N(x) dx. \tag{A.6}$$

Let g be an integrable function. Multiplying (A.6) by $e^{E(\sigma+it)}g(t)$ and integrating over t we obtain, after a change of the order of integration,

$$\int_{-\infty}^{\infty} \frac{e^{\psi_E(\sigma+it)}}{\sigma+it} g(t)dt = \sqrt{2\pi}\int_{-\infty}^{\infty} e^{\sigma(E-x)}\widehat{g}(x-E)N(x)dx, \tag{A.7}$$

where we have set

$$\psi_E(s) = \phi(s) + Es. \tag{A.8}$$

Shifting the integration variable in the right hand side of (A.7) we rewrite (A.7) as the following basic identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\psi_E(\sigma+it)}}{\sigma+it} g(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\sigma x}\widehat{g}(x)N(x+E)dx. \tag{A.9}$$

We take the function g to be of the form $g(t) = f(t/T)$, where f is continuous in the interval $[-1,1]$ and zero outside it, $f(0) = 1$, and where $T > 0$ is a number which will be chosen shortly. We let $L(\sigma)$ denote the left hand side of (A.9), i.e.

$$L(\sigma) = \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{\psi_E(\sigma+it)}}{\sigma+it} f(t/T)dt.$$

For $0 < \delta < T$, we decompose $L(\sigma)$ into two parts

$$\begin{aligned} L(\sigma) &= \frac{1}{2\pi}\int_{|t|\leq\delta} \frac{e^{\psi_E(\sigma+it)}}{\sigma+it} f(t/T)dt + \frac{1}{2\pi}\int_{\delta<|t|<T} \frac{e^{\psi_E(\sigma+it)}}{\sigma+it} f(t/T)dt \\ &\equiv L^{(1)}(\sigma) + L^{(2)}(\sigma), \end{aligned}$$

and analyze them separately.

So far the considerations have been quite general, and we will now start making specific choices. Pick any $\Delta > 0$ (which we will eventually want to make arbitrarily large), and choose $\sigma_0 > 0$ such that the triangle $T(\Delta, \sigma_0)$ defined in assumption (γ) does not contain nonreal roots of $\mathrm{Im}\phi'(s)$. Take E sufficiently large so that $\sigma_E < \sigma_0$. To simplify the notation, σ_E will be denoted by σ in the throughout the remainder of this proof. Furthermore, take E large enough so that

$$\frac{\Delta}{\sqrt{\sigma^2\phi''(\sigma)}} \leq 1, \tag{A.10}$$

which is possible by assumption (α). Set $T = \sigma\Delta$. The choice of δ will be made shortly.

To analyze $L^{(1)}(\sigma)$ we expand $\psi_E(s)$ around $s = \sigma$ (in the following the subscript E in ψ_E will be suppressed):

$$\psi(\sigma+it) = \psi(\sigma) - 1/2\ \phi''(\sigma)t^2 - 1/6\ \phi'''(\theta)it^3,$$

for a θ belonging to the line segment which joins $\sigma - i\delta$ and $\sigma + i\delta$. By assumption (β) we have,

$$|\phi'''(\theta)t| \leq \Big|\frac{\sigma\phi'''(\sigma)}{\phi''(\sigma)}\Big| \frac{\delta}{\sigma}\,\phi''(\sigma) = o(1)\phi''(\sigma),$$

if $\delta/\sigma = o(1)$, as $E \to \infty$. We now make the following choice of δ:

$$\delta = \left(\frac{\sigma^2}{\phi''(\sigma)}\right)^{1/4}. \tag{A.11}$$

Then, by assumption (α), $\delta/\sigma = (\sigma^2\phi''(\sigma))^{-1/4} = o(1)$, and consequently

$$\psi(\sigma + it) = \psi(\sigma) - 1/2\ \phi''(\sigma)(1 + o(1))t^2.$$

Therefore,

$$\begin{aligned} L^{(1)}(\sigma) &= \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{1}{\sigma} f(0) e^{\psi(\sigma)} e^{-1/2\ \phi''(\sigma)(1+o(1))t^2} \\ &= \frac{e^{\psi(s)}}{2\pi\sqrt{\sigma^2\phi''(\sigma)}} \int_{-\delta(\phi''(\sigma))^{1/2}}^{\delta(\phi''(\sigma))^{1/2}} e^{-1/2\ (1+o(1))t^2}\, dt\ \ (1 + o(1)). \end{aligned}$$

But $\delta(\phi''(\sigma))^{1/2} = (\sigma^2\phi''(\sigma))^{1/4} \to \infty$, as $E \to \infty$, and so the gaussian integral above becomes an integral over entire $\mathbb{R}$ in this limit. As a result,

$$L^{(1)}(\sigma) = \big(2\pi\sigma^2\phi''(\sigma)\big)^{-1/2} e^{\psi(\sigma)}\big(1 + o(1)\big).$$

We now turn to the analysis of $L^{(2)}(\sigma)$. We wish to show that this term is much smaller than the previous one. Indeed,

$$\begin{aligned} \big|L^{(2)}(\sigma)\big| &\leq T\ \sup|f(t/T)|\,\frac{1}{\sigma} \sup_{\delta<|t|<T} \big|e^{\psi(\sigma+it)}\big| \\ &= O(1)\,\Delta \sup_{\delta<|t|<T} e^{\mathrm{Re}\,\psi(\sigma+it)}. \end{aligned}$$

Assumption (γ) implies that the above supremum is attained at $|t| = \delta$. To see this, we consider the function

$$t \to \mathrm{Re}\,\psi(\sigma + it) = \mathrm{Re}\,\phi(\sigma + it) + E\sigma.$$

The critical points of this function satisfy

$$0 = \frac{d}{dt}\,\mathrm{Re}(\phi(\sigma + it) + E\sigma) = \mathrm{Im}\,\phi'(\sigma + it).$$

Hence there are no critical points in the interval $\delta < |t| < T = \sigma\Delta$, and the function attains its maximum value at an endpoint. Consequently, using a Taylor expansion as in the analysis of $L^{(1)}(\sigma)$,

$$\left|L^{(2)}(\sigma)\right| \le O(1)\,\Delta\, e^{\mathrm{Re}\,\psi(\sigma\pm i\delta)} = O(1)\,\Delta\, e^{\psi(\sigma)-1/2\phi''(\sigma)\delta^2(1+o(1))}.$$

It is easy to see that, with our choices of δ and Δ, we have

$$\Delta e^{-1/2\phi''(\sigma)\delta^2} \ll \left(\sigma^2\phi''(\sigma)\right)^{-1/2}$$

and so $L^{(2)}(\sigma) = o(1)\, L^{(1)}(\sigma)$. This concludes the analysis of the left hand side of (A.9).

The asymptotic behavior of $L(\sigma)$ turns out to be independent of the choice of function f. In the following we study the right hand side of (A.9) which will be denoted by $R(\sigma)$. We shall make suitable choices of f in order to get bounds on $N(E)$ from above and from below.

Lemma A.2. *Define*

$$\tilde{N}(E) := \left(2\pi\sigma^2\phi''(\sigma)\right)^{-1/2} e^{\sigma E+\phi(\sigma)}. \tag{A.12}$$

Then

$$\tilde{N}(E+O(1)/\sigma) = \tilde{N}(E)\big(1+o(1)\big).$$

Proof. Let σ_1 be the unique solution of the equation $-\phi'(\sigma_1) = E + O(1)/\sigma$. Taylor expanding ϕ' around σ yields

$$\sigma_1 = \sigma + \frac{O(1)}{\sigma\phi''(\sigma)},$$

since $1/\sigma\phi''(\sigma) \ll \sigma$. It then follows readily that $\sigma_1 = \sigma\big(1+o(1)\big)$. In a similar fashion, we conclude that $\phi''(\sigma_1) = \phi''(\sigma)\big(1+o(1)\big)$, and $\psi(\sigma_1) = \psi(\sigma) + o(1)$. Inserting these expressions into the definition of $\tilde{N}(E+O(1)/\sigma)$ completes the proof. □

Choice 1. Set

$$f(t) = \begin{cases} 1-|t|, & \text{if } |t| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transform of f is

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(x/2)}{x/2}\right)^2,$$

and thus the right hand side of (A.9) is

$$R(\sigma) = \frac{T}{2\pi}\int_{\mathbb{R}} e^{-x\sigma}\left(\frac{\sin(Tx/2)}{Tx/2}\right)^2 N(E+x)\,dx. \tag{A.13}$$

The integrand of (A.13) is positive and so, for any Λ,

$$\begin{aligned} R(\sigma) &\geq \frac{T}{2\pi}\int_{-\Lambda}^{\Lambda} e^{-x\sigma}\left(\frac{\sin(Tx/2)}{Tx/2}\right)^2 N(E+x)\,dx \\ &\geq N(E-\Lambda)\,e^{-\sigma\Lambda}\,\frac{1}{2\pi}\int_{-T\Lambda}^{T\Lambda}\left(\frac{\sin(x/2)}{x/2}\right)^2 dx, \end{aligned}$$

where we have used the monotonicity of $N(x)$. Now take $\Lambda = 1/(\sigma\sqrt{\Delta})$. With this choice, $\sigma\Lambda = 1/\sqrt{\Delta}$, and the exponential term in the above formula tends to 1, as $\Delta \to \infty$. Similarly, $T\Lambda = \sqrt{\Delta}$ and the integral over $(-T\Lambda, T\Lambda)$ tends to the integral (equal to 1) over all of $\mathbb{R}$. This yields the inequality

$$R(\sigma) \geq N(E-\Lambda)\big(1+o(1)\big).$$

Replacing E by $E-\Lambda$ and using Lemma A.2, we conclude that

$$N(E) \leq \tilde{N}(E)\big(1+o(1)\big).$$

Choice 2. Set

$$f_1(t) = \begin{cases} \frac{1}{2i\mu}\left(e^{i\mu|t|} - e^{-i\mu|t|}\right), & \text{if } |t| \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu = 2k\pi$, $0 < k \in \mathbb{Z}$. Let

$$f_2(t) = \begin{cases} 1-|t|, & \text{if } |t| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and take $f = f_1 + f_2$. The Fourier transform of f is

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(x/2)}{x/2}\right)^2 \frac{\mu^2}{\mu^2 - x^2},$$

and so $\widehat{f}(x) < 0$, for $|x| > \mu$. Consequently,

$$\begin{aligned} R(\sigma) &\leq \frac{T}{2\pi}\int_{|Tx|\leq\mu} e^{-x\sigma}\left(\frac{\sin(Tx/2)}{Tx/2}\right)^2 \frac{\mu^2}{\mu^2-x^2}\,N(E+x)\,dx \\ &\leq N(E+\mu/T)\,e^{\mu\sigma/T}\,\frac{1}{2\pi}\int_{-\mu}^{\mu}\left(\frac{\sin(x/2)}{x/2}\right)^2 \frac{\mu^2}{\mu^2-x^2}\,dx, \end{aligned}$$

by monotonicity. Now take k to be the integer part of $[\sqrt{\Delta}]$. With this choice, the integral above tends to 1, as $\Delta \to \infty$. Also, $\mu\sigma/T \sim \Delta^{-1/2} \to 0$ and the exponential term tends to 1. Since $\mu/T \sim 1/\sigma\sqrt{\Delta}$, we can use Lemma A.2 to replace E by $E + \mu/T$. This yields

$$N(E) \geq \tilde{N}(E)\big(1+o(1)\big),$$

and concludes the proof of the theorem. $\square$

Corollary A.3. *With the above assumptions we have*

$$N\big(E + O(1)\big) = N(E)\big(1 + o(1)\big).$$

Proof. This follows directly from Theorem A.1 and Lemma A.2. □

Appendix B. Some examples of Kronecker systems

The theorem below provides a source of examples of Kronecker systems satisfying the assumptions of Theorem A.1.

Theorem B.1. *Let $\omega_n = An^\alpha(1 + \mu_n)$, where $A > 0$ and $\alpha \geq 1$ are constant, and where $\mu_n = o(1)$, as $n \to \infty$. Then $\phi(s)$ satisfies the assumptions of Theorem A.1.*

Proof. Assumptions (1) – (3) of Appendix A are clearly satisfied, and so it is sufficient to verify assumptions (α) – (γ).

Assumption (α) is a consequence of the following equalities:

$$-\sigma\phi'(\sigma) = \sum_{n=1}^{\infty} f(\sigma\omega_n),$$

$$\sigma^2\phi''(\sigma) = \sum_{n=1}^{\infty} f(\sigma\omega_n),$$

where the functions f and g are given by

$$f(x) = \frac{xe^{-x}}{1 - e^{-x}}, \qquad g(x) = \frac{x^2e^{-x}}{(1 - e^{-x})^2}.$$

Since both $f(x)$ and $g(x)$ increase monotonically 1 as $x \searrow 0$, the claim follows.

To prove (β), we note that

$$-\sigma^3\phi'''(\sigma) \leq \theta(\sigma),$$

and

$$\sigma^2\phi''(\sigma) \geq C_\epsilon\theta((1 - \epsilon)\sigma), \text{ for some } 0 < \epsilon < 1,$$

where $\theta(\sigma)$ is defined in (II.8). Since $\omega_n = An^\alpha(1 + \mu_n)$ implies that $\theta(\sigma) = C\sigma^{-1/\alpha}(1 + o(1))$, as $\sigma \to 0$, we conclude that

$$\left|\frac{\sigma\phi'''(\sigma)}{\phi''(\sigma)}\right| \leq O(1)\, \frac{\theta(\sigma)}{\theta((1 - \epsilon)\sigma)} = O(1),$$

as $\sigma \to 0$.

Finally, assumption (γ) is verified in the following lemma. □

Lemma B.2. *Under the assumptions of Theorem B.1,*

$$\operatorname{Im}\phi'(\sigma+ix\sigma)=C_{A,\alpha}\sigma^{-\beta}x\big(h(x)+o(1)\big), \tag{B.1}$$

for $x\in\mathbb{R}$, as $\sigma\to 0$, uniformly in $|x|\le\Delta$. Here $\beta=1+\alpha^{-1}$, and $h(x)$ is a function such that $h(x)\neq 0$, for all $x\in\mathbb{R}$.

Proof. Explicitly,

$$\operatorname{Im}\phi'(\sigma+ix\sigma)=\sum_{n\ge 1}\frac{\lambda_n\, e^{-\sigma\lambda_n}\sin(x\sigma\lambda_n)}{1+e^{-2\sigma\lambda_n}-2e^{-\sigma\lambda_n}\cos(x\sigma\lambda_n)}. \tag{B.2}$$

We will analyze this expression in two steps.
Step 1. Assume first that $\lambda_n=An^\alpha$, and set $u_n=(A\sigma)^{1/\alpha}n$. Then

$$\operatorname{Im}\phi'(\sigma+ix\sigma)=A^{-1/\alpha}\sigma^{-\beta}x\sum_{n\ge 1}\psi(u_n,x)\Delta u_n, \tag{B.3}$$

where

$$\psi(u,x)=\frac{1}{2s}\frac{u^\alpha\sin(xu^\alpha)}{\cosh u^\alpha-\cos xu^\alpha}, \tag{B.4}$$

and where $\Delta u_n=u_n-u_{n-1}=(A\sigma)^{1/\alpha}$. The sum in (B.3) is a Riemann sum of the integral

$$\frac{1}{s}\int_0^\infty\frac{u^\alpha\sin(xe^{-u^\alpha}u^\alpha)}{1+e^{-2u^\alpha}-2e^{-u^\alpha}\cos(xu^\alpha)}\,du=C_{A,\alpha}h(x), \tag{B.5}$$

where $C_{A,\alpha}=\alpha^{-1}\Gamma(\beta)\zeta(\beta)$, and where

$$h(x)=(1+x^2)^{\beta/2}\,\frac{\sin(\beta\arctan x)}{x}. \tag{B.6}$$

Note that $h(x)\neq 0$, if $\alpha\ge 1$. We claim that the difference of the Riemann sum in (B.3) and the integral (B.5) is $o(1)$, as $\sigma\to 0$. Indeed, this difference can be written as

$$\sum_{n\ge 1}\int_{u_{n-1}}^{u_n}\big(\psi(u_n,x)-\psi(u,x)\big)du,$$

which can readily be bounded by

$$A^{1/\alpha}\sigma^{\beta}\sum_{n\ge 1}\max_{u_{n-1}\le u\le u_n}\big|\frac{\partial}{\partial u}\psi(u,x)\big|. \tag{B.7}$$

Using the fact that, uniformly in x,

$$\big|\frac{\partial}{\partial u}\psi(u,x)\big|\le\begin{cases}O(1)u^{-1}, & \text{if } 0<u\le 1;\\ O(1)e^{-(1-\epsilon)u}, & \text{if } u>1,\end{cases} \tag{B.8}$$

(with $0 < \epsilon < 1$), we can bound (B.7) by

$$
\begin{aligned}
& O(1)\sigma^{\beta} \sum_{1 \leq n \leq (A\sigma)^{-1/\alpha}} (A\sigma)^{-1/\alpha} n^{-1} + O(1)\sigma^{\beta} \sum_{n > (A\sigma)^{-1/\alpha}} e^{-(1-\epsilon)(A\sigma)^{1/\alpha} n} \\
& = O(1)\sigma \log(A\sigma)^{-1/\alpha} + O(1)\sigma \\
& = o(1),
\end{aligned}
$$

and our claim follows.

Step 2. In the general case, we write $\sigma\lambda_n = u_n(1+\mu_n)$, with u_n as before. We now claim that the difference

$$
\sum_{n \geq 1} \big(\psi(u_n(1+\mu_n), x) - \psi(u_n, x)\big)\Delta u_n \tag{B.9}
$$

is $o(1)$, as $\sigma \to 0$. Indeed, using (B.8) we can bound (B.9) by

$$
\begin{aligned}
& \sum_{n \geq 1} u_n |\mu_n| \Delta u_n \max_{u \in [u_{n-1}, u_n]} \Big|\frac{\partial}{\partial u}\psi(u, x)\Big| \\
& \leq O(1)\sigma^{1/\alpha} \sum_{1 \leq \Lambda} 1 + O(1)\sigma^{2/\alpha} \sum_{n > \Lambda} \mu_n n e^{-(1-\epsilon)(A\sigma)^{1/\alpha} n} \\
& O(1)\sigma^{1/\alpha}\Lambda + O(1) \sup_{n > \Lambda} \mu_n,
\end{aligned}
$$

where $\Lambda > 0$ is arbitrary. Choosing e.g. $\Lambda = \sigma^{-1/(2\alpha)}$ we conclude that the above expression is $o(1)$, as $\sigma \to 0$. $\square$

References

[BC] Bost, J.P., Connes, A.: Hecke Algebras, Type III Factors and Phase Transitions with Spontaneous Symmetry Breaking in Number Theory, preprint (1995)

[BR] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics II*, Springer Verlag (1981)

[C1] Colin de Verdiere, Y.: Ergodicite et fonctions propres du Laplacien, *Comm. Math. Phys.*, **102**, 497–502 (1985)

[C2] Connes, A.: *Noncommutative Geometry*, Academic Press (1994)

[GJ] Glimm, J., Jaffe, A.: Boson quantum field models, in *Mathematics of Contemporary Physics*, R.Streater, ed., Academic Press (1972)

[HR] Hewitt, E., Ross, K.A.: *Abstract Harmonic Analysis*, Springer Verlag (1963)

[I] Ingham, A.: A Tauberian theorem for partitions, *Ann. Math.*, **42**, 1075–1090 (1941)

[JLW] Jaffe, A., Lesniewski, A., Wisniowski, M.: Deformations of super-KMS functionals, *Comm. Math. Phys.*, **121**, 527–540 (1989)

[J] Julia, B.: Statistical Theory of Numbers, in *Number Theory and Physics*, Les Houchs Winter School, J.-M. Luck, P. Moussa and M. Waldschmidt editors, Springer Verlag (1990)

[K] Kastler, D.: Cyclic cocycles from graded KMS functionals, *Comm. Math. Phys.*, **121**, 345–350 (1989)

[P] Postnikow, A.: *Introduction to analytic number theory*, Am. Math. Soc. (1988)

[S1] Schnirelman, A.: Ergodic properties of the eigenfunctions, *Usp. Math. Nauk*, **29**, 181–182 (1974)

[S2] Spector, D.: Supersymmetry and the Möbius inversion function, *Comm. Math. Phys.*, **127**, 239–252 (1990)

[Z] Zelditch, S.: Quantum ergodicity of $\mathbb{C}^*$ dynamical systems, *Comm. Math. Phys.*, **177**. 507–528 (1996)