ON THE DYNAMICS OF A GENERAL UNITARY OPERATOR

OLIVER KNILL

Abstract. A unitary operator U acting on the weakly compact unit ball B of a Hilbert space H gives a topological dynamical system (B, U). We prove that for every ergodic U-invariant measure m, the dynamical system (B, U, m) is conjugated to a strictly ergodic group translation. Especially, the topological entropy of (B, U) is zero. As an application, it follows that any flow on the unit ball given by a Schrödinger equation $ih\psi = L\psi$ has zero topological entropy.

1. Introduction

We consider here the problem to find the set of invariant measures of a linear unitary flow U_t on a Hilbert space H. Important examples of such dynamical systems are the Schrödinger flow $U_t\psi = e^{itL}\psi$ defined by $i\dot{\psi} = L\psi$ or the Koopman flow $f_t(x) = U_t f = f(T_t x)$ on $L^2(X, m)$ associated to an invariant measure m of a differential equation $\dot{x} = f(x)$ on a manifold X. As a motivation for the problem one can say that the existence of invariant measures for U_t allows to make statements about recurrence of a wave ψ in quantum mechanics or a measurable set $f = 1_A$ in classical mechanics. Examples in both settings show that a linear unitary dynamical system is in general non trivial.

In quantum mechanics, an orbit ψ_t of the time evolution can be observed by measuring the probabilities $|(\phi, U_t\psi)|^2$ that the wave ψ is in the pure state ϕ at time t. For a differential equation with invariant measure m, one can observe $m(A \cap T_t(C)) = (1_A, U_t 1_C)$, the probability with which an orbit $T_t(x) = x_t$ starting in the set C hits an other set A at time t. It is therefore in both cases natural to look for invariant Borel measures on H with respect to the weak topology. The set of invariant measures of the flow is contained in the set of invariant measures of a time one map so that one can consider the general problem, to determine the set of invariant measures of the topological dynamical system (B, U), where U is an arbitrary unitary operator and B is the weakly compact unit ball. We are not aware that this problem has been addressed somewhere already. In the two cases, when U has only absolutely continuous spectrum (i.e. scattering states in quantum mechanics, the mixing property in classical mechanics) or discrete spectrum (i.e. bound states in quantum mechanics, group translation in classical mechanics), the answer to the problem is quite easy. However, both discrete and absolutely continuous spectrum are exceptional in a Baire category sense: a generic unitary operator U or a generic Koopman operator U_T has purely singular continuous spectrum [14, 15, 3, 10]. Singular continuous spectrum appears in the quantum mechanics of solid state physics [6], especially when the potential takes only finitely many values (see for example [8]). In classical mechanics, it appears, whenever a topological shift is embedded because many shift invariant measures have purely singular continuous spectrum [10].

2. Topological dynamics of a unitary operator

Given a unitary operator U on a separable Hilbert space H. The unit ball $B \subset H$ is compact in the weak topology and $U : B \to B$ is a homeomorphism. For $\phi \in H$, let H_ϕ be the cyclic subspace of ϕ and let $B_\phi = \{||\phi|| \leq 1\}$ be the unit ball in H_ϕ. For $\phi, \psi \in H$, the complex measure $\mu = \mu_{\psi,\phi}$ on $\mathbf{T} = \{|z| = 1\}$ is defined by the functional $f \mapsto (\psi, f(U)\phi)$ on $C(\mathbf{T})$.

Lemma 2.1. *The dynamics of U on B_ϕ is topologically conjugated to a shift on a compact, weakly closed subset in $l^\infty(\mathbf{Z})$.*

Proof. Consider
$$F : B_\phi \to l^\infty(\mathbf{Z}, \mathbf{C}),\ \psi \mapsto \{(\phi, U^n\psi)\}_{n\in\mathbf{Z}} ,$$
which maps ψ to the Fourier series of the complex measure $\mu_{\phi,\psi}$ on $\mathbf{T}$. F is injective, because if $(\phi, U^n\psi_1) = (\phi, U^n\psi_2)$ for all n, then $(U^{-n}\phi, \psi_1) = (U^{-n}\phi, \psi_2)$ for all n so that $\psi_1 = \psi_2$ in H_ϕ. If

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$l^\infty(\mathbf{Z})$ has the weak topology, then F is continuous and F is therefore a homeomorphism onto its image $Y_\phi = F(B_\phi)$. Moreover, F conjugates $U : B_\phi \to B_\phi$ to the shift on Y_ϕ. $\square$

For $\phi \in H$, denote by $\mu_\phi = \mu_{\phi,\phi}$ the spectral measure of ϕ. If μ_ϕ has only atoms, then ϕ is called discrete. If μ_ϕ has no atoms, then ϕ is called continuous. If μ_ϕ is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$, then ϕ is called absolutely continuous. If ϕ is both orthogonal to the discrete vectors and all absolutely continuous vectors, it is called singular continuous. (For refereneces on the spectral theory of unitary operators, see ([13] Chapter II) or [5] Appendix), for the topological dynamics and ergodic theory see [7, 16, 12]).

Proposition 2.2. *a) Assume ϕ is discrete. Then U restricted to the weak closure of the orbit of ϕ is a strictly ergodic dynamical system, which is topologically conjugated to a group translation on a compact topological group.*
b) If ϕ is absolutely continuous. Then $U^n\phi \to 0$.
c) Given $\phi \in H$. There exists a dense G_δ of unitary operators for which ϕ is singular continuous and such that ϕ is recurrent in the sense that there exists a sequence $n_k \to \infty$ such that $U^{n_k}\phi \to \phi$.
d) There is a dense set ψ in the continuous subspace of U, such that the orbit of ψ has 0 as an accumulation point.

Proof. a) The Fourier transform of $\mu_\phi = \sum_j a_j \delta(e^{i\lambda_j})$ is the Bohr almost periodic sequence $F(\phi) = \hat{\mu}_n \sim \sum_j a_j e^{i\lambda_j n}$. The translates of the sequence $F(\phi)$ have a compact closure $X_\phi \subset Y_\phi \subset l^\infty(\mathbf{Z})$ and the shift T acts as a group translation on the compact hull X_ϕ. The claim follows from Lemma 2.1.
An alternative proof is obtained by diagonalizing U so that U becomes a direct sum of operators U_i on the complex plane, where all U_i are acting by multiplication with some λ_i.
b) If μ_ϕ is absolutely continuous then $\mu_{\phi,\psi}$ is absolutely continuous ([13]). The Riemann-Lebesgue lemma assures that $\hat{\mu}_n = (\phi, U^n\psi) \to 0$. Because this holds for all ϕ, we know that $U^n\psi \to 0$ in the weak topology.
c) A dense G_δ of measures is rigid, that is, it has the property that for a subsequence $\hat{\mu}_{n_k} \to 1$ [3]. A dense G_δ set of operators has all vectors singular continuous [14]. Let $\mathcal{U}$ be the set of all unitary operators with the strong operator topology. The map $\mathcal{U} \to M(\mathbf{T})$ attaching to U the spectral measure $\mu_{\psi,U}$ is continuous and surjective. A given vector ψ is therefore rigid for a generic U in the sense that $(\psi, U^{n_k}\psi) \to 1$ for some sequence n_k, which implies $U^{n_k}\psi$ converges to ψ.
d) A point ψ is called weakly wandering, if there exists a sequence k_i such that all vectors $U^{k_i}\psi$ are orthogonal. Weakly wandering points are dense in the subspace of all continuous vectors by a theorem of Krengel ([11]). $\square$

Remark 2.3. It would be interesting to know whether (B, U) can be transitive if the Hilbert space H has dimension > 1. If U has a two dimensional discrete subspace H_p, then there can not exist a dense orbit because it would also give a dense orbit on the unit sphere of the discrete subspace, where the dynamics is minimal on the closure of an orbit. This contradicts that an initial point in an eigenspace stays there and is not dense on the sphere. If U has some absolutely continuous spectrum, then (B, U) can not be transitive because it would also be transitive on the absolutely continuous subspace which would contradict Proposition (2.2) b). For the transitivity of (B, U), it is necessary that all spectral measures are purely singular continuous.

3. Ergodic theory of a unitary operator

We look now at ergodic properties of (B, U), that is, we determine the U-invariant measures m on the compact space B.

Theorem 3.1. *Given a unitary operator U on a Hilbert space H with unit ball B. Every U-invariant measure m defines a uniquely ergodic system* $(\text{supp}(m), U, m)$ *which is topologically conjugated to a group translation.*

Proof.
(i) Two vectors $\phi, \psi \in H$ define the complex measure $\mu = \mu_{\phi,\psi}$. By Wiener's theorem,

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |(\psi, U^k\phi)|^2 = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |\hat{\mu}_k|^2 = \sum_{x\in\mathbf{T}} |\mu(\{x\})|^2 .$$

(Wiener's theorem is usually formulated for real probability measures on $\mathbf{T}$ (see [9]) but holds for any complex measures on $\mathbf{T}$.) Especially, the right hand side is zero, for ϕ, ψ in the continuous subspace H_c

because then, μ is a continuous measure ($\mu = \mu_{\phi,\psi}$ is absolutely continuous to both μ_ψ and μ_ϕ [13]).

(ii) For fixed ψ, the map $f : B \to \mathbf{R}$, $f(\phi) = |(\psi,\phi)|^2$ is continuous. Let m be a U-invariant measure on B. Then $f \in L^1(B,m)$. By Birkhoff's ergodic theorem,

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |(\psi, U^k\phi)|^2 = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} f(U^k\phi) \to \int_B f(\phi)\; dm(\phi) = \int_B |(\psi,\phi)|^2\; dm(\phi) .$$

(iii) Let $H = H_d \oplus H_c$ be the orthogonal decomposition of H into the discrete and continuous subspace with respect to U. Every U-invariant measure m defines a U-invariant measure m_c on H_c by push-forward $m_c(A) = m(P_c^{-1}A)$, where $P_c : H \to H_c$ is the projection. Let B_c be the unit ball H_c. By (i) and (ii), $\int_{B_c} |(\psi,\phi)|^2\; dm_c(\phi) = 0$ for all $\psi \in H_c$. This implies that m-almost all vectors $\phi \in H_c$ are all orthogonal to the given vector $\psi \in H_c$. Taking a countable dense set $\{\psi_n\}_{n\in\mathbf{N}}$ in B_c, m almost all vectors ϕ are orthogonal to all these vectors and so to every vector in B_c. Therefore, the support of m_c is contained in $\{0\} \subset B_c$ or $m_c = 0$. Especially, every invariant measure m has its support on the subspace H_d and the claim follows from Proposition 2.2 a). □

Remark 3.2. It follows that U has continuous spectrum, if and only if every U-invariant measure m is located on $\{0\}$. All nontrivial invariant measures have their support away from $\{0\}$. Because a dense set of the support $\mathrm{supp}(\mu)$ of m is on the sphere, every ergodic U-invariant measure has its support contained in a sphere $S_r = \{||\phi|| = r\}$.

Remark 3.3. The mean ergodic theorem in Hilbert space deals with averages $n^{-1}\sum_{k=1}^{n} U^k\psi$. Invariant measures are constructed by taking any measure ρ and taking accumulation points of $n^{-1}\sum_{k=1}^{n} \rho(U^k)$. Assume U has some point spectrum not containing 1. Because 1 is not an eigenvalue, there is no nonzero fixed point of U and the mean ergodic theory gives $n^{-1}\sum_{k=1}^{n} U^k\psi \to 0$ while $n^{-1}\sum_{k=1}^{n} \delta(U^k\psi)$ converges to a nontrivial invariant measure.

Corollary 3.4. *The topological entropy of $U : B \to B$ is zero.*

Proof. The metric entropy of every invariant measure m is zero. By Goodman's variational theorem in ergodic theory, the topological entropy is the supremum over all metric entropies and is vanishing too. □

Remark 3.5. While U is an isometry with respect to the norm (for which B is not compact), it is not an isometry in the weak topology and it is not apriory clear, that the topological entropy of U is vanishing.

4. Additional remarks and questions

4.1. A measure on B which is invariant under a $\mathbf{R}$ group action $t \mapsto U_t$, is also invariant under the $\mathbf{Z}$ action $n \mapsto U_1^n$ and Theorem 3.1 shows that U_t is measure theoretically conjugated to a flow on a compact topological group. Especially: "there is no chaos for the Schrödinger flow": for every measure m on B which is invariant under the Schrödinger equation $ih\dot{\psi} = \Delta\psi + V\psi$, the measure theoretical flow (B, U_t) is measure theoretically conjugated to a group translation and the flow has zero topological entropy.

4.2. Let G be a topological group and let $g \mapsto U_g$ be a unitary representation of G on a Hilbert space H. For a measure m which is invariant under all U_g, the dynamical system (B, U_g) with time G is topologically conjugated to a group translation. If U is only a contraction on a Hilbert space, then Wiener's theorem still holds (compare [11]) and Theorem 3.1 extends to all contraction semi-groups.

4.3. While unitary operators are not interesting for the invariant subspace problem because there are always plenty of invariant subspaces given as images of spectral projections, it is interesting to ask for the structure of invariant measures of (H, A), if A is a general linear operator and H is equipped with the weak topology. In the compactification $H \cup \{\infty\}$, there must be some invariant measures. Are still all nontrivial invariant measures having their support on eigenspaces belonging to eigenvalues on the unit circle? The nontriviality of such dynamical systems is illustrated by the open question, whether there exists a system (H, A), where every nonzero vector ψ has a dense orbit. (A positive answer would of course solve the invariant subspace problem). Rolevicz constructed a system (H, A), which is transitive in the norm topology of H ([2]).

4.4. A unitary operator is called reductive, if the lattice of invariant subspaces of U and U^* coincide. By a result of Wermer ([17]), the reductive unitary operators are exactly those which do not contain a subspace on which U is the shift. If follows that a unitary operator is reductive if it has no Lebesgue spectrum, so that a generic unitary operator is reductive. A unitary Koopman operator U_T is reductive if and only T has positive metric entropy because a transformation with positive entropy contains a Bernoulli shift as a factor [4].

4.5. In quantum mechanics, one is also interested in the dynamics $A \mapsto UAU^*$, induced on the space of bounded operators. Because $V : A \mapsto UAU^*$ is a unitary operator on the Hilbert space of Hilbert-Schmidt operators $\mathcal{B}_2$ (see [1]), Theorem 3.1 implies that for every invariant measure m on $\mathcal{B}_2$, the dynamical system $(\sup(m), V)$ is topologically conjugated to a group translation. If U has continuous spectrum, then V has also continuous spectrum ([1]). It is also evident that if U has some point spectrum, then V contains the same point spectrum. We don't know, whether the inner automorphism $V : A \mapsto UAU^*$ acting on the weakly compact unit ball of the algebra of *all* operators has zero topological entropy.

4.6. Point spectrum is unstable, because operators with singular continuous spectrum form a dense G_δ in the group of all unitaries [14]. The theory of rank one perturbations also illustrates, how point spectrum can be distroyed [15]. This should be compared to the situation in Hamiltonian mechanics, where quasi-periodic motion disappears by small perturbations because of the generic ergodicity results of Oxtoby-Ulam or Katok-Stepin. In ergodic theory, a small perturbation of an automorphism with discrete spectrum renders it continuous by Halmos weak mixing theorem. However, these are quite general perturbations. Quasi-periodic motion is often observed in experiments both in Hamiltonian mechanics as well as in quantum mechanics. KAM theory gives an explanation of that fact in some circumstances. It would be interesting to know of criteriums for a topological space of selfadjoint Schrödinger operators to contain some open set of operators with point spectrum.

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Division of Physics, Mathematics and Astronomy, Caltech, 91125 Pasadena, CA, USA