Limit Set of Trajectories of the Coupled Viscous Burgers' Equations *

July 28, 1996

Janpou Nee
Institute of Mathematics, Academia Sinica
Taipei, 11529, Taiwan, Republic of China.
and
Jinqiao Duan
Department of Mathematical Sciences
Clemson University
Clemson, South Carolina 29634, USA.

Abstract

In this letter, a coupled system of viscous Burgers' equations with zero Dirichlet boundary conditions and appropriate initial data is considered. For the well-known single viscous Burgers' equation with zero Dirichlet boundary conditions, the zero equilibrium is the unique global exponential point attractor. A similar property is shown for the coupled Burgers' equations, i.e., trajectories starting with initial data which is not too large approach the zero equilibrium as time goes to infinity. This "approaching" or convergence is not necessarily exponentially fast, unlike the single viscous Burgers' equation.

Key words: Burgers' equation, long time behavior, attractor, phase space analysis, ω-limit set

*Address for correspondence: Dr. J. Duan, Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29634, USA. E-mail address: duan@math.clemson.edu; Tel: (864)656-2730; Fax: (864)656-5230.

1 Introduction

We consider the following coupled Burgers' equations

$$\begin{cases} u_t = u_{xx} - uu_x - a(uv)_x, \\ v_t = v_{xx} - vv_x - b(uv)_x, \end{cases} \tag{1.1}$$

together with Dirichlet boundary conditions

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ v(0,t) = v(1,t) = 0, \end{cases} \tag{1.2}$$

and appropriate initial conditions

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \tag{1.3}$$

for $0 < x < 1, t > 0$. Here a, b are constants.

This coupled system, derived by Esipov [1], is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity. The constants a, b depend on the system parameters such as the Péclet number, the Stokes velocity of particles due to gravity, and the Brownian diffusivity. In [1] Esipov reported numerical simulations for (1.1)-(1.3) and compared the results with experimental data.

In this letter, we consider dynamical aspect of the coupled system. We show that trajectories (orbits of solutions) of this coupled system approach the zero equilibrium as $t \to +\infty$ when the initial data $u(x,0)$, $v(x,0)$, is not too large in some Sobolev norm. That is, the ω-limit set of these trajectories is the zero equilibrium.

2 Single viscous Burgers' equation

We first recall an interesting property about ω-limit set for the well-known (single) viscous Burgers' equation ([2]), i.e. equations in (1.1) without the nonlinear coupling term

$$w_t = w_{xx} - ww_x, \tag{2.1}$$

with boundary and initial conditions

$$\begin{cases} w(0,t) = w(1,t) = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{2.2}$$

The interest in the Burgers' equation arises because it is a simple one dimensional analog of the Navier-Stokes equation. The importance of the Burgers' equation is due to the nonlinear convection term uu_x.

In the following, $L^2(0,1)$, $L^\infty(0,1)$, $H_0^1(0,1)$ and $H_0^2(0,1)$ are the usual Sobolev spaces, while $C(0,1)$ is the space of continuous functions. We denote by $||\cdot||$ the usual $L^2(0,1)$ norm. All integrals $\int$ are with respect to $x \in [0,1]$, unless specified otherwise.

Multiplying the equation (2.1) by w and integrating over $x \in [0,1]$, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 = -\|w_x\|^2 \leq -\|w\|^2, \tag{2.3}$$

where we have used the Poincaré inequality

$$\int_0^1 w^2 dx \leq \int_0^1 w_x^2 dx, \tag{2.4}$$

in the last step. Thus by the Gronwall inequality ([3], p.88), we further have

$$\|w(x,t)\|^2 \leq \|w(x,0)\|^2 e^{-2t}, \quad t > 0. \tag{2.5}$$

This means that all trajectories converge in the $L^2(0,1)$- norm to the zero equilibrium exponentially fast, i.e., the ω-limit set ([4]) of every trajectory is the zero equilibrium. The zero equilibrium is the global point attractor. See [5] and [6] for further results in this regard. This property also holds for higher dimensional viscous Burgers type convection-diffusion equations ([7]).

3 ω-limit set of the coupled viscous Burgers' equations

In this section, we show that trajectories of the coupled viscous Burgers' equations (1.1)-(1.3), whose initial data is not too large in $H_0^1(0,1)$-norm, converge to the zero equilibrium in the max-norm. This convergence, though, is not necessarily exponentially fast, unlike the situation for the single viscous Burgers' equation.

Let

$$I(t) = \frac{1}{2}(||u||^2 + ||v||^2), \tag{3.1}$$

$$J(t) = \frac{1}{2}(||u_x||^2 + ||v_x||^2), \tag{3.2}$$

$$K(t) = \frac{1}{2}(||u_{xx}||^2 + ||v_{xx}||^2). \tag{3.3}$$

For initial data $u(x,0), v(x,0) \in H_0^1(0,1)$, local (-in-time) existence and uniqueness of classical solutions to (1.1)-(1.3) can be easily shown by the usual semigroup method; cf. [8], Theorems 3.3.3, 3.3.4 and 3.5.2. Global existence and uniqueness of classical solutions then follows once we show a priori that the solutions do not become unbounded in $H_0^1(0,1)$-norm at finite positive time. Moreover, the solutions, whenever they exist, are in $H_0^k(0,1), k = 1, 2, \cdots$; (cf. [8], p.73). Note that $I(t) \leq J(t) \leq K(t)$ whenever u, v exist, by the Poincaré inequality.

We will show that $J(t) \to 0$ as $t \to 0$, when $J(0)$ is bounded by some constants depending only on the system parameters a, b in (1.1).

Multiplying the first and second equation in (1.1) by $-u_{xx}$ and $-v_{xx}$, respectively, adding the two resulting equations and then integrating over $x \in [0,1]$, we obtain

$$\begin{aligned}
\frac{dJ}{dt} &= \int -u_{xx}^2 + uu_xu_{xx} + a(uv)_xu_{xx} - v_{xx}^2 + vv_xv_{xx} + b(uv)_xu_{xx} \\
&= \int -u_{xx}^2 - \frac{u_x^3}{2} - v_{xx}^2 - \frac{v_x^3}{2} + (a+b)(uv)_x(u_{xx} + v_{xx}) \\
&\leq \int -u_{xx}^2 - \frac{u_x^3}{2} - v_{xx}^2 - \frac{v_x^3}{2} \\
&\quad +(|a| + |b|)(\int (u_xv + uv_x)^2)^{\frac{1}{2}}(\int (u_{xx} + v_{xx})^2)^{\frac{1}{2}} \\
&\leq -2K - \int \frac{u_x^3}{2} - \int \frac{v_x^3}{2} + (|a| + |b|)(\int 2(u_x^2v^2 + u^2v_x^2))^{\frac{1}{2}}(2K)^{\frac{1}{2}} \\
&\leq -2K - \int \frac{u_x^3}{2} - \int \frac{v_x^3}{2} \\
&\quad +2(|a| + |b|)\{\sqrt{\int u_x^4 \int v^4} + \sqrt{\int u^4 \int v_x^4}\}^{\frac{1}{2}}K^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

Now we further estimate the right hand side of (3.4) term by term.

From the fact that

$$u^2 = 2\int_0^x uu_xdx \leq 2||u|| \cdot ||u_x||,$$

for $u \in H_0^1(0,1)$, we get the so-called Agmon inequality

$$||u||_\infty^2 \leq 2||u|| \cdot ||u_x||, \tag{3.5}$$

where $||u||_\infty$ is the L^∞-norm.

Observe, using the Cauchy-Schwarz inequality ([9], p.183) and the above Agmon inequality

$$\begin{aligned}
\int u_x^3 dx &= \int u_x^2 du = -2\int u u_x u_{xx} dx \\
&\leq 2\|u\|_\infty (\int u_x^2)^{\frac{1}{2}} (\int u_{xx}^2)^{\frac{1}{2}} \\
&\leq 2\sqrt{2}(\int u^2)^{\frac{1}{4}} (\int u_x^2)^{\frac{1}{4}} (\int u_x^2)^{\frac{1}{2}} (\int u_{xx}^2)^{\frac{1}{2}} \\
&= 2\sqrt{2}(\int u^2)^{\frac{1}{4}} (\int u_x^2)^{\frac{3}{4}} (\int u_{xx}^2)^{\frac{1}{2}} \\
&\leq 8 I^{\frac{1}{4}} J^{\frac{3}{4}} K^{\frac{1}{2}} \\
&\leq 8 J K^{\frac{1}{2}} \qquad (3.6)
\end{aligned}$$

In the last step, we have used the fact that $I(t) \leq J(t)$. The same inequality holds for $\int v_x^3 dx$.

Similarly,

$$\begin{aligned}
\int u_x^4 dx &= \int u_x^3 du = -3\int u u_x^2 u_{xx} dx \\
&\leq 3\|u\|_\infty (\int u_x^4)^{\frac{1}{2}} (\int u_{xx}^2)^{\frac{1}{2}} \\
&\leq 3\sqrt{2}(\int u^2)^{\frac{1}{4}} (\int u_x^2)^{\frac{1}{4}} (\int u_x^4)^{\frac{1}{2}} (\int u_{xx}^2)^{\frac{1}{2}}. \qquad (3.7)
\end{aligned}$$

Thus

$$\begin{aligned}
(\int u_x^4 dx)^{\frac{1}{2}} &\leq 3\sqrt{2}(\int u^2)^{\frac{1}{4}} (\int u_x^2)^{\frac{1}{4}} (\int u_{xx}^2)^{\frac{1}{2}} \\
&\leq 6\sqrt{2} I^{\frac{1}{4}} J^{\frac{1}{4}} K^{\frac{1}{2}} \\
&\leq 6\sqrt{2} J^{\frac{1}{2}} K^{\frac{1}{2}}. \qquad (3.8)
\end{aligned}$$

The same inequality holds for $\int v_x^4 dx$.

Moreover, note that

$$u^4 = (\int_0^x 2u u_x dx)^2 \leq 4\int u^2 dx \int u_x^2 dx.$$

So we have

$$\int u^4 dx \leq 4\int u^2 dx \int u_x^2 dx \leq 16 IJ \leq 16 J^2. \qquad (3.9)$$

The same estimate holds for $\int v^4 dx$.

Substituting (3.6), (3.8) and (3.9) into (3.4), and using the *Young's inequality* ([3], p.108), we finally get

$$\begin{aligned} \frac{dJ}{dt} &\leq -2K + 8JK^{\frac{1}{2}} + 8\sqrt{3}2^{\frac{1}{4}}(|a|+|b|)J^{\frac{3}{4}}K^{\frac{3}{4}} \\ &\leq -2K + \frac{\epsilon}{2}K + \frac{1}{2\epsilon}64J^2 + \frac{3\epsilon}{4}K + \frac{1}{4\epsilon^3}[8\sqrt{3}2^{\frac{1}{4}}(|a|+|b|)]^4 J^3 \end{aligned} \quad (3.10)$$

for $\epsilon > 0$. Taking $\epsilon = \frac{4}{5}$ and noting that $J(t) \leq K(t)$, we now have

$$\begin{aligned} \frac{dJ}{dt} &\leq -K + 40J^2 + 36000(|a|+|b|)^4 J^3 \\ &\leq -J + 40J^2 + 36000(|a|+|b|)^4 J^3. \end{aligned} \quad (3.11)$$

The comparison equation

$$\frac{dJ}{dt} = -J + 40J^2 + 36000(|a|+|b|)^4 J^3 \equiv f(J) \quad (3.12)$$

has fixed points 0 and

$$J_+ = \frac{-1 + \sqrt{1 + 9000(|a|+|b|)^4}}{1800(|a|+|b|)^4}.$$

The third fixed point is negative and is discarded since $J(t)$ is always nonnegative by definition (3.2). We calculate $f'(0) < 0$ and $f'(J_+) > 0$. So fixed point 0 is stable while J_+ is unstable; see [10], p.187 or [11], p.8. That is, for the comparison equation (3.12), $J(t) \to 0$ if $J(0) < J_+$, while $J(t) \to \infty$ if $J(0) > J_+$. Due to the standard comparison result for ordinary differential inequalities and equations ([12], P.69), the $J(t)$ satisfying the differential inequality (3.11) also approach zero as t goes to infinity, when $J(0) < J_+$.

We have thus shown that, if the initial data satisfies $\frac{1}{2}(\|u_0'(x)\|^2 + \|v_0'(x)\|^2) < J_+$, the corresponding classical solutions and hence trajectories exist for all $t > 0$, since the $H_0^1(0,1)$-norm, i.e., $J(t)$, in this case is bounded. Moreover these trajectories approach the zero equilibrium as $t \to \infty$. The zero equilibrium is the ω-limit set of these trajectories. We remark that this convergence is not necessarily exponentially fast, unlike the single Burgers' equation. If $\frac{1}{2}(\|u_0'(x)\|^2 + \|v_0'(x)\|^2) > J_+$, however, we cannot conclude any thing about the corresponding trajectories, based on the above dynamical system style analysis.

Note that $\|u_x\|$, $\|v_x\|$ are actually $H_0^1(0,1)$-norms of u and v due to the Poincaré inequality, and note also that $H_0^1(0,1)$ is embeded in $C(0,1)$. So the above convergence of trajectories holds in the max-norm in $C(0,1)$.

Therefore we obtain the following theorem

Theorem 1 *Assume that* $u_0(x), v_0(x)$ *are in* $H_0^1(0,1)$ *and satisfy*

$$\frac{1}{2}(\|u_0'(x)\|^2 + \|v_0'(x)\|^2) < \frac{-1+\sqrt{1+45(|a|+|b|)^4}}{225(|a|+|b|)^4}.$$

Then the global unique classical solutions of the coupled system (1.1)-(1.2)-(1.3) exist, and the corresponding trajectories (orbits of solutions) approach the zero equilibrium in max-*norm. That is, the zero equilibrium is the* ω-*limit set of these trajectories.*

Acknowledgement. We would like to thank Vince Ervin for useful comments of this work.

References

[1] S. E. Esipov, Coupled Burgers equations: A model of polydispersive sedimentation, *Phys. Rev. E* **52** (1995), 3711-3718.

[2] J. M. Burgers, Applications of a model system to illustrate some points of the statistical theory of free turbulence, *Proc. Roy. Neth. Acad. Sci. Amsterdam* **43** (1940), 2-12.

[3] R. Temam, *"Infinite-Dimensional Dynamical Systems in Mechanics and Physics,"* Springer-Verlag, New York, 1988.

[4] J. K. Hale, *"Asymptotic Behavior of Dissipative Systems"*, American Math. Soc., (1988).

[5] H. V. Ly, K. D. Mease and E. S. Titi, Some remarks on distributed and boundary control of the viscous Burgers' equation, *Numerical Functional Analysis and Optimization* (submitted), 1996.

[6] J. A. Burns and S. Kang, A control problem for Burgers' equation with bounded input/output, *Nonlinear Dynamics*, **2** (1991), pp. 235-262.

[7] A. T. Hill and E. Suli, Dynamics of a nonlinear convection-diffusion equation in multidimensional bounded domains, *Proc. R. Soc. Edinburgh* **125A** (1995), 439-448.

[8] D. Henry, *"Geometric Theory of Semilinear Parabolic Equations"*, Springer-Verlag, Berlin, 1981.

[9] M. Renardy and R. C. Rogers, *"An Introduction to Partial Differential Equations,"* Springer-Verlag, New York, 1993.

[10] M. W. Hirsch and S. Smale, "*Differential Equations, Dynamical Systems and Linear Algebra*", Academic Press, 1974.

[11] S. Wiggins, "*Introduction to Applied Nonlinear Dynamical Systems and Chaos*", Springer-Verlag, New York, 1990.

[12] W. Walter, "*Differential and Integral Inequalities*", Springer-Verlag, New York, 1970.