Statistical Solutions of the Navier-Stokes Equations on the Phase Space of Vorticity and the Inviscid Limits

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Abstract

Using the methods of Foias [6] and Vishik-Fursikov [10], we prove the existence and uniqueness of both spatial and space-time statistical solutions of the Navier-Stokes equations on the phase space of vorticity. Here the initial vorticity is in Yudovich space and the initial measure has finite mean enstrophy. We show under further assumptions on the initial vorticity that the statistical solutions of the Navier-Stokes equations converge weakly and the inviscid limits are the corresponding statistical solutions of the Euler equations.

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1 INTRODUCTION

Statistical study of the Navier-Stokes equations was initiated by Hopf [7] for the purpose of describing turbulent flow and developed as a coherent mathematical theory by Foias [6], Vishik-Fursikov [10] and others. Roughly speaking, a statistical solution is a probability measure concentrated on the individual solution space associated with the initial value problem for the Navier-Stokes equations.

In this paper we are concerned with the statistical solutions of the Navier-Stokes equations on the phase space of vorticity. Their corresponding initial measure concentrated on $\mathbf{Y}$ of initial vorticity, where $\mathbf{Y}=L^1 \cap L_c^\infty$ is the Yudovich space. There exists a classical theory of existence and uniqueness for individual solutions in the setting [11]. Several important statistical equilibrium theories ([8],[9]) are for vorticity in this phase space. Recently we have established the inviscid limit results for individual solutions corresponding to initial vorticity in $\mathbf{Y}$ ([4],[5]). So it is natural to study the statistical solutions and their inviscid limits related to this phase space.

We will consider both spatial and space-time statistical solutions. Foias [6] and Vishik-Fursikov [10] define spatial and space-time statistical solutions on velocity spaces and prove their existence and uniqueness, respectively. The classical Galerkin approximation plays a basic role in their proof of existence. We adopt their definitions to define spatial and space-time statistical solutions on the phase space of vorticity corresponding to initial measure μ on $\mathbf{Y}$. Thanks to the recent results of the existence and uniqueness of individual solutions with L^1 initial data ([1],[2]), we are able to construct explicitly the defined statistical solutions without appealing to Galerkin approximation. Our proofs of uniqueness are similar to theirs. The inviscid limit results of the statistical solutions are obtained with more regularity assumptions on initial measures.

Finally we remark that because there is no homogeneous Borel measure on $L^p \setminus \{0\}(1 \leq p < \infty)$ the phase space $\mathbf{Y}$ as a subspace of L^p does not support any homogeneous measure except the trivial one. It would be interesting to introduce physically proper spaces of vorticity on which a homogeneous measure can concentrate.

2 PRELIMINARIES

We consider the two dimensional Navier-Stokes and Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u,$$

$$\nabla \cdot u = 0.$$

The kinematic viscosity ν is a positive number in the case of the Navier-Stokes equations; it equals zero for the Euler equations. The vorticity

$$\omega(x,t) = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

obeys the nonlinear advection-diffusion equation

$$(\partial_t + u \cdot \nabla - \nu \Delta)\, \omega = 0. \tag{2.1}$$

The vorticity equation can be viewed as the basic evolution equation. In this formulation the velocity is computed from the vorticity via the Biot-Savart law:

$$u = K * \omega \tag{2.2}$$

where

$$K(x) = \frac{1}{2\pi} \nabla^{\perp} \log(|x|)\,.$$

We consider the initial vorticity in the space

$$\mathbf{Y} = L^1(R^2) \cap L^{\infty}_c(R^2)$$

of bounded functions with compact support and the norm on $\mathbf{Y}$ is the sum of L^1 and L^∞ norms.

For initial vorticity in $\mathbf{Y}$ there exists a unique global in time weak solution of the Euler equations, as shown by Yudovich [11]. The well-posedness for the Navier-Stokes equations with L^1 initial data has also been established. More precisely,

Theorem 2.1 *Let the initial vorticity $\omega_0 \in L^1(\mathbb{R}^{\not\models})$. Then there exist unique C^∞ functions ω and u on $\mathbb{R}^{\not\models} \times \mathbb{R}_+$ which satisfy equation (2.1) and (2.2).*

Furthemore, the operators $S : \omega_0 \longmapsto \omega$ and its derivatives $\partial_t^k \nabla^\alpha S$ (for every integer k and double-index α)are continuous maps as follows

$$S: \quad L^1(\mathbb{R}^{\nvDash}) \longmapsto \mathbb{C}(\overline{\mathbb{R}}_+, \mathbb{L}^{\nVdash}(\mathbb{R}^{\nvDash})) \cap \mathbb{C}(\mathbb{R}_+, \mathbb{W}^{\nVdash,\nVdash} \cap \mathbb{W}^{\nVdash,\infty})$$

$$\partial_t^k \nabla^\alpha S: \quad L^1(\mathbb{R}^{\nvDash}) \longmapsto \mathbb{C}(\mathbb{R}_+, \mathbb{L}^{\nVdash}(\mathbb{R}^{\nvDash}) \cap \mathbb{L}^{\infty}(\mathbb{R}^{\nvDash})),$$

In particular, every $\omega_0 \in L^1(\mathbb{R}^{\nvDash})$ determines a continuous trajectory $S\omega_0 \in L^1(\mathbb{R}^{\nvDash})$, which depends continuously on ω_0.

This theorem has been recently proved by Ben-Artzi [1] and Brezis[2]. We will especially use the continuity of the operator S.

We will also need the following estimates and exponential decay results, which are proved in [4].

Theorem 2.2 *Let $\omega_0 \in \mathbf{Y}$ be the initial vorticity and ω be the corresponding solution of the Navier-Stokes equations. Then for all $t \geq 0$,*

$$\|\omega(\cdot,t)\|_{L^p} \leq \|\omega_0\|_{L^p}, \qquad 1 \leq p \leq \infty,$$

$$\|u(\cdot,t)\|_{L^\infty} \leq U \equiv \sqrt{\|\omega_0\|_{L^1}\|\omega_0\|_{L^\infty}}$$

where u is the velocity corresponding to ω. Furthermore, if the support of the initial vorticity is included in the disk

$$\{x : |x| \leq L\}$$

Then the vorticity satisfies

$$|\omega(x,t)| \leq \|\omega_0\|_{L^\infty} e^{-\sqrt{1+\frac{(Ux)^2}{\nu^2}}} \tag{2.3}$$

for all $x \in Q_t$,

$$Q_t = \{x : |x| \geq C(L + \frac{\nu}{U} + Ut)\}$$

The following energy estimates will also be used.

Proposition 2.3 *Let $\omega_0 \in \mathbf{Y}$ be the initial vorticity and ω be the solution of the Navier-Stokes equations. Then for each $T \geq 0$,*

$$\max_{0\leq t\leq T} \|\omega(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^T \|\omega(\cdot,\tau)\|_{H^1}^2 d\tau \leq \|\omega_0\|_{L^2}^2 \tag{2.4}$$

$$\max_{0\leq t\leq T} \|\frac{d\omega}{dt}(\cdot,t)\|_{H^{-2}} \leq (U+\nu)\|\omega_0\|_{L^2} \tag{2.5}$$

Proof. We obtain from multiplying the Navier-Stokes equations of vorticity by ω and integrating over $\mathbb{R}^{\not\models}$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^{\not\models}} |\omega(x,t)|^2 dx + \int_{\mathbb{R}^{\not\models}} (u \cdot \nabla\omega) \cdot \omega dx = \nu \int_{\mathbb{R}^{\not\models}} (\Delta\omega) \cdot \omega dx$$

We are able to integrate the above terms by parts because of Theorem 2.2 and eventually obtain (2.4). The inequality (2.5) is obtained after multiplying the Navier-Stokes equations by $v \in H^2(\mathbb{R}^{\not\models})$ and integrating by parts.

We will use $\mathcal{B}(\mathcal{X})$ to denote the σ- algebra of a nonempty set X. Often we will use the following basic measure transform lemma without mentioning.

Lemma 2.4 *Let X_i be a space with σ-algebra $\mathcal{B}_\rangle$ for $i = 1, 2$ and*

$$S : X_1 \longmapsto X_2$$

be a measurable mapping and μ be a measure on X_1. Define

$$A^*\mu(\varpi) = \mu(S^{-1}\varpi), \qquad \forall \varpi \in \mathcal{B}_\in,$$

Then we have

$$\int g(u)\, dA^*\mu(u) = \int g(Su_0)\, d\mu(u_0)$$

if either $g(Su_0)$ is $\mu(du_0)$-integrable or $g(u)$ is $A^*\mu$- integrable.

3 SPACE-TIME STATISTICAL SOLUTIONS

We will also use the functional spaces V and Z:

$$V \equiv \{\omega : \omega \in L^2([0, T], H^1) \cap L^\infty([0, T], L^2), \quad \frac{d\omega}{dt} \in L^\infty([0, T], H^{-2})\},$$

$$Z \equiv L^2([0, T], L^2) \cap C([0, T], H^{-2})$$

with the corresponding norms

$$\|\omega\|_V = \|\omega\|_{L^2([0,T],H^1)} + \|\omega\|_{L^\infty([0,T],L^2)} + \|\frac{d\omega}{dt}\|_{L^\infty([0,T],H^{-2})}$$

$$\|\omega\|_Z = \|\omega\|_{L^2([0,T],L^2)} + \|\omega(\cdot, t)\|_{L^\infty([0,T],H^{-2})}$$

Clearly, V is continuously embedded in Z.

Proposition 2.3 implies that the solution (vorticity) ω of the Navier-Stokes equations with initial vorticity ω_0 in $\mathbf{Y}$ is in V.

Let $\mu(\omega_0)$ be the initial probability measure concentrated on $\mathbf{Y}$ satisfying

$$\int_{\mathbf{Y}} \|\omega_0\|^2_{L^2} d\mu(\omega_0) < \infty. \tag{3.1}$$

The space-time statistical solution of the Navier-Stokes equations on the phase space of vorticity can be defined by adopting the definition of Vishik and Furshikov [10].

Definition 3.1 *A space-time statistical solution of the Navier-Stokes equations corresponding to the initial measure μ is a probability measure P on Z such that*

(i) *P is supported on V, i.e., $P(V) = 1$;*

(ii) *There exists a set W closed in V such that*

$$W \in \mathcal{B}(\mathcal{Z}), \quad \mathcal{P}(\mathcal{W}) = \infty,$$

and W includes the solutions of the Navier-Stokes equations;

(iii) *Measure P and μ are related by the formula*

$$P(\gamma_0^{-1}\varpi_0) = \mu(\varpi_0), \quad \forall \varpi_0 \in \mathcal{B}(\mathbf{Y})$$

where $\gamma_0^{-1}\varpi_0 = \{\omega : \omega \in Z, \gamma_0\omega \in \varpi_0\}$ *and* $\gamma_0(\omega) = \omega(0)$*;*

(iv) *The inequality holds*

$$\int (\nu\|\omega\|^2_{L^2([0,T],H^1)} + \|\omega(t)\|^2_{L^2} + \|\omega\|^2_{L^\infty([0,T],L^2)} \tag{3.2}$$

$$+\|\frac{d\omega}{dt}\|_{L^\infty([0,T],H^{-2})})dP(\omega) \le C\int_{\mathbf{Y}} \|\omega_0\|^2_{L^2} d\mu(\omega_0), \quad \forall t \in [0,T]$$

where C is constant independent of ν.

We define a probability measure on Z

$$P(\varpi) = \mu(S^{-1}\varpi), \qquad \forall \varpi \in \mathcal{B}(\mathcal{Z}) \tag{3.3}$$

where $S^{-1}\varpi$ is the preimage of ϖ and S is the solution operator of the Navier-Stokes equations defined in Theorem 2.1. The following proposition shows that $S : \mathbf{Y} \longmapsto Z$ is continuous. Therefore $S^{-1}\varpi \in \mathcal{B}(\mathbf{Y})$ for any $\varpi \in \mathcal{B}(\mathcal{Z})$, that is, (3.3) is well defined.

Proposition 3.2 $S : \mathbf{Y} \longmapsto Z$ *is continuous.*

Proof Let $\omega_1, \omega_2 \in \mathbf{Y}$ and $\omega_{max} = \max\{\|\omega_1\|_{L^\infty}, \|\omega_2\|_{L^\infty}\}$. By the definition of the norm on Z,

$$\|S\omega_1 - S\omega_2\|^2_Z = \int_0^T \|S\omega_1(\cdot,\tau) - S\omega_2(\cdot,\tau)\|^2_{L^2} d\tau$$

$$+ \max_{0\le t\le T} \|S\omega_1(\cdot,t) - S\omega_2(\cdot,t)\|^2_{H^{-2}}$$

Theorem 2.2 and the continuity of the emdedding $L^2(\mathbb{R}^{\not{\mathbb{E}}}) \longmapsto \mathbb{H}^{-\not{\mathbb{E}}}(\mathbb{R}^{\not{\mathbb{E}}})$ imply that

$$\|S\omega_1 - S\omega_2\|^2_Z \le 2\omega_{max} \int_0^T \|S\omega_1(\cdot,\tau) - S\omega_2(\cdot,\tau)\|_{L^1} d\tau$$

$$+2\omega_{max} \max_{0\leq t\leq T} \|S\omega_1(\cdot,t) - S\omega_2(\cdot,t)\|_{L^1}$$

The above estimate and Theorem 2.1, i.e. the continuity of

$$S : L^1(\mathbb{R}^{\nvDash}) \longmapsto \mathbb{C}(\overline{\mathbb{R}}_+, \mathbb{L}^{\nVdash}(\mathbb{R}^{\nvDash}))$$

imply the continuity of $S : \mathbf{Y} \longmapsto Z$, which concludes the proof of this proposition.

We now prove that P is a space-time statistical solution:

Theorem 3.3 *The measure P defined in (3.3) is a space-time statistical solution of the Navier-Stokes equations with initial measure μ in the sense of Definition 3.1.*

Proof We first prove the inequality (3.2). By integrating inequalities (2.4),(2.5) of Proposition 2.3 with respect to $d\mu(\omega_0)$, we obtain

$$\int \left(\max_{0\leq t\leq T} \|\omega(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^T \|\omega(\cdot,\tau)\|_{H^1}^2 d\tau \right) dP(\omega) \leq C_1 \int \|\omega_0\|_{L^2}^2 d\mu(\omega_0)$$

$$\int \|\frac{d\omega}{dt}\|_{L^\infty([0,T],H^{-2})} dP(\omega) \leq C_2 \int_{\mathbf{Y}} \|\omega_0\|_{L^2}^2 d\mu(\omega_0)$$

for some constant C_1 and C_2. In particular, these estimates imply (3.2) and

$$\int \|\omega\|_V \, dP(\omega) < \infty,$$

that is, $P(V) = 1$.

We can define, thanks to Theorem 2.1,

$$W = S\mathbf{Y}$$

Clearly, W consists of the solutions of the Navier-Stokes equations. Using the idea in [10], we can prove that W is closed in V and $W \in \mathcal{B}(\mathcal{Z})$. Furthermore,

$$P(W) = P(S\mathbf{Y}) = \mu(S^{-1}(S\mathbf{Y})) = \mu(\mathbf{Y}) = 1.$$

Let $\varpi \in \mathcal{B}(\mathbf{Y})$. As remarked in [10],

$$\gamma_0^{-1}\varpi \equiv \{\omega \in Z : \gamma_0\omega \in \varpi\} \in \mathcal{B}(\mathcal{Z})$$

Since $S^{-1}\gamma_0^{-1}\varpi$ is the preimage of $\gamma_0^{-1}\varpi$ with respect to $S : \mathbf{Y} \longmapsto Z$,

$$S^{-1}\gamma_0^{-1}\varpi = \varpi$$

$$P(\gamma_0^{-1}\varpi) = \mu(S^{-1}\gamma_0^{-1}\varpi) = \mu(\varpi)$$

which implies (iii) of Definition 3.1.

Thus we have showed that the measure defined in (3.3) is a space-time statistical solution as in Definition 3.1.

Following the idea of Vishik and Fursikov ([10]), we can also prove that the probability measure P defined in (3.3) is the unique space-time statistical solution corresponding to the initial measure μ.

Theorem 3.4 *The space-time statistical solution P in the sense of Definition 3.1 is uniquely determined by the initial measure μ.*

Proof. First we can show that for any $Q \in \mathcal{B}(\mathcal{Z})$

$$\gamma_0(Q \cap S\mathbf{Y}) \in \mathcal{B}(\mathbf{Y})$$

The uniqueness of the solution of the Navier-Stokes equations implies

$$\gamma_0^{-1}(\gamma_0(Q \cap S\mathbf{Y})) \cap S\mathbf{Y} = Q \cap S\mathbf{Y}$$

Using the above inequality and $P(S\mathbf{Y}) = 1$,

$$P(Q) = P(Q \cap S\mathbf{Y}) = P(\gamma_0^{-1}(\gamma_0(Q \cap S\mathbf{Y})) \cap S\mathbf{Y})$$

$$= P(\gamma_0^{-1}(\gamma_0(Q \cap S\mathbf{Y}))) = \mu(\gamma_0(Q \cap S\mathbf{Y}))$$

That is, P is uniquely determined by μ.

4 INVISCID LIMIT OF SPACE-TIME STATISTICAL SOLUTIONS

Let μ be the initial probability measure satisfying (3.1) and P^{NS} be the space-time statistical solution obtained in the previous section. To show the inviscid limit (as $\nu \to 0$), we need to make further assumptions on the initial data:

$$\int |\nabla \omega_0| dx < \infty, \qquad \int \|\omega_0\|_{H^1} d\mu(\omega_0) < \infty \tag{4.1}$$

and we have

Proposition 4.1 *Assume that the initial vorticity ω_0 and measure μ satisfy assumption (3.1). Then the individual solution (vorticity) $\omega^{(NS)}$ and the statistical solution $P^{(NS)}$ of the Navier-Stokes equations with initial vorticity ω_0 and respectively, measure μ satisfy*

$$\int \left(\max_{0\le t\le T} \|\omega^{(NS)}(\cdot,t)\|^2_{H^1} + \nu \int_0^T \|\omega^{(NS)}(\cdot,\tau)\|^2_{H^2} d\tau \right) dP^{(NS)}(\omega^{(NS)}) \le c_1 \tag{4.2}$$

$$\int \|\frac{d\omega^{(NS)}}{dt}\|_{L^\infty(0,T;H^{-2})} dP^{(NS)}(\omega^{(NS)}) \le c_2 \tag{4.3}$$

where c_1, c_2 are constants independent of ν for small ν.

Proof Since $\omega^{(NS)}$ decays at infinity (see Theorem 2.2), We can show by the standard energy estimate that

$$\max_{0\le t\le T} \|\omega^{(NS)}(\cdot,t)\|^2_{H^1} + \nu \int_0^T \|\omega^{(NS)}(\cdot,\tau)\|^2_{H^2} d\tau \le c_1 \tag{4.4}$$

where c_1 does not depend on ν but may depend on T. We obtain (4.2) by integrating both side of (4.4) with respect to $d\mu(\omega_0)$. (4.3) is an easy consequence of Proposition 2.3.

The statistical solution of the Euler equations on the phase space of vorticity can be defined through Definition 3.1 by formally setting $\nu = 0$. We can construct the space-time statistical solution of the Euler equations as the inviscid limit of $P^{(NS)}$ of the Navier-Stokes equations.

Theorem 4.2 *Let the initial vorticity ω_0 and measure μ satisfy the assumption (3.1) and $P^{(NS)}$ be the statistical solution of the Navier-Stokes equations constructed in the previous sections. Then there exists a subsequence of $P^{(NS)}$ (still denoted by $P^{(NS)}$) such that its inviscid limit $P^{(E)}$ exists and $P^{(E)}$ is the statistical solution of the Euler equations. Furthermore, $P^{(E)}$ satisfies the estimates:*

$$\int \max_{0\leq t\leq T} \|\omega^{(E)}(\cdot, t)\|_{H^1}^2 \, dP^{(E)}(\omega^{(E)}) \leq c_1 < \infty$$

$$\int \|\frac{d\omega^{(E)}}{dt}\|_{L^\infty([0,T];H^{-2})} dP^{(E)}(\omega^{(E)}) \leq c_2 < \infty$$

where $\omega^{(E)}$ is the vorticity of the Euler equations with initial vorticity ω_0.

The following Prokhorov 's weak compactness result plays an important role in the proof of Theorem 4.2.

Lemma 4.3 *Let X_1 and X_2 be two Banach spaces such that X_2 is separable and X_1 is compactly imbedded in X_2. Assume that M is a family of probability measures defined on $\mathcal{B}(\mathcal{X}_\in)$ with support on $\mathcal{B}(\mathcal{X}_\infty)$. If for any $\mu \in M$, $\|\cdot\|_{X_1}$ is $\mu-$measurable and*

$$\sup_{\mu\in M} \int \|f\|_{X_1} \, d\mu(f) < \infty,$$

then M is weakly compact.

Proof of Theorem 4.2 We just sketch the proof. Estimates (4.2),(4.3) of Proposition 4.1 imply

$$\int_V \|\omega\|_V \, dP^\nu(\omega) \leq C,$$

where C is independent of ν. Since V is compactly imbedded in Z , we can use Prokhorov's theorem. That is, $P^{(NS)}$ is weakly compact in Z and for a subsequence converges to some probability measure $P^{(E)}$ on Z. We can check that $P^{(E)}$ is the statistical solution of the Euler equation. The two estimates for $P^{(E)}$ hold because of the bounds in the estimates (4.2) ,(4.3) are uniform for small ν.

5 SPATIAL STATISTICAL SOLUTIONS

We will use BC to denote the space of real bounded continuous functions on $L^2(\mathbb{R}^{\not\models})$ and $\mathcal{C}_\in$ to denote the space of functions Φ on $L^2(\mathbb{R}^{\not\models})$ such that

$$\|\Phi\|_{\mathcal{C}_\in} \equiv \sup_{\omega} \frac{|\Phi(\omega)|}{1+\|\omega\|_{L^2}^2} < \infty.$$

Let μ be a Borel probability measure concentrated on $\mathbf{Y}$ and satisfy

$$\int_{\mathbf{Y}} \|\omega_0\|_{L^2}^2 \, d\mu(\omega_0) < \infty. \tag{5.1}$$

The spatial statistical solution of the Navier-Stokes equations on the vorticity phase space is defined as follows. A quite similar definition of spatial statistical solutions on the velocity space was given by Foias in [6].

Definition 5.1 *A family of Borel probability measures $\{\mu_t\}_{0\le t\le T}$ on $L^2(\mathbb{R}^{\not\models})$ is called a spatial statistical solution of the Navier-Stokes equations on the phase space of vorticity corresponding to μ if it satisfies*

(i)

$$t \longmapsto \int \psi(\omega)\, d\mu_t(\omega) \textit{is measurable on } [0,\, T] \textit{ for all } \psi \in BC,$$

(ii)

$$\int \|\omega\|_{L^2}^2 \, d\mu_t(\omega) \in L^\infty([0,\, T]),$$

(iii)

$$\int \|\omega\|_{H^1}^2 \, d\mu_t(\omega) \in L^1([0,\, T]),$$

(iv)

$$\int_0^T \int_{L^2} [-\Phi_t^{'}(t,\omega) + \nu((\omega, \Phi_\omega^{'}(t,\omega)))$$
$$+(u\cdot\nabla\omega, {\Phi'}_\omega(t,\omega)]\, d\mu_t(\omega)\, dt = \int_{\mathbf{Y}} \Phi(0,\omega)\, d\mu(\omega),$$

for all $\Phi(t,\omega) = r(t)\phi(\omega)$ *with* $r(t) \in C_0^\infty([0,T))$ *and* $\phi \in \mathcal{J}$, *where* u *is the corresponding velocity of* ω *and* $\mathcal{J}$ *is a class of real functions* ϕ *defined on* H^1 *satisfying*

(A) $|\phi(\omega)| \le c_1 + c_2\|\omega\|_{H^1}$, *for any* $\omega \in H^1$ *and some constants* c_1, c_2 *which may depend on* ϕ;

(B) ϕ *is Frechet* L^2*-differentiable in the direction of* H^1 *i.e.,* $\exists \phi'_\omega \in L^2$ *such that for* $v \in H^1$

$$\frac{1}{\|v\|_{H^1}}|\phi(\omega+v) - \phi(\omega) - (\phi'_\omega, v)| \to 0, \quad as \quad \|v\|_{H^1} \to 0;$$

(C) ϕ' *is continuous from* H^1 *to* H^1 *and* ϕ' *is bounded.*

We shall now prove the existence of the spatial statistical solutions. Let $\omega(\cdot,t) = S(t)\omega_0(\cdot)$ be the unique solution (vorticity) of the Navier-Stokes equations corresponding to the initial data $\omega_0 \in \mathbf{Y}$,where $S(t)$ is the solution operator defined in Theorem 2.1. We now define a Borel probability measure on $L^2(\mathbb{R}^{\not\vDash})$

$$\mu_t(\varpi) = \mu(S(t)^{-1}\varpi), \quad \forall \varpi \in \mathcal{B}(\mathcal{L}^{\in}) \tag{5.2}$$

where $\mathcal{B}(\mathcal{L}^{\in})$ stands for the $\sigma-$algebra of $L^2(\mathbb{R}^{\not\vDash})$.

Proposition 5.2 *For all* $t \in [0,T]$, μ_t *in (5.2) is well defined.*

Proof. We only need to show that for any $t \in [0,T]$, $S(t) : \mathbf{Y} \longmapsto L^2$ is continuous. In fact, for any $t \in [0,T]$ and $\omega_1, \omega_2 \in \mathbf{Y}$,

$$\|S(t)\omega_1 - S(t)\omega_2\|_{L^2}^2 \le 2\tilde{\omega}\|S(t)\omega_1 - S(t)\omega_2\|_{L^1}$$

where $\tilde{\omega} = \max\{\|\omega_1\|_{L^\infty}, \|\omega_2\|_{L^\infty}\}$. This estimate and the continuity of

$$S : L^1(\mathbb{R}^{\not\vDash}) \longmapsto \mathbb{C}(\overline{\mathbb{R}}_+, \mathbb{L}^{\not\Vdash}(\mathbb{R}^{\not\vDash}))$$

imply that $S(t) : \mathbf{Y} \longmapsto L^2$ is continuous.

Theorem 5.3 *The family of probability measures* $\{\mu_t\}_{0\le t\le T}$ *defined in (5.2) is a spatial statistical solution of the Navier-Stokes equations on the phase space of vorticity.*

Proof. We need to check that μ_t given by (5.2) satisfies $(i), (ii), (iii)$ and (iv) of Definition 5.1. First we check (i). By the definition of $\mu_t(\omega)$,

$$\int \psi(\omega)\, d\mu_t(\omega) = \int \psi(S(t)\omega_0)\, d\mu(\omega_0)$$

for $\psi \in BC$. The continuity of ψ and S imply the continuity of $\int \psi(\omega)\, d\mu_t(\omega)$ and thus its measurability on $[0, T]$.

(ii) and (iii) are easy consequences of the estimates in Proposition 2.3.

Now we prove (iv). First we show that functional

$$g(\Phi) = \int_0^T \int_{L^2} [-\Phi_t'(t,\omega) + \nu((\omega, \Phi_\omega'(t,\omega))) + (u \cdot \nabla\omega, \Phi_\omega'(t,\omega)]\, d\mu_t(\omega)\, dt$$

makes sense. For $\Phi(t,\omega) = r(t)\phi(\omega)$ with $r(t) \in C_0^\infty([0, T))$ and $\phi \in \mathcal{J}$,

$$\Phi_t'(t,\omega) = r'(t)\phi(\omega), \quad \Phi_\omega'(t,\omega) = r(t)\phi'(\omega)$$

are continuous from $[0, T] \times H^1$ to $\mathbb{R}$. Clearly, $u = K * \omega$ as a function of t and ω is continuous from $[0, T] \times H^1$ to $\mathbb{R}$. Thus,

$$h(\Phi) = \Phi_t'(t,\omega) + \nu((\omega, \Phi_\omega'(t,\omega))) + (u \cdot \nabla\omega, \Phi_\omega'(t,\omega)$$

is continuous from $[0, T] \times H^1$ to $\mathbb{R}$. Furthermore, the definition of Φ and the fact that $\|u\|_{L^\infty} \le U$ (see Proposition 2.2) lead to

$$|h(\Phi)| \le c_1 + c_2\|\omega\|_{H^1} + c_3\nu\|\omega\|_{H^1} + c_4 U\|\omega\|_{H^1}$$

where $c_1 - c_4$ are constants depending on ϕ (but independent of t, ω). (iii) and the above estimate imply that $g(\Phi)$ makes sense.

We have from multiplying the Navier-Stokes equations of vorticity by $r(t)\phi'(\omega)$ with $r(t) \in C_0^\infty([0, T))$ and $\phi \in \mathcal{J}$

$$\frac{d}{dt}(r(t)\phi(\omega)) - r'(t)\phi(\omega) + (u \cdot \nabla\omega, r(t)\phi'(\omega)) = \nu(\Delta\omega, r(t)\phi'(\omega))$$

after integrating with respect to t,

$$-\int_0^T \phi(\omega)r'(t)\, dt + \int_0^T r(t)(u \cdot \nabla\omega, \phi'(\omega))\, dt \tag{5.3}$$

$$= \int_0^T r(t)(\Delta\omega, \phi'(\omega))dt + r(0)\phi(\omega_0)$$

Due to the results of Theorem 2.1 and Theorem 2.2, more precisely, the fact that

$$\nabla\omega : \quad L^1(\mathbb{R}^{\not\models}) \longmapsto \mathbb{C}(\mathbb{R}_+, \mathbb{L}^{\not\Vdash} \cap \mathbb{L}^\infty(\mathbb{R}^{\not\models}))$$

vanishes at infinity, we integrate by parts to obtain

$$(\Delta\omega, \phi'(\omega)) = -((\omega, \phi'(\omega))) \tag{5.4}$$

We obtain (iv) by integrating (5.3) with respect to $d\mu(\omega_0)$ and using (5.4).

Our next goal is to show that the spatial statistical solution of the Navier-Stokes equations on the phase space of vorticity is uniquely determined by the initial probability measure μ. Here we assume that the initial measure μ has bounded support in $\mathbf{Y}$, i.e., for some constant a

$$supp\mu \subset \{\omega \in \mathbf{Y} : \|\omega\|_{\mathbf{Y}} \leq a\}$$

The definition of μ_t in (5.2) implies that for all $t \in [0, T]$

$$supp\mu_t \subset \{\omega \in L^2 : \|\omega\|_{L^2} \leq r\} \equiv B_1$$

for some constant $r > 0$.

We shall need the following lemma, whose proof is quite similiar to that given by Foias [6].

Lemma 5.4 *Let $\{\mu_t\}_{0\leq t\leq T}$ be a family of Borel probability measures on L^2satisfying (i),(ii),(iii) and let μ be a probability on $\mathbf{Y}$ such that (5.1) holds. Then the following two conditions are equivalent:*

(iv) $\{\mu_t\}_{0<t<T}$ *satisfies the equation:*

$$\int_0^T \int_{L^2} [-\Phi_t^{'}(t,\omega) + \nu((\omega, \Phi_\omega'(t,\omega)))$$

$$+(u\cdot\nabla\omega, \phi_\omega^{'}(t,\omega)]d\mu_t(\omega)dt = \int_{\mathbf{Y}} \Phi(0,\omega)d\mu(\omega)$$

for all $\Phi(t,\omega) = r(t)\phi(\omega)$ with $r(t) \in C_0^\infty([0, T))$ and $\phi \in \mathcal{J}$;

(iv′) $\{\mu_t\}_{0<t<T}$ *satisfies the equation:*

$$\int_{L^2}\Phi(t,\omega)\,d\mu_t(\omega)+\int_0^t\int_{L^2}[-\Phi_s'(s,\omega)+\nu((\omega,\Phi_\omega'(s,\omega)))$$

$$+(u\cdot\nabla\omega,\Phi_\omega'(s,\omega))]\,d\mu_s(\omega)\}\,ds=\int_{\mathbf{Y}}\Phi(0,\omega)\,d\mu(\omega)$$

for all $t\in(0,T)$ *and* $\Phi\in\mathcal{J}_\infty$*, where* $\mathcal{J}_\infty$ *is the class of real functions defined on* $[0,T]\times H^1$ *satisfying:*

(A_1) $\Phi(t,\omega)$ *is continuous in* $(t,\omega)\in[0,T]\times H^1$*,*

$$|\Phi_t'(t,\omega)|\le c_1+c_2\|\omega\|_{H^1}$$

for some constants c_1, c_2*.*

(B_1) $\Phi(t,\omega)$ *is Frechet* L^2 *differentiable in the direction of* H^1*,*

(C_1) $\Phi_\omega'(\cdot,\cdot)$ *is continuous from* $[0,T]\times H^1$ *to* H^1 *and is bounded.*

Clearly, $\mathcal{J}_\infty$ contains $\mathcal{J}$.

We can now state the uniqueness theorem and the idea of its proof is from [6].

Theorem 5.5 *Suppose that* μ *has bounded support in* $\mathbf{Y}$*. Then any space statistical solution of the Navier-Stokes equations on the phase space of vorticity with bounded support in* B_1 *is uniquely determined by* μ*.*

Proof. Let $\{\tilde{\mu}_t\}_{0\le t\le T}$ be another statistical solution satisfying the conditions in the theorem. First we show that for any $t\in[0,T]$ and $\phi\in\mathcal{J}$,

$$\int_{L^2}\phi(\omega)\,d\tilde{\mu}_t(\omega)=\int_{L^2}\phi(\omega)\,d\mu_t(\omega).$$

Let $\phi\in\mathcal{J}$ and $\Phi(\tau,\omega)=\phi(S(t-\tau)\omega)$, $\tau\in[0,t]$, where S is the solution operator of the Navier-Stokes equations defined in Theorem 2.1. It is easy to check that $\Phi\in\mathcal{J}_\infty$. By Lemma 5.4, the space statistical solution $\{\tilde{\mu}_t\}$ should satisfy

$$\int_{L^2}\Phi(t,\omega)\,d\tilde{\mu}_t(\omega)+\int_0^t\int_{L^2}[-\Phi_s'(s,\omega)+\nu((\omega,\Phi_\omega'(s,\omega)))$$

$$+(u\cdot\nabla\omega,\Phi'_\omega(s,\omega))]\,d\tilde{\mu}_s(\omega)\,ds=\int_{\mathbf{Y}}\Phi(0,\omega)\,d\mu(\omega)$$

since $\Phi(\tau,\omega)=\phi(S(t-\tau)\omega)$,

$$\int_{L^2}\phi(\omega)\,d\tilde{\mu}_t(\omega)-\int_{\mathbf{Y}}\phi(S(t)\omega)\,d\mu(\omega) \tag{5.5}$$

$$=\int_0^t\int_{L^2}[-\Phi'_s(s,\omega)+\nu((\omega,\Phi'_\omega(s,\omega)))+(u\cdot\nabla\omega,\Phi'_\omega(s,\omega)]\,d\tilde{\mu}_s(\omega)\,ds$$

but

$$\Phi'_s(s,\omega)=(\phi'(S(t-s)\omega),\frac{d}{ds}(S(t-s)\omega)) \tag{5.6}$$

The right hand side of (5.5) actually becomes zero after we replace Φ'_s by the formula (5.6) in it . Thus,

$$\int_{L^2}\phi(\omega)\,d\tilde{\mu}_t(\omega)=\int_{\mathbf{Y}}\phi(S(t)\omega)\,d\mu(\omega)$$

On the other hand, by the definition of $\{\mu_t\}_{0\le t\le T}$,

$$\int_{L^2}\phi(\omega)\,d\mu_t(\omega)=\int_{\mathbf{Y}}\phi(S(t)\omega)\,d\mu(\omega)$$

Thus for all $t\in[0,\,T]$ and $\phi\in\mathcal{J}$,

$$\int_{L^2}\phi(\omega)\,d\tilde{\mu}_t(\omega)=\int_{L^2}\phi(\omega)\,d\mu_t(\omega)$$

We obtain by using the result of Lemma 5.6 below

$$\int_{B_1}\Psi(\omega)\,d\tilde{\mu}_t(\omega)=\int_{B_1}\Psi(\omega)\,d\mu_t(\omega)$$

for all $t\in[0,\,T]$ and $\Psi\in\mathfrak{C}(B_1)$, where $\mathfrak{C}$ denotes the space of all continuous real functionals on B_1 with respect to the weak topology on L^2. Since on B_1 the Borel sets with respect to L^2 weak topology coincide with those with respect to the usual L^2 topology. Thus,

$$\tilde{\mu}_t(\varpi)=\mu_t(\varpi),\quad\text{for any Borel set}\quad\varpi\subset B_1,$$

Since the supports of both measures are also included in B_1, we have for all $t \in [0, T]$,

$$\tilde{\mu}_t = \mu_t,$$

which concludes the proof of this theorem.

We've used the following lemma in the proof.

Lemma 5.6 *The set* $\{\Phi(\cdot)|B_1, \Phi \in \mathcal{J}\}$ *is dense in* $\mathfrak{C}(\mathfrak{B}_1)$.

This lemma can be found in [6].

6 INVISCID LIMIT OF SPATIAL STATISTICAL SOLUTIONS

In this section we prove the existence of the inviscid limit of the spatial statistical solutions constructed in the previous sections and furthermore we show that this inviscid limit is the space statistical solutions of the Euler equations. The idea of the proofs for these results comes from Foias [6]. Our proof is also similar to that of Chae [3] for the inviscid limit of statistical solutions defined on the phase space of velocity .

We need to make further assumptions on the initial data:

$$\int |\nabla \omega_0| dx < \infty, \qquad \int \|\omega_0\|_{H^1} d\mu(\omega_0) < \infty \tag{6.1}$$

and we have with these assumptions

Proposition 6.1 *Let ω_0 be the initial vorticity satisfying (5.1) and $\omega^{(NS)}$ be the corresponding solution (vorticity) of the Navier-Stokes equations. Assume that $\{\mu_t^{(NS)}\}_{0\le t\le T}$ be the spatial statistical solution of the Navier-Stokes equations with initial μ obtained in the previous sections. Then*

$$\sup_{0\le t\le T} \int \|\omega^{(NS)}\|_{H^1}^2 d\mu_t^{(NS)} + \nu \int_0^T \int \|\omega^{(NS)}(\cdot, t)\|_{H^2}^2 d\mu_t^{(NS)} dt \le C \tag{6.2}$$

for some constant C independent of ν.

The proof of this inequality is similiar to that of Proposition 4.1.

The definition of the statistical solution of the Euler equations on the phase space of vorticity is obtained from that of the Navier-Stokes equations by formally taking $\nu = 0$ and restricting the test functions Φ to $\mathcal{J}_\in$, a subclass of $\mathcal{J}$. $\mathcal{J}_\in$ consists of functions of the type

$$\Phi(t,\omega) = r(t)\phi(\omega), \quad r(t) \in C_0^\infty([0,T))$$

and $\phi(\omega) = \psi((\omega, g_1), (\omega, g_2), \cdots, (\omega, g_k))$, where $\psi \in C^1(\mathbb{R}^\urcorner)$ has bounded first derivatives and for $1 \le i \le k, \quad g_i \in H^1$.

Our main results are included in the following theorem.

Theorem 6.2 *Let μ be a Borel probability measure on* $\mathbf{Y}$ *satisfying (5.1) and* $\{\mu_t^{(NS)}\}_{0\le t\le T}$ *be the corresponding spatial statistical solutions of the Navier-Stokes equations on the phase space of vorticity . If we further assume that* ω_0 *and* μ *satisfy (6.1), then there exists a subsequence* $\{\mu_t^{(NS)}\}_{0\le t\le T}$ *(we use the same notation for this subsequence) and a family of Borel probability measures* $\{\mu_t\}_{0\le t\le T}$ *on* L^2 *such that*

$$t \longmapsto \int \Phi(\omega)\, d\mu_t(\omega) \quad \textit{is measurable on } [0,T] \quad \textit{for all } \Phi \in \mathcal{C}_\in$$

$$\lim_{\nu\to 0}\int_0^T \int \Phi(t,\omega)\, d\mu_t^{(NS)} dt = \int_0^T \int \Phi(t,\omega)\, d\mu_t dt, \quad \textit{for all } \Phi \in L^1(0,T;\mathcal{C}_\in).$$

Furthermore, $\{\mu_t\}_{0\le t\le T}$ *is a spatial statistical solution of the Euler equation on the phase space of vorticity corresponding to* μ*.*

To prove this theorem, we need the lemma:

Lemma 6.3 *Let* $\{\mu_t^\nu\}$ *be a family of Borel probability measures such that*

(a) $t \longmapsto \int \Phi(\omega)\, d\mu_t^\nu(\omega)$ *is measurable on* $[0,T]$, $\forall \Phi \in BC$,

(b) *For some constant* C, $\sup_{0\le t\le T} \int \|\omega\|_{H^1}^2\, d\mu_t^\nu(\omega) \le C, \quad \forall \nu > 0$,

(c) $|\omega|^2$ *is uniformly integrable with respect to* μ_t^ν*, i.e. for* $\forall \epsilon > 0$*, there exists a* $r_\epsilon > 0$ *such that*

$$\int_{\{\omega : |\omega| > r_\epsilon\}} |\omega|^2\, d\mu_t^\nu(\omega) \le \epsilon.$$

Then there exists a family of Borel probability measures $\{\mu_t\}$ *such that*

$$t \longmapsto \int \Phi(\omega)\, d\mu_t(\omega) \quad \textit{is measurable on } [0,T]\ , \quad \forall \Phi \in BC$$

$$\lim_{\nu\to 0}\int_0^T \int \phi(t,\omega)\, d\mu_t^\nu(\omega)\, dt = \int_0^T \int \phi(t,\omega)\, d\mu_t dt, \quad \forall \phi \in L^1(0,T;\mathcal{C}_\in)$$

Proof of Theorem 6.2 We sketch the proof. It is easy to check that the statistical solution of the Navier-Stokes equations $\{\mu_t^{(NS)}\}$ satisfy all the conditions in the lemma. By applying this lemma, we obtain a family of probability measures $\{\mu_t\}$ as the inviscid limit of $\{\mu_t^{(NS)}\}$. We can show that $\{\mu_t\}$ is actually a spatial statistical solution of the Euler equations on the phase space of vorticity. In fact we only need to prove equality (iv) in Definition 5.1 with $\nu = 0$ and $\Phi \in \mathcal{J}_\in$. The idea of showing (iv) is to check that each term of the intergrand in (iv) is in $L^1(0, T; \mathcal{C}_\in)$ and then use the limit equality

$$\lim_{\nu \to 0} \int_0^T \int \Phi(t, \omega) d\mu_t^{(NS)} dt = \int_0^T \int \Phi(t, \omega) d\mu_t dt, \quad \forall \Phi \in L^1(0, T; \mathcal{C}_\in)$$

Further details are omitted.

Proof of Lemma 6.3. We consider the functional

$$F^\nu(\Phi) = \int_0^T \int_{L^2} \Phi(t, \omega) d\mu_t^\nu(\omega) dt, \qquad \Phi \in L^1(0, T; \mathcal{C}_\in)$$

This functional is well-defined because $\int_{L^2} \phi(\omega) d\mu_t(\omega)$ is measurable on $[0, T]$ for any $\phi \in \mathcal{C}_\in$. This can be seen from assumption (a) and the fact that any $\phi \in \mathcal{C}_\in$ can be written as the limit of $\phi_k = \min(\phi, k)$, which is in BC.

Furthermore, we show that $F^\nu \in (L^1(0, T; \mathcal{C}_\in))'$:

$$|F^\nu(\Phi)| \le \int_0^T \int \|\Phi(t, \cdot)\|_{\mathcal{C}_\in}(1 + \|\omega\|^2) d\mu_t^\nu(\omega) dt \le C\|\Phi\|_{L^1(0,T;\mathcal{C}_\in)}$$

where C does not depend on ν and we've used inequality (6.2) in the above. By the Banach-Alaoglu theorem there exists a $F \in (L^1(0, T; \mathcal{C}_\in))'$ such that for a subsequence

$$F^\nu \to F \quad \text{in the weak-}* \text{ sense in } (L^1(0, T; \mathcal{C}_\in))'.$$

Let λ be a strong lifting of $L^\infty([0, T])$([6]). By the integral representation theorem, there exists a family $\{F_t\}_{0 \le t \le T} \subset (\mathcal{C}_\in)'$ such that

$$F(\Phi) = \int_0^T < F_t, \Phi(t) > dt, \qquad \forall \Phi \in L^1(0, T; \mathcal{C}_\in)$$

$$\sup_{0<t<T} \|F_t\| = \|F\|$$

$$\lambda(F.(\Phi))(t) = F_t(\Phi), \quad \forall \Phi \in \mathcal{C}_{\in}, \quad \forall \sqcup \in [/, \mathcal{T}]$$

Next we want to show that for any $t \in [0, T]$ there is a Borel probability measure μ_t on L^2 such that

$$< F_t, \Phi >= \int_{L^2} \Phi(\omega)\, d\mu_t(\omega), \quad \forall \Phi \in BC$$

The idea of proof is to use Daniell's theorem (see Lemma 6.4 below). First we can prove that for any $\Phi \in BC$:

$$| < F_t, \Phi > | \leq \sup_{B_r} |\Phi| + \frac{c}{r^2} |\Phi|_{BC}, \quad \forall r > 0, \text{a.e. } t \in [0, T] \tag{6.3}$$

where B_r is the ball of radius r in L^2. For the lifted family F_t the above estimate holds for all $t \in [0, T]$. Let $\Phi_m \geq 0$ be a sequence in BC such that $\Phi_m \to 0$ pointwise as $m \to \infty$. By the Dini's theorem,

$$\sup_{B_r} |\Phi_m| \to 0, \quad m \to \infty$$

Thus, by letting $r \to \infty$ in (6.3),

$$\lim_{m\to\infty} < F_t, \Phi_m >= 0, \quad \forall t \in [0, T]$$

Daniell' theorem then implies that for all $t \in [0, T]$ there exists a Borel measure μ_t

$$< F_t, \Phi >= \int \Phi(\omega)\, d\mu_t(\omega), \quad \Phi \in BC, \quad t \in [0, T]$$

It is easy to see that μ_t is actually a probability measure. By taking $\Phi(t, \omega) = r(t) \in L^1([0, T])$ in

$$\lim_{\nu\to 0} \int_0^T \int \Phi(t, \omega)\, d\mu_t^\nu(\omega)\, dt = \int_0^T < F_t, \Phi(t) > dt,$$

we obtain $< F_t, 1 >= 1$, that is, $\int d\mu_t(\omega) = 1$.

Thus we have proved that there exists a Borel probability measure μ_t such that

$$\int_0^T \int \Phi(t,\omega)\,d\mu_t^\nu(\omega)\,dt \to \int_0^T \int \Phi(t,\omega)\,d\mu_t(\omega)\,dt \quad \text{as } \nu \to 0 \tag{6.4}$$

for all $\Phi \in L^1(0,T;BC)$.

We can prove that the above limit equality (6.4) actually holds for a broader class of test functions $\Phi \in L^1(0,T;\mathcal{C}_\in)$. The main idea of showing this extension is to approximate $\mathcal{C}_\in$ functions by BC functions and use the uniform integrability of $\|\omega\|_{L^2}$ with respect to μ_t^ν. We won't give more details.

We have used the Daniell's theorem, which states

Lemma 6.4 *Let F be a positive linear function on BC satisfying the Daniell's condition: if $\{\phi_m\}$ is a sequence in BC which deceases to zero pointwise, then $\lim_{m\to\infty} < F, \phi_m >= 0$. Then there exists a Borel measure P on L^2 such that*

$$< F, \phi >= \int \phi dP, \quad \forall \phi \in BC.$$

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