BILLIARDS IN A POTENTIAL: VARIATIONAL METHODS, PERIODIC ORBITS AND KAM TORI

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Abstract. We consider the classical motion of a particle in a plane domain, under the influence of a perpendicular magnetic field and a smooth potential, with elastic reflections on the walls of the domain. We discuss a variational method for finding periodic orbits, determining their stability and proving the existence of KAM tori. This method is applied to a circular scatterer in crossed electromagnetic fields, where we prove the existence of a set of bound states with positive measure, for sufficiently small electric field and low or moderate magnetic field.

1. Introduction

Classical billiards are popular models for various physical systems, in fields ranging from mechanics of systems with impacts and ergodic theory to semiclassical methods in quantum chaos. In particular, billiards in a magnetic field appear to be relevant for the study of transport properties in mesoscopic systems, diamagnetism and the quantum Hall effect (see for instance [T]).

Periodic orbits play an important role in the billiard dynamics. They often strongly influence the structure of phase space: Elliptic orbits are usually surrounded by KAM tori, which prevent the system from being ergodic. On the other hand, hyperbolic orbits are often accompanied by homoclinic tangles which make the dynamics non–integrable. Furthermore, periodic orbits are of fundamental importance in semi–classical methods.

In practice, it is often difficult to construct periodic orbits. In this work, we discuss a variational method which simplifies the computation of such orbits, at least for not too large periods. More specifically, we will consider billiards in a plane domain, with a perpendicular magnetic field and an in–plane potential. The case without a potential has been previously discussed in [BK].

In Section 2, we construct from the action a generating function which contains all the necessary information on the billiard dynamics. It allows us to define canonically conjugate variables, to compute the location of periodic orbits, and to determine their linear and non–linear stability.

In Section 3, we apply these methods to the billiard outside a circular scatterer, the potential being given by a uniform electric field. This model is of basic interest for the Lorenz gas in a magnetic field, and was studied in the low–magnetic–field limit in [BHHP]. It has been shown that a particle drifting in from infinity will leave the scatterer again with probability one. However, it may happen that in spite of the drift due to the electric field, the particle remains trapped in the vicinity of

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the scatterer for infinite positive and negative times. Here we use our variational method to find stable orbits of period 2, proving the existence of a set of such bound states with positive measure, for sufficiently small electric field and small or *moderate* magnetic field. We also obtain an estimation of the critical electric field beyond which there is no trapping.

2. General billiards

2.1. **Billards in a potential.**

We consider the classical motion of a particle in a connected domain Q of the plane. This domain is not necessarily bounded, nor simply connected. We assume its boundary ∂Q to consist of one or several simple closed curves, which will be, unless otherwise specified, piecewise $\mathcal{C}^2$. A convenient parametrization of ∂Q is given by its arclength:

$$\mathbf{x}(s) = (X(s), Y(s)), \quad ds^2 = dX^2 + dY^2, \tag{1}$$

so that the unit tangent vector has the form $\mathbf{t}(s) = (X'(s), Y'(s))$.

Inside the domain Q, the billiard flow is defined by the Lagrangian

$$\mathcal{L} := \frac{1}{2}m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) + V(\mathbf{x}), \tag{2}$$

where m and q denote mass and charge of the billiard particle, $V(\mathbf{x})$ is a smooth ($\mathcal{C}^1$) scalar potential, and $\mathbf{A}(\mathbf{x}) := \frac{1}{2}B(-y, x)$ the vector potential for a uniform magnetic field B in symmetric gauge.

The dynamics is defined in the following way: the particle evolves in Q according to the Lagrange equations until it hits the boundary, where there is a change of velocity direction specified by the law of specular reflection (i.e., the component tangent to ∂Q remains the same, while the normal component changes sign).

The flow is defined on the three–dimensionnal manifold of constant energy $E = \frac{1}{2}m\dot{\mathbf{x}}^2 + V(\mathbf{x})$. We will mostly consider orbits that hit the boundary repeatedly, which can be described by a "bouncing map": to each collision, we assign two variables (describing for example position and direction of velocity), and the map describes the evolution of these quantities from one collision to the next.

2.2. **Generating functions.**

Consider two points $\mathbf{x}_0 = \mathbf{x}(s_0)$ and $\mathbf{x}_1 = \mathbf{x}(s_1)$ on the boundary ∂Q. If there exists a trajectory γ, solution of the Lagrange equations, connecting $\mathbf{x}_0$ to $\mathbf{x}_1$ (Fig.1), we can define the (reduced) action along γ:

$$F(\mathbf{x}_0, \mathbf{x}_1) := \int_\gamma \mathbf{p} \cdot \mathrm{d}\mathbf{x} \tag{3}$$

$$\mathbf{p} := \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}} + \mathbf{A}(\mathbf{x}) = (m\dot{x} - \frac{1}{2}qBy, m\dot{y} + \frac{1}{2}qBx). \tag{4}$$

We know from analytical mechanics that for infinitesimal variations of the end points $\mathbf{x}_0$ and $\mathbf{x}_1$, the change in the action is given by

$$\mathrm{d}F = -\mathbf{p}_0 \cdot \mathrm{d}\mathbf{x}_0 + \mathbf{p}_1 \cdot \mathrm{d}\mathbf{x}_1, \tag{5}$$

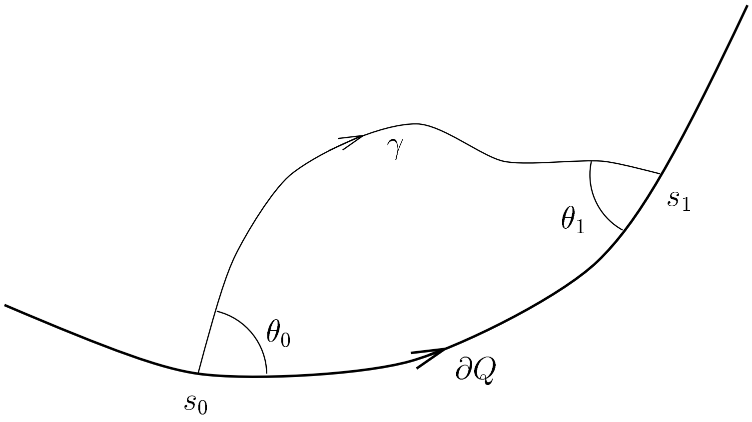

FIGURE 1. Trajectory γ between two collisions with the boundary ∂Q occuring at $s = s_0$ and $s = s_1$.

where $\mathbf{p}_i$ is the momentum (4) evaluated at the end point $\mathbf{x}_i$. In particular, for variations along the boundary,

$$\mathrm{d}F = -\mathbf{p}_0 \cdot \mathbf{t}(s_0)\mathrm{d}s_0 + \mathbf{p}_1 \cdot \mathbf{t}(s_1)\mathrm{d}s_1 = -p_0\mathrm{d}s_0 + p_1\mathrm{d}s_1 \tag{6}$$

where $p_i = \mathbf{p}_i \cdot \mathbf{t}(s_i)$ denotes the tangent momentum.

Thus, if we define the *generating function*

$$G(s_0, s_1) := F(\mathbf{x}(s_0), \mathbf{x}(s_1)), \tag{7}$$

it will have the property

$$\frac{\partial G}{\partial s_0} = -p_0, \qquad \frac{\partial G}{\partial s_1} = p_1. \tag{8}$$

Instead of p, it is often more convenient to use the tangent velocity $u = \cos\theta\,|\mathbf{v}|$ as a conjugate variable (Fig.1). This can be achieved by defining, instead of (7),[1]

$$G(s_0, s_1) := \frac{1}{m}F(\mathbf{x}(s_0), \mathbf{x}(s_1)) + \frac{qB}{2m}\int_{s_0}^{s_1} Y(s)X'(s) - X(s)Y'(s)\mathrm{d}s \tag{9}$$

so that (8) becomes

$$\frac{\partial G}{\partial s_0} = -u_0, \qquad \frac{\partial G}{\partial s_1} = u_1. \tag{10}$$

Up to now, we have assumed that there is exactly one trajectory γ connecting the points $\mathbf{x}_0$ and $\mathbf{x}_1$. This is not necessarily true. For some values of s_0 and s_1, there may be no such trajectory, either because there is no solution to the Lagrange equations, or because the solutions would leave the domain Q ("ghost" orbits). This would impose restrictions on the domain of definition of $G(s_0, s_1)$. On the other hand, there may also be *several* trajectories connecting the same end points. In this case, we would have several "branches" of generating functions $G_i(s_0, s_1)$, $i = 0, \dots, N(s_0, s_1)$. From the implicit function theorem, we expect each branch G_i to be a smooth function of its arguments (as smooth as the boundary), except at some special points, where several branches meet.

[1]It should be clear that the results will not be affected by any linear transformation of G. It is also possible to use another parametrization of ∂Q than its arclength, if the definition of $\mathbf{t}$ is changed accordingly.

For example, in the case where there is only a magnetic field[2] ($V(\mathbf{x}) = 0$), there is no solution when the distance beween $\mathbf{x}_0$ and $\mathbf{x}_1$ is larger than twice the Larmor radius, and there are two solutions when it is smaller (provided the two arcs are compatible with the geometry of Q). When both distances are equal, these two determinations meet in a square root singularity [BK].

2.3. **Periodic orbits.**

Once we have made a choice for the generating function, we have automatically a pair of canonically conjugate variables, that we will denote $x = (s, u)$. For each initial condition (s_0, u_0) such that the trajectory returns to the boundary of the billiard, we can determine the coordinates (s_1, u_1) of the next collision, which defines the *bouncing map*

$$T : (s_0, u_0) \mapsto (s_1, u_1). \tag{11}$$

The conjugacy of s and u means that this map is area preserving (see next section). In the special case where $\frac{\partial s_1}{\partial u_0}$ has always the same sign, T is a *twist map* [Me] and has a unique generating function.

An *orbit* of the map is a sequence $\{x_n | x_{n+1} = T(x_n)\}$, where n belongs to $\mathbb{Z}$ if the particle returns indefinitely to the boundary, or to a smaller subset if it never returns to the boundary after a finite number of bounces. A *periodic orbit of period* n of the map is an orbit such that $x_{i+n} = x_i \, \forall i$. It is obtained by searching a fixed point x^* of T^n: $T^n(x^*) = x^*$. To localize a periodic orbit, we meet in general the following technical problem: The exact expression of T is often difficult to compute, since it involves some implicit condition of intersection between orbit and boundary. The iterates T^n of T are even harder to compute, so that its fixed points are nearly impossible to find, when there is no special symmetry to help us.

An alternative is to use a variational method. To this end, we define the *n-point generating function*

$$G^{(n)}(s_0, s_1, \dots, s_{n-1}) \coloneqq G(s_0, s_1) + G(s_1, s_2) + \dots + G(s_{n-1}, s_0). \tag{12}$$

Let us consider an orbit of period n (assuming for the moment that the generating function is unique). If G is defined and differentiable for each orbit segment between consecutive collisions, then the law of specular reflection takes the form

$$\frac{\partial G^{(n)}}{\partial s_i} = 0, \quad i = 0, \dots, n-1. \tag{13}$$

In other words, the total action along the orbit is stationary.

Conversely, if $(s_0, \dots, s_{n-1})$ is a stationary point of $G^{(n)}$, then there exists a periodic orbit connecting these points, provided $G(s_i, s_{i+1})$ is defined for each i (that means in particular that we have to exclude "ghost" orbits, which would leave the billiard domain Q).

It is thus possible to find almost all[3] orbits of period n by computing the stationary points of a function. The advantage of this method is that once the expression of G is known, it is not difficult to compute $G^{(n)}$. Equation (13) is a system of n nonlinear algebraic equations for n variables, which is easier to solve than the equation $T^n(x^*) = x^*$, be it analytically or numerically (note that each line of the system contains only three different variables). Moreover, the *existence* of stationary

[2] In this case, G can be given a simple geometric interpretation [BK].

[3] All orbits but those containing arclengths where the generating function is singular.

points can be sometimes deduced from topological properties [Me]. Finally, the solution of (13) immediately gives the abscissas of the n collision points, and we shall see that the stability of the orbit can be directly related to these quantities, without having to compute the u_i.

In the more general case where there are several determinations of G, the existence of a periodic orbit implies the stationarity of a combination of the form

$$\sum_{i=0}^{n-1} G_{\sigma(i)}(s_i, s_{i+1}), \quad s_n = s_0, \quad \sigma(i) = 1, 2, \dots, N(s_i, s_{i+1}), \tag{14}$$

and, conversely, any admissible stationary point of one of the above functions corresponds to a periodic orbit (because of the different ranges of the $\partial_j G_i$, only some of these equations will actually admit solutions).

2.4. **Linear stability.**

The linearized bouncing map can be obtained directly from the generating function. To do this, we differentiate equation (10), giving

$$\begin{aligned} \mathrm{d}u_0 &= -G_{20}\mathrm{d}s_0 - G_{11}\mathrm{d}s_1 \\ \mathrm{d}u_1 &= G_{11}\mathrm{d}s_0 + G_{02}\mathrm{d}s_1 \end{aligned} \qquad G_{nm} := \frac{\partial^{n+m} G}{\partial s_0^n \, \partial s_1^m}(s_0, s_1). \tag{15}$$

Inverting this system with respect to $\mathrm{d}s_1$, $\mathrm{d}u_1$, we obtain the Jacobian matrix of the map (11), which is defined by $\mathrm{d}x_1 = T'\mathrm{d}x_0$:

$$T'(s_0, s_1) = -\frac{1}{G_{11}} \begin{pmatrix} G_{20} & 1 \\ G_{20}G_{02} - G_{11}^2 & G_{02} \end{pmatrix}. \tag{16}$$

Note that, as announced, this matrix has unit determinant, so that the bouncing map expressed in these coordinates is area preserving.

After n bounces, occuring at the arclengths $s_0, s_1, \dots, s_n$, the chain derivation rule gives $\mathrm{d}x_n = S_n \mathrm{d}x_0$, where the *stability matrix* S_n is given by

$$S_n(s_0, \dots, s_n) := T'(s_n, s_{n-1}) T'(s_{n-1}, s_{n-2}) \dots T'(s_1, s_0). \tag{17}$$

Of course, for multiply defined generating functions, this definition has to be changed accordingly.

The linear stability of a periodic orbit with period n depends on the eigenvalues λ_+, λ_- of S_n, because $\mathrm{d}x_{kn} = S_n^k \, \mathrm{d}x_0$. Since $\lambda_+\lambda_- = \det S_n = 1$, we have three cases, depending on the value of $t = \frac{1}{2} \operatorname{Tr} S_n$:

- If $|t| > 1$, the eigenvalues are reciprocal real numbers, $\lambda_\pm = \operatorname{sign}(t)\, \mathrm{e}^{\pm \operatorname{Argch} t}$. The periodic orbit is *hyperbolic* (inverse hyperbolic if $t < -1$), and in the vicinity the map acts like a contraction in one direction and a stretching in another one, thus the orbit is unstable.
- If $|t| < 1$, the eigenvalues are conjugate complex numbers on the unit circle, $\lambda_\pm = \mathrm{e}^{\pm \mathrm{i} \operatorname{Arccos} t}$. The orbit is *elliptic* and linearly stable, since the map acts like a rotation in its neighborhood.
- If $|t| = 1$, the eigenvalues are equal to ± 1. The orbit is *parabolic.*

The type of periodic orbit can sometimes be related to the nature of the stationary point of G [MM].

2.5. **Nonlinear stability and KAM theory.**

If one includes the effect of nonlinear terms in a neighborhood of a periodic orbit, hyperbolic orbits remain unstable, as a consequence of the center manifold theorem.

The case of elliptic orbits is more subtle. In fact, the KAM theorem implies that they are generically stable. To see this, we have to compute the Birkhoff normal form to lowest order. We begin with a few preliminary steps:

1. For two consecutive collisions at s_0, s_1, expansion of (10) gives (assuming the map is sufficiently differentiable)
$$\begin{aligned} \delta u_0 &= -\sum_{n+m\leqslant 3} \frac{1}{n!m!} G_{n+1\,m} \delta s_0^n \delta s_1^m + \mathcal{O}(4) \\ \delta u_1 &= \sum_{n+m\leqslant 3} \frac{1}{n!m!} G_{n\,m+1} \delta s_0^n \delta s_1^m + \mathcal{O}(4), \end{aligned} \tag{18}$$
where G_{nm} is given in (15), and $\mathcal{O}(4)$ denotes terms of fourth order in δs_0, δs_1.
2. Inverting the first series with respect to δs_1 and replacing this in the second equation, we can express $\delta x_1 = (\delta s_1, \delta u_1)$ as a function of $\delta x_0 = (\delta s_0, \delta u_0)$ to order 3.
3. Composing these expansions along the orbit, we get
$$\delta x_n = S_n \delta x_0 + b(\delta x_0) + \mathcal{O}(4), \tag{19}$$
where $b(\delta x)$ is a polynomial with terms of order 2 and 3.
4. A linear change of variables $z = \alpha\,\delta s + \beta\,\delta u$, $\alpha, \beta \in \mathbb{C}$ transforms (19) into
$$z_n = \mathrm{e}^{\mathrm{i}\,\varphi} z_0 + \sum_{2\leqslant n+m\leqslant 3} b_{nm} z_0^n \overline{z}_0^m + \mathcal{O}(4), \tag{20}$$
where $\varphi = \operatorname{Arccos} t$ is the rotation angle of the linear part.

The Birkhoff normal form is obtained from (20) by eliminating a maximum of terms of order 2 and 3. If the normal form is not degenerate, Moser's theorem [Mo] can be used to show existence of an invariant neighborhood of the periodic orbit, implying its stability in the sense of Liapunov. We summarize these results in the following way:

Theorem 1. *Let the map (20) be measure–preserving and $\mathcal{C}^5$ in a neighborhood of $z_0 = 0$. Assume that φ is such that $(\mathrm{e}^{\mathrm{i}\,\varphi})^3 \neq 1$ and $(\mathrm{e}^{\mathrm{i}\,\varphi})^4 \neq 1$ and define*
$$C := b_{20} b_{11} \frac{1 - 2\,\mathrm{e}^{\mathrm{i}\,\varphi}}{\mathrm{e}^{\mathrm{i}\,\varphi}(\mathrm{e}^{\mathrm{i}\,\varphi} - 1)} + \frac{|b_{11}|^2}{1 - \mathrm{e}^{\mathrm{i}\,\varphi}} + \frac{2\,|b_{02}|^2}{\mathrm{e}^{-\mathrm{i}\,\varphi}(\mathrm{e}^{3\,\mathrm{i}\,\varphi} - 1)} + b_{21}. \tag{21}$$
Then if the non–degeneracy condition
$$\operatorname{Im}(C\,\mathrm{e}^{-\mathrm{i}\,\varphi}) \neq 0 \tag{22}$$
is satisfied, there exists a neighborhood of 0 which is invariant under the map (20).

Proof: We first carry out three successive changes of variables:

1. If $(\mathrm{e}^{\mathrm{i}\,\varphi})^3 \neq 1$, we may introduce a new variable w defined by
$$z = w + \sum_{n+m=2} h_{nm} w^n \overline{w}^m, \qquad h_{nm} = \frac{b_{nm}}{\mathrm{e}^{\mathrm{i}\,\varphi}(\mathrm{e}^{(n-m-1)\,\mathrm{i}\,\varphi} - 1)},$$

transforming the map into

$$w_1 = \mathrm{e}^{\mathrm{i}\,\varphi}\, w_0 + \sum_{n+m=3} c_{nm} w_0^n \overline{w}_0^m + \mathcal{O}(4),$$

where an explicit calculation shows $c_{21} =: C$ to be given by (21).

2. If $(\mathrm{e}^{\mathrm{i}\,\varphi})^4 \neq 1$, a similar change of variables eliminates terms of order 3, except the term $c_{21} \left|w_0\right|^2 w_0$, which is resonant, so that we get

$$\omega_1 = \mathrm{e}^{\mathrm{i}\,\varphi}\, \omega_0 + C \left|\omega_0\right|^2 \omega_0 + \mathcal{O}(4).$$

3. Introducing polar coordinates $\omega = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\,\theta}$, $C = |C|\,\mathrm{e}^{\mathrm{i}\,\psi}$ finally yields the Birkhoff normal form

$$\begin{aligned} \theta_1 &= \theta_0 + \varphi + |C| \sin(\psi - \varphi)\rho_0 + \mathcal{O}(\rho_0^{3/2}) \\ \rho_1 &= \rho_0 + \mathcal{O}(\rho_0^2) \end{aligned}$$

The result follows from Moser's theorem, which can be applied in a strip $\epsilon < \rho < 2\epsilon \ll 1$, provided $|C| \sin(\psi - \varphi) \neq 0$, which is equivalent to (22): the existence of a KAM torus encircling the periodic point shows its interior to be invariant under the map. □

3. A scattering system

3.1. Definition of the system.

We now particularize to the case where the scalar potential is given by a uniform in–plane electric field, so that the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} &:= \frac{1}{2} m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) + q\boldsymbol{\mathcal{E}} \cdot \mathbf{x}, \\ \mathbf{A}(\mathbf{x}) &= \frac{1}{2} B(-y, x), \quad \boldsymbol{\mathcal{E}} = (0, \mathcal{E}). \end{aligned} \tag{23}$$

The billiard domain Q is defined as the exterior of a circle of radius r, centered at the origin, parametrized by $\mathbf{x}(s) = (r\cos s, r \sin s)$, $s \in [0, 2\pi)$.

The cyclotron frequency $\Omega = |qB|/m$ and the drift velocity $\overline{v} = |\mathcal{E}/B|$ allow us to define dimensionless variables, by introducing a new time $\psi = \Omega t$, a new length unit r and an energy unit $mr^2\Omega^2$. The Lagrangian thus becomes (with the sign conventions $qB < 0$, $q\mathcal{E} < 0$)

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{x}}^2 + \frac{1}{2}(y\dot{x} - x\dot{y}) - \varepsilon y, \tag{24}$$

where the dimensionless parameter $\varepsilon = \overline{v}/\Omega r$ measures the strength of the electric field.

The trajectories are cycloids of the form

$$\begin{aligned} x(\psi) &= a + \varepsilon\psi + \rho\cos(\psi - \bar{\psi}) \\ y(\psi) &= b + \rho \sin(\psi - \bar{\psi}) \end{aligned} \tag{25}$$

where ρ reduces to the Larmor radius when $\varepsilon = 0$. The coefficients a, b, ρ and $\bar{\psi}$, which can be expressed in terms of initial conditions, change after each collision with the scatterer. However, the energy $E = \frac{1}{2}(\varepsilon^2 + 2\varepsilon b + \rho^2)$ being conserved, we may introduce a new dimensionless parameter $\mu = \sqrt{2E - \varepsilon^2}$, describing the energy of the particle, such that the width of the cycloid becomes $\rho = \sqrt{\mu^2 - 2\varepsilon b}$.

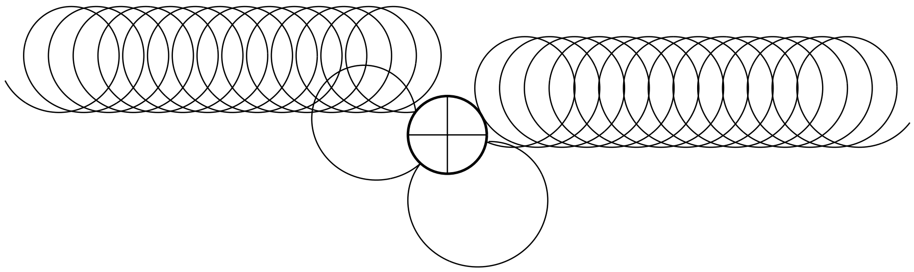

FIGURE 2. A trajectory scattered off the hard disc. Trajectories coming in from infinity leave the scatterer again with probability one. However, some orbits may form "bound states" which are indefinitely bouncing on the scatterer.

3.2. **Generating function.**

We will describe the trajectory γ between two consecutive collisions, with abscissas s_0 and s_1, by the equation

$$z(\psi) = x(\psi) + \mathrm{i}\, y(\psi) = a + \mathrm{i}\, b + \varepsilon\psi + \sqrt{\mu^2 - 2\varepsilon b}\, \mathrm{e}^{\mathrm{i}\,\psi}, \qquad \psi_0 \leqslant \psi \leqslant \psi_1. \tag{26}$$

To make use of the symmetries of the problem, we will introduce the variables $s_\pm := \frac{1}{2}(s_1 \pm s_0) \pmod{2\pi}$. The generating function will be defined by

$$G := 2\int_\gamma \mathbf{p}\cdot \mathrm{d}\mathbf{x} = 2\,\mathrm{Re}\int_{\psi_0}^{\psi_1} \overline{p}\,\dot z\,\mathrm{d}\psi, \qquad p = \dot z - \frac{1}{2}\,\mathrm{i}\,z, \tag{27}$$

for the convenience of notation. The relations between the parameters defining the trajectory and the arclenghs $s_\pm$ involve implicit equations which we will solve perturbatively. It turns out that the generating function has to be known at second order in ε. The result is

Proposition 1. *There are positive constants c_1, c_2 and ε_0, such that for $c_1\varepsilon < s_- < \pi - c_1\varepsilon$, $\mu > 1 + c_2\varepsilon$ and $0 \leqslant \varepsilon < \varepsilon_0$, the generating function of the bouncing map is an analytic function of $s_\pm$, μ and ε, and admits the expansion*

$$\begin{aligned} G(s_-,s_+) = {}& \mu^2\Delta\psi - 2(C+R)S - 2\varepsilon\,[2S + (C+R)\Delta\psi]\sin s_+ \\ &+ \varepsilon^2\left[\left(\frac{2(C+R)^2 S}{\mu^2 R} + 2\frac{C+R}{R}\Delta\psi + \frac{\mu^2}{2RS}\Delta\psi^2\right)\sin^2 s_+ - \frac{R}{2S}\Delta\psi^2\right] \\ &+ \mathcal{O}(\varepsilon^3), \end{aligned} \tag{28}$$

where C, S, R, $\Delta\psi$ denote functions of s_- alone:

$$\begin{aligned} &C(s_-) := \cos s_-, \quad S(s_-) := \sin s_-, \quad R(s_-) := \sqrt{\mu^2 - S(s_-)^2}, \\ &\Delta\psi(s_-) := 2\pi - \mathrm{Arccos}\left[1 - \frac{2}{\mu^2}S(s_-)^2\right]. \end{aligned} \tag{29}$$

PROOF: Integration of (27) gives

$$\begin{aligned} G = {}& (\mu^2 - \varepsilon b + 2\varepsilon^2)(\psi_1 - \psi_0) + \sqrt{\mu^2 - 2\varepsilon b}\,[(2\varepsilon + b)(\cos\psi_1 - \cos\psi_0) \\ & - a(\sin\psi_1 - \sin\psi_0) - \varepsilon(\psi_1\sin\psi_1 - \psi_0\sin\psi_0)] \end{aligned} \tag{30}$$

The four parameters a, b, ψ_0 and ψ_1 are related to $s_\pm$ by the equations $z(\psi_j) = \mathrm{e}^{\mathrm{i}\, s_j}$, $j = 0, 1$. For $\varepsilon = 0$, they have the solution

$$\begin{gathered} a^{(0)} + \mathrm{i}\, b^{(0)} = (C \pm R)\, \mathrm{e}^{\mathrm{i}\, s_+}, \\ \mathrm{e}^{\mathrm{i}\, \psi_0^{(0)}} = -\frac{1}{\mu}(\pm R + \mathrm{i}\, S)\, \mathrm{e}^{\mathrm{i}\, s_+}, \qquad \mathrm{e}^{\mathrm{i}\, \psi_1^{(0)}} = -\frac{1}{\mu}(\pm R - \mathrm{i}\, S)\, \mathrm{e}^{\mathrm{i}\, s_+}. \end{gathered} \tag{31}$$

The two solutions $\pm$ correspond to a long or a short skip. In the case $\mu > 1$, only the $+$ solution is admissible.
For positive ε, we have to solve

$$\begin{aligned} 2a + \varepsilon(\psi_1 + \psi_0) + \sqrt{\mu^2 - 2\varepsilon b}\,(\cos\psi_1 + \cos\psi_0) &= 2C\cos s_+ \\ \varepsilon(\psi_1 - \psi_0) + \sqrt{\mu^2 - 2\varepsilon b}\,(\cos\psi_1 - \cos\psi_0) &= -2S\sin s_+ \\ 2b + \sqrt{\mu^2 - 2\varepsilon b}\,(\sin\psi_1 + \sin\psi_0) &= 2C\sin s_+ \\ \sqrt{\mu^2 - 2\varepsilon b}\,(\sin\psi_1 - \sin\psi_0) &= 2S\cos s_+. \end{aligned} \tag{32}$$

From these relations, we can in a first step eliminate all nonlinear functions of the parameters in (30), giving

$$\begin{aligned} G = {} & (\mu^2 - \varepsilon b - \varepsilon C\sin s_+)(\psi_1 - \psi_0) - \varepsilon S\cos s_+(\psi_1 + \psi_0) \\ & - 2(2\varepsilon + b)S\sin s_+ - 2aS\cos s_+. \end{aligned} \tag{33}$$

It remains to express all variables as functions of $s_\pm$ by inverting the relations (32). We do this perturbatively, by using the implicit function theorem. (32) is of the form $\Phi(x,\varepsilon) = 0$. For $\varepsilon = 0$, the solution of $\Phi(x^{(0)}, 0) = 0$ is given by (31). The implicit function theorem assures that for small positive ε, there is an analytic solution to (32) provided $\det\left[\partial_x\Phi(x^{(0)},0)\right] = -8RS \neq 0$. This is true under the assumptions of the theorem (when s_1 is close to s_0, there can be trajectories encircling the scatterer several times). The solution can be computed by the recurrence $x^{(n+1)} = x^{(n)} - \left[\partial_x\Phi(x^{(0)},0)\right]^{-1}\Phi(x^{(n)},\varepsilon)$ to second order. The calculation is tedious but straightforward[4], and replacing the solution in (33), we obtain the conclusion of the proposition. □

Remarks

1. In this proposition, we discussed only the case $\mu > 1 + \mathcal{O}(\varepsilon)$. When $\mu < 1$, there are in general two determinations (see (31)) for the generating function, corresponding to a long or a short skip, if $\sin s_- < \mu - \mathcal{O}(\varepsilon)$. A singularity arises where the two determinations meet.
2. Since the system (32) is invariant under the symmetry transformation

   $$(s_+, a, b, \psi_0, \psi_1) \mapsto (\pi - s_+, -a - \varepsilon\pi, b, \pi - \psi_1, \pi - \psi_0),$$

   the generating function has the property

   $$G(s_-, \pi - s_+) = G(s_-, s_+) \tag{34}$$

   on its entire domain of definition.

[4]The calculations can be quite easily implemented with computer algebra, using the derivation rules $C' = -S$, $S' = C$, $R' = -SC/R$, $\Delta\psi' = -2C/R$.

3. When $\varepsilon = 0$, the generating function depends only on s_-: $G|_{\varepsilon=0} = g(s_-)$. This means that the zero–electric–field limit is integrable: From equations (10), we obtain that the bouncing map takes the form $s_1 = s_0 + \Omega(u_0)$, $u_1 = u_0$, where the frequency Ω is given by $\frac{1}{2}g'(\Omega/2) = u_0$.

   The introduction of an electric field will perturb this map with terms uniformly bounded by a constant times ε, provided we exclude initial conditions with a very small normal velocity, for which the hypothesis on s_- is not satisfied. Applying KAM theorems to this map would already allow us to conclude as for the existence of trapped orbits, but we will try to obtain better estimates on the critical electric field by studying orbits of period 2 with our variational method.

3.3. **Orbits of period 2: rigorous results.**

The 2-point generating function is given by

$$
\begin{aligned}
G^{(2)}(s_-,s_+) &= G(s_-,s_+) + G(\pi - s_-, \pi + s_+) \\
&= 2\mu^2\Delta\psi - 4RS - 4\varepsilon C\Delta\psi \sin s_+ \\
&\quad + 2\varepsilon^2\left[\frac{2(C^2+R^2)S}{\mu^2 R} + 2\Delta\psi + \frac{\mu^2}{2RS}\Delta\psi^2\right]\sin^2 s_+ - \varepsilon^2\frac{R}{S}\Delta\psi^2 \\
&\quad + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{35}
$$

By using the implicit function theorem (and the symmetries of the problem), one shows that $G^{(2)}$ has the following two pairs of stationary points[5], corresponding to two orbits of period 2:

1. Minima at $s_+ = 0, \pi$ and $s_- = \pi/2$. The associated orbit hits the scatterer at $s = \pi/2$ and $3\pi/2$ (Fig.3a).
2. Saddle points at $s_+ = \pi/2, 3\pi/2$ and $s_- = \pi/2 \mp \delta_-\varepsilon + \mathcal{O}(\varepsilon^3)$, where

$$
\delta_-(\mu) = \frac{\Delta\psi(\pi/2)}{2R(\pi/2)} = \frac{2\pi - \operatorname{Arccos}(1 - 2/\mu^2)}{2\sqrt{\mu^2-1}}. \tag{36}
$$

   The corresponding orbit hits the disc at $s = \delta_-\varepsilon + \mathcal{O}(\varepsilon^3)$ and $\pi - \delta_-\varepsilon + \mathcal{O}(\varepsilon^3)$ (Fig.3b).

From the results of [MM], we expect the first orbit to be hyperbolic, and the second elliptic or inverse hyperbolic. This can be checked by a direct calculation of $t = \frac{1}{2}\operatorname{Tr} S_2$, using (17).

1. Orbit $s \in \{\pi/2, 3\pi/2\}$: $t = 1 + \tau(\mu)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$, where

$$
\tau(\mu) = 2\left[\frac{4}{\mu^2} + \frac{4\Delta\psi(\pi/2)}{\sqrt{\mu^2-1}} + \Delta\psi(\pi/2)^2\right], \tag{37}
$$

$$
\Delta\psi(\pi/2) = 2\pi - \operatorname{Arccos}(1 - 2/\mu^2). \tag{38}
$$

   Since $\tau(\mu) > 0$, this orbit is hyperbolic for small positive ε.
2. Orbit $s \in \{\delta_-\varepsilon + \mathcal{O}(\varepsilon^3), \pi - \delta_-\varepsilon + \mathcal{O}(\varepsilon^3)\}$: $t = 1 - \tau(\mu)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$, for the same $\tau(\mu)$. Here, the orbit is elliptic for small positive ε (and may become inverse hyperbolic for larger ε).

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[5]We needed to compute G at second order in ε because the denivellation between the stationary points is of order ε.

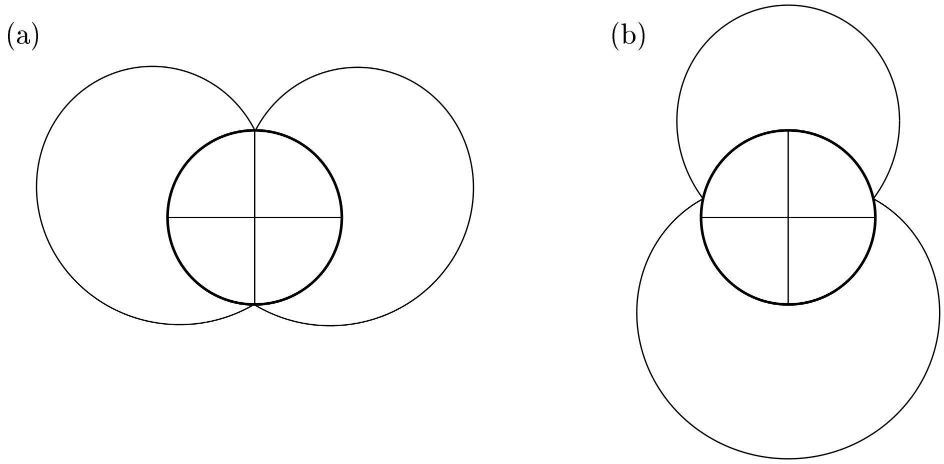

FIGURE 3. Orbits of period 2, for $\varepsilon = 0.1$ and $\mu = 1.5$: (a) hyperbolic orbit, (b) elliptic orbit.

We can now apply Theorem 1 in order to show the existence of an invariant neighborhood of the elliptic periodic orbit, which will correspond physically to bound states of the scattering system.

Theorem 2. *There are positive constants c_3 and ε_1 such that for almost all $\varepsilon \in [0, \varepsilon_1)$, $\mu > 1 + c_3\varepsilon$, the scattering system has a set of bound states with positive measure.*

PROOF: When $\varepsilon = 0$, the assertion follows from the integrability of the map. When $\varepsilon > 0$, we have to check the hypotheses of theorem 1, treating with some care the limit $\varepsilon \to 0$.
The first condition is that we avoid the resonance values $\mathrm{e}^{3\,\mathrm{i}\,\varphi} = 0$, $\mathrm{e}^{4\,\mathrm{i}\,\varphi} = 0$, where $\varphi = \operatorname{Arccos} t = \sqrt{2\tau}\varepsilon + \mathcal{O}(\varepsilon^3)$ (that is, we must have $t \neq \pm 1, 0, -\frac{1}{2}$). Since $t = 1 - \varepsilon^2\tau(\mu) + \mathcal{O}(\varepsilon^3)$ with $\tau'(\mu) \neq 0$, both derivatives of t with respect to μ and ε do not vanish in a neighborhood of $\varepsilon = 0$. Hence, the resonance values are only reached exceptionnaly, on a set of measure 0 (with respect to $\mathrm{d}\mu\,\mathrm{d}\varepsilon$).
The second thing to check is the non-degeneracy condition (22). First of all, we have to show that the limit $\varepsilon \to 0$ is well–defined, although the denominators in (21) vanish, and the stability matrix S_2 becomes non–diagonalizable. To see this, we first note that S_2 has the structure

$$S_2 = \begin{pmatrix} 1 + \varepsilon u - \varepsilon^2\tau & 2/R - \varepsilon^2 w \\ \varepsilon^2 v & 1 - \varepsilon u - \varepsilon^2\tau \end{pmatrix} + \mathcal{O}(\varepsilon^3),$$

where u, v, w are known functions of μ. The linear part of (19) can be diagonalized by the change of variables $z = \varepsilon^\nu \left[(u + \mathrm{i}\sqrt{2\tau})\delta s - \varepsilon v\,\delta u + \mathcal{O}(\varepsilon^3)\right]$, where ν can always be chosen in such a way that C is finite when $\varepsilon \to 0$.
To avoid checking the non–degeneracy condition for arbitrary μ, we can study the low–magnetic–field limit. With $\eta = 1/\mu$ and $\epsilon = 2\pi\varepsilon$, the rescaled

generating function

$$\begin{aligned}\widetilde{G} &:= \frac{1}{2}\left[\eta G - \frac{2\pi}{\eta}\right] \\ &= -2\sin s_- - \epsilon\sin s_+ - \frac{1}{4}\epsilon^2\frac{\cos^2 s_+}{\sin s_-} + \mathcal{O}(\epsilon^3) + \mathcal{O}(\eta)\end{aligned} \tag{39}$$

is analytic in a neighborhood of the real–η axis for $0 < \eta < 1 - \mathcal{O}(\epsilon)$. In this limit, condition (22) is relatively easily checked since by symmetry, the quadratic terms of $b(\delta x)$ are equal to zero. $C(\eta,\epsilon)$ being an analytic function on the considered set, with $\lim_{\eta,\epsilon\to 0} C(\eta,\epsilon) \neq 0$, it cannot vanish on an set of positive measure. □

3.4. **Further observations.**

Beyond what is proved rigorously in the present work, we can make several remarks. First of all, KAM theory breaks down for the resonance values of ε and μ, where $e^{i\varphi}$ is a cubic or quartic root of unity. This does not *necessarily* imply that the orbit becomes unstable (the behaviour depends on nonlinear terms). Numerically, it seems that the elliptic orbit of period 2 is always stable for $\mu > 1$ and small ε.

The dependence of the dynamics on ε becomes clear on figure 4. For small ε, the structure of phase space is quite typical for a slightly perturbed integrable map. The measure of the elliptic islands vanishes in the limit $\varepsilon \to 0$, because all periodic orbits become parabolic. Thus, our method is not optimal in this limit, better results can be obtained by applying Moser's theorem to the explicit expression of the bouncing map, establishing the existence of invariant curves winding around phase space. On the other hand, for larger ε, only a few elliptic islands remain. Hence, the method we use to prove existence of bound states allows us to control a larger domain of electric fields.

We did not try to determine precisely the critical value of ε beyond which perturbation theory no longer works. It is clear that for sufficiently strong electric fields, orbits of period 2 cease to exist. But assuming that the critical value is not too small, one may estimate from the equation $t = 1 - \varepsilon^2\tau(\mu) + \mathcal{O}(\varepsilon^3)$ the value of ε where $t = -1$ and the period-2 orbit loses stability. If we neglect terms of order ε^3, we get

$$\varepsilon \simeq \hat{\varepsilon}(\mu) = \sqrt{\frac{2}{\tau(\mu)}} = \left[\frac{4}{\mu^2} + \frac{4\Delta\psi(\pi/2)}{\sqrt{\mu^2-1}} + \Delta\psi(\pi/2)^2\right]^{-1/2}. \tag{40}$$

This estimation is in good agreement with numerics, which show the period–2 orbit undergoing a period doubling bifurcation near $\hat{\varepsilon}(\mu)$.

The function $\hat{\varepsilon}(\mu)$ is increasing and converges towards $\frac{1}{2\pi}$ as $\mu \to \infty$. In this limit, the estimation (40) can be proved to be exact. Indeed, if ϕ denotes the direction, measured from the potitive–x axis, of the velocity before a collision with the scatterer, and β denotes the impact parameter, one obtains from simple geometry [BHHP] the map

$$\begin{aligned}\phi_1 &= \phi_0 + \Omega(\beta_0) \\ \beta_1 &= \beta_0 - \epsilon\sin\phi_1,\end{aligned} \qquad \Omega(\beta) = \pi - 2\,\mathrm{Arcsin}(\beta), \tag{41}$$

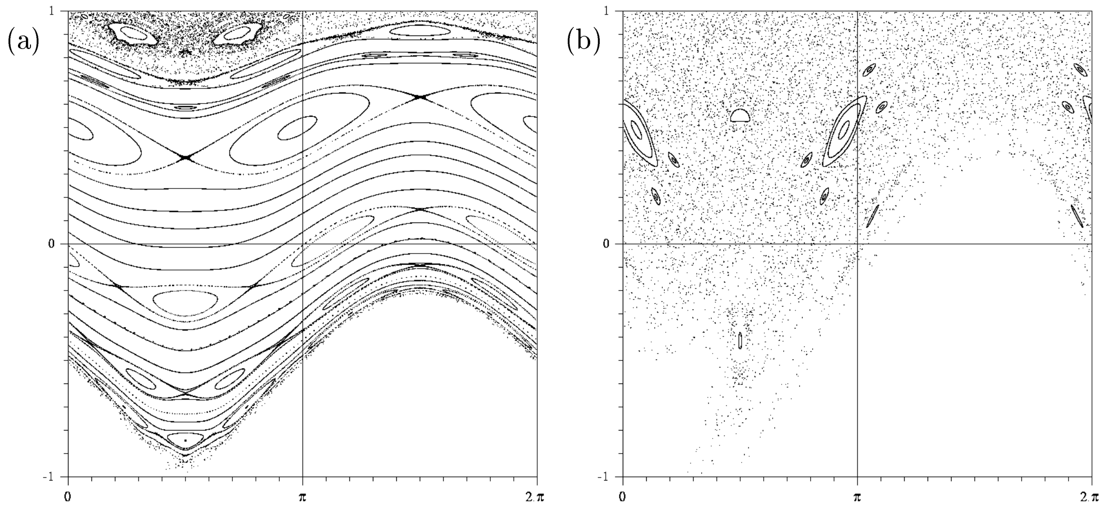

FIGURE 4. Phase portraits of the bouncing map ($\mu = 2$), with the arclength on the horizontal axis and the tangent velocity on the vertical axis. (a) $\varepsilon = 0.05$: Chains of elliptic islands and hyperbolic points are separated by KAM tori winding around phase space. Near the upper boundary, there are more chaotic orbits, which correspond to forward skipping trajectories sometimes encircling the scatterer several times. The empty region near the lower boundary corresponds to backward glancing orbits which escape. (b) $\varepsilon = 0.12$: The invariant curves winding around phase space have disappeared. There remain a few elliptic islands, in particular around the easily recognized orbit of period 2. Between these islands is a stochastic sea of chaotic orbits which remain trapped for a considerable time, but ultimately miss the scatterer and escape.

where $\epsilon = 2\pi\varepsilon$. This map is compatible with the generating function (39) obtained in the limit $\mu \to \infty$ (with $s = \phi + \pi - \text{Arcsin}(\beta)$ and $u = \beta$). One easily checks that the period–2 orbit $\phi = 0, \pi$, $\beta = 0$ is elliptic for $0 < \epsilon < 1$, i.e. for $0 < \varepsilon < \frac{1}{2\pi} = \hat{\varepsilon}(\infty)$.

On the other hand, in the limit $\mu \to 1_+$, $\hat{\varepsilon}(\mu)$ goes to zero, reflecting the fact that orbits of period 2 cease to exist for smaller μ. When the magnetic field increases beyond that value one could study the stability of orbits with higher period in order to prove the existence of bound states. It seems however that the measure of trapped orbits goes to zero when $\mu \to 0$ ($B \to \infty$).

Finally, one could wonder if bound states also exist for more general, not necessarilly rotationally symmetric scatterers. The behaviour will depend on the dynamics in zero electric field. If elliptic islands are present in this limit (this is the case, for instance, for an elliptical or a rectangular scatterer in not too strong magnetic field), we expect that these islands will survive perturbation by a small electric field. If the billiard is ergodic when $\varepsilon = 0$, the question on what effect the addition of an electric field will have remains open.

4. Conclusion

The introduction of a generating function considerably simplifies several problems arising in billiards with a potential and a magnetic field. In particular, periodic orbits can be found by a variational method, and their stability can be determined. This requires of course the generating function to be known explicitly, and still the calculations become rapidly more complex for increasing period. However, to solve some questions as existence of elliptic islands, it is often sufficient to construct one orbit of small period.

This method has been illustrated on a scattering billiard, where the existence, for small electric and not too high magnetic fields, of an elliptic orbit of period 2 implies bound states of positive measure. Since we computed the generating function perturbatively, for small electric field, we were not able to delineate exactly the domain of existence of elliptic islands, but it was possible to obtain estimations which agree with numerics.

An interesting problem which remains open is the converse question: is there a critical electric field beyond which bound states have zero measure? From intuitive and numerical observations, it seems that the answer is positive, but we have for the moment no method for proving this rigorously.

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