THE CAUCHY PROBLEM FOR CLASSICAL FIELD EQUATIONS WITH GHOST AND FERMION FIELDS

T. SCHMITT

Abstract. Using a supergeometric interpretation of field functionals, we show that for quite a large class of systems of nonlinear field equations with anticommuting fields, infinite-dimensional supermanifolds (smf) of classical solutions can be constructed. Such systems arise in classical field models used for realistic quantum field theoretic models. In particular, we show that under suitable conditions, the smf of smooth Cauchy data with compact support is isomorphic with an smf of corresponding classical solutions of the model.

Contents

Special thanks to the late German Democratic Republic who made this research possible by continuous financial support over twelve years.

1. Systems of field equations

1.1. **Introduction.** The investigation of the field equations belonging to a quantumfield theoretical model as classical nonlinear wave equations has a long history, dating back to Segal [11]; cf. also [6], [5], [2], [16]. Usually, Dirac fields have been considered in the obvious way as sections of a spinor bundle, as e. g. in [2].

On the other hand, the rise of supersymmetry made the question of an adequate treatment of the fermion fields urgent — supersymmetry and supergravity do not work with commuting fermion fields. The same applies to ghost fields: BRST symmetry, which now arouses a considerable interest among mathematicians (cf. [9]), simply does not exist with commuting ghost fields.

The anticommutivity required from fermion and ghost fields is often implemented by letting these fields have their values in the odd part of an auxiliary Grassmann algebra, as e. g. in [3], [8]. However, in [14], we have raised our objections against the use of such an algebra, at least as a fundamental tool. (In 3.11, we will show how to derive Grassmann-valued solutions from our approach, which in some sense provides a "universal", intrinsic solution.)

As we have argued in [14], a satisfactory description of fermion and ghost fields is possible in the framework of infinite-dimensional supergeometry: the totality of configurations on space-time should not be considered as a set but as an infinite-dimensional supermanifold (smf), and the totality of classical solutions should be a sub-supermanifold. While in [13], [14], we have developed the necessary supergeometric machinery, this paper will combine it with old and new techniques in non-linear wave equations in order to implement this point of view.

In this paper, we consider only two characteristic examples; a systematic application of our results to a large class of classical field theories will be given in the successor paper.

Even if the dream of the old, heroic days to construct a quantum field theory rigorously by direct geometric quantization of the symplectic manifold of classical solutions (cf. [11]) has turned out to be too naive, since it ignores the apparently intrinsic necessity of renormalization, we nevertheless hope that our construction sheds somewhat more light onto the geometry of classical field theories. Perhaps, the dream mentioned will come true some day in a refined variant (cf. also Remark 1.5.1).

1.2. The Φ^4 toy model.

1.2.1. *Classical solutions of Sobolev class.* We start with the usual toy model of every physicist working on quantum field theory, namely the purely bosonic Φ^4 theory on Minkowski $\mathbb{R}^{1+3}$, with the equation of motion

$$\big({\partial_t}^2 - \sum_{a=1}^{3} {\partial_a}^2\big)\phi + m^2\phi + 4q\phi^3 = 0, \tag{1.2.1}$$

where $m, q \geq 0$. In the usual first-order form, the field equations are $L_1[\phi_1, \phi_2] = L_2[\phi_1, \phi_2] = 0$ where

$$L_1[\Phi_1, \Phi_2] := \partial_t\Phi_1 - \Phi_2, \quad L_2[\Phi_1, \Phi_2] := \partial_t\Phi_2 - \sum_{a=1}^{3} {\partial_a}^2\Phi_1 + m^2\Phi_1 + 4q\Phi_1^3. \tag{1.2.2}$$

It is well-known (cf. e. g. [10, X.13]) that for given Cauchy data $(\phi_1^{\mathrm{Cau}}, \phi_2^{\mathrm{Cau}}) \in H_{k+1}(\mathbb{R}^3) \oplus H_k(\mathbb{R}^3)$ (here H_k is the standard Sobolev space with order $k > 1$, in order to ensure the algebra property of H_{k+1} under pointwise multiplication), there exists a unique solution $\phi = (\phi_1, \phi_2) \in C(\mathbb{R}, H_{k+1}(\mathbb{R}^3)) \otimes \mathbb{R}^2 \subseteq C(\mathbb{R}^4) \otimes \mathbb{R}^2$ of the Cauchy problem

$$L_1[\phi] = L_2[\phi] = 0, \quad \phi_1(0) = \phi_1^{\mathrm{Cau}}, \quad \phi_2(0) = \phi_2^{\mathrm{Cau}},$$

and that the arising nonlinear map

$$\check{\Phi}^{\mathrm{sol}} : M_k^{\mathrm{Cau}} := H_{k+1}(\mathbb{R}^3) \oplus H_k(\mathbb{R}^3) \to C(\mathbb{R}, H_{k+1}(\mathbb{R}^3)) \otimes \mathbb{R}^2 =: M_k^{\mathrm{cfg}}, \quad (\phi_1^{\mathrm{Cau}}, \phi_2^{\mathrm{Cau}}) \mapsto (\phi_1, \phi_2)$$

is continuous.

As a special case of the general results of this paper, it will turn out that this map is in fact real-analytic, and that its image is a submanifold of the Frèchet manifold $C(\mathbb{R}, H_k(\mathbb{R}^3)) \otimes \mathbb{R}^2$. Its Taylor expansion at zero arises as the solution of the "formal Cauchy problem" to find a formal power series $\Phi^{\rm sol}[\Phi_1^{\rm Cau}, \Phi_2^{\rm Cau}] \in \mathcal{P}_f(H_{k+1}(\mathbb{R}^3) \oplus H_k(\mathbb{R}^3);\ C(\mathbb{R}, H_{k+1}(\mathbb{R}^3)) \otimes \mathbb{R}^2)$ (cf. [13]) with

$$L_1[\Phi^{\rm sol}] = L_2[\Phi^{\rm sol}] = 0, \quad \Phi_1^{\rm sol}(0) = \Phi_1^{\rm Cau}, \quad \Phi_2^{\rm sol}(0) = \Phi_2^{\rm Cau}$$

where $\Phi_1^{\rm Cau}, \Phi_2^{\rm Cau}$ are "functional variables". This problem is readily solved by recursion over the degree; one finds $\Phi^{\rm sol} = (\Phi_1^{\rm sol}, \partial_t \Phi_1^{\rm sol})$, with $\Phi_1^{\rm sol} = \sum_{m\geq 0} \Phi^{\rm sol}_{1,(2m+1)}$,

(1.2.3)

$$\begin{aligned}
\Phi^{\rm sol}_{1,(1)}(t,y) &= \Phi_1^{\rm free}(t,y) := \textstyle\int_{\mathbb{R}^3} dx \left(\partial_t \mathcal{A}(t,y-x)\Phi_1^{\rm Cau}(x) + \mathcal{A}(t,y-x)\Phi_2^{\rm Cau}(x)\right), \\
\Phi^{\rm sol}_{1,(3)}(t,y) &= 4q \textstyle\int_{\mathbb{R}\times\mathbb{R}^3} dsdx \Phi_1^{\rm free}(s,x)^3 G(t,s,y-x), \\
\Phi^{\rm sol}_{1,(5)}(t,y) &= 48q^2 \textstyle\int dsdx \Phi_1^{\rm free}(s,x)^2 G(t,s,y-x) \int ds'dx' \Phi_1^{\rm free}(s',x')^3 G(s,s',x-x'), \\
\Phi^{\rm sol}_{1,(7)}(t,y) &= \textstyle\int dsdx G(t,s,y-x) \Big(192 q^3 \Phi_1^{\rm free}(s,x) \left(\int ds" dx" \Phi_1^{\rm free}(s",x")^3 G(s,s",x-x")\right)^2 \\
&\quad + 576 q^3 \Phi_1^{\rm free}(s,x)^2 \textstyle\int ds'dx' \Phi_1^{\rm free}(x')^2 G(s,s',x-x') \int ds"dx" \Phi_1^{\rm free}(s",x")^3 G(s',s",x'-x")\Big)
\end{aligned}$$

etc. Here $\mathcal{A}(t,x)$ is the Pauli-Jordan exchange function given by its spatial Fourier transform as

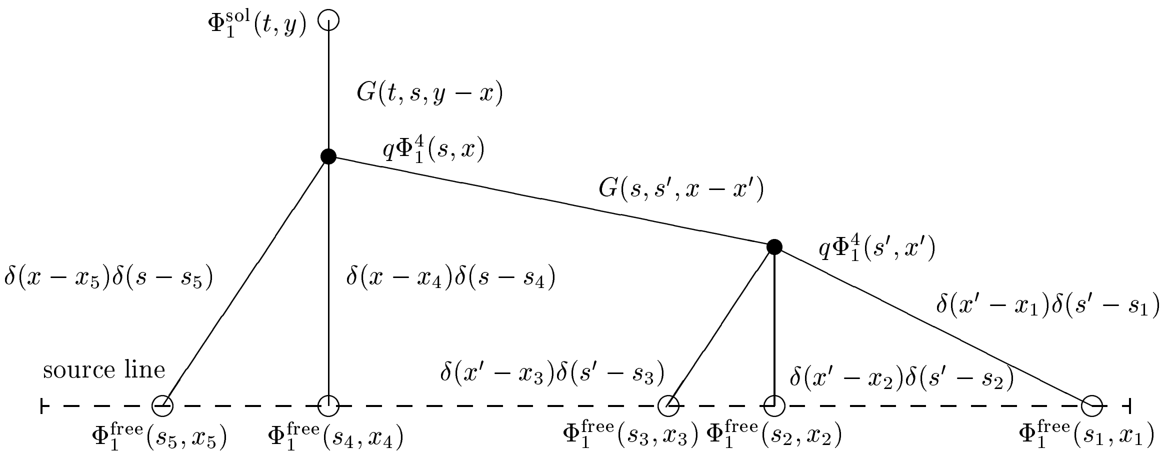
$$\hat{\mathcal{A}}(t,p) = \frac{\sin(\sqrt{m^2+p^2}t)}{(2\pi)^{3/2}\sqrt{m^2+p^2}}, \tag{1.2.4}$$

and the Green function $G(t,s,x)$ is chosen such that its Cauchy data vanish; explicitly,

$$G(t,s,x) = (\theta(t-s) - \theta(-s))\mathcal{A}(s,x)$$

where $\theta(\cdot)$ is the Heavyside step function.

The terms of $\Phi_1^{\rm sol}$ correspond to certain Feynman-like tree diagrams; for instance, $\Phi^{\rm sol}_{1,(5)}$ corresponds to the diagram



The general results of this paper (cf. Thm. 2.2.1, Thm. 2.6.1, Thm. 3.6.1) now imply:

Corollary 1.2.1. *(i) For all $c > 0$ there exists $\theta_c > 0$ such that $\Phi^{\rm sol}$, viewed as power series with values in the Banach space $C([-\theta_c, \theta_c], H_{k+1}(\mathbb{R}^3)) \otimes \mathbb{R}^2$, converges on the c-fold unit ball of $M_k^{\rm Cau}$.*

(ii) Fixing Cauchy data $\phi^{\text{Cau}} = (\phi_1^{\text{Cau}}, \phi_2^{\text{Cau}}) \in M_k^{\text{Cau}}$ and a lifetime $\theta > 0$, there exists a neighbourhood U of zero in M_k^{Cau} such that the translation $\Phi^{\text{sol}}[\Phi^{\text{Cau}} + \phi^{\text{Cau}}]$ (which is only defined for a sufficiently short target time) "prolongates" to a uniquely determined power series $\Phi^{\text{sol}}_{\phi^{\text{Cau}}}[\Phi^{\text{Cau}}]$ which converges on U and solves the field equations.

(iii) The image of the map $\check{\Phi}^{\text{sol}} : \phi^{\text{Cau}} \mapsto \Phi^{\text{sol}}[\phi^{\text{Cau}}]$ is a submanifold $M_k^{\text{sol}} \subseteq M_k^{\text{cfg}}$. Moreover, the map

$$\alpha : M_k^{\text{cfg}} \to M_k^{\text{cfg}}, \quad \phi \mapsto (\alpha_1(\phi), \partial_t \alpha_1(\phi)), \quad \alpha_1(\phi) := \phi_1 + \Phi_1^{\text{sol}}[\phi(0)] - \Phi_1^{\text{free}}[\phi(0)]$$

is an automorphism of M_k which satisfies $\alpha \circ \Phi^{\text{free}} = \Phi^{\text{sol}}$. □

Of course, $\Phi^{\text{sol}}_{\phi^{\text{Cau}}}[\Phi^{\text{Cau}}]$ is just the Taylor expansion of the map Φ^{sol} at ϕ^{Cau}. Note that U may shrink with growing θ; this is connected with the fact that the target M_k of the map $\check{\Phi}^{\text{sol}}$ is only a Frèchet space. This indicates that in Minkowski models, there is no way to work entirely in the framework of Banach spaces.

1.2.2. *Critique and improvement.* Now, viewing M_k^{sol} as "the" manifold of classical solutions of our model has the severe defect that we do not know whether it is Lorentz invariant in a reasonable sense; probably, it is not.

An obvious way out is to use smooth Cauchy data and configurations. Thm. 3.8.1 now yields:

Corollary 1.2.2. *The restriction of $\check{\Phi}^{\text{sol}}$ to shooth configuration extends to a real-analytic map*

$$\check{\Phi}^{\text{sol}} : M_{C^\infty}^{\text{Cau}} := C^\infty(\mathbb{R}^3) \otimes \mathbb{R}^2 \to C^\infty(\mathbb{R}^4) \otimes \mathbb{R}^2 =: M_{C^\infty}^{\text{cfg}}.$$

Its image $M_{C^\infty}^{\text{sol}}$, which is precisely the set of all smooth solutions of the field equations, is a submanifold of the Frèchet manifold $M_{C^\infty}^{\text{cfg}}$. □

Since the reduction to a first-order system is not Lorentz-invariant, $M_{C^\infty}^{\text{cfg}}$ is not the adequate configuration space for the purposes of quantum field theory. One should use instead of it the *covariant configuration space*

$$M_{C^\infty} := C^\infty(\mathbb{R}^4)$$

and compose $\check{\Phi}^{\text{sol}}$ with the projection $M_{C^\infty}^{\text{cfg}} \to M_{C^\infty}$ on the first component; we get:

Corollary 1.2.3. *$\check{\Phi}_1^{\text{sol}}$ restricts to a real-analytic map*

$$\check{\Phi}_1^{\text{sol}} : M_{C^\infty}^{\text{Cau}} \to M_{C^\infty}.$$

Its image, which is precisely the set of all smooth solutions of the original second order field equation (1.2.1)*, is a Lorentz-invariant submanifold of the Frèchet manifold M_{C^∞}.* □

However, while the absence of any growth condition in spatial direction does not cause trouble in the construction, due to finite propagation speed, it causes difficulties in the subsequent investigation of differential-geometric structures on the image $M_{C^\infty}^{\text{sol}}$: Every continuous seminorm $p \in \text{CS}(C^\infty(\mathbb{R}^{d+1}))$ is compactly supported, i. e. there exists some $\Omega \Subset \mathbb{R}^{d+1}$ such that $p(f) = 0$ once $f|_\Omega = 0$. This simplifies some proofs (cf. 3.4), but turns into a vice when looking onto the superfunctions on M: For each superfunction $K \in \mathcal{O}(M_{C^\infty})$, there exists some compact $\Omega \Subset \mathbb{R}^{d+1}$ such that for the coefficient functions $K_{k|l}$ of the Taylor expansion at the origin we have $\operatorname{supp} K_{k|l} \subseteq \prod^{k+l} \Omega$; analogously for superfunctions on $M_{C^\infty}^{\text{Cau}}$. Roughly spoken, $K[\Phi|\Psi]$ is influenced only by the "values" of the fields on the finite region Ω. In particular, the energy at a given time instant is not a well-defined superfunction; only the energy in a finite space-time region is so.

What is still worse, the symplectic structure on $M_{C^\infty}^{\text{Cau}} \cong M_{C^\infty}^{\text{sol}}$ which one expects (cf. [14, 1.12.4] and the successor of this paper) simply does not make sense; only the induced Poisson structure does.

Thus, it seems reasonable to use only smooth Cauchy data with compact support, i. e. of test function quality: $M^{\text{Cau}} := C_0^\infty(\mathbb{R}^3) \otimes \mathbb{R}^2$. The target space M^{cfg} of configurations has to be chosen such that the image of $\check{\Phi}^{\text{sol}}$ still is the set of all classical solutions in M. Simply taking all smooth functions on $\mathbb{R}^4$ which are spatially compactly supported would violate Lorentz invariance. However, if we additionally suppose that the spatial support grows only with light speed then everything is OK: $M^{\mathrm{cfg}} = C_{\mathrm{c}}^{\infty}(\mathbb{R}^4) \otimes \mathbb{R}^2$ where $C_{\mathrm{c}}^{\infty}(\mathbb{R}^{d+1})$ is the space of all $f \in C_c^{\infty}(\mathbb{R}^{d+1})$ such that there exists $R > 0$ with $f(t,x) = 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ with $|x| \geq |t| + R$. Also, the corresponding covariant configuration space is now $M = C_{\mathrm{c}}^{\infty}(\mathbb{R}^4)$. (Note that this is only a strict inductive limes of Frèchet spaces.)

Thm. 3.6.1 now yields:

Corollary 1.2.4. $\check{\Phi}_1^{\mathrm{sol}}$ *restricts to a real-analytic map* $\check{\Phi}_1^{\mathrm{sol}} : M^{\mathrm{Cau}} \to M$. *Its image* M^{sol}, *which is precisely the set of all those smooth solutions of the original second order field equation, which have at any time spatially compact support, is a Lorentz-invariant submanifold of the manifold* M. $\square$

(Of course, M^{sol} might miss to contain some interesting classical solutions; but, at any rate, it comes locally arbitrarily close to them.)

1.3. **The program of this paper.** We start with fixing in 1.5 a class of systems of classical nonlinear wave equations in Minkowski space $\mathbb{R}^{d+1}$ which is wide enough to describe the field equations of many usual models, like e. g. Φ^4, quantum electrodynamics, Yang-Mills theory with usual gauge-breaking term, Faddeev-Popov ghosts, and possibly minimally coupled fermionic matter. The novelty in our equations is the appearance of anticommuting fields; in describing the system, they simply appear as anticommuting variables generating a differential power series algebra. However, it is no longer obvious what a solution of our system should be. In fact, as argued in [14], there are no longer "individual" solutions (besides purely bosonic ones, with all fermionic components put to zero); but it is sensible to look for *families* of solutions parametrized by supermanifolds. In particular, solutions with values in Grassmann algebras can be reinterpreted as such families (cf. 3.11).

We call a system in our class *complete* iff the underlying bosonic equations admit all-time solutions. In that case, there is a *universal solution family* from which every other solution family arises in a unique way by pullback. We will construct this universal solution family by generalizing the map $\check{\Phi}^{\mathrm{sol}}$ of Cor. 1.2.4 to a *morphism of supermanifolds*

(1.3.1)
$$\check{\Xi}^{\mathrm{sol}} : M^{\mathrm{Cau}} = \{\text{smf of Cauchy data at } t = 0\} \longrightarrow \{\text{smf of configurations on space-time}\} = M^{\mathrm{cfg}}.$$

For the construction of this morphism, we follow the usual scheme of solving non-linear evolution equations: First, one shows the existence of short-time solutions, and then the existence of all-time solutions.

The necessary supergeometric machinery has been provided in [13], [14].

Turning to the functional spaces needed, a reasonable choice for the Cauchy data is the test function space $\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau}} := \mathcal{D}(\mathbb{R}^d)$; for the configurations we take the space $\mathcal{E}_{\mathrm{c}}$ of all those $f \in C^{\infty}(\mathbb{R}^{d+1})$ the support of which on every time slice is compact and grows only with light velocity (cf. 3.1 for details).

Now we associate to a given model a *configuration supermanifold*, or more precisely, the *supermanifold of smooth configurations with causally growing spatially compact support*, which is the linear smf modelled over the "naive configuration space",

$$M^{\mathrm{cfg}} = \mathrm{L}(\mathcal{E}_{\mathrm{c}} \otimes V);$$

here V is the target space for the fields. The standard coordinate (cf. [14, 2.5]) of this linear smf will be denoted by Ξ.

Also, we need the *supermanifold of compactly supported smooth Cauchy data* which is the linear smf

$$M^{\mathrm{Cau}} = \mathrm{L}(\mathcal{D}(\mathbb{R}^d) \otimes V)$$

with the standard coordinate being denoted by Ξ^{Cau}.

We will not use the standard methods of operator semigroups in Hilbert space. Instead of this, our exposition of infinite-dimensional supergeometry given in [15] and [13] suggests, and makes here in fact necessary, another, more direct approach: we expand the solution in a formal, "functional" power series in the Cauchy data, and then we show convergence on Sobolev spaces for small times.

Thus, we construct a *formal solution* $\Xi^{\rm sol}[\Xi^{\rm Cau}]$ of the field equations, which is a formal power series (cf. [13, 2.3]) in the Cauchy data $\Xi^{\rm Cau}$ (in fact, its terms can be interpreted as belonging to certain tree diagrams).

Next we show that $\Xi^{\rm sol}[\Xi^{\rm Cau}](t)$ is for small times t an *analytic* power series on an arbitrarily large multiple of the unit ball of the Sobolev function space $H_k(\mathbb{R}^d)$ for $k > d/2$. That is, for given $c > 0$, there exists $t(c) > 0$ with $\Xi^{\rm sol}[\Xi^{\rm Cau}](t) \in \mathcal{P}(H_k(\mathbb{R}^d), \|\cdot\|/c;\ H_k(\mathbb{R}^d))$ (cf. [13, 3.2]) for $|t| = t(c)$. Loosely said, this has the consequence that there exist short time solutions of the field equations: Given bosonic Cauchy data $\phi^{\rm Cau}$ of Sobolev norm $< c$, a corresponding solution with lifetime $\geq t(c)$ exists and is given by $\Xi^{\rm sol}[\phi^{\rm Cau}]$. Cf. Cor. 2.4.5.

So far for the short-time solution; next we observe that a classical solution $\phi \in C([0,\theta], H_k(\mathbb{R}^d)\otimes V_{\bar 0})$ of the underlying bosonic equations with $\theta > 0$ can serve as "staircase" to prolongate the formal solution to an analytic solution in a neighbourhood of the Cauchy data of ϕ. That is, $\Xi^{\rm sol}[\phi(0) + \Xi^{\rm Cau}](t,\cdot)$ is analytic up to time θ (and, in fact, some epsilon beyond).

For proceeding, we have to suppose that the system is *causal*, i. e. that the influence functions have their support in the light cone. In that case, one can ascend from Sobolev spaces to spaces of smooth functions.

For a causal and complete model, the formal solution $\Xi^{\rm sol}[\Xi^{\rm Cau}]$ is the Taylor expansion of a *superfunction* $\Xi^{\rm sol} \in \mathcal{O}^{\mathcal{E}_c\otimes V}(M^{\rm Cau})$ at zero, and this superfunction determines the smf morphism (1.3.1) wanted. This morphism identifies $M^{\rm Cau}$ with a sub-smf $M^{\rm sol}$ of $M^{\rm cfg}$ which we call the *supermanifold of classical solutions*. The name is justified by the fact that given a morphism $\phi : Z \to M^{\rm cfg}$, i. e. a Z-family of configurations, it factors through $M^{\rm sol}$ iff we have $\phi^*(L_i[\Xi]) = 0$, i. e. $Z \xrightarrow{\phi} M^{\rm cfg}$ is a Z-family of solutions.

As a variant, we also construct the version $\check{\Xi}^{\rm sol} : M^{\rm Cau}_{C^\infty} \to M^{\rm cfg}_{C^\infty}$ which arises by admitting *all* smooth configurations and *all* smooth Cauchy data. By the reasons mentioned in the preceding section, this is not the functional-analytic quality of main interest.

Another variant arises by considering fluctuations around a fixed bosonic "background" configuration which solves the bosonic field equations; cf. 3.9.

For the construction of the sub-smf $M^{\rm sol}$ cut out by the field equations, the most obvious idea would be to form the ideal subsheaf $\mathcal{J}$ of the structure sheaf $\mathcal{O}_{M^{\rm cfg}}$ generated by the superfunctions $L_i[\Xi](x)$, where x varies over space-time. Of course, the ideal sheaf algebraically generated by these infinitely many elements is too small, and one should pass to a suitably completed ideal sheaf. The main difficulty, however, is that even if a reasonable sub-smf $M^{\rm sol}$ exists, there is no a priori guaranty that it is equal to the ringed space $(\operatorname{supp} \mathcal{O}/\mathcal{J},\ \mathcal{O}/\mathcal{J})$. This is due to a typical infinite-dimensional phenomenon: There is no general "non-linear Hahn-Banach Theorem", even for a complex-analytic function on an open subset of a closed linear subspace of a Banach space it may happen that there is not even locally an extension to a complex-analytic function defined on an open subset of the ambient space. Therefore, the approach via ideal sheaves should not play the primary role. Instead of this, the definition given in [14, 2.12] avoids these difficulties: Given an smf M and some family A of superfunctions on it, the *sub-smf N cut out by A* is, if it exists, uniquely characterized by the requirement that all elements of A restrict on N to zero, and every smf morphisms $Z \to M$ which pullbacks all elements of A to zero factors through N. Assertion (v) of Thm. 3.6.1 implements this point of view.

A posteori, it turns out that $M^{\rm sol}$ is a split sub-smf of $M^{\rm cfg}$, and thus we could get it as $(\operatorname{supp} \mathcal{O}/\mathcal{J}$, $\mathcal{O}/\mathcal{J})$; but this observation does not help in its construction.

Even in the case of a purely bosonic model, where all our supermanifolds turn into ordinary real-analytic manifolds modelled over locally convex spaces, two non-trivial assertions follow from our theory:

First, for any smooth Cauchy datum there exists a short-time solution, and the latter varies real-analytically with the Cauchy data.

Second, if for any compactly carried smooth Cauchy datum, the existence of an all-time solution of Sobolev class can be guaranteed, it lies automatically in $\mathcal{E}_c$ (cf. Lemma 3.3.2), and in that case, the all-time solution depends real-analytically on the Cauchy data.

On the other hand, in a purely fermionic model, like e. g. the Gross-Neveu model, the all-time solution can be guaranteed to exist a priori; however, there are no non-trivial "individual" solutions, only families of them.

1.4. **Preliminaries and notations.** Let us shortly recall some notions and conventions from [13], [14]. We follow the usual conventions of $\mathbb{Z}_2$-graded algebra: All vector spaces will be $\mathbb{Z}_2$-graded, $E = E_{\mathbf{0}} \oplus E_{\mathbf{1}}$ (decomposition into *even* and *odd* part); for the *parity* of an element, we will write $|e| = \mathbf{i}$ for $e \in E_{\mathbf{i}}$. In multilinear expressions, parities add up; this fixes parities for tensor product and linear maps. (Note that space-time, being not treated as vector space, remains ungraded. On the other hand, "classical" function spaces, like Sobolev spaces, are treated as purely even.)

First Sign Rule: Whenever in a complex multilinear expression two adjacent terms a, b are interchanged the sign $(-1)^{|a||b|}$ has to be introduced.

In order to get on the classical level a correct model of operator conjugation in the quantized theory we also have to use the additional rules of the hermitian calculus developed in [12]. That is, the role of real supercommutative algebras is taken over by *hermitian supercommutative algebras*, i. e. complex supercommutative algebras R together with an involutive antilinear map $\bar{\cdot} : R \to R$ (*hermitian conjugation*) such that

$$\overline{rs} = \overline{s} \cdot \overline{r}$$

for $r, s \in R$ holds. Note that *the real elements of a hermitian algebra do in general not form a subalgebra*, i. e. R is not just the complexification of a real algebra. More general, all real vector spaces have to be complexified before its elements may enter multilinear expressions.

The essential ingredient of the hermitian framework is the

Second Sign Rule: If conjugation is applied to a bilinear expression in the terms a, b (i. e. if conjugation is resolved into termwise conjugation), either a, b have to be rearranged backwards, or the expression acquires the sign factor $(-1)^{|a||b|}$. Multilinear terms have to be treated iteratively.

Turning to supergeometry, a calculus of real-analytic infinite-dimensional supermanifolds (smf's) has been constructed by the present author in [15, 13]. Here we note that it assigns to every real $\mathbb{Z}_2$-graded locally convex space ($\mathbb{Z}_2$-lcs) $E = E_{\mathbf{0}} \oplus E_{\mathbf{1}}$ a *linear supermanifold* $\mathrm{L}(E)$ which is essentially a ringed space $\mathrm{L}(E) = (E_{\mathbf{0}}, \mathcal{O})$ with underlying topological space $E_{\mathbf{0}}$ while the structure sheaf $\mathcal{O}$ might be thought very roughly of as a kind of completion of $\mathcal{A}(\cdot) \otimes \Lambda E_{\mathbf{1}}^*$; here $\mathcal{A}(\cdot)$ is the sheaf of real-analytic functions on $E_{\mathbf{0}}$ while $\Lambda E_{\mathbf{1}}^*$ is the exterior algebra over the dual of $E_{\mathbf{1}}$.

The actual definition of the structure sheaf treats even and odd sector much more on equal footing than the tensor product ansatz above: Given a second real $\mathbb{Z}_2$-graded (lcs) F, one defines the space $\mathcal{P}(E;\ F)$ of *F-valued power series on E* as the set of all formal sums $u = \sum_{k,l \geq 0} u_{(k|l)}$ where $u_{(k|l)} : \prod^k E_{\mathbf{0}} \times \prod^l E_{\mathbf{1}} \to F \otimes_{\mathbb{R}} \mathbb{C}$ is a jointly continuous, multilinear map which is symmetric on $E_{\mathbf{0}}$ and alternating on $E_{\mathbf{1}}$. Now one defines the sheaf $\mathcal{O}^F(\cdot)$ of *F-valued superfunctions* on $E_{\mathbf{0}}$: an element of $\mathcal{O}^F(U)$ where $U \subseteq E_{\mathbf{0}}$ is open is a map $f : U \to \mathcal{P}(E;\ F)$, $x \mapsto f_x$, which satisfies a certain "coherence" condition which makes it sensible to interpret f_x as the Taylor expansion of f at x.

Now the structure sheaf of our ringed space $\mathrm{L}(E)$ is simply $\mathcal{O}(\cdot) := \mathcal{O}^{\mathbb{R}}(\cdot)$; it is a sheaf of hermitian supercommutative algebras, and each $\mathcal{O}^F(\cdot)$ is a module sheaf over $\mathcal{O}(\cdot)$.

Actually, in considering more general smf's than superdomains, one has to enhance the structure of a ringed space slightly, in order to avoid "fake morphisms" (not every morphism of ringed spaces is a morphism of supermanifolds). What matters here is that the enhancement is done in such a way that the following holds (cf. [14, Thm. 2.8.1]):

Lemma 1.4.1. *Given an $\mathbb{Z}_2$-lcs F and an arbitrary smf Z, the set of morphisms $Z \to \mathrm{L}(F)$ is in natural 1-1-correspondence with the set*

$$\mathcal{M}^F(Z) := \mathcal{O}^F(Z)_{0,\mathbb{R}}.$$

(Here the subscript stands for the real, even part.) The correspondence works as follows: There exists a distinguished element $x \in \mathcal{M}^F(\mathrm{L}(F))$ called the standard coordinate*, and one assigns to $\mu : Z \to \mathrm{L}(F)$ the pullback $\hat{\mu} := \mu^*(x)$.* □

This is the infinite-dimensional version of the fact that if $F = \mathbb{R}^{m|n}$ then a morphism $Z \to \mathrm{L}(\mathbb{R}^{m|n})$ is known by knowing the pullbacks of the coordinate superfunctions, and these can be prescribed arbitrarily as long as parity and reality are OK.

The most straightforward way to do the enhancement mentioned is a chart approach; since the supermanifolds we are going to use are actually all superdomains, and only the morphisms between them are non-trivial, we need not care here for details.

If E, F are spaces of generalized functions on $\mathbb{R}^d$ which contain the test functions as dense subspace then the Schwartz kernel theorem tells us that the multilinear forms $u_{(k|l)}$ are given by their integral kernels, which are generalized functions. Thus one can apply rather suggestive integral writings (cf. [13]) quite analogous to that used in (1.2.3): The general form of a power series in $\mathbb{R}^d$ is

$$(1.4.1)\quad K[\Phi|\Psi] = \sum_{k,l\geq 0} \frac{1}{k!l!} \sum_{I,J} \int_{\mathbb{R}^{d(k+l)}} dx_1 \cdots dx_k dy_1 \cdots dy_l \\ K^{i_1,\ldots,i_k|j_1,\ldots,j_l}(x_1,\ldots,x_k|y_1,\ldots,y_l)\Phi_{i_1}(x_1)\cdots\Phi_{i_k}(x_k)\Psi_{j_1}(y_1)\cdots\Psi_{j_l}(y_l)$$

where we have used collective indices $i = 1,\ldots,N_{\mathbf{0}}$ and $j = 1,\ldots,N_{\mathbf{1}}$ for the real components of bosonic and fermionic fields, respectively. The *coefficient functions* $K^{i_1,\ldots,i_k|j_1,\ldots,j_l}(x_1,\ldots,x_k|y_1,\ldots,y_l)$ are distributions which can be supposed to be symmetric in the pairs $(x_1,i_1),\ldots,(x_k,i_k)$ and anti-symmetric in $(y_1,j_1),\ldots,(y_l,j_l)$. Of course, they have to satisfy also certain growth and smoothness conditions. However, what matters here is that the Φ's and Ψ's can be formally treated as commuting and anticommuting fields, respectively; in fact, after establishing the proper calculational framework, the writing (1.4.1) is sufficiently correct. Also, it is possible to substitute power series into each other under suitable conditions. Cf. [13] for a detailed exposition.

We conclude with some additional preliminaries. It will be convenient to work not with the bidegrees $(k|l)$ of forms but with *total degrees*: For any formal power series $K \in \mathcal{P}_f(E;\ F)$ set for $m \geq 0$

$$K_{(m)} := \sum_{k=0}^{m} K_{(k|m-k)}, \qquad K_{(\leq m)} := \sum_{n=0}^{m} K_{(n)}.$$

Thus $K = \sum_{m\geq 0} K_{(m)}$.

Let E be a $\mathbb{Z}_2$-lcs and $p \in \mathrm{CS}(E)$ be a continuous seminorm on E; let $U \subseteq E$ be the unit ball of p. Also, suppose that F is a $\mathbb{Z}_2$-graded Banach space. We will use often the suggestive notation

$$\mathcal{P}(E,U;\ F) := \mathcal{P}(E,p;\ F)$$

(cf. [13] for the definition of the r. h. s.) for the *space of power series converging on* U. Indeed, every element $K \in \mathcal{P}(E,p;\ F)$ is "a function element on $U \cap E_{\mathbf{0}}$", i. e. it is the Taylor expansion at zero of a uniquely determined superfunction $K \in \mathcal{O}^F(U \cap E_{\mathbf{0}})$ within the superdomain $\mathrm{L}(E)$.

As usual, we call a power series (in the finite-dimensional sense) in even and odd variables, $P[y|\eta] = \sum P_{\mu\nu} y^\mu \eta^\nu \in \mathbb{C}[[y_1,\ldots,y_m|\eta_1,\ldots,\eta_n]]$, *entire* iff for all $R > 0$ there exists $C > 0$ such that

$$|P_{\mu\nu}| \leq CR^{-|\mu|}$$

for all μ,ν. The following Lemma is elementary.

Lemma 1.4.2. *Let $P[y|\eta]$ be an entire power series of lower degree $\geq l \geq 0$. Then, for any $R > 0$ there exists some $C_R > 0$ such that if A is a real $\mathbb{Z}_2$-graded commutative Banach algebra, and*

$$y'_1,\ldots,y'_m \in A_{\mathbf{0}}, \quad \eta'_1,\ldots,\eta'_n \in A_{\mathbf{1}}, \quad \|y'_i\| \leq R, \quad \|\eta'_i\| \leq R$$

then

$$\|P(y'_1,\ldots,y'_m|\eta'_1,\ldots,\eta'_n)\| \leq C_R \cdot \max\{\|y'_1\|^l,\ldots,\|y'_m\|^l,\|\eta'_1\|^l,\ldots,\|\eta'_n\|^l\}.$$

$\square$

Since the calculus of differential polynomials of [13] is insufficient to formulate e. g. the exponential self-interaction $\exp \underline{\Phi}$ of the Liouville model, we consider *differential power series* instead. We set

$$\mathbb{C}[[\partial^* \underline{\Xi}]] := \bigcup_{n>0} \mathbb{C}[[(\partial^\nu \underline{\Xi}_i)_{i=1,\dots,N,\ \nu\in\mathbb{Z}_+^d,\ |\nu|\le n}]]$$

where, as usual, $\partial^\nu := \partial_1^{\nu_1}\cdots\partial_d^{\nu_d}$. As in [13], the underlined letters $\underline{\Xi}, \underline{\Phi}, \underline{\Psi}$ denote the indeterminates of an algebra of differential polynomials or differential power series, while the non-underlined letters Ξ, Φ, Ψ denote superfunctions or their Taylor expansions.

As usual, we will write

$$xy := \sum_{a=1}^d x_a y_a, \quad x^2 := xx, \quad |x| := \sqrt{x^2}$$

for $x, y \in \mathbb{R}^d$.

All Fourier transformations will be with respect to the spatial coordinates: For $f \in \mathcal{S}(\mathbb{R}^d)$, we set

$$\hat{f}(p) = \mathcal{F}_{x\to p} f(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} dx\, \mathrm{e}^{-\mathrm{i}\, px} f(x);$$

the extension to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is done as usual.

1.5. **Systems of field equations.** In order to fix a system of field equations we need the following data:

I. The *space dimension* $d \ge 0$; the points of space-time $\mathbb{R}^{d+1} = \mathbb{R}\times\mathbb{R}^d$ will be labelled $(t, x) = (t, x_1, \dots, x_d)$.

II. The numbers $N_{\mathbf{0}}$, $N_{\mathbf{1}}$ of *bosonic*, commuting, and *fermionic*, anticommuting, field components, respectively; write $N := N_{\mathbf{0}} + N_{\mathbf{1}}$ for the total number of field components.

Thus, the setup (cf. [13, 2.2]) for superfunctionals on the fields in space-time will be $(d+1, V)$ with the target space

$$V := \mathbb{R}^{N_{\mathbf{0}}|N_{\mathbf{1}}}.$$

For the tuple of real field components, we will write as in [13]

$$\Xi = (\Xi_1, \dots, \Xi_N) = (\Phi_1, \dots, \Phi_{N_{\mathbf{0}}} | \Psi_1, \dots, \Psi_{N_{\mathbf{1}}});$$

this will be also the functional coordinate on the configuration smf M^{cfg}.

Turning to the Cauchy data, the setup for superfunctionals on them will be (d, V). We will use the functional coordinates

$$\Xi^{\mathrm{Cau}} = (\Xi_1^{\mathrm{Cau}}, \dots, \Xi_N^{\mathrm{Cau}}) = (\Phi_1^{\mathrm{Cau}}, \dots, \Phi_{N_{\mathbf{0}}}^{\mathrm{Cau}} | \Psi_1^{\mathrm{Cau}}, \dots, \Psi_{N_{\mathbf{1}}}^{\mathrm{Cau}})$$

for the fields at $t = 0$.

III. The vector $\tau = (\tau_i)_{i=1}^N \in \mathbb{Z}_+^N$ of *smoothness offsets*; its role will become clear below.

IV. The *field equations*, which are given as real, even, entire differential power series of the form

$$L_i[\underline{\Xi}] = \partial_t \underline{\Xi}_i + \sum_{j=1}^N K_{ij}(\partial_x)\underline{\Xi}_j + \Delta_i[\underline{\Xi}], \qquad \Delta_i[\underline{\Xi}] \in \mathbb{C}[[\partial^* \underline{\Xi}]]_{\mathbf{0},\mathbb{R}}.$$

Here $K_{ij}(\partial_x)$ is a real differential operator with constant coefficients and containing only spatial derivatives, called the *kinetic operator*, and $\Delta_i[\underline{\Xi}]$ is a real, entire differential power series of lower degree ≥ 2 which is even and odd for $i = 1, \dots, N_{\mathbf{0}}$ and $i = N_{\mathbf{0}} + 1, \dots, N_{\mathbf{0}} + N_{\mathbf{1}}$, respectively, called the *interaction term*. We now specify our requirements onto these terms.

The matrix-valued function

$$\hat{A} : \mathbb{R}\times\mathbb{C}^d \to \mathbb{C}^{N\times N}, \quad \hat{A}^\Psi(t, p) := (2\pi)^{-d/2} \exp(-K(\mathrm{i}\, p)t) \tag{1.5.1}$$

satisfies the spatially Fourier-transformed and complexified free field equations,

$$\frac{d}{dt}\hat{A}(t,p) + K(\mathrm{i}\,p)\hat{A} = 0, \quad \hat{A}(0,p) = (2\pi)^{-d/2} 1_{N\times N}.$$

(The reason for the notations $\hat{A}$ will become clear in the next section.) We require that there exist $t_0 > 0$, $C > 0$ such that

$$\left\|\hat{A}_{ij}(t,p)\right\| \le C(1+|p|)^{\tau_i - \tau_j}, \tag{1.5.2}$$

for $p \in \mathbb{R}^d$, $t \in [-t_0, t_0]$.

Remark . Obviously, the estimate (1.5.2) implies hyperbolicity of the kinetic operators, i. e. for all $p \in \mathbb{R}^d$, the matrix $K(-\mathrm{i}\,p)$ has only imaginary eigenvalues.

Define the *smoothness degree* of a differential power series $P = P[\Xi]$ by

$$\tau(P) := \min_{k,\nu}\{\tau_k - |\nu| : \quad \frac{\partial}{\partial(\partial^\nu \Xi_k)} P \neq 0 \text{ for some } \nu \in \mathbb{Z}_+{}^n\};$$

thus, $\tau(\partial^\nu \Xi_k) = \tau_k - l$, and the smoothness degree of P is just the infimum of the smoothness degrees of the variables which enter it. Of course, if P is constant we set $\tau(P) = \infty$.

We have to state a *smoothness condition*: For all $i = 1, \ldots, N$, we require that

$$\tau_i \le \tau(\Delta_i). \tag{1.5.3}$$

Remark 1.5.1. Thus, we will assume that numerical values for the coupling constants, as well as for the masses, have been fixed. In view of the necessity of renormalization, seemingly intrinsic for any quantization procedure, it might be sensible to allow these "constants" to vary. Instead of the solution supermanifold to be constructed we will then get a bundle of solution smfs over the domain $U \subseteq \mathbb{R}^N$ of all tuples of coupling constants and masses for which the system is complete (cf. 3.5 below). Moreover, the total space of this bundle will carry a Poisson structure which induces on each fibre a symplectic structure; perhaps, this is the right object to quantize.

Definition 1.5.2. A *system of field equations (s.o.f.e.)* is a quadruple $(d, N_{\mathbf{0}}|N_{\mathbf{1}}, \tau, (L_i[\Xi]))$ which satisfies the requirements given above.

Given a s.o.f.e., the *underlying bosonic s.o.f.e.* is given by $(d, N_{\mathbf{0}}|0, (\tau_1, \ldots, \tau_{N_{\mathbf{0}}}), (L_i[\underline{\Phi}|0]))$, and the *underlying free s.o.f.e.* by $(d, N_{\mathbf{0}}|N_{\mathbf{1}}, \tau, (L_i^{\mathrm{free}}[\Xi]))$ with $L_i^{\mathrm{free}}[\Xi] := (\partial_t + K(\partial_x))\Xi_j$.

We will use matrix writing; in particular, we set $\Delta = (\Delta_1, \ldots, \Delta_N)^{\mathrm{T}}$ and $L = (L_1, \ldots, L_N)^{\mathrm{T}}$.

Remarks . (1) Usually, the smoothness offsets save that smoothness information which would be otherwise lost in reducing a temporally higher-order system to a temporally first-order one.

(2) The smoothness condition is rather constraining; it excludes e. g. the Korteweg-de Vries equation as well as the nonlinear Schrödinger equations. Fortunately, it is satisfied (for a suitable choice of smoothness offsets) for apparently all wave equations occurring in quantum-field theoretical models.

1.6. **Function spaces.** In order to keep legibility, we need a certain systematics in the notations: The superscript "$^{\mathrm{Cau}}$" will qualify a space as space of Cauchy data, and thus living on the Cauchy hyperplane $\mathbb{R}^d$; otherwise, it lives either on $\mathbb{R}^{d+1}$, or, if the notation is qualified with an argument I, on $I \times \mathbb{R}^d$. Also, the superscript V qualifies as being a space of V-valued functions.

Our main technical tool will be the standard Sobolev spaces: For real $k > d/2$, let $\mathcal{H}_k^{\mathrm{Cau}} := H_k(\mathbb{R}^d)$ be the space of all $f \in L_2(\mathbb{R}^d)$ for which $(1+|p|)^k \hat{f}(p)$ is square-integrable. Recall that by the Sobolev Embedding Theorem, $H_k(\mathbb{R}^d) \subseteq C^{k'}(\mathbb{R}^d)$ where k' is the maximal integer with $k' < k - d/2$. Also, $\mathcal{H}_k^{\mathrm{Cau}}$ is a Banach algebra under pointwise multiplication, cf. [18]. (In fact, the consideration of Sobolev spaces with integer index k would be sufficient for our purposes.) We equip $H_k(\mathbb{R}^d)$ with the norm

$$\|f\|_{H_k(\mathbb{R}^d)} := c\int dp (1+|p|)^{2k} \left|\hat{f}(p)\right|^2$$

where the constant $c > 0$ is chosen minimal with the property that $\|\cdot\|_{H_k(\mathbb{R}^d)}$ is submultiplicative.

The corresponding space of vector-valued Cauchy data is

$$\mathcal{H}_k^{\mathrm{Cau},V} := \bigoplus_{i=1}^{N_0} H_{k+\tau_i}(\mathbb{R}^d) \oplus \bigoplus_{i=N_0+1}^{N_0+N_1} \Pi H_{k+\tau_i}(\mathbb{R}^d) \tag{1.6.1}$$

Set

$$\mu := \max_{i=1,\ldots,N} \max\left\{1, \tau_i - \tau(\Delta_i), \max_{j=1,\ldots,N} (\tau_i - \tau_j + \operatorname{ord} K_{ij}(\partial_x))\right\} \tag{1.6.2}$$

where $\operatorname{ord} K_{ij}(\partial_x)$ is the order of the differential operator ($= -\infty$ if $K_{ij} = 0$). The number μ will bound the loss of spatial smoothness for each temporal derivative of the solutions (cf. Prop. 2.4.1).

For integer $l \geq 0$ such that $k - \mu l > d/2$, set

$$\mathcal{H}_k^l(I) := \bigcap_{n=0}^{l} C^l(I, H_{k-\mu n}(\mathbb{R}^d))$$

(the intersection taken within $C(I \times \mathbb{R}^d)$), equipped with the topology defined by the seminorms

$$\|\xi\|_{\mathcal{H}_k^l([a,b])} := \sum_{n=0}^{l} \sup_{t\in[a,b]} \frac{1}{n!} \|\partial_t^n \xi(t)\|_{H_{k-\mu n}(\mathbb{R}^d)} \tag{1.6.3}$$

where $a, b \in I$, $a < b$. We will use mainly the special case that I is a bounded closed interval; then $\mathcal{H}_k^V(I)$ is a Banach space, and this observation justifies the notation on the l. h. s. of (1.6.3). By straightforward computation, one has:

Lemma 1.6.1. *(i) $\mathcal{H}_k^l(I)$ is a Banach algebra under pointwise multiplication, and the norm $\|\cdot\|_{\mathcal{H}_k^l([a,b])}$ is submultiplicative.*

(ii) We have a continuous embedding $\mathcal{H}_k^l(I) \subseteq C^l(I \times \mathbb{R}^d)$. □

For $k - \mu l > d/2$ as above, set

$$\mathcal{H}_k^{l,V}(I) := \bigoplus_{i=1}^{N_0} \mathcal{H}_{k+\tau_i}^l(I) \oplus \bigoplus_{i=N_0+1}^{N_0+N_1} \Pi\mathcal{H}_{k+\tau_i}^l(I) \tag{1.6.4}$$

where $a, b \in I$, $a < b$. Again, $\mathcal{H}_k^{l,V}(I)$ is a Banach space if I is a bounded closed interval.

Note that all the spaces qualified with a time definition interval I are not admissible in the sense of [13, 3.1] unless $I = \mathbb{R}$. This should not bother us since they appear as target of formal power series, not as their source.

For better orientation, a table of spaces of configurations and Cauchy data is given in the Appendix.

1.7. **Influence functions and Green functions.** It follows from (1.5.2) that for fixed $t \in \mathbb{R}$, the matrix entries $\hat{A}_{ij}(t,\cdot)$ are the spatial Fourier transforms of elements $A_{ij}(t,\cdot) \in \mathcal{S}'(\mathbb{R}^d)$; let $A(t,\cdot)$ be the corresponding matrix.

For $t \in \mathbb{R}$, let, as usual, $\theta(t)$ be 1, 1/2, and 0, for $t > 0$, $t = 0$, and $t < 0$, respectively. Also, define $f : \mathbb{R}^2 \to \{-1, 0, 1\}$ by

$$f(t,s) := \theta(s)\theta(t-s) - \theta(-s)\theta(s-t) = \theta(t-s) - \theta(-s). \tag{1.7.1}$$

In particular, $f(t,s) = 1$ iff $0 < s < t$, $f(t,s) = -1$ iff $t < s < 0$, and $f(t,s)$ vanishes unless $0 \leq s \leq t$ or $t \leq s \leq 0$. We define the matrix of *Green functions* $G = (G_{ij}) \in \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^d)^{N\times N}$ by

$$G(t,s,x) = f(t,s)A(t-s,x). \tag{1.7.2}$$

Remark . The usual retarded and advanced Green functions are given by

$$D_{\mathrm{ret}}(t,x) = \theta(t)A(t,x), \quad D_{\mathrm{adv}}(t,x) = -\theta(-t)A(t,x);$$

conversely,

$$G(t,s,x) = \theta(s)D_{\mathrm{ret}}(t-s,x) + \theta(-s)D_{\mathrm{adv}}(t-s,x).$$

Altogether, we have

$$(1.7.3)\qquad \begin{array}{ll} (\partial_t + K(\partial_x))A(t,x) = 0, & A(0,x) = \delta(x)\cdot 1_{N\times N}, \\ (\partial_t + K(\partial_x))G(t,s,x) = -\delta(t-s)\delta(x)\cdot 1_{N\times N}, & G(0,s,x) = 0. \end{array}$$

Lemma 1.7.1. *(i) Given $\xi^{\mathrm{Cau}} \in \mathcal{H}_k^{\mathrm{Cau},V}$ and $g \in \mathcal{H}_k^{0,V}(\mathbb{R})$, the inhomogeneous linear Cauchy problem to find $\xi \in \mathcal{S}'(\mathbb{R}^{d+1}) \otimes V$ with*

$$(\partial_t + K(\partial_x))\xi + g = 0, \quad \xi(0) = \xi^{\mathrm{Cau}}$$

has the unique solution $\xi = \mathrm{A}\xi^{\mathrm{Cau}} + \mathrm{G}g$ where the operators A, G are given by

$$\mathrm{A}\xi^{\mathrm{Cau}}(t,y) := \int_{\mathbb{R}^d} dx\, A(t,y-x)\xi^{\mathrm{Cau}}(x), \quad \mathrm{G}g(t,y) := \int_{\mathbb{R}\times\mathbb{R}^d} dsdx\, G(t,s,y-x)g(s,x).$$

(ii) There exists a constant C_1 such that we have for any $\theta > 0$

$$\left\| \mathrm{A}\xi^{\mathrm{Cau}}] \right\|_{\mathcal{H}_k^{0,V}([-\theta,\theta])} \le C_1 \left\| \xi^{\mathrm{Cau}} \right\|_{\mathcal{H}_k^{\mathrm{Cau},V}}.$$

(iii) There exists a constant $C_2 > 0$ such that

$$\| \mathrm{G}g \|_{\mathcal{H}_k^{0,V}([-\theta,\theta])} \le C_2\theta \, \|g\|_{\mathcal{H}_k^{0,V}([-\theta,\theta])}$$

for all $\theta > 0$.

(iv) The assignment $(\xi^{\mathrm{Cau}}, g) \mapsto \mathrm{A}\xi^{\mathrm{Cau}} + \mathrm{G}g$ defines a continuous linear map

$$\mathcal{H}_k^{\mathrm{Cau},V} \oplus \mathcal{H}_k^{0,V}(\mathbb{R}) \to \mathcal{H}_k^{0,V}(\mathbb{R}).$$

Proof. (i) is standard.

Ad (ii). It follows from (1.5.2) that $\mathrm{A}\xi^{\mathrm{Cau}}(t,\cdot) \in \mathcal{H}_k^{\mathrm{Cau},V}$ for all t; however, we have to show that $\mathbb{R} \to \mathcal{H}_k^{\mathrm{Cau},V}$, $t \mapsto \mathrm{A}\xi^{\mathrm{Cau}}(t,\cdot)$ is continuous. We may assume that the Fourier transform of ξ^{Cau} has compact support; now observe that for any $R > 0$ we have $\sup_{|p|\le R} \left\| \hat{A}(t,p) - \hat{A}(t_0,p) \right\| \to 0$ for $t \to t_0$ where $\|\cdot\|$ is any matrix norm. The assertion follows.

Ad (iii). We have $\mathrm{G}g(t,y) = \int ds f(t,s)(\mathrm{A}g(s,\cdot))(t-s,y)$ and hence

$$\begin{aligned} \|\mathrm{G}g(t,\cdot)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} &\le \int_{-\theta}^{\theta} ds \, \|(\mathrm{A}g(s,\cdot))(t-s,\cdot)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} \\ &\le C_1 \int_{-\theta}^{\theta} ds \, \|g(s,\cdot)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} \le 2\theta C_1 \|g\|_{\mathcal{H}_k^{0,V}([-\theta,\theta])} \end{aligned}$$

which implies the assertion.

(iv) is an obvious corollary. □

2. Formal solution and solution families

2.1. The formal Cauchy problem. In the following, we consider the Cauchy problem for the field equations on the formal power series level. That is, we fix some $k > d/2$ and consider the problem to determine a formal power series in the sense of [13, 2.3],

$$\Xi^{\mathrm{sol}} = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}_f(\mathcal{H}_k^{\mathrm{Cau},V};\ \mathcal{H}_k^{0,V}(\mathbb{R}))_{\mathbf{0},\mathbb{R}}$$

such that

$$(2.1.1)\qquad L[\Xi^{\mathrm{sol}}] = 0, \quad \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}](0,\cdot) = \Xi^{\mathrm{Cau}}(\cdot).$$

We call this the *formal Cauchy problem.* Its solution Ξ^{sol}, which we call the *formal solution of the s.o.f.e.* will be the power series expansion at the zero configuration of the solution of the "analytic Cauchy problem", cf. Cor. 2.4.3.

Splitting (2.1.1) into total degrees we get for $n \ge 0$

$$(2.1.2)\qquad (\partial_t + K(\partial_x))\Xi^{\mathrm{sol}}_{(n)} + \Delta[\Xi^{\mathrm{sol}}_{(<n)}]_{(n)} = 0,$$

and hence, using Lemma 1.7.1.(i), $\Xi^{\mathrm{sol}}_{(0)} = 0$. Moreover, using Lemma 1.7.1.(i) again, $\Xi^{\mathrm{sol}}_{(1)}$ is the formal solution of the free s.o.f.e.:

$$\Xi^{\mathrm{sol}}_{(1)} = \Xi^{\mathrm{free}}[\Xi^{\mathrm{Cau}}] := \mathrm{A}\Xi^{\mathrm{Cau}}. \tag{2.1.3}$$

Finally, for $n > 1$, the $\Xi_{(n)}$ are recursively determined by the linear Cauchy problem consisting of (2.1.2) and the initial conditions $\Xi^{\mathrm{sol}}_{(n)}(0) = 0$. We get:

Theorem 2.1.1. *There exists a uniquely determined solution $\Xi^{\mathrm{sol}} = \sum_{n\geq 1} \Xi^{\mathrm{sol}}_{(n)}$ to the formal Cauchy problem. The homogeneous components $\Xi^{\mathrm{sol}}_{(n)}$ are recursively given by* (2.1.3) *and*

$$\Xi^{\mathrm{sol}}_{(n)} = \mathrm{G}\Delta[\Xi^{\mathrm{sol}}_{(<n)}]_{(n)} \tag{2.1.4}$$

for $n \geq 2$. Moreover, Ξ^{sol} satisfies the integral equation

$$\Xi^{\mathrm{sol}} = \Xi^{\mathrm{free}} + \mathrm{G}\Delta[\Xi^{\mathrm{sol}}]. \tag{2.1.5}$$

□

Remarks . (1) Note that (2.1.4) makes sense due to Lemma 1.7.1.(ii).

(2) Of course, Ξ^{sol}, considered as element of the bigger space $\mathcal{P}_f(\mathcal{E}^{\mathrm{Cau},V}_c;\ \mathcal{D}'(\mathbb{R}^{d+1})\otimes V)$, is independent of k.

(3) Neither the smoothness condition (1.5.3) nor the analyticity of the Δ_i have been used up to now.

2.2. **Short-time analyticity of the solution.**

Theorem 2.2.1. *Let be given s.o.f.e., and fix k with $k > d/2$. Let $U \subset \mathcal{H}^{\mathrm{Cau},V}_k$ denote the unit ball. For any $c > 0$ there exists $\theta = \theta_c > 0$ such that*

$$\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{H}^{\mathrm{Cau},V}_k, cU;\ \mathcal{H}^{0,V}_k([-\theta,\theta])). \tag{2.2.1}$$

Proof. We begin with the estimation of the free solution. From Lemma 1.7.1.(ii) we have:

Corollary 2.2.2. *There exists a constant C_1 such that for all $\theta, c > 0$,*

$$\left\|\Xi^{\mathrm{free}}[\Xi^{\mathrm{Cau}}]\right\| \leq C_1 c \quad \textit{within } \mathcal{P}(\mathcal{H}^{\mathrm{Cau},V}_k, cU;\ \mathcal{H}^{0,V}_k([-\theta,\theta])).$$

□

The idea of the proof of the Theorem is the following: For $n \geq 0$, we have from (2.1.5)

$$\Xi^{\mathrm{sol}}_{(\leq n+1)} = \Xi^{\mathrm{free}} + \mathrm{G}\Delta[\Xi^{\mathrm{sol}}_{(\leq n)}]_{(\leq n+1)}. \tag{2.2.2}$$

We will show that for sufficiently small $\theta > 0$ we have for all $n \geq 0$ the estimate

$$\left\|\Xi^{\mathrm{sol}}_{(\leq n)}\right\| \leq 2C_1 c \quad \text{within } \mathcal{P}(\mathcal{H}^{\mathrm{Cau},V}_k, cU;\ \mathcal{H}^{0,V}_k([-\theta,\theta])). \tag{2.2.3}$$

Passing to the limit $n \to \infty$ we get assertion (i).

We estimate the interaction term. From Lemma 1.4.2 one gets:

Lemma 2.2.3. *Given $C' > 0$, there exists $C" > 0$ with the following property: If E is a $\mathbb{Z}_2$-lcs, $p \in \mathrm{CS}(E)$ and the power series $\Xi' = (\Phi'|\Psi') \in \mathcal{P}(E, p;\ \mathcal{H}^{0,V}_k([-\theta,\theta]))$ satisfies $\|\Xi'\| < C'$, then*

$$\|\Delta[\Xi']\| \leq C" \quad \textit{within } \mathcal{P}(E, p;\ \mathcal{H}^{0,V}_k([-\theta,\theta])).$$

□

We now prove (2.2.3) by induction on n. In view of (2.1.3), the start of induction, $n = 1$, is settled by (2.2.2). Now, for $n \geq 1$, we find from 2.2.2 that within $\mathcal{P}(\mathcal{H}^{\mathrm{Cau},V}_k, cU;\ \mathcal{H}^{0,V}_k([-\theta,\theta]))$

$$\left\|\Xi^{\mathrm{sol}}_{(\leq n+1)}\right\| \leq \left\|\Xi^{\mathrm{free}}\right\| + \left\|\mathrm{G}\left(\Delta[\Xi^{\mathrm{sol}}_{(\leq n)}]_{(\leq n+1)}\right)\right\|.$$

Using (2.2.2) and Lemma 1.7.1.(iii), this becomes

$$\leq C_1 c + C_3\theta \left\|\Delta[\Xi^{\mathrm{sol}}_{(\leq n)}]_{(\leq n+1)}\right\|;$$

because of the hypotheses of induction, the Lemma applies with $C' := 2C_1c$, yielding

$$\left\|\Xi^{\mathrm{sol}}_{(\leq n+1)}\right\| \leq C_1c + C_3\theta C",$$

and the assertion of induction, $\left\|\Xi^{\mathrm{sol}}_{(\leq n+1)}\right\| \leq 2C_1c$, is satisfied for $\theta \leq C_1c/(C_3C")$.

The theorem is proved. □

2.3. **Configuration families.** Let I be a connected subset of $\mathbb{R}$ with non-empty kernel, and let Z be an arbitrary smf. A *configuration family of quality $\mathcal{H}^l_k$ parametrized by Z with time definition domain I* (or *Z-family over I*, for short) is an even, real superfunction Ξ' on Z with values in the locally convex space $\mathcal{H}^{l,V}_k(I)$:

$$\Xi' = (\Phi'|\Psi') \in \mathcal{M}^{\mathcal{H}^{l,V}_k(I)}(Z)$$

(we recall that $\mathcal{M}$ denotes the real, even part of the sheaf $\mathcal{O}$). Thus, Ξ' encodes a N-tuple $\Xi' = (\Xi'_1, \ldots, \Xi'_N)$ of superfunctions $\Xi'_i \in \mathcal{O}^{\mathcal{H}^l_{k+\tau_i}(I)}(Z)_{\mathbb{R}}$ on Z, with the Φ'_i and Ψ'_j being even and odd, respectively.

Beginning with 3.3, we will also consider smooth families, i. e. families with values in the spaces $\mathcal{E}_{\mathrm{c}}$ and $\mathcal{E}$ to be defined later on. (On the other hand, one might consider also still more general families which have values in spaces of generalized functions; however, it is then not clear how to define families of solutions).

Now, given an smf morphism $\pi : Z' \to Z$ we can assign to every Z-family Ξ' its *pullback* $\Xi" := \pi^*(\Xi')$ which is a Z'-family. In fact, the process of passing from Ξ' to $\Xi"$ means in family language nothing but a *change of parametrization* (cf. [14, 1.11]).

Fixing a Z-family Ξ', the field strengthes $\Xi'_i(t,x) = \delta_{(t,x)} \circ \Xi'_i$ for $(t,x) \in \mathbb{R}^{d+1}$ are scalar superfunctions on Z. More generally, we define the *value at* Ξ' of any superfunctional $K \in \mathcal{O}^F(M^{\mathrm{cfg}})$ as the pullback of K along $\check{\Xi}'$:

$$K[\Xi'] := \check{\Xi}'^*(K) \in \mathcal{O}^F(Z).$$

For instance, in case Z is a point, the value $K[\Xi']$ of an F-valued superfunctional K at a Z-family Ξ' is an element of $F_{\mathbb{C}}$; thus, for a scalar functional $K \in \mathcal{O}(M^{\mathrm{cfg}})$, it is simply a complex number (which, however, is zero for all odd K, and, in particular, for the fermionic field strengthes).

If Z is $0|n$-dimensional then the value $K[\Xi']$ of $K \in \mathcal{O}^F(M^{\mathrm{cfg}})$ at a Z-family Ξ' is an element of $F_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda_n$ where $\Lambda_n = \mathbb{C}[\zeta_1, \ldots, \zeta_n]$ is a finite-dimensional Grassmann algebra; thus, for a scalar functional $K \in \mathcal{O}(M^{\mathrm{cfg}})$, it is a "Grassmann number" $K[\Xi'] \in \Lambda_n$.

2.4. **Solution families.** If $\Xi' \in \mathcal{M}^{\mathcal{H}^{l,V}_k(I)}(Z)$ is a Z-family of configurations then, using Lemma 1.6.1.(i) and (1.5.3), $\Delta[\Xi'] \in \mathcal{O}^{\mathcal{H}^{0,V}_k(I)}(Z)$ is well-defined. It follows that $L[\Xi'] \in \mathcal{O}^{\mathcal{D}'(I\times\mathbb{R}^d)\otimes V}(Z)$ is well-defined, since the derivatives needed exist at least in the distributional sense.

We call Ξ' a *Z-family of solutions*, or *solution family* for short, if $L[\Xi'] = 0$. Obviously, every pullback of a solution family is a solution family.

Of course, the universal family Ξ is not a family of solutions. However, we will show in Thm. 3.6.1 that in the case of a complete s.o.f.e., the formal solution will define a family of solutions Ξ^{sol} of quality $\mathcal{E}_{\mathrm{c}}$ which is universal for this quality, i. e. every other solution family of quality $\mathcal{E}_{\mathrm{c}}$ will be a pullback of Ξ^{sol}.

If $Z = P$ is a point then a Z-family of solutions is just an element $\phi \in \mathcal{H}^{l,V}_k(I)_{\mathbf{0}}$ which solves the field equations of the underlying bosonic s.o.f.e. in the usual sense. We call ϕ also a *trajectory*.

The *Cauchy data* of a family $\Xi' \in \mathcal{M}^{\mathcal{H}^{l,V}_k(I)}(Z)$ with $I \ni 0$ is the element $\Xi'(0) \in \mathcal{M}^{\mathcal{H}^{\mathrm{Cau},V}_k}(Z)$.

The condition $L[\Xi'] = 0$ makes also sense if Ξ' is only a power series $\Xi' \in \mathcal{P}(E;\ \mathcal{H}^{l,V}_k(I))$ where E is an arbitrary $\mathbb{Z}_2$-lcs; if it is satisfied we call Ξ' a *solution power series.*

Of course, the Taylor expansions $\Xi"_z$ $(z \in Z)$ of any solution family $\Xi" \in \mathcal{M}^{\mathcal{H}^{l,V}_k(I)}(Z)$ are solution power series. Conversely, if I is compact then the target $\mathcal{H}^{l,V}_k(I)$ is a Banach space, and hence for any element $\Xi' \in \mathcal{P}(E;\ \mathcal{H}^{l,V}_k(I))$ there exists some $p \in \mathrm{CS}(E)$ such that $\Xi' \in \mathcal{P}(E,p;\ \mathcal{H}^{l,V}_k(I))$; by [13, Prop. 3.5.2], any solution power series defines a solution family $\Xi' \in \mathcal{M}^{\mathcal{H}^{l,V}_k(I)}(U)$ on the open unit ball U within the superdomain $\mathrm{L}(\hat{E}_p)$. Hence we can switch rather freely between solution families and solution power series.

Spatial Sobolev quality implies to a certain degree temporal differentiabilty:

Proposition 2.4.1. *Every Z-family Ξ' of solutions of quality $\mathcal{H}_k^{0,V}$ is in fact of quality $\mathcal{H}_k^{l,V}$ for every $l \geq 0$ with $k > \mu l + d/2$, where μ is given by (1.6.2).*

Proof. First, we show the corresponding assertion for solution power series. We apply induction on l; for $l = 0$, there is nothing to prove. For the step from l to $l+1$, it is sufficient to show that $\Xi' \in \mathcal{P}(E;\ \mathcal{H}_k^{l,V}(I))$ implies

$$\partial_t^{l+1}\Xi' \in \mathcal{P}(E;\ \mathcal{H}_{k-\mu(l+1)}^{0,V}(I)) \tag{2.4.1}$$

for all $i = 1, \ldots, N$. From the i-th field equation we get

$$\partial_t^{l+1}\Xi_i' = -{\partial_t}^l \sum\nolimits_j K_{ij}(\partial_x)\Xi_j' - {\partial_t}^l \Delta_i[\Xi'].$$

While clearly ${\partial_t}^l K_{ij}(\partial_x)\Xi_j' \in \mathcal{P}(E;\ \mathcal{H}^0_{k+\tau_j - \mathrm{ord}\, K_{ij}(\partial_x) - \mu l}(I)) \subseteq \mathcal{P}(E;\ \mathcal{H}^0_{k+\tau_i-\mu(l+1)}(I))$, we get from Lemma 1.6.1.(i) that $\Delta_i[\Xi'] \in \mathcal{P}(E;\ \mathcal{H}^l_{k+\tau(\Delta_i)}(I))$ for all j. The assertion (2.4.1) follows.

Now, given a solution family $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{0,V}(I)}(Z)$, it follows that its Taylor expansions Ξ_z' lie in $\mathcal{M}^{\mathcal{H}_k^{l,V}(I)}(Z)$. Using [14, Prop. 2.4.4], we get the result. $\square$

Z-families of Sobolev quality are uniquely determined by their Cauchy data:

Theorem 2.4.2. *Let be given a solution family $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{0,V}(I)}(Z)$ with $k > d/2$.*

(i) If $I \ni 0$ then the integral equation

$$\Xi' = \Xi^{\mathrm{free}}[\Xi'(0)] + \mathrm{G}\Delta[\Xi'] \tag{2.4.2}$$

holds.

(ii) Suppose that $\Xi" \in \mathcal{M}^{\mathcal{H}_k^{0,V}(I)}(Z)$ is another solution family such that for some $t_0 \in I$ we have $\Xi'(t_0) = \Xi"(t_0)$. Then $\Xi' = \Xi"$.

This can be proved with the ideas of the proof of Thm. 3.3.1 below, so the proof will not be repeated here.

It follows that the Taylor expansions of any solution family arises from the formal solution by inserting the appropiate Cauchy data:

Corollary 2.4.3. *A Z-family $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{l,V}(I)}(Z)$ of configurations with $k > \mu l + d/2$ over $I \subseteq \mathbb{R}$ is a solution family iff for each $t_0 \in I$ there exists some $\epsilon > 0$ such that*

$$\Xi'(t) = \Xi^{\mathrm{sol}}[\Xi'(t_0)](t - t_0)$$

for $t \in I \cap [t_0 - \epsilon, t_0 + \epsilon]$. $\square$

Conversely, Ξ^{sol} produces local solution families:

Corollary 2.4.4. *Let be given a family of Cauchy data, i. e. an element $\Xi^{\mathrm{Cau'}} \in \mathcal{M}^{\mathcal{H}_k^{\mathrm{Cau},V}}(Z)$. For any compact subset $K \subseteq \mathrm{space}\, Z$ there exists a neighbourhood $U \supset K$ in $\mathrm{space}\, Z$, some $\theta > 0$ and and a U-family*

$$\Xi_U' \in \mathcal{M}^{\mathcal{H}_k^{0,V}([-\theta,\theta])}(U) \tag{2.4.3}$$

of solutions such that $\Xi_U'(0) = \Xi^{\mathrm{Cau'}}$.

Proof. It suffices to consider the case that $K = \{z\}$ is a point. In that case, Ξ_U' can be constructed explicitly as follows: Identify a neighbourhood of z with a superdomain $V \subseteq \mathrm{L}(E)$ such that z becomes the origin, and choose $p \in \mathrm{CS}(E)$ such that $\Xi_z^{\mathrm{Cau'}} \in \mathcal{P}(E, p;\ \mathcal{H}_k^{\mathrm{Cau},V})_{\mathbf{0},\mathbb{R}}$; let c be the norm of this element. By Thm. 2.2.1, there exists $\theta > 0$ such that $\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V}, \|\cdot\|/(c+1);\ \mathcal{H}_k^{0,V}([-\theta,\theta]))$. Then

$$\Xi_z' := \Xi^{\mathrm{sol}}[\Xi_z^{\mathrm{Cau'}}] \in \mathcal{P}(E, p;\ \mathcal{H}_k^{0,V}([-\theta,\theta]))_{\mathbf{0},\mathbb{R}},$$

and, letting U be the unit ball of p in $E_{\mathbf{0}}$ and using [13, Prop. 3.5.2], this element determines the solution family (2.4.3) wanted. □

In particular, we get that trajectories over short times always exist, i. e. that the even, bosonic field equations are short-time solvable in the usual sense:

Corollary 2.4.5. *Given bosonic Cauchy data $\phi^{\mathrm{Cau}} \in \mathcal{H}_k^{\mathrm{Cau},V}(\mathbb{R}^d)_{\mathbf{0}}$ there exists $\theta > 0$ and a unique trajectory $\phi \in \mathcal{H}_k^{0,V}([-\theta,\theta])_{\mathbf{0}}$ with $\phi(0) = \phi^{\mathrm{Cau}}$. It is given by*

$$\phi = \Xi^{\mathrm{sol}}[\phi^{\mathrm{Cau}}|0].$$

□

Remark . At this stage, we have no information on whether ϕ can be extended to an all-time trajectory. In fact, we have to suppose this later on, defining in this way the completeness of a s.o.f.e.

Obviously, if $\Xi' \in \mathcal{M}^{\mathcal{E}^V(I)}(Z)$ is a solution family then so is every temporal translate $\Xi'(\cdot + t_0) \in \mathcal{M}^{\mathcal{E}^V(I+t_0)}(Z)$ with $t_0 \in \mathbb{R}$. (A general discussion of symmetry transformations will be given in the successor paper.) This gives the possibility to "splice" solution families: From Thm. 2.4.2.(ii) and Thm. 2.2.1 we get

Corollary 2.4.6. *If $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{l,V}(I)}(Z)$ and $\Xi" \in \mathcal{M}^{\mathcal{H}_k^{l,V}(I')}(Z)$ are solution families with $0 \in I'$ such that $\Xi'(t_0) = \Xi"(0)$ for some $t_0 \in I$ then there exists a unique solution family $\Xi^{\mathrm{splice}} \in \mathcal{M}^{\mathcal{H}_k^{l,V}(I_1)}(Z)$ with $I_1 := I \cup (t_0 + I')$ such that $\Xi^{\mathrm{splice}}|_I = \Xi'$, $\Xi^{\mathrm{splice}}|_{t_0+I'} = \Xi"(\cdot + t_0)$.* □

2.5. **Lifetime intervals.** Fix a $\mathbb{Z}_2$-lcs E and $p \in \mathrm{CS}(E)$. A priori, the space $\mathcal{P}(E,p;\ \mathcal{H}_k^{l,V}(I))$ is defined only if I is closed and bounded (since only in that case, the target is a Banach space); we extend the definition to any connected subset $I \subseteq \mathbb{R}$ with non-empty kernel by

$$\mathcal{P}(E,p;\ \mathcal{H}_k^{l,V}(I)) := \varprojlim \mathcal{P}(E,p;\ \mathcal{H}_k^{l,V}([a,b])) \subseteq \mathcal{P}(E;\ \mathcal{H}_k^{l,V}(I)).$$

For shortness, we will write again $\|\cdot\|_F$ for the norms in $\mathcal{P}(E,p;\ F)$ where F is one of the Banach spaces $\mathcal{H}_k^l(I)$, $\mathcal{H}_k^{l,V}(I)$, $\mathcal{H}_k^{\mathrm{Cau},V}$ with I being closed.

The following Lemma is a standard idea in nonlinear wave equations.

Lemma 2.5.1. *Let $k > d/2$, let be given a solution power series $\Xi' \in \mathcal{P}(E,p;\ \mathcal{H}_{k+1}^{l,V}(I))_{\mathbf{0},\mathbb{R}}$ where either $I = [0,b)$ with $0 < b < \infty$, or $I = (b,0]$ with $0 > b > -\infty$, and suppose*

$$\sup_{t\in I} \|\Xi'(t)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} < \infty. \tag{2.5.1}$$

Then we have also

$$\sup_{t\in I} \|\Xi'(t)\|_{\mathcal{H}_{k+1}^{\mathrm{Cau},V}} < \infty.$$

Proof. We treat only the case $I = [0,b)$ with $0 < b < \infty$. It is sufficient to prove that there exists some $\theta \in (0,b)$ such that $\|\Xi'\|_{\mathcal{H}_{k+1}^{0,V}([\theta,t])}$ remains bounded with varying $t \in [\theta,b)$, which is equivalent with boundedness of $\|\partial_a\Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])}$ for all $a = 1,\dots,d$.

We use a time-shifted variant of (2.4.2): Fix $\theta \in (0,b)$. We have $\Xi' = \Xi^{\mathrm{free}}[\Xi'(\theta)] + \Theta$ with

$$\Theta(t) := \mathrm{G}(\Delta[\Xi'](\cdot + \theta))(t - \theta).$$

Hence

$$\|\partial_a\Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])} \leq \left\|\partial_a\Xi^{\mathrm{free}}[\Xi'(\theta)]\right\|_{\mathcal{H}_k^{0,V}([\theta,t])} + \|\partial_a\Theta\|_{\mathcal{H}_k^{0,V}([\theta,t])}.$$

Using partial integration, $\partial_a\Theta(t) := \mathrm{G}(\partial_a\Delta[\Xi'](\cdot + \theta))(t - \theta)$. Using Lemma 1.7.1.(iii) we get

$$\|\partial_a\Theta\|_{\mathcal{H}_k^{0,V}([\theta,t])} \leq K_1(b-\theta)\,\|\partial_a\Delta[\Xi']\|_{\mathcal{H}_k^{0,V}([\theta,t])} \tag{2.5.2}$$

for all $t \in [\theta, b)$ with some $K_1 > 0$ which depends only on the underlying free s.o.f.e. Now

$$\partial_a \Delta_i[\Xi'] = \sum_{i,j,\nu} \partial_a \partial^\nu \Xi'_j \cdot \frac{\partial}{\partial(\partial^\nu \Xi_j)} \Delta_i[\Xi'] \tag{2.5.3}$$

where the sum runs over those $i, j = 1, \ldots, N$ and $\nu \in {\mathbb{Z}_+}^d$ for which $\frac{\partial}{\partial(\partial^\nu \Xi_j)} \Delta_i[\Xi] \neq 0$. Using Lemma 1.6.1.(i),

$$\|\partial_a \Delta[\Xi']\|_{\mathcal{H}_k^{0,V}([\theta,t])} \leq \sum_{i,j,\nu} \left\|\partial_a \partial^\nu \Xi'_j\right\|_{\mathcal{H}^0_{k+\tau(\Delta_i)}([\theta,t])} \cdot \left\| \frac{\partial}{\partial(\partial^\nu \Xi_j)} \Delta[\Xi'] \right\|_{\mathcal{H}_k^{0,V}([\theta,t])} \tag{2.5.4}$$

with the same range of the sum as in (2.5.3). Now, for the (i, j, ν) which enter (2.5.3), we have $\tau(\Delta_i) \leq \tau_j - |\nu|$, and hence

$$\left\|\partial_a \partial^\nu \Xi'_j\right\|_{\mathcal{H}^0_{k+\tau(\Delta_i)}([\theta,t])} \leq \left\|\partial_a \Xi'_j\right\|_{\mathcal{H}^0_{k+\tau(\Delta_i)+|\nu|}([\theta,t])} \leq \left\|\partial_a \Xi'_j\right\|_{\mathcal{H}^0_{k+\tau_j}([\theta,t])} \leq \|\partial_a \Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])}\,.$$

From (2.5.1) we get that the second factor on the r. h. s. of (2.5.4) is bounded by a constant K_2, so that

$$\|\partial_a \Delta[\Xi']\|_{\mathcal{H}_k^{0,V}([\theta,t])} \leq K_2 \cdot \|\partial_a \Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])}\,. \tag{2.5.5}$$

Putting (2.5.2), (2.5.5) together, and using also Cor. 2.2.2, we get an estimate

$$\|\partial_a \Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])} \leq K_3 \|\Xi'(\theta)\|_{\mathcal{H}_{k+1}^{\mathrm{Cau},V}} + K_1 K_2 (b-\theta) \, \|\partial_a \Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])}\,.$$

Now, fixing $\theta := b - 1/(2K_1K_2)$, we get $\|\partial_a \Xi'\|_{\mathcal{H}_k^{0,V}([\theta,t])} \leq 2K_3 \|\Xi'(\theta)\|_{\mathcal{H}_{k+1}^{\mathrm{Cau},V}}$ for all $t \in [\theta, b)$ and the assertion. $\square$

Proposition 2.5.2. *Let be given a $\mathbb{Z}_2$-lcs E, a seminorm $p \in \mathrm{CS}(E)$, and an element*

$$\Xi^{\mathrm{Cau'}} \in \mathcal{P}(E, p;\ \mathcal{H}_k^{\mathrm{Cau},V})_{\mathbf{0},\mathbb{R}} \tag{2.5.6}$$

where $k > d/2$.

(i) There exists a uniquely determined pair $(I_{\max}, \Xi'_{\max})$ where $I_{\max} \subseteq \mathbb{R}$ is connected and open with $I_{\max} \ni 0$, and

$$\Xi'_{\max} \in \mathcal{P}(E, p;\ \mathcal{H}_k^{0,V}(I_{\max}))_{\mathbf{0},\mathbb{R}} \tag{2.5.7}$$

is a solution power series which has $\Xi^{\mathrm{Cau'}}$ as Cauchy data, $\Xi'_{\max}(0) = \Xi^{\mathrm{Cau'}}$, and is maximal with this property: If $(I", \Xi")$ is another pair where $I"_{\max} \subseteq \mathbb{R}$ is connected and open with $I"_{\max} \ni 0$, and $\Xi" \in \mathcal{P}(E, p;\ \mathcal{H}_k^{0,V}(I"))$ is a solution power series which has $\Xi^{\mathrm{Cau'}}$ as Cauchy data then $I" \subseteq I_{\max}$, and $\Xi'_{\max}|_{I"} = \Xi"$.

We call $\Xi'_{\max}$ the maximal solution power series *belonging to the Cauchy data* (2.5.6).

(ii) If $a := \inf_{t \in I_{\max}} t > -\infty$ then, for any $\epsilon > 0$

$$\limsup_{t \to a} \|\Xi'_{\max}(t)\|_{\mathcal{H}_{d/2+\epsilon}^{\mathrm{Cau},V}} = \infty;$$

likewise, if $b := \sup_{t \in I_{\max}} t < \infty$ then

$$\limsup_{t \to b} \|\Xi'_{\max}(t)\|_{\mathcal{H}_{d/2+\epsilon}^{\mathrm{Cau},V}} = \infty.$$

Proof. It follows from Thm. 2.2.1 and Thm. 2.4.2 that there exists at least a connected subset with non-empty kernel $I_{\max} \subseteq \mathbb{R}$ with $I_{\max} \ni 0$ and a maximal element (2.5.7); we have to prove that $I_{\max}$ is open. Let $b := \sup_{t \in I_{\max}} t$, and assume that $b \in I_{\max}$. Then $\Xi'_{\max}(b) \in \mathcal{P}(E, p;\ \mathcal{H}_k^{\mathrm{Cau},V})$ is well-defined; let c' be its norm. By Thm. 2.2.1, we have

$$\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]|_{[-\theta,\theta]} \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V}, \|\cdot\|/c';\ \mathcal{H}^{l,V}([-\theta,\theta]))$$

with some $\theta > 0$. Now, using [13, Prop. 3.3],

$$\Xi"(\cdot) := \Xi^{\mathrm{sol}}[\Xi'_{\max}(b)](\cdot - b) \in \mathcal{P}(E, p;\ \mathcal{H}^{l,V}([b-\theta, b+\theta]))$$

is a solution power series, and its Cauchy data at time b agree with that of $\Xi'_{\max}$. Hence it can be spliced (cf. Cor. 2.4.6) with $\Xi'_{\max}$ to another solution power series which shows that $\Xi'_{\max}$ was not maximal.

Likewise, one shows that $I_{\max} \not\ni \inf_{t\in I_{\max}} t$, which yields the assertion.

Ad (ii): Assume that $b < \infty$, and that there exist $\epsilon > 0$, $c' > 0$, and $b' \in I_{\max}$ such that $\|\Xi'_{\max}(t)\|_{d/2+\epsilon} < c'$ for all $t \in I_{\max} \cap [b', \infty)$. Again, we have

$$\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]|_{[-\theta,\theta]} \in \mathcal{P}(\mathcal{H}^{\mathrm{Cau},V}_{d/2+\epsilon}, \|\cdot\|/c';\ \mathcal{H}^{l,V}_{d/2+\epsilon}([-\theta,\theta]))$$

with some $\theta > 0$. Now, choosing some $t \in I_{\max}$ with $t > b'$ and $t + \theta > b$ and applying the splicing technique once again, we get, using also the previous Lemma, again a contradiction. $\square$

It is sensible to call $I_{\max}$ the *lifetime interval* of the Cauchy data $\Xi^{\mathrm{Cau}'}$. It follows from assertion (ii) that the lifetime does not depend on the choice of $k > d/2$ as long as (2.5.6) holds. However, it depends on c; we indicate this notationally by writing $\Xi'_{\max,c}$, $I_{\max,c}$. What happens if we allow c to vary?

Proposition 2.5.3. *Let be given a $\mathbb{Z}_2$-graded lcs E and an element*

$$\Xi^{\mathrm{Cau}'} \in \mathcal{P}(E;\ \mathcal{H}^{\mathrm{Cau},V}_k)_{\mathbf{0},\mathbb{R}}. \tag{2.5.8}$$

where $k > d/2$.

(i) There exists a connected open subset $I_{\max} \subseteq \mathbb{R}$ with $I_{\max} \ni 0$ and a uniquely determined solution power series $\Xi'_{\max} \in \mathcal{P}(E;\ \mathcal{H}^{0,V}_k(I_{\max}))_{\mathbf{0},\mathbb{R}}$ which is maximal with the given Cauchy data (2.5.8) *(in the analogous sense to that of Prop. 2.5.2.(i)).*

(ii) In case E is a $\mathbb{Z}_2$-graded Banach space we have $I_{\max} = \bigcup_{c>0} I_{\max,c}$ where $I_{\max,c}$ is the lifetime interval of (2.5.8) *viewed as element* (2.5.6).

(iii) For arbitrary E, $I_{\max} = \bigcap_p I_{\max,p}$, where the intersection runs over all those $p \in \mathrm{CS}(E)$ for which $\Xi^{\mathrm{Cau}'} \in \mathcal{P}(\hat{E}_p;\ \mathcal{H}^{\mathrm{Cau},V}_k)$, and $I_{\max,p}$ is the lifetime interval of that element.

(iv) If $a := \inf_{t\in I_{\max}} t > -\infty$ then, for any $\epsilon > 0$

$$\limsup_{t\to a} \|\Xi'_{\max}[0](t)\|_{\mathcal{H}^{\mathrm{Cau},V}_{d/2+\epsilon}} = \infty$$

where $\Xi'_{\max}[0] = (\Xi'_{\max})_{(0|0)}$ is the absolute term of $\Xi'_{\max}$; likewise for $b := \sup_{t\in I_{\max}} t < \infty$.

(v) $I_{\max}$ is equal to the lifetime interval of

$$\Xi^{\mathrm{Cau}'}[0] \in (\mathcal{H}^{\mathrm{Cau},V}_k)_{\mathbf{0}} = \mathcal{P}(\mathbf{0}, 0;\ \mathcal{H}^{\mathrm{Cau},V}_k)_{\mathbf{0},\mathbb{R}} \tag{2.5.9}$$

in the sense of Prop. 2.5.2. Here $\mathbf{0}$ is the Banach space consisting of zero alone.

Again, the lifetime interval does not depend on the choice of $k > d/2$ as long as (2.5.8) holds.

Proof. The proof is quite analogous to that of the preceding Proposition, using again the material of [13, 3.3]. $\square$

Remarks . (1) (2.5.9) encodes Cauchy data for the bosonic field equations in the ordinary, non-super sense. Thus, (v) says roughly that the full field equations are solvable as long as the underlying bosonic equations are solvable.

(2) Even for Banach E, there is no guaranty that the power series $\Xi'_{\max}$ is the Taylor series of a morphism $\Xi'_{\max} : U \to \mathrm{L}(\mathcal{H}^{0,V}_k(I_{\max}))$ with some neighbourhood $U \subseteq \mathrm{L}(E)$ of zero.

(3) We could try to ascend from power series to genuine families of Cauchy data and solutions. However, since our primary interest is not the Sobolev quality but the quality $\mathcal{E}_{\mathrm{c}}$, we will do so only in Thm. 3.4.3.

(4) For every $k > d/2$, the function

$$l_+ : (\mathcal{H}^{\mathrm{Cau},V}_k)_{\mathbf{0}} \to \mathbb{R}_+ \cup \infty$$

which assigns to every ϕ^{Cau} the supremum of its lifetime interval is lower semicontinous.

Indeed, fix ϕ^{Cau}, and let $s < l_+(\phi^{\mathrm{Cau}})$. By Prop. 2.5.3, there exists a solution power series $\Xi' \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V}, \delta;\ \mathcal{H}_k^{0,V}([0,s]))$ with $\Xi'[0](0) = \phi^{\mathrm{Cau}}$, $\delta > 0$. It follows that l_+ is $> s$ in the δ-neighbourhood of ϕ^{Cau}, proving our claim.

2.6. **Long-time analyticity.** Avoiding the notion of lifetime interval, the contents of Prop. 2.5.3.(v) can be rephrased by saying that a trajectory can serve as a "staircase" for showing long-time analyticity in a small neighbourhood of its Cauchy data.

Theorem 2.6.1. *Let be given a trajectory $\phi \in \mathcal{H}_k^{0,V}(I)_{\mathbf{0}}$ with $I = [-\theta', \theta]$, $0 \le \theta', \theta < \infty$, $0 < \theta' + \theta$, and let $\phi^{\mathrm{Cau}} := \phi(0)$ be its Cauchy data. Then there exists a closed interval I' with $I \subseteq (I')^o$ and a solution power series*

$$\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}} = \Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V};\ \mathcal{H}_k^{0,V}(I'))_{\mathbf{0},\mathbb{R}}$$

such that

$$\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}](0) = \Xi^{\mathrm{Cau}} + \phi(0). \tag{2.6.1}$$

Remarks . (1) Recalling that ϕ is uniquely determined by its Cauchy data, the notation is sufficiently correct. It is also introduced to get notational coherence with [13, 3.5]: in case of a complete s.o.f.e., we will construct in Thm. 3.6.1 a superfunctional Ξ^{sol} the Taylor expansions of which will be the elements $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}$.

(2) For small times t, $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}|_{[-t,t]\cap I}$ is simply the translation (cf. [13, 3.3])

$$\mathrm{t}_{\phi^{\mathrm{Cau}}}\left(\Xi^{\mathrm{sol}}|_{[-t,t]\cap I}\right) \tag{2.6.2}$$

of Ξ^{sol} by the Cauchy data of ϕ; this is for sufficiently small t well-defined due to Thm. 2.2.1. Thus, $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}$ is a prolongation of (2.6.2) to the whole time definition interval of ϕ (and, in fact, some δ beyond). Because of Thm. 2.4.2.(ii), the absolute term $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[0|0] = \phi$ of $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}$ is just the trajectory given.

Although this Theorem is a Corollary of Prop. 2.5.3, we give also a direct proof:

Proof. We prove this assuming $\theta' = 0 < \theta$; the case $0 = \theta < \theta'$ is handled quite analogously, and the general case follows using Cor. 2.4.6. Let $c := 1 + \|\phi\|_{\mathcal{H}_k^{0,V}(I)}$. By Thm. 2.2.1, we can find some integer $m > 0$ such that

$$\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}]|_{[-\theta/m,\theta/m]} \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V}, \|\cdot\|/c;\ \mathcal{H}_k^{0,V}([-\theta/m,\theta/m])).$$

Define for $i := 0, \ldots, m-1$ recursively

$$\Xi^{(i)} \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V};\ \mathcal{H}_k^{0,V}([(i-1)\theta/m, (i+1)\theta/m]))$$

as follows:

$$\Xi^{(0)}[\Xi^{\mathrm{Cau}}] := \mathrm{t}_{\phi^{\mathrm{Cau}}}\left(\Xi^{\mathrm{sol}}|_{[-\theta/m,\theta/m]}\right)[\Xi^{\mathrm{Cau}}] = \Xi^{\mathrm{sol}}[\phi^{\mathrm{Cau}} + \Xi^{\mathrm{Cau}}],$$

$$\Xi^{(i)}[\Xi^{\mathrm{Cau}}](t) := \Xi^{\mathrm{sol}}\left[\Xi^{(i-1)}[\Xi^{\mathrm{Cau}}]((i-1)\theta/m)\right](t - (i-1)\theta/m) \quad \text{for } i \ge 1.$$

However, we have to show that these insertions are legal. The crucial point is to show by induction, using Thm. 2.4.2.(ii), that

$$\Xi^{(i-1)}[0]((i-1)\theta/m) = \phi((i-1)\theta/m)$$

for $i \ge 1$; the legality now follows from [13, 3.3].

Now it follows from Thm. 2.4.2.(ii) again that the $\Xi^{(i)}$'s agree on their temporal overlaps,

$$\Xi^{(i)}|_{[i\theta/m,(i+1)\theta/m]} = \Xi^{(i+1)}|_{[i\theta/m,(i+1)\theta/m]} \quad \text{within } \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V};\ \mathcal{H}_k^{0,V}([i\theta/m,(i+1)\theta/m])).$$

By Cor. 2.4.6, we can glue these elements together to an element $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}$ with

$$\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}|_{[(i-1)\theta/m,(i+1)\theta/m]} = \Xi^{(i)}.$$

Setting $I' := [-\theta/m, \theta(1 + 1/m)]$, one checks that all requirements are satisfied. $\square$

In particular, the Theorem always applies to the trivial trajectory $\phi = 0$:

Corollary 2.6.2. *For any $\theta > 0$ and k, l with $k > \mu l + d/2$,*

$$\Xi^{\mathrm{sol}}|_{[-\theta,\theta]} \in \mathcal{P}(\mathcal{H}_k^{\mathrm{Cau},V};\ \mathcal{H}_k^{l,V}([-\theta,\theta])).$$

□

Remarks . (1) Thus, for an arbitrary long finite time interval $[-\theta,\theta]$, $\Xi^{\mathrm{sol}}|_{[-\theta,\theta]}$ is analytic for sufficiently small Cauchy data. Note, however, that there need not be a common domain for the Cauchy data on which $\Xi^{\mathrm{sol}}(t)$ is analytic for all times.

(2) Thus, even for s.o.f.e.'s which are not complete in the sense of 3.5 below, there is still a one-parameter group of time evolution which acts on the smf germ $(M^{\mathrm{Cau}}, 0)$.

(3) The result could also be proved directly by modifying the proof of Thm. 2.2.1. Also, one needs there only a non-vanishing convergence radius of Δ, not its entireness.

(4) Using Thm. 2.4.2.(ii) we get the group property of the formal solution: For $s, t > 0$ we have

$$\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}](t) = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}](s)](t-s).$$

(5) At this stage, we could already construct a solution smf $M^{\mathrm{sol}}_{H_k}$ within $\mathrm{L}(C(\mathbb{R}, \mathcal{H}_k^{0,V}(\mathbb{R}^d)))$ for each $k > d/2$. However, this would not be very useful since this configuration smf is not Poincaré invariant. It cannot be excluded that $M^{\mathrm{sol}}_{H_k}$ is nevertheless Poincaré invariant but we cannot prove this. Therefore we are using the Sobolev quality only for intermediate steps, and in the end we are interested in the qualities $\mathcal{E}$ and $\mathcal{E}_{\mathrm{c}}$, which are Poincaré invariant.

3. Causality and the supermanifold of classical solutions

3.1. **Spaces of smooth functions.** In this section, we study the consequences of finite propagation speed, as it holds in classical field theories.

For the quality of the Cauchy data on the initial hyperplane we choose the test function space $\mathcal{D}(\mathbb{R}^d)$; this gives us a maximal reservoir of superfunctions. (Of course, with this choice we might miss some interesting classical solutions; but, at any rate, we come locally arbitrary close to them, and we avoid quite a lot of technical and rhetorical difficulties.) We now need a function space on $\mathbb{R}^{d+1}$ such that $\mathcal{D}(\mathbb{R}^d)$ is the corresponding space of Cauchy data.

The most naive choice $\mathcal{D}(\mathbb{R}^{d+1})$ of compactly supported functions is not suitable since it contains no nontrivial solutions of the field equations. On the other hand, if we require from $f \in C^\infty(\mathbb{R}^{d+1})$ that it have on every time slice compact support then the resulting space is not Poincaré invariant. However, if we additionally require these supports to grow with time maximally with light velocity then everything works. (Cf. also 3.7 for a variation of this idea.)

Thus, for $r \geq 0$, let

$$\mathbb{V}_r := \{(t,x) \in \mathbb{R}^{d+1} : \quad |x| \leq r + |t|\},$$

and let temporarily $C^\infty_{\mathbb{V}_r}(\mathbb{R}^{d+1})$ be the closed subspace of $C^\infty(\mathbb{R}^{d+1})$ which consists of all those elements which have support in $\mathbb{V}_r$. Set

$$\mathcal{E}_{\mathrm{c}} = \bigcup_{r>0} C^\infty_{\mathbb{V}_r}(\mathbb{R}^{d+1})$$

and equip it with the inductive limit topology. This is a strict inductive limes of Frèchet spaces, and hence complete. Also, $\mathcal{D}(\mathbb{R}^{d+1})$ is dense in $\mathcal{E}_{\mathrm{c}}$; hence $\mathcal{E}_{\mathrm{c}}$ is admissible in the sense of [13, 3.1]. Moreover, one easily shows that the subspace $\mathcal{E}_{\mathrm{c}}$ of $C^\infty(\mathbb{R}^{d+1})$ is invariant under the standard action of the Poincaré group $\mathfrak{P}$, and that the arising action $\mathfrak{P} \times \mathcal{E}_{\mathrm{c}} \to \mathcal{E}_{\mathrm{c}}$ is continuous.

For later use, we need a technical notion: Given a seminorm $p \in \mathrm{CS}(\mathcal{D}(\mathbb{R}^{d+1}))$, we define the *support of* p, denoted by $\operatorname{supp} p$, as the complement of the set of all x which have a neighbourhood $U \ni x$ such that $\operatorname{supp}\varphi \subseteq U$ implies $p(\varphi) = 0$. Obviously, $\operatorname{supp} p$ is closed; using partitions of unity one shows that $\operatorname{supp}\varphi \subseteq \mathbb{R}^{d+1} \setminus \operatorname{supp} p$ implies $p(\varphi) = 0$.

For every $p \in \mathrm{CS}(C^\infty(\mathbb{R}^{d+1}))$, $\operatorname{supp} p$ is compact (where we have silently restricted p to $\mathcal{D}(\mathbb{R}^{d+1})$). On the other hand:

Lemma 3.1.1. *Given* $p \in \mathrm{CS}(\mathcal{E}_{\mathrm{c}})$, $\quad \operatorname{supp} p \cap \mathbb{V}_r$ *is compact for all* $r \geq 0$. ☐

We set

$$\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau}} := \mathcal{D}(\mathbb{R}^d), \quad \mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V} := \mathcal{D}(\mathbb{R}^d, V), \quad \mathcal{E}_{\mathrm{c}}^{V} := \mathcal{E}_{\mathrm{c}} \otimes V;$$

thus, the smf's of Cauchy data and of configurations,

$$M^{\mathrm{Cau}} = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V}), \quad M^{\mathrm{cfg}} = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{V})$$

are now well-defined. Analogously, we set $\mathcal{E} := C^\infty(\mathbb{R}^{d+1})$ and

$$\mathcal{E}^{\mathrm{Cau}} := C^\infty(\mathbb{R}^d), \quad \mathcal{E}^{\mathrm{Cau},V} := C^\infty(\mathbb{R}^d, V), \quad \mathcal{E}^{V} := C^\infty(\mathbb{R}^{d+1}, V).$$

3.2. **Causality.** We call the s.o.f.e. under consideration *causal* iff the function $\hat{A}$ defined in (1.5.1) satisfies the following estimate: for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\left\| \hat{A}(t, p + \mathrm{i}\, y) \right\| \leq C_\epsilon \exp((1+\epsilon)\,|yt|),. \tag{3.2.1}$$

for all $t \in \mathbb{R}$, $\quad p, y \in \mathbb{R}^d$.

For $(s,x), (t,y) \in \mathbb{R}^{d+1}$ we will write $(s,x) \leq (t,y)$ and $(t,y) \geq (s,x)$ iff (t,y) lies in the forward light cone of (s,x), i.e. $(t-s)^2 \geq (y-x)^2$.

Lemma 3.2.1. *Suppose that the s.o.f.e. is causal.*

(i) We have

$$\operatorname{supp} A \subseteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^d : \quad |x| \leq |t|\},$$

$$\begin{aligned}\operatorname{supp} G(t,s,x) = \{(t,s,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d :& \\ (s \geq 0 \text{ and } 0 \leq (t-s, y-x)) \text{ or } (s \leq 0 \text{ and } 0 \geq (t-s, y-x))\}&.\end{aligned} \tag{3.2.2}$$

(ii) The assignment $(\xi^{\mathrm{Cau}}, g) \mapsto \mathrm{A}\xi^{\mathrm{Cau}} + \mathrm{G}g$ *of Lemma 1.7.1 restricts to a continuous linear map*

$$\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V} \oplus \mathcal{E}_{\mathrm{c}}^{V} \to \mathcal{E}_{\mathrm{c}}^{V}.$$

(iii) Let $p = (s', x') \in \mathbb{R}^{d+1}$ *be a point with* $s' \neq 0$, *and set*

$$\Omega(p) = \begin{cases} \{(s,x) \in \mathbb{R}^{d+1} : \ (s,x) < (s',x'),\ 0 < s\} & \text{if } s' > 0, \\ \{(s,x) \in \mathbb{R}^{d+1} : \ (s,x) > (s',x'),\ 0 > s\} & \text{if } s' < 0, \end{cases} \tag{3.2.3}$$

$$\mathcal{J}(p) = \{x \in \mathbb{R}^d : \ |x - x'| < |s'|\}. \tag{3.2.4}$$

Also, let $I = [0, s']$, *and let be given a formal power series* $\Xi' \in \mathcal{P}_f(E;\ \mathcal{H}_k^{0,V}(I))$ *with* $k > d/2$ *and some* $\mathbb{Z}_2$*-lcs* E *which satisfies*

$$(\partial_t + K(\partial_x))\Xi'|_{\Omega(p)} = 0, \quad \Xi'(0)|_{\mathcal{J}(p)} = 0.$$

Then $\Xi'(p) = 0$.

Proof. (i) is an immediate consequence of the Paley-Wiener theorem while (ii) follows by standard techniques.

Ad (iii). By standard techniques (cf. e. g. the proof of Thm. 3.3.1 for one possibility), one proves this for ordinary functions; then one looks at the coefficient functions of Ξ'. ☐

3.3. **Solution families in the causal case.** We will call any element $\Xi' \in \mathcal{M}^{\mathcal{E}_c^V}(Z)$ a *Z-family of quality $\mathcal{E}_c$*. Because of the inclusion $\mathcal{E}_c^V \subseteq \mathcal{H}_k^{l,V}(\mathbb{R})$, such an element can be viewed as family of quality $\mathcal{H}_k^l$ with time definition domain $\mathbb{R}$ for all k, l with $k > d/2 + \mu l$.

On the other hand, we will need also *families of quality $\mathcal{E}$*, i. e. elements $\Xi' \in \mathcal{M}^{\mathcal{E}^V}(Z)$; the notions *family of solutions* and *pullback* make still obvious sense for them.

One family of quality $\mathcal{E}_c$ is given a priori, namely the M^{cfg}-family

$$\Xi = (\Phi|\Psi) \in \mathcal{M}^{\mathcal{E}_c^V}(M^{\mathrm{cfg}})$$

where, we recall, $M^{\mathrm{cfg}} = \mathrm{L}(\mathcal{E}_c^V)$ is the smf of configurations of quality $\mathcal{E}_c$, and Ξ is the standard coordinate (cf. [14, 2.5]).

Ξ is in fact the *universal family of quality $\mathcal{E}_c$*: Given an arbitrary Z-family Ξ' of quality $\mathcal{E}_c$, it defines by Lemma 1.4.1 a *classifying morphism*

$$\check{\Xi}' : Z \to M^{\mathrm{cfg}}, \quad \hat{\check{\Xi}}' = \Xi'$$

and Ξ' arises from Ξ just by pullback: $\Xi' = \check{\Xi}'^*(\Xi)$.

Remarks . (1) In the language of category theory, this means that the cofunctor

$$\{\text{supermanifolds}\} \to \{\text{sets}\}, \qquad Z \mapsto \mathcal{M}^{\mathcal{E}_c}(Z),$$

is represented by the object M^{cfg} with the universal element Ξ.

(2) Of course, universal families exist also for other time definition domains and qualities: One simply takes functional coordinates on the linear supermanifolds over the corresponding locally convex function spaces. However, we have no use for them.

For a superfunction with values in continuous functions on $\mathbb{R}^{d+1}$, i. e. $K \in \mathcal{O}^{C(\mathbb{R}^{d+1})}(Z)$, let the *target support of K* be defined as

$$\text{t-supp}\, K := \text{Closure}\Big(\{x \in \mathbb{R}^{d+1} : \ K(x) \neq 0\}\Big),$$

where, of course, $K(x) = \delta_x \circ K$. This should not be confused with the support of a power series as defined in [13, 3.11].

For a causal s.o.f.e., we have the following strengthening of Thm. 2.4.2:

Theorem 3.3.1. *Suppose that the s.o.f.e. is causal. Let be given a point $p = (s', x') \in \mathbb{R}^{d+1}$ with $s' \neq 0$, and let $\Omega(p), \mathcal{J}(p)$ be as in (3.2.3), (3.2.4). Also, let $I = [0, s']$ if $s' > 0$ and $I = [s', 0]$ if $s' < 0$, respectively.*

(i) Let be given a Z-family $\Xi' \in \mathcal{M}^{\mathcal{H}_k^{l,V}(I)}(Z)$ of configurations with $k > d/2$, and suppose that

$$L[\Xi']|_{\Omega(p)} = 0 \tag{3.3.1}$$

within $\mathcal{M}^{\mathcal{D}'(\Omega(p))\otimes V}(Z)$ (i. e., loosely said, Ξ' "is a solution on the open space-time domain $\Omega(p)$"). Then Ξ' satisfies the integral equation

$$\Xi'(t, y) = \Xi^{\mathrm{free}}[\Xi'(0)](t, y) + \mathrm{G}\Delta[\Xi'](t, y) \tag{3.3.2}$$

within $\mathcal{O}^V(Z)$ for all $(t, y) \in \Omega(p)$.

(ii) Let be given two Z-families $\Xi', \Xi" \in \mathcal{M}^{\mathcal{H}_k^{0,V}(I)}(Z)$ with $k > d/2$, and suppose that

$$L[\Xi']|_{\Omega(p)} = L[\Xi"]|_{\Omega(p)} = 0, \quad (\Xi'(0) - \Xi"(0))\,|_{\mathcal{J}(p)} = 0.$$

Then $(\Xi' - \Xi")\,|_{\Omega(p)} = 0$.

(iii) Suppose that for some $r > 0$

$$\text{t-supp}\, \Xi'(0) \subseteq r\, \mathbb{B}^d,$$

i. e. $\Xi'(0)|_{\mathbb{R}^d \setminus r\,\mathbb{B}^d} = 0$. Then

$$\text{t-supp}\, \Xi'(t) \subseteq (r + |t|)\, \mathbb{B}^d$$

for all $t \in I$.

Proof. Ad (i). Set temporarily $F := \Xi' - \Xi^{\text{free}}[\Xi'(0)] - \mathrm{G}\Delta[\Xi'] \in \mathcal{O}^{\mathcal{H}_k^{l,V}(I)}(Z)$. We find for $(t,y) \in \Omega(p)$, using (3.3.1),

$$(\partial_t + K(\partial_x))F(t,y) = -\Delta[\Xi'](t,y) + (\partial_t + K(\partial_x))\mathrm{G}\Delta[\Xi'](t,y).$$

Using (1.7.3), this vanishes. By Lemma 3.2.1.(iii), $F(t,y) = 0$.

Ad (ii). We may suppose Z to be a superdomain $Z \subseteq \mathrm{L}(E)$; pick one $z \in Z$. Now we may choose some $r \in \mathrm{CS}(E)$ such that the relevant Taylor expansions $\Xi'_z, \Xi"_z$ lie in the Banach space $\mathcal{P}(E, r;\ \mathcal{H}_k^{0,V}(I))$.

After a spatial translation, we may assume $x' = 0$, so that the closure of $\mathcal{J}(p)$ is the closed ball $s'\,\mathbb{B}^d$. Also, we may assume $s' > 0$; otherwise, the following arguments have to be "mirrored".

Suppose there exists some $t_1 \in I$ with $(\Xi'_z - \Xi")_z(t_1)|_{(s'-t_1)\,\mathbb{B}^d} \neq 0$. Now the set $\{t \in [0,t_1] : (\Xi' - \Xi")_z(t)|_{(s'-t)\,\mathbb{B}^d} = 0\}$ is easily seen to be closed; let t_2 be its maximum. By passing to the shifted families $\Xi'_z(\cdot - t_2)$, $\Xi"_z(\cdot - t_2)$, we may assume $t_2 = 0$.

From (3.3.2) and the hypotheses we get with $\Theta := \Xi" - \Xi'$ that

$$\Theta_z(t,y) = \mathrm{G}(\Delta[\Xi'_z + \Theta_z] - \Delta[\Xi'_z])(t,y) \tag{3.3.3}$$

for $(t,y) \in \Omega(p)$ where the integral over s runs effectively over $[0,t]$.

Our problem is that (3.3.3) does not hold for all (t,y). Therefore, we have to use temporarily the standard Sobolev space on the closed ball $c\,\mathbb{B}^d$: For integer $k > d/2$, let

$$H_k(c\,\mathbb{B}^d) := \Big\{ f \in L_2(\mathbb{R}^d) : \quad \operatorname{supp} f \subseteq c\,\mathbb{B}^d,\ \|f\| := \sum\nolimits_{\nu \in \mathbb{Z}_+^d,\ |\nu| \leq k} \|\partial^\nu f\|_{L_2} < \infty \Big\};$$

for non-integer $k > d/2$, define $H_k(c\,\mathbb{B}^d)$ by interpolation. It is well-known (cf. e. g. [17]) that there exists bounded linear operators $\mathrm{E}_c : H_k(c\,\mathbb{B}^d) \to H_k(\mathbb{R}^d)$ which are right inverses to restriction, i. e. $\mathrm{E}_c(f)|_{c\,\mathbb{B}^d} = f$. In fact, having choosen E_1, we may and will set $\mathrm{E}_c(f)(x) := \mathrm{E}_1(f(c\cdot))(x/c)$. Now these operators yield a bounded linear operator

$$\mathrm{E} : \mathcal{H}_k^{0,V}([0, \frac{s'}{2}]) \to \mathcal{H}_k^{0,V}([0, \frac{s'2}{2}]), \quad \mathrm{E}(f)(t,x) := \mathrm{E}_{s'-t}(f(t,\cdot))(x);$$

thus, $\mathrm{E}(f)$ depends only on the restriction $f|_{\Omega(p)}$ (of course, the role of $s'/2$ could be played by any number in $(0,s')$).

Now, using the support property (3.2.2) of the Green functions we get from (3.3.3)

$$\Theta_z|_{\Omega(p)} = \mathrm{G}(\Delta[\Xi'_z + \mathrm{E}(\Theta_z)] - \Delta[\Xi'_z])|_{\Omega(p)}. \tag{3.3.4}$$

For shortness, we will write again $\|\cdot\|_F$ for the norms in $\mathcal{P}(E, r;\ F)$ where F is one of the Banach spaces $\mathcal{H}_k^l(I)$, $\mathcal{H}_k^{\mathrm{Cau},V}$.

Using Lemma 1.7.1.(iii) we get that there exists a constant $C_1 > 0$ such that

$$\|\mathrm{G}(\Delta[\Xi'_z + \mathrm{E}(\Theta_z)] - \Delta[\Xi'_z])(t)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} \leq C_1\,|t|\,\|\Delta[\Xi'_z + \mathrm{E}(\Theta_z)] - \Delta[\Xi'_z]\|_{\mathcal{H}_k^{0,V}([0,t])}$$

for $t \in [0, s'/2]$. Because of (3.3.4), we have a fortiori,

$$\|\Theta_z(t)\|_{H_{k+1}((s'-t)\,\mathbb{B}^d)} \leq C_1\,|t|\,\|\Delta[\Xi'_z + \mathrm{E}(\Theta_z)] - \Delta[\Xi'_z]\|_{\mathcal{H}_k^{0,V}([0,t])}\,;$$

because of the continuity of E this implies

$$\|\mathrm{E}(\Theta_z(t))\|_{\mathcal{H}_k^{\mathrm{Cau},V}} \leq C_2\,|t|\,\|\Delta[\Xi'_z + \mathrm{E}(\Theta_z)] - \Delta[\Xi'_z]\| \tag{3.3.5}$$

with some $C_2 > 0$. To estimate the r. h. s. of this, we introduce temporarily new indeterminates $\underline{\Theta}_i$, with $i = 1, \ldots, N$, and $|\underline{\Theta}_i| = |\underline{\Xi}_i|$. Working in the power series algebra $\mathbb{C}[[\partial^*\underline{\Xi}, \partial^*\underline{\Theta}]]$, we can expand

$$\Delta_j[\underline{\Xi} + \underline{\Theta}] - \Delta_j[\underline{\Xi}] = \sum_{k=1}^{N} \sum_{|\nu| \leq \tau_k - \tau(\Delta_j)} B_{\nu,jk}[\underline{\Xi}]\partial^\nu \underline{\Theta}_k + R_j[\underline{\Xi}, \underline{\Theta}]$$

where both $B_{\nu,jk}[\underline{\Xi}]$, $R_j[\underline{\Xi}, \underline{\Theta}]$ are entire, and $R_j[\underline{\Xi}, \underline{\Theta}]$ is in $\underline{\Theta}$ of lower degree ≥ 2. Using also Lemma 1.4.2 we get that there exists $C_3 > 0$ with

$$\|\Delta[\Xi'_z + \mathrm{E}(\Theta_z)] - \Delta[\Xi'_z]\| \leq C_3\,\|\mathrm{E}(\Theta_z)\| \tag{3.3.6}$$

(both norms in $\mathcal{P}(E, r;\ \mathcal{H}_k^{0,V}([0,t]))$) for $t \in [0, s'/2]$. Putting (3.3.5), (3.3.6) together we get

$$\|\mathrm{E}(\Theta_z(t))\|_{\mathcal{H}_k^{\mathrm{Cau},V}} \leq C_4\,|t|\,\left\|\mathrm{E}(\Theta_z|_{[0,t]})\right\|_{\mathcal{H}_k^{0,V}([0,t])}$$

for $t \in [0, s'/2]$ with $C_4 := C_2 C_3$. Now, for (say) $t < 1/(2C_4)$, this estimate implies $\left\|\mathrm{E}(\Theta_z)|_{[0,t]}\right\| = 0$, which yields a contradiction to our assumptions.

Ad (iii). This follows because $\Xi'' := 0$ is a solution family. □

Lemma 3.3.2. *Suppose that the s.o.f.e. is causal. Let be given a trajectory $\phi \in \mathcal{H}_k^{0,V}(I)_{\mathbf{0}}$ such that $\phi(0) \in (\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V})_{\mathbf{0}}$.*

(i) We have

$$\phi \in C^\infty(I \times \mathbb{R}^d) \otimes V_{\mathbf{0}}. \tag{3.3.7}$$

Moreover, if $\operatorname{supp}\phi(0) \subseteq r\,\mathbb{B}^d$ *for some $r > 0$ then*

$$\operatorname{supp}\phi(t) \subseteq (r + |t|)\,\mathbb{B}^d \tag{3.3.8}$$

for all $t \in I$.

(ii) In particular, if $I = \mathbb{R}$ then $\phi \in (\mathcal{E}_{\mathrm{c}}^V)_{\mathbf{0}}$.

Proof. Ad (i). By Prop. 2.5.3 and Prop. 2.4.1, we have $\phi \in \mathcal{H}_{k'}^{l,V}(I)_{\mathbf{0}}$ for all k', l with $k' > \mu l + d/2$. On the other hand, by Lemma 1.6.1.(ii), we have a continuous embedding $\bigcap_{k,l:\ k > \mu l + d/2} \mathcal{H}_k^{l,V}(I) \subseteq C^\infty(I \times \mathbb{R}^d) \otimes V$, which proves (3.3.7). (3.3.8) is a special case of Thm. 2.4.2.(iii).

Ad (ii). Obvious. □

3.4. **Analyticity with targets $\mathcal{E}$ and $\mathcal{E}_{\mathrm{c}}$.** Causality will provide the deux ex machina, which allows to conclude from Sobolev continuity to continuity in the quite different topologies of $\mathcal{E}$ and $\mathcal{E}_{\mathrm{c}}$.

We begin with some technical preparations. Given a bounded open set $\Omega \Subset \mathbb{R}^{d+1}$, we denote by $\mathcal{J}(\Omega) \Subset \mathbb{R}^d$ the *causal influence domain of* Ω on the Cauchy hyperplane, i. e. the set of all $x \in \mathbb{R}^d$ such that $(0, x)$ lies in the twosided light cone of a point in Ω.

For $\Omega \Subset \mathbb{R}^{d+1}$, $l \geq 0$, define the seminorm $q_{l,\Omega} \in \mathrm{CS}(\mathcal{E}^V)$ by

$$q_{l,\Omega}(\xi) = \sum_{i=1}^N \sup_{(t,x)\in\Omega} \sum_{\nu \in \mathbb{Z}_+^{d+1},\ |\nu| \leq l} |\partial^\nu \xi_i(t,x)|;$$

thus $\operatorname{supp} q = \Omega$ (cf. 3.1).

For $J \Subset \mathbb{R}^d$, $k \geq 0$, define the seminorm $p_{k,J} \in \mathrm{CS}(\mathcal{E}^{\mathrm{Cau},V})$ by

$$p_{k,J}(\xi^{\mathrm{Cau}}) := \sum_{i=1}^N \sup_{x\in J} \sum_{\nu \in \mathbb{Z}_+^d,\ |\nu| \leq k} \left|\partial^\nu \xi_i^{\mathrm{Cau}}(x)\right|;$$

thus, $\operatorname{supp} p_{k,J} = J$.

Lemma 3.4.1. *Suppose that the s.o.f.e. is causal. Fix Cauchy data $\phi^{\mathrm{Cau}} \in (\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V})_{\mathbf{0}}$ the lifetime interval of which is the whole time axis $\mathbb{R}$, so that by Lemma 3.3.2, there exists an all-time trajectory $\phi \in (\mathcal{E}_{\mathrm{c}}^V)_{\mathbf{0}}$ with these Cauchy data.*

(i) For $\Omega \Subset \mathbb{R}^{d+1}$, $l \geq 0$, let $k > \mu l + d/2 + \max\{\tau_1, \ldots, \tau_N\}$. Then, for all $\epsilon > 0$, the power series $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ given by Thm. 2.6.1 satisfies a $(q_{l,\Omega}, C_\epsilon p_{k,J_\epsilon})$-estimate (cf. [13, 3.1]*) with some $C_\epsilon > 0$, where $J_\epsilon = U_\epsilon(\mathcal{J}(\Omega))$ is the ϵ-neighbourhood of $\mathcal{J}(\Omega)$.*

(ii) Let $q \in \mathrm{CS}(\mathcal{E}^V)$ be arbitrary. Then there exists $k > 0$ such that for all $\epsilon > 0$, $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ satisfies the $(q, C_\epsilon p_{k,J_\epsilon})$-estimate with some $C_\epsilon > 0$, where $J_\epsilon = U_\epsilon(\mathcal{J}(\operatorname{supp} q))$.

Proof. Ad (i). Let $I \subseteq \mathbb{R}$ be the projection of Ω onto the time axis. By Lemma 1.6.1.(ii), there exists a constant C_1 such that

$$q_{l,\Omega}(\varphi) \leq C_1 \cdot \|\varphi|_I\|_{\mathcal{H}_k^l(I)}$$

for $\phi \in \mathcal{H}^l_k(I)$. Combining this with the Sobolev analyticity of $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ given by Thm. 2.6.1, there exists a constant C_2 such that we have for $r, s \geq 0$, $\varphi^1, \dots, \varphi^r \in (\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})_{\mathbf{0}}$, $\psi^1, \dots, \psi^s \in (\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})_{\mathbf{1}}$

$$q_{l,\Omega}\Big(\Big\langle\Big(\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}\Big)_{r|s}, \bigotimes_{m=1}^{r} \varphi^m \otimes \bigotimes_{n=1}^{s} \Pi\psi^n\Big\rangle\Big) \leq C_2 \cdot \prod_{m=1}^{r} \|\varphi^m\|_{\mathcal{H}^{\mathrm{Cau},V}_k} \cdot \prod_{n=1}^{s} \|\psi^n\|_{\mathcal{H}^{\mathrm{Cau},V}_k}$$

(cf. [13, 3.1] for the notation on the l. h. s.). Now choose some buffer function $h \in \mathcal{D}'(\mathbb{R}^d)$ with $\operatorname{supp} h \subseteq J_\epsilon$ and $h|_{\mathcal{J}(\Omega)} = 1$. By causality (cf. Thm. 3.3.1.(ii)), we have $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]|_\Omega = \Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[h\Xi^{\mathrm{Cau}}]|_\Omega$, and hence

$$\begin{aligned} q_{l,\Omega}\Big(\Big\langle\Big(\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}\Big)_{r|s}, \bigotimes_{m=1}^{r} \varphi^m \otimes \bigotimes_{n=1}^{s} \Pi\psi^n\Big\rangle\Big) &= q_{l,\Omega}\Big(\Big\langle\Big(\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}\Big)_{r|s}, \bigotimes_{m=1}^{r} (h\varphi^m) \otimes \bigotimes_{n=1}^{s} \Pi(h\psi^n)\Big\rangle\Big) \\ &\leq C_2 \cdot \prod_{m=1}^{r} \|h\varphi^m\|_{\mathcal{H}^{\mathrm{Cau},V}_k} \cdot \prod_{n=1}^{s} \|h\psi^n\|_{\mathcal{H}^{\mathrm{Cau},V}_k} \end{aligned}$$

But obviously $\|h\cdot\|_{\mathcal{H}^{\mathrm{Cau},V}_k}$ is estimated from above by $C_\epsilon p_{k,J_\epsilon}(\cdot)$ with some $C_\epsilon > 0$, and the assertion follows.

Ad (ii). Since the collection of all $q_{l,\Omega}$ defines the topology of $\mathcal{E}^V$, there exist l, C',and $\Omega' \Subset \mathbb{R}^{d+1}$ such that $q \leq C' q_{l,\Omega'}$. However, Ω' may be larger than $\operatorname{supp} q$. Choose a buffer function $g \in \mathcal{D}(\mathbb{R}^{d+1})$, $g \geq 0$, with $g|_{\operatorname{supp} q} = 1$, $\operatorname{supp} g \subseteq J_{\epsilon/2}$. Then

$$q(\cdot) = q(g\cdot) \leq C' q_{l,\Omega'}(g\cdot) \leq C'_\epsilon q_{l,J_{\epsilon/2}}(\cdot)$$

with some $C'_\epsilon > 0$. The assertion now follows from (i). □

Proposition 3.4.2. *Suppose that the s.o.f.e. is causal. Let be given bosonic Cauchy data* $\phi^{\mathrm{Cau}} \in (\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})_{\mathbf{0}}$ *the lifetime interval of which is the whole time axis* $\mathbb{R}$. *Then* $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ *is an analytic power series from* $\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}}$ *to* $\mathcal{E}^V_{\mathrm{c}}$*:*

$$\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}};\ \mathcal{E}^V_{\mathrm{c}})_{\mathbf{0},\mathbb{R}}.$$

Proof. Let be given a seminorm $q \in \mathrm{CS}(\mathcal{E}^V_{\mathrm{c}})$. With standard methods one constructs for $i > 0$ buffer functions $f_i \in C^\infty(\mathbb{R}^{d+1})$ with $f_i|_{\mathbb{V}_{i-1}} = 0$, $f_i|_{\mathbb{R}^{d+1}\setminus\mathbb{V}_i} = 1$ where the $\mathbb{V}_i$ are as in 3.1. Set for convenience $f_0 := 1$. For the seminorms $q_i := q((f_i - f_{i+1})\cdot) \in \mathrm{CS}(\mathcal{E}^V_{\mathrm{c}})$ we get

$$q(\varphi) \leq \sum_{i \geq 0} q_i(\varphi) \tag{3.4.1}$$

for all $\phi \in \mathcal{E}^V_{\mathrm{c}}$, where in fact only finitely many terms on the r. h. s. are non-zero. Now

$$\operatorname{supp} q_i \subseteq \mathbb{V}_{i+1} \cap \operatorname{supp} q$$

which is by Lemma 3.1.1 compact. Also, for $i \geq 1$, we have $(f_i - f_{i+1})|_{\mathbb{V}_{i-1}} = 0$ and hence

$$\operatorname{supp} q_i \cap \mathbb{V}_{i-1} = \emptyset. \tag{3.4.2}$$

Because of (3.4.2), we have $\mathcal{J}(\operatorname{supp} q_i) \subseteq \{x \in \mathbb{R}^d :\ \ \|x\| \geq i-1\}$ for $i \geq 1$; hence, setting $J_i := \{x \in \mathbb{R}^d :\ \ \|x\| \geq i-2\}$, Lemma 3.4.1.(ii) yields for each i numbers $C_i > 0$, $k_i \geq 0$ such that $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ satisfies a $(q_i, C_i p_{k_i,J_i})$-estimate.

It follows that for each $\varphi \in \mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}}$, the sum

$$p(\varphi) := \sum_i C_i p_{k_i,J_i}(\varphi)$$

has only finitely many nonvanishing terms; using [7, Thm. 15.4.1], we have $p := p(\cdot) \in \mathrm{CS}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})$. It follows directly from the definition of the (q,p)-estimates (cf. [13, 3.1]) and (3.4.1) that the $(q_i, C_i p_{k_i,J_i})$-estimates for $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ imply the (q,p)-estimate wanted. □

Theorem 3.4.3. *Suppose that the s.o.f.e. is causal. Let be given an smf Z and a superfunction $\Xi^{\text{Cau'}} \in \mathcal{M}^{\mathcal{E}_c^{\text{Cau},V}}(Z)$ (this encodes an smf morphism $(\Xi^{\text{Cau'}})^\vee : Z \to M^{\text{Cau}}$, i. e. a family of Cauchy data).*

Suppose that for each $z \in Z$, there exists a smooth all-time trajectory $\phi_z \in (\mathcal{E}_c^V)_{\mathbf{0}}$ with $\phi_z(0) = \Xi^{\text{Cau'}}(z)$. Then there exists a unique Z-family of solutions $\Xi' \in \mathcal{M}^{\mathcal{E}_c^V}(Z)$ which has $\Xi^{\text{Cau'}}$ as its Cauchy data, i. e. $\Xi'(0) = \Xi^{\text{Cau'}}$. The Taylor expansion of Ξ' at z is given by

$$\Xi'_z = \Xi^{\text{sol}}_{\phi_z(0)}[\Xi^{\text{Cau'}}_z - \phi_z(0)] \tag{3.4.3}$$

where $\Xi^{\text{sol}}_{\phi_z(0)}$ is given by Thm. 2.6.1. Note that the insertion is defined since the power series inserted has no absolute term.

Also, the underlying map of the arising smf morphism $\breve{\Xi}' : Z \to M^{\text{cfg}}$ is $z \mapsto \phi_z$.

Proof. We may assume that $Z \subseteq \mathrm{L}(E)$ is a superdomain. Using Prop. 3.4.2, we get a map

$$\text{space}\, Z \ni z \mapsto \Xi'_z \in \mathcal{P}(E;\ \mathcal{E}_c^V)$$

where Ξ'_z is defined by (3.4.3). We have to show that this is an element of $\mathcal{M}^{\mathcal{E}_c^V}(Z)$. This task is simplified by remarking that the set of all functionals

$$\delta^i_{(t,x)} : \mathcal{E}_c^V \to \mathbb{R}, \quad \xi = (\xi_i) \mapsto \xi_i(t,x), \tag{3.4.4}$$

with $i = 1, \dots, N$ and $(t,x) \in \mathbb{R}^{d+1}$ is strictly separating in the sense of [14, Prop. 2.4.4]; it follows that it is sufficient to check that for each $\delta^i_{(t,x)}$, the assignment

$$\text{space}\, Z \ni z \mapsto \delta^i_{(t,x)} \circ \Xi'_z \in \mathcal{P}(E;\ \mathbb{R})$$

is an element of $\mathcal{O}(Z)$ (note that $\delta^i_{(t,x)}$ is even for $i \le N_{\mathbf{0}}$ and odd otherwise). Thus it is sufficient to prove: Fix $i, (t,x)$ and $z \in Z$. There exists $p \in \mathrm{CS}(E)$ such that $\delta^i_{(t,x)}\Xi'_z \in \mathcal{P}(E,p;\ \mathbb{R})$, and for $z' \in Z,\ p(z'-z) < 1$, we have

$$\mathrm{t}_{z'-z}\, \delta^i_{(t,x)}\Xi'_z = \delta^i_{(t,x)}\Xi'_{z'}. \tag{3.4.5}$$

Indeed, set (say) $k := d/2 + 1$, and $H := \mathcal{H}^{0,V}_k([-|t|-1, |t|+1])$. Choose $p \in \mathrm{CS}(E)$ such that $\Xi^{\text{Cau'}}_z - \phi_z(0) \in \mathcal{P}(E,p;\ \mathcal{H}^{\text{Cau},V}_k)$; since there is no absolute term we may assume by dilating p that $C\left\|\Xi^{\text{Cau'}}_z - \phi_z(0)\right\| < 1$. The composite (3.4.3) is now defined in the sense of [13, Prop. 3.3], and we get $\Xi'_z \in \mathcal{P}(E,p;\ H)$.

Choose by Thm. 2.6.1 some $C > 0$ such that

$$\Xi^{\text{sol}}_{\phi_z(0)}[\Xi^{\text{Cau}}] \in \mathcal{P}(\mathcal{H}^{\text{Cau},V}_k, C\left\|\cdot\right\|;\ H).$$

For $z' \in Z,\ p(z'-z) < 1$, we have

$$\Xi^{\text{Cau'}}_{z'} = \mathrm{t}_{z'-z}\, \Xi^{\text{Cau'}}_z \in \mathcal{P}(E, cp;\ H), \quad c := 1 - p(z'-z),$$

and hence

$$\mathrm{t}_{z'-z}\, \Xi'_z = \Xi^{\text{sol}}_{\phi_z(0)}[\mathrm{t}_{z'-z}(\Xi^{\text{Cau'}}_z - \phi_z(0))] = \Xi^{\text{sol}}_{\phi_z(0)}[\Xi^{\text{Cau'}}_{z'} - \phi_z(0)] \in \mathcal{P}(E, cp;\ H);$$

since translation is an algebra homomorphism, this element is a solution power series. Using (2.6.1), we find its Cauchy data as

$$\mathrm{t}_{z'-z}\, \Xi'_z(0) = \Xi^{\text{Cau'}}_{z'}.$$

On the other hand, we have also the element $\Xi'_{z'} \in \mathcal{P}(E, c'p;\ H)$ with some $c' > 0$ which yields a solution family $\Xi'_{z'} \in \mathcal{M}^H(V')$ where V' is the c'-fold multiple of the unit ball of p in $\mathrm{L}(E)$. Since it has the same Cauchy data as $\mathrm{t}_{z'-z}\, \Xi'_z$, we get from Thm. 2.4.2.(ii) that $\mathrm{t}_{z'-z}\, \Xi'_z = \Xi'_{z'}$ within $\mathcal{P}(E;\ H)$. A fortiori, we have (3.4.5). $\square$

We need the power series $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}]$ also for the case that $\phi^{\mathrm{Cau}} \in \mathcal{E}^{\mathrm{Cau},V}$ has no longer compact support, so that the proofs of Thm. 2.6.1 and Prop. 2.5.3 break down.

We call a bosonic Cauchy datum $\phi^{\mathrm{Cau}} \in (\mathcal{E}^{\mathrm{Cau},V})_{\mathbf{0}}$ *approximable* if there exists a sequence of compactly supported bosonic Cauchy data $\phi^{\mathrm{Cau}}_{(n)} \in (\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})_{\mathbf{0}}$, $n \in \mathbb{Z}_+$, such that $\phi^{\mathrm{Cau}}_{(n)}|_{n\,\mathbb{B}^d} = \phi^{\mathrm{Cau}}|_{n\,\mathbb{B}^d}$ for all i, and each $\phi^{\mathrm{Cau}}_{(n)}$ has the whole time axis as lifetime interval, i. e., there exists an all-time solution $\phi_{(n)} \in (\mathcal{E}^V_{\mathrm{c}})_{\mathbf{0}}$ of the bosonic field equations with these Cauchy data.

Lemma 3.4.4. *Suppose that the s.o.f.e. is causal. Given an approximable bosonic Cauchy datum $\phi^{\mathrm{Cau}} \in (\mathcal{E}^{\mathrm{Cau},V})_{\mathbf{0}}$, there exists a solution power series $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V};\ \mathcal{E}^V)_{\mathbf{0},\mathbb{R}}$ such that*

$$\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}](0) = \Xi^{\mathrm{Cau}} + \phi(0). \tag{3.4.6}$$

Proof. Composing the power series $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}_{(n)}} \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}};\ \mathcal{E}^V_{\mathrm{c}})$ given by (ii) and Prop. 3.4.2 with the projection $\mathcal{E}^V_{\mathrm{c}} \to C^\infty(n\,\mathbb{B}^{d+1}) \otimes V$ we get a sequence of power series

$$\Xi_{(n)} := \Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}_{(n)}}|_{n\,\mathbb{B}^{d+1}} \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}};\ C^\infty(n\,\mathbb{B}^{d+1}) \otimes V)_{\mathbf{0},\mathbb{R}}.$$

Because of Thm. 3.3.1.(ii) and (i), the restrictions of $\Xi_{(n+1)}$ and $\Xi_{(n)}$ onto $n\,\mathbb{B}^{d+1}$ coincide. Hence there exists a power series $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}};\ \mathcal{E}^V)$ whose restriction onto $n\,\mathbb{B}^{d+1}$ is $\Xi_{(n)}$. It is clear that this is a solution power series which satisfies (3.4.6); the fact that it is actually analytic with respect to the source space $\mathcal{E}^{\mathrm{Cau},V}$ follows from Lemma 3.4.1.(i) and the construction. □

We also have the analogon of Thm. 3.4.3:

Theorem 3.4.5. *Suppose that the s.o.f.e. is causal. Let be given an smf Z and a superfunction $\Xi^{\mathrm{Cau}'} \in \mathcal{M}^{\mathcal{E}^{\mathrm{Cau},V}}(Z)$. Suppose that for each $z \in Z$, the Cauchy datum $\phi^{\mathrm{Cau}'}(z) \in (\mathcal{E}^{\mathrm{Cau},V})_{\mathbf{0}}$ is approximable.*

Then there exists a unique Z-family of solutions $\Xi' \in \mathcal{M}^{\mathcal{E}^V}(Z)$ which has $\Xi^{\mathrm{Cau}'}$ as its Cauchy datum. Again, the Taylor expansion of Ξ' at z is given by (3.4.3), where this time, $\Xi^{\mathrm{sol}}_{\phi_z(0)}$ is given by Lemma 3.4.4.

Proof. Quite analogous to that of Thm. 3.4.3; instead of the functionals $\delta^i_{(t,x)}$, one could use also the seminorms $p_{l,\Omega}$. □

Finally, we will need a variant of Prop. 3.4.2 which describes compactly supported local excitations around a classical solution:

Proposition 3.4.6. *Suppose that the s.o.f.e. is causal. Let be given an approximable bosonic Cauchy datum $\phi^{\mathrm{Cau}} \in (\mathcal{E}^{\mathrm{Cau},V})_{\mathbf{0}}$, and let $\phi := \Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[(0,0),0] \in (\mathcal{E}^V)_{\mathbf{0}}$ be the corresponding solution. Then the power series*

$$\Xi^{\mathrm{exc}}_{\phi}[\Xi^{\mathrm{Cau}}] := \Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}[\Xi^{\mathrm{Cau}}] - \phi$$

satisfies $\Xi^{\mathrm{exc}}_{\phi} \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}};\ \mathcal{E}^V_{\mathrm{c}})_{\mathbf{0},\mathbb{R}}$.

Proof. First one shows the analogon of Lemma 3.4.1 for $\Xi^{\mathrm{exc}}_{\phi}$; in the proof, one replaces the Sobolev analyticity of $\Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}}$ by the fact that $\Xi^{\mathrm{exc}}_{\phi} \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V};\ \mathcal{E}^V)$, and the necessary causality property is again provided by Thm. 3.3.1.(ii). Having this, the proof of Prop. 3.4.2 carries over. □

3.5. **Completeness.** Loosely said, we call a s.o.f.e. complete iff the underlying bosonic s.o.f.e. is globally solvable:

Theorem 3.5.1. *For a causal s.o.f.e., the following conditions are equivalent:*

(i) For every smooth solution $\phi \in C^\infty((a,b) \times \mathbb{R}^d) \otimes V_{\mathbf{0}}$ of the underlying bosonic field equations on a bounded open time interval (a,b) such that $\operatorname{supp}\phi(t)$ is compact for all $t \in (a,b)$ there exists a Sobolev index $k > d/2$ such that

$$\sup_{t\in(a,b)} \|\phi(t)\|_{\mathcal{H}^{\mathrm{Cau},V}_k} < \infty. \tag{3.5.1}$$

(ii) The underlying bosonic equations are all-time solvable with quality $\mathcal{E}_{\mathrm{c}}$:

Given bosonic Cauchy data $\phi^{\mathrm{Cau}} \in (\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V})_{\mathbf{0}}$ there exists an element $\phi \in (\mathcal{E}_{\mathrm{c}}^{V})_{\mathbf{0}}$ with these Cauchy data which solves the field equations.

(iii) The underlying bosonic equations are all-time solvable with quality $\mathcal{E}$:

Given bosonic Cauchy data $\phi^{\mathrm{Cau}} \in (\mathcal{E}^{\mathrm{Cau},V})_{\mathbf{0}}$ there exists an element $\phi \in (\mathcal{E}^{V})_{\mathbf{0}}$ with these Cauchy data which solves the field equations.

If these conditions are satisfied we call the s.o.f.e. complete.

Remarks . (1) Of course, in case of completeness, (3.5.1) holds for all $k > d/2$ for every ϕ as in (i).

(2) The solutions provided in (iii), (iv) are necessarily uniquely determined. We need no information about their continuous dependence on the initial data since our theory yields automatically real-analytic dependence.

(3) It would be nice to add the following conditions to the list:

(iv) The underlying bosonic equations are all-time solvable with some Sobolev quality $k > d/2$:

Given bosonic Cauchy data $\phi^{\mathrm{Cau}} \in (\mathcal{H}_k^{\mathrm{Cau},V})_{\mathbf{0}}$ there exists an element $\phi \in \mathcal{H}_k^{0,V}(\mathbb{R})_{\mathbf{0}}$ with these Cauchy data which solves the field equations.

(v) The solvability assertion of (iv) holds for all Sobolev orders $k > d/2$.

However, at the time being, we cannot exclude the possibility that even for a complete s.o.f.e., there exist bosonic Cauchy data $\phi^{\mathrm{Cau}} \in (\mathcal{H}_k^{\mathrm{Cau},V})_{\mathbf{0}} \setminus (\mathcal{E}^{\mathrm{Cau},V})_{\mathbf{0}}$ with finite lifetime t_0; for any sequence $\phi_{(i)}^{\mathrm{Cau}} \in (\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V})_{\mathbf{0}}$ converging to ϕ^{Cau} within $\mathcal{H}_k^{\mathrm{Cau},V}$, the corresponding sequence of solutions $\phi_{(i)} \in \mathcal{E}_{\mathrm{c}}^{V}$ satisfies

$$\lim_{i\to\infty} \left\|\phi_{(i)}(t_0)\right\|_{\mathcal{H}_k^{\mathrm{Cau},V}} = \infty.$$

(4) The notion "completeness" has been chosen by analogy with the usual completeness of flows (i. e. local one-parameter groups of automorphisms) on manifolds. Indeed, any s.o.f.e. determines a time evolution flow on the smf M^{Cau}, and it is complete iff this flow is complete.

However, if making that rigorous, we have to circumvent the difficulty that we are using a real-analytic calculus of superfunctions while our flow is only differentiable in time direction. Therefore, our flow is defined as an smf morphism $F : U \to M^{\mathrm{Cau}}$ where $\mathrm{space}(U) \subseteq \mathrm{space}(M^{\mathrm{Cau}}) \times \mathbb{R}$ is open, but the topology and the smf structure of U are induced from the open embedding $U \subseteq M^{\mathrm{Cau}} \times \mathbb{R}_d$ where $\mathbb{R}_d$ is the real axis equipped with the discrete topology and viewed as a zero-dimensional smf. Explicitly, one chooses a map $\theta(\cdot) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for any $c > 0$, we have (2.2.1) with $\theta := \theta(c)$, and one sets $\mathrm{space}(U) := \bigcup_{c>0} cV \times (-\theta(c), \theta(c))$ where V is the open unit ball in $\mathrm{space}(M^{\mathrm{Cau}})$. Now F is given by $\hat{F}|_{M^{\mathrm{Cau}} \times \{t\}} = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}](t)$.

Analogously, every s.o.f.e. determines flows on the smf's $\mathrm{L}(\mathcal{H}_k^{\mathrm{Cau},V})$ of Cauchy data of Sobolev quality for all $k > d/2$; however, as we saw in the previous remark, their completeness seems to be a stronger condition than completeness of the s.o.f.e.

Proof of the Theorem. (ii)⇒(i) follows from Thm. 2.4.2.(ii).

(i)⇒(ii) follows from Prop. 2.5.3 and Lemma 3.3.2.

(ii)⇒(iii): For each $(t, x) \in \mathbb{R}^{d+1}$ choose a buffer function $g \in \mathcal{D}(\mathbb{R}^d)$ which is equal to one in some neighbourhood of $\mathcal{J}(\{(t, x)\})$, and let $\phi(t, x) := \Xi^{\mathrm{sol}}[g\phi^{\mathrm{Cau}}](t, x) \in V$. It follows from Thm. 3.3.1 again that this does not depend on the choice of g; hence $\phi : \mathbb{R}^{d+1} \to V$ is well-defined. It also follows directly from the construction that $\phi \in (\mathcal{E}^V)_{\mathbf{0}}$ is the trajectory wanted.

(iii)⇒(i): Given ϕ as in (i), it extends by (iii) to a solution $\phi \in {\mathcal{E}^V}_{\mathbf{0}}$. Let (say) $c = (a + b)/2$, and choose R with $\mathrm{supp}\,\phi(c) \subseteq \{x \in \mathbb{R}^d : \ |x| \leq R\}$. By Lemma 3.3.2, we get $\mathrm{supp}\,\phi(t) \subseteq \{x \in \mathbb{R}^d : |x| \leq R + |t - c|\}$; thus, $\phi \in \mathcal{E}_{\mathrm{c}}^V$, and the assertion becomes obvious. □

We conclude with a fairly simple sufficient criterion for completeness. First note that the definition (1.6.1) of $\mathcal{H}_k^{\mathrm{Cau},V}$ makes sense for all $k \geq 0$; however, the assertions of Lemma 1.6.1 are valid only for $k > d/2$.

Proposition 3.5.2. *Suppose that a causal s.o.f.e. satisfies the following additional conditions:*

(i) We have

$$\sup_{t\in(a,b)} \|\phi(t)\|_{\mathcal{H}_0^{\mathrm{Cau},V}} < \infty$$

for every trajectory $\phi \in C^\infty((a,b)\times\mathbb{R}^d)\otimes V_{\mathbf{0}}$ *for which* $\operatorname{supp}\phi(t)$ *is compact for all* $t \in (a,b)$.

(ii) Let $k_0 \in \mathbb{Z}$ *be minimal such that* $k_0+1 > d/2$. *There exists a monotonously increasing function* $F : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|\partial_a \Delta[\phi^{\mathrm{Cau}}|0]\|_{\mathcal{H}_k^{\mathrm{Cau},V}} \le \|\phi^{\mathrm{Cau}}\|_{\mathcal{H}_{k+1}^{\mathrm{Cau},V}} F(\|\phi^{\mathrm{Cau}}\|_{\mathcal{H}_k^{\mathrm{Cau},V}})$$

for all $\phi^{\mathrm{Cau}} \in (\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V})_{\mathbf{0}}$, $a = 1,\dots,d$, $k = 0,\dots,k_0$.

Then the s.o.f.e. is complete.

Proof. Let ϕ be a trajectory as in (i). We will prove inductively that (3.5.1) holds for for all $k = 0,\dots,k_0$, the start being given by (i). For the step, we mimick the proof of Lemma 2.5.1: From the time-shifted integral equation, we have for $0 \le \theta < t$

$$\|\partial_a\phi\|_{\mathcal{H}_k^{0,V}([\theta,t])} \le C_1\|\partial_a\phi'(\theta)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} + C_2(b-\theta)\,\|\partial_a\Delta[\phi|0]\|_{\mathcal{H}_k^{0,V}([\theta,t])}\,;$$

investing the hypothesis (ii), the r. h. s. becomes

$$\le C_1\|\partial_a\phi'(\theta)\|_{\mathcal{H}_k^{\mathrm{Cau},V}} + C_2(b-\theta)(\|\phi\|_{\mathcal{H}_k^{0,V}([\theta,t])} + \|\partial_a\phi\|_{\mathcal{H}_k^{0,V}([\theta,t])})F(\|\phi\|_{\mathcal{H}_k^{0,V}([\theta,t])})$$

which again implies the assertion. □

3.6. **The main Theorem.**

Theorem 3.6.1. *Suppose that the s.o.f.e. is causal and complete.*

(i) The formal solution Ξ^{sol} *is the Taylor expansion at zero of a unique superfunctional*

$$\Xi^{\mathrm{sol}} \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}^V}(M^{\mathrm{Cau}}), \tag{3.6.1}$$

and (3.6.1) *is an* M^{Cau}*-family of solutions of quality* $\mathcal{E}_{\mathrm{c}}$.

(ii) Consider the arising smf morphism (cf. Lemma 1.4.1)

$$\check{\Xi}^{\mathrm{sol}} : M^{\mathrm{Cau}} \to M^{\mathrm{cfg}}.$$

Its underlying map assigns to each bosonic Cauchy datum ϕ^{Cau} *the unique trajectory* ϕ *with* $\phi(0) = \phi^{\mathrm{Cau}}$.

(iii) The smf morphism $\alpha : M^{\mathrm{cfg}} \to M^{\mathrm{cfg}}$ *given by*

$$\hat{\alpha}[\Xi] := \Xi + \Xi^{\mathrm{sol}}[\Xi(0)] - \Xi^{\mathrm{free}}[\Xi(0)] \tag{3.6.2}$$

is an automorphism of M^{cfg} *which makes the diagram*

$$\begin{tikzcd} & M^{\mathrm{Cau}} \arrow[dl, "\check{\Xi}^{\mathrm{free}}"'] \arrow[dr, "\check{\Xi}^{\mathrm{sol}}"] & \\ M^{\mathrm{cfg}} \arrow[rr, "\alpha"'] & & M^{\mathrm{cfg}} \end{tikzcd}$$

commutative.

(iv) The image of the $\check{\Xi}^{\mathrm{sol}}$ *is a split sub-smf which we call the* smf of classical solutions, *or, more exactly, the* smf of smooth classical solutions with spatially compact support, *and denote by* $M^{\mathrm{sol}} \subseteq M^{\mathrm{cfg}}$.

(v) M^{sol} *has the following universal property: Recall that fixing an smf* Z *we have a bijection between* Z*-families* Ξ' *of configurations of quality* $\mathcal{E}_{\mathrm{c}}$ *with time definition interval* $\mathbb{R}$, *and morphisms* $\check{\Xi}' : Z \to M^{\mathrm{cfg}}$. *Now* Ξ' *is a solution family iff* $\check{\Xi}'$ *factors to* $\check{\Xi}' : Z \to M^{\mathrm{sol}} \subseteq M^{\mathrm{cfg}}$.

In this way, we get a bijection between Z*-families* Ξ' *of solutions of quality* $\mathcal{E}_{\mathrm{c}}$ *with time definition interval* $\mathbb{R}$, *and morphisms* $\check{\Xi}' : Z \to M^{\mathrm{sol}}$.

(vi) The underlying manifold $\widetilde{M^{\rm sol}}$ identifies with the set of all bosonic configurations which satisfy the field equations with all fermion fields put to zero:

$$\widetilde{M^{\rm sol}} = \{\phi \in (\mathcal{E}^V_{\rm c})_{\bar 0} : \quad L_i[\phi|0] = 0, \quad i = 1, \ldots, N\}.$$

Remarks . (1) In the language of category theory, assertion (v) means that the cofunctor

$$\{\text{supermanifolds}\} \to \{\text{sets}\}, \qquad Z \mapsto \{Z\text{-families of solutions}\}$$

is represented by the object $M^{\rm sol}$ with the universal element $\Xi^{\rm sol}$. Thus, (3.6.1) is the *universal family of solutions of quality* $\mathcal{E}_{\rm c}$: Every other family of solutions of this quality arises uniquely as pullback from (3.6.1).

(2) Note that $M^{\rm sol}$ is still a linear smf which is, however, in a non-linear way embedded into $M^{\rm cfg}$. In the successor paper we will show that, once the action of the Lorentz group on V has been fixed, the sub-smf $M^{\rm sol}$ is invariant under the arising action of the Poincaré group on $M^{\rm cfg}$; the other data α, $\check\Xi^{\rm free}$, $\check\Xi^{\rm sol}$ are not (they are only invariant under the Euclidian group of $\mathbb{R}^d$).

(3) The superfunction (3.6.1) is uniquely characterized by the conditions (2.1.1). The initial value condition can be recoded supergeometrically to the fact that the composite morphism

$$M^{\rm Cau} \xrightarrow{\check\Xi^{\rm sol}} M^{\rm cfg} \xrightarrow{\pi} M^{\rm Cau}$$

is the identity; here π is the projection onto the Cauchy data:

$$\hat\pi[\Xi] = \Xi(0). \tag{3.6.3}$$

(4) The Taylor expansion of (3.6.2) at zero is just

$$\hat\alpha[\Xi]_{(0,0)} = \Xi + \Xi^{\rm sol}_{(\geq 3)}[\Xi(0)].$$

(5) Although a Inverse Function Theorem is valid for morphisms of smf's with Banach model space (it will be given in the next part of [15]), we cannot apply it to α since the model space $\mathcal{E}_{\rm c}$ is far away from being Banach. Fortunately, a purely soft method well-known in category theory will turn out to be sufficient to conclude that α is even globally isomorphic.

(6) In qft slang, the homomorphism

$$\mathcal{O}^F(M^{\rm cfg}) \to \mathcal{O}^F(M^{\rm sol}), \qquad K[\Xi] \mapsto K[\Xi^{\rm sol}] \tag{3.6.4}$$

is called *restriction of classical observables onto the mass shell* (the latter term comes from free field theory). It follows from ass. (iii) and the proof given below that (3.6.4) is surjective.

(7) Let us comment on the fact that completeness depends only on the underlying bosonic s.o.f.e.: Mathematically, this is an analogon of several theorems in supergeometry that differential-geometric tasks, like trivializing a fibre bundle, or presenting a closed form as differential, are solvable iff the underlying smooth tasks are solvable.

Physically, our interpretation is somewhat speculative: In the bosonic sector, the classical field theory approximates the behaviour of coherent states, and completeness excludes that "too many" particles may eventually assemble at a space-time point, making the state non-normable. On the fermionic side, apart from the non-existence of genuine coherent states, it is the Pauli principle which automatically prevents such an assembly.

Proof of the main Theorem. Ad (i), (ii). Apply Thm. 3.4.3 with $Z := M^{\rm Cau}$ and $\Xi^{\rm Cau'} = \phi^{\rm Cau}$.

Ad (iii). The commutativity of the diagram is clear; we show that α is isomorphic. Let

$$\mathcal{E}^{\rm free}_{\rm c} := \{\xi \in \mathcal{E}^V_{\rm c} : \quad (\partial_t + K(\partial_x))\xi = 0\}, \quad \mathcal{E}^{\rm zero}_{\rm c} := \{\xi \in \mathcal{E}^V_{\rm c} : \quad \xi(0) = 0\}.$$

Using Lemma 3.2.1.(iii), we have a direct decomposition

$$\mathcal{E}^V_{\rm c} = \mathcal{E}^{\rm free}_{\rm c} \oplus \mathcal{E}^{\rm zero}_{\rm c}, \tag{3.6.5}$$

with the corresponding projections given by

$$\mathrm{pr}^{\rm free}(\xi) := \Xi^{\rm free}[\xi(0)], \quad \mathrm{pr}^{\rm zero} := 1 - \mathrm{pr}^{\rm free}$$

(we recall that due to linearity, the insertion makes sense, as in Cor. 2.2.2).

Now (3.6.5) yields an identification $M^{\mathrm{cfg}} = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{V}) = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{free}}) \times \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{zero}})$, with the corresponding projection morphisms being $\mathrm{L}(\mathrm{pr}^{\mathrm{free}})$, $\mathrm{L}(\mathrm{pr}^{\mathrm{zero}})$, and α becomes the composite

$$M^{\mathrm{cfg}} = \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{free}}) \times \mathrm{L}(\mathcal{E}_{\mathrm{c}}^{\mathrm{zero}}) \xrightarrow{(\check{\Xi}^{\mathrm{sol}}\circ\pi)\times\mathrm{L}(\subseteq)} M^{\mathrm{cfg}} \times M^{\mathrm{cfg}} \xrightarrow{\mathrm{L}(+)} M^{\mathrm{cfg}} \tag{3.6.6}$$

with π as in (3.6.3). As often in supergeometry, it is convenient to look at the point functor picture, i. e. we look how α acts on Z-families of configurations: For any smf Z we get a map

$$\mathrm{Mor}(Z, M^{\mathrm{cfg}}) \to \mathrm{Mor}(Z, M^{\mathrm{cfg}}), \quad \xi \mapsto \alpha \circ \xi, \tag{3.6.7}$$

and our assertion follows once we have shown that this is always an isomorphism. (Indeed, it is sufficient to take $Z = M^{\mathrm{cfg}}$, $\xi = \mathrm{Id}$.)

Now α acts on $\xi \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}}(Z) = \mathrm{Mor}(Z, M^{\mathrm{cfg}})$ by

$$\xi = \xi^{\mathrm{free}} + \xi^{\mathrm{zero}} \mapsto \Xi^{\mathrm{sol}}[\xi^{\mathrm{free}}(0)] + \xi^{\mathrm{zero}}.$$

We show injectivity of (3.6.7): If $\alpha \circ \xi = \alpha \circ \xi'$ then, taking Cauchy data at both sides, we get that ξ^{free}, $(\xi')^{\mathrm{free}}$ have the same Cauchy data; hence $\xi^{\mathrm{free}} = (\xi')^{\mathrm{free}}$, and the hypothesis now implies $\xi = \xi'$.

We show surjectivity of (3.6.7): Given $\xi \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}}(Z)$, its preimage is given by $\xi^{\mathrm{free}} + \xi^{\mathrm{zero}}$ with

$$\xi^{\mathrm{zero}} := \xi - \Xi^{\mathrm{sol}}[\xi(0)], \quad \xi^{\mathrm{free}} := \Xi^{\mathrm{free}}[\xi(0)].$$

Assertion (iii) is proved.

Assertion (iv) follows from (iii), and (vi) follows from (ii) (or by considering P-families where P is a point).

Finally, (v) follows from:

Proposition 3.6.2. *A Z-family Ξ' of configurations of quality $\mathcal{E}_{\mathrm{c}}$ with time definition interval $\mathbb{R}$ is a solution family iff the corresponding morphism $\check{\Xi}'$ coincides with the composite*

$$Z \xrightarrow{\check{\Xi}'} M^{\mathrm{cfg}} \xrightarrow{\pi} M^{\mathrm{Cau}} \xrightarrow{\check{\Xi}^{\mathrm{sol}}} M^{\mathrm{cfg}}.$$

Here π is the projection onto Cauchy data, $\hat{\pi} = \Xi(0)$.

Proof. "$\Leftarrow$" follows since Ξ^{sol} is a solution family while "$\Rightarrow$" follows using Thm. 2.4.2. $\square$

The Theorem is proved. $\square$

From Thm. 3.3.1.(ii) we get a "supergeometric" formulation of causality:

Proposition 3.6.3. *Let be given an smf Z, two morphisms $Z \rightrightarrows M^{\mathrm{Cau}}$, and an open set $U \subseteq \mathbb{R}^d$ such that the composites*

$$Z \rightrightarrows M^{\mathrm{Cau}} \xrightarrow{\mathrm{proj}} \mathrm{L}(C^\infty(U) \otimes V)$$

coincide. Let $U' := \{(t,x) \in \mathbb{R}^{d+1} : \mathcal{J}(\{(t,x)\}) \subseteq U\}$. Then the composites

$$Z \rightrightarrows M^{\mathrm{Cau}} \xrightarrow{\check{\Xi}^{\mathrm{sol}}} M^{\mathrm{cfg}} \xrightarrow{\mathrm{proj}} \mathrm{L}(C^\infty(U') \otimes V)$$

coincide. $\square$

3.7. **Variation: The space $\mathcal{E}_{\mathrm{t}}$.** There is a slightly larger space than $\mathcal{E}_{\mathrm{c}}$ such that $\mathcal{D}(\mathbb{R}^d)$ still appears as space of corresponding Cauchy data.

Let $\mathcal{E}_{\mathrm{t}}$ be the space of all $f \in C^\infty(\mathbb{R}^{d+1})$ such that for every Poincaré transformation $l : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ and every time $\theta > 0$, $\mathrm{supp}\, l^*(f) \cap [-\theta, \theta] \times \mathbb{R}^d$ is compact. It is sufficient to check this for Lorentz transformations l only. We will equip $\mathcal{E}_{\mathrm{t}}$ with the topology defined by the seminorms

$$\|f\|_{l,\theta,w,k} := \sup_{x\in\mathbb{R}^d,\ |t|\le\theta} w(x) \sum_{|\nu|\le k} |\partial^\nu l^*(f)(t,x)|$$

where l is a Lorentz transformation, $\theta > 0$, $k > 0$, and w is a non-negative continuous function on $\mathbb{R}^d$.

We have a continuous inclusion $\mathcal{E}_{\mathrm{c}} \subseteq \mathcal{E}_{\mathrm{t}}$ which is proper for $d \ge 1$. For instance, for $f(t,x) := \sum_{n\in\mathbb{Z}_+} \varphi(t-n, x-2^n p)$ where $p \in \mathbb{R}^d \setminus 0$, and $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$, the intersection of $\mathrm{supp}\, f$ with any space-like hyperplane in $\mathbb{R}^{d+1}$ is compact; hence $f \in \mathcal{E}_{\mathrm{t}} \setminus \mathcal{E}_{\mathrm{c}}$.

We get a morphism of the corresponding configuration smf's

$$M^{\mathrm{cfg}} \xrightarrow{\mathrm{L}(\subseteq)} M_{\mathrm{t}} := \mathrm{L}(\mathcal{E}_{\mathrm{t}}^V)$$

where $\mathcal{E}_{\mathrm{t}}^V := \mathcal{E}_{\mathrm{t}} \otimes V$. The map $\pi_{\mathrm{t}} : \mathcal{E}_{\mathrm{t}}^V \to \mathcal{E}_{\mathrm{t}}^{\mathrm{Cau},V}$ (assignment of Cauchy data) is still well-defined, continuous, and surjective; thus, the morphism of assignment of Cauchy data, $M^{\mathrm{cfg}} \xrightarrow{\mathrm{L}(\pi)} M^{\mathrm{Cau}}$, factors to $M^{\mathrm{cfg}} \xrightarrow{\mathrm{L}(\subseteq)} M_{\mathrm{t}} \xrightarrow{\mathrm{L}(\pi_{\mathrm{t}})} M^{\mathrm{Cau}}$.

Proposition 3.7.1. *Suppose that the s.o.f.e. is causal and complete. For any smf Z, the map*

$$\mathcal{M}^{\mathcal{E}_{\mathrm{c}}^V}(Z) \xrightarrow{\subseteq} \mathcal{M}^{\mathcal{E}_{\mathrm{t}}^V}(Z),$$

maps the solution families of quality $\mathcal{E}_{\mathrm{c}}$ bijectively onto the solution families of quality $\mathcal{E}_{\mathrm{t}}$. Thus, the composite embedding $M^{\mathrm{sol}} \subseteq M^{\mathrm{cfg}} \xrightarrow{\mathrm{L}(\subseteq)} M_{\mathrm{t}}$ makes M^{sol} the solution smf for the quality $\mathcal{E}_{\mathrm{t}}$.

Proof. Given a solution family $\Xi' \in \mathcal{M}^{\mathcal{E}_{\mathrm{t}}^V}(Z)$, its Cauchy data $\pi_{\mathrm{t}}\Xi' \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V}}(Z)$ determine a solution family $\Xi" := \Xi^{\mathrm{sol}}[\pi_{\mathrm{t}}\Xi'] \in \mathcal{M}^{\mathcal{E}_{\mathrm{c}}^V}(Z)$; by Thm. 2.4.2.(ii), we have $\Xi' = \Xi"$. □

3.8. **The smf of classical solutions without support restriction.** It takes not much additional effort to lift the constraints on the supports of solution families, considering arbitrary smooth solution families. The appropiate smf's of Cauchy data and configurations are

$$M_{C^\infty}^{\mathrm{Cau}} := \mathrm{L}(\mathcal{E}^{\mathrm{Cau},V}), \quad M_{C^\infty}^{\mathrm{cfg}} := \mathrm{L}(\mathcal{E}^V).$$

Theorem 3.8.1. *Suppose the s.o.f.e. is causal and complete.*

The formal solution Ξ^{sol} is the Taylor series at zero of a unique superfunctional $\Xi^{\mathrm{sol}} \in \mathcal{M}^{\mathcal{E}^V}(M_{C^\infty}^{\mathrm{Cau}})$, which in turn determines an smf morphism

$$\check{\Xi}^{\mathrm{sol}} : M_{C^\infty}^{\mathrm{Cau}} = \mathrm{L}(\mathcal{E}^{\mathrm{Cau},V}) \to \mathrm{L}(\mathcal{E}^V) = M_{C^\infty}^{\mathrm{cfg}}.$$

Its underlying map assigns to each bosonic Cauchy datum ϕ^{Cau} the unique trajectory ϕ with $\phi(0) = \phi^{\mathrm{Cau}}$.

(3.6.2) *defines an smf automorphism $\alpha : M_{C^\infty}^{\mathrm{cfg}} \to M_{C^\infty}^{\mathrm{cfg}}$ which satisfies $\alpha \circ \check{\Xi}^{\mathrm{free}} = \check{\Xi}^{\mathrm{sol}}$ again.*

The image of $\check{\Xi}^{\mathrm{sol}}$ is a split sub-smf which we call the smf of smooth classical solutions without support restriction, *and denote by $M_{C^\infty}^{\mathrm{sol}} \subseteq M_{C^\infty}^{\mathrm{cfg}}$.*

A Z-family Ξ' of quality $\mathcal{E}$ is a solution family iff the corresponding morphism $\check{\Xi}' : Z \to M_{C^\infty}^{\mathrm{cfg}}$ factors to $\check{\Xi}' : Z \to M_{C^\infty}^{\mathrm{sol}} \subseteq M_{C^\infty}^{\mathrm{cfg}}$.

In this way, we get a bijection between Z-families Ξ' of solutions of quality $\mathcal{E}$, and morphisms $\check{\Xi}' : Z \to M_{C^\infty}^{\mathrm{sol}}$.

Proof. The proof is quite analogous to that of Thm. 3.6.1; Lemma 3.4.4 provides the needed Taylor expansions, and Thm. 3.4.5 shows that they fit together. □

We get a commutative diagram of smf's

$$\begin{array}{ccccc}
M^{\mathrm{Cau}} & \xrightarrow{\check{\Xi}^{\mathrm{sol}}} & M^{\mathrm{sol}} & \xrightarrow{\subseteq} & M^{\mathrm{cfg}} \\
\subseteq\downarrow & & \subseteq\downarrow & & \subseteq\downarrow \\
M_{C^\infty}^{\mathrm{Cau}} & \xrightarrow{\check{\Xi}^{\mathrm{sol}}} & M_{C^\infty}^{\mathrm{sol}} & \xrightarrow{\subseteq} & M_{C^\infty}^{\mathrm{cfg}}.
\end{array}$$

3.9. **Local excitations.** A further variant arises by considering compactly supported excitations of a classical solution; in particular, it is applicable for situations with spontaneous symmetry breaking, like the Higgs mechanism:

Theorem 3.9.1. *Suppose that the s.o.f.e. is causal and complete, and fix a trajectory $\phi \in \mathcal{E}^V{}_{\mathbf{0}}$; let ϕ^{Cau} be its Cauchy data.*

(i) The superfunctional

$$\Xi_\phi^{\mathrm{exc}}[\Xi^{\mathrm{Cau}}] := \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}} + \phi^{\mathrm{Cau}}] - \phi, \tag{3.9.1}$$

which lies a priori in $\mathcal{M}^{\mathcal{E}^V}(M^{\mathrm{Cau}}_{C^\infty})$*, restricts to a superfunctional*

$$\Xi^{\mathrm{exc}}_\phi[\Xi^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{E}^V_{\mathrm{c}}}(M^{\mathrm{Cau}}).$$

(ii) Consider the arising smf morphism $\check{\Xi}^{\mathrm{exc}}_\phi : M^{\mathrm{Cau}} \to M^{\mathrm{cfg}}$*. The smf morphism* $\alpha_\phi : M^{\mathrm{cfg}} \to M^{\mathrm{cfg}}$ *given by*

$$\widehat{\alpha_\phi}[\Xi] := \Xi + \Xi^{\mathrm{sol}}[\Xi(0) + \phi^{\mathrm{Cau}}] - \Xi^{\mathrm{free}}[\Xi(0)] - \phi$$

is an automorphism of M^{cfg} *which satisfies* $\alpha_\phi \circ \check{\Xi}^{\mathrm{free}} = \check{\Xi}^{\mathrm{exc}}_\phi$*.*

(iii) The image of $\check{\Xi}^{\mathrm{exc}}_\phi$ *is a split sub-smf which we call the* smf of excitations around the trajectory ϕ*, and denote by* $M^{\mathrm{exc}}_\phi \subseteq M^{\mathrm{cfg}}$*.*

(iv) M^{exc}_ϕ *has the following universal property: Given a* Z*-family* $\Xi' \in \mathcal{M}^{\mathcal{E}^V_{\mathrm{c}}}(Z)$*, the corresponding morphism* $\check{\Xi}' : Z \to M^{\mathrm{cfg}}$ *factors through* M^{exc}_ϕ *iff the* Z*-family* $\Xi' + \phi \in \mathcal{M}^{\mathcal{E}^V}(Z)$ *is a solution family.*

Proof. Ad (i). First, we prove that for $\phi^{\mathrm{Cau}'} \in (\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})_{\mathbf{0}}$, we have

$$\Xi^{\mathrm{exc}}_\phi[\Xi^{\mathrm{Cau}}]_{\phi^{\mathrm{Cau}'}} \in \mathcal{P}(\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}};\ \mathcal{E}^V_{\mathrm{c}}). \tag{3.9.2}$$

Indeed,

$$\Xi^{\mathrm{exc}}_\phi[\Xi^{\mathrm{Cau}}]_{\phi^{\mathrm{Cau}'}} = \Xi^{\mathrm{sol}}_{\phi^{\mathrm{Cau}}+\phi^{\mathrm{Cau}'}}[\Xi^{\mathrm{Cau}}] - \phi.$$

On the other hand, setting

$$\phi" := \Xi^{\mathrm{sol}}[\phi^{\mathrm{Cau}} + \phi^{\mathrm{Cau}'}|0],$$

it follows from Thm. 3.3.1 that $\phi" - \phi \in (\mathcal{E}^V_{\mathrm{c}})_{\mathbf{0}}$. Now (3.9.2) follows from Prop. 3.4.6.

Now, we use again the strictly separating set (cf. [14, Prop. 2.4.4]) of linear functionals (3.4.4). Since all these functionals extend onto $\mathcal{E}^V$, it follows simply from Thm. 3.8.1 that the elements (3.9.2) fit together to the superfunction wanted.

The proofs of the remaining assertions are quite analogous to those for Thm. 3.6.1; in (3.6.6) and the following formulas, one simply replaces Ξ^{sol} by Ξ^{exc}_ϕ. ☐

Remark . This theorem yields new information only if the ϕ^{Cau} are not compactly carried. If they are, i. e. $\phi^{\mathrm{Cau}} \in (\mathcal{E}^{\mathrm{Cau},V}_{\mathrm{c}})_{\mathbf{0}}$, then (3.9.1) is already a priori defined as element of $\mathcal{M}^{\mathcal{E}^V_{\mathrm{c}}}(M^{\mathrm{Cau}})$, and M^{exc}_ϕ can be identified with M^{sol}.

3.10. **Other generalizations.** For a non-causal s.o.f.e., one can still construct solution supermanifolds of Sobolev quality. Adapting the proof of Thm. 3.6.1 we have:

Corollary 3.10.1. *Let be given a (perhaps non-causal) s.o.f.e., and fix* k, l *with* $k > d/2+\mu l$*. Suppose that*

$$\sup_{t\in(a,b)} \|\phi(t)\|_{\mathcal{H}^{\mathrm{Cau},V}_k} < \infty \tag{3.10.1}$$

holds for every trajectory $\phi \in \mathcal{H}^{l,V}_k(I)_{\mathbf{0}}$*. Then the formal solution is the Taylor expansion at zero of a uniquely determined superfunction* $\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{M}^{\mathcal{H}^{l,V}_k(\mathbb{R})}(\mathrm{L}(\mathcal{H}^{\mathrm{Cau},V}_k))$ *which is the universal solution family for the quality* $\mathcal{H}^l_k$*. The image of the corresponding smf morphism* $\check{\Xi}^{\mathrm{sol}} : \mathrm{L}(\mathcal{H}^{\mathrm{Cau},V}_k) \to \mathcal{H}^{l,V}_k(\mathbb{R}))$ *is a sub-supermanifold which is called the* supermanifold of solutions of quality $\mathcal{H}^l_k$*.* ☐

One can get rid of the k-dependence by taking the intersections over all k equipped with the projective limes topology:

$$\mathcal{H}^{\mathrm{Cau},V}_\infty := \bigcap_{k>0} \mathcal{H}^{\mathrm{Cau},V}_k, \qquad \mathcal{H}^V_\infty(\mathbb{R}) := \bigcap_{k,l>0} \mathcal{H}^{l,V}_k(\mathbb{R}).$$

One then gets a morphism

$$\check{\Xi}^{\mathrm{sol}} : \mathrm{L}(\mathcal{H}^{\mathrm{Cau},V}_\infty) \to \mathrm{L}(\mathcal{H}^V_\infty(\mathbb{R}))$$

provided (3.10.1) holds for every trajectory $\phi \in \mathcal{H}^V_\infty(\mathbb{R})_{\mathbf{0}}$.

Note that, roughly spoken, $\mathcal{H}_\infty^{\mathrm{Cau},V}$, $\mathcal{H}_\infty^V(\mathbb{R})$ "lie between" the Schwartz spaces $\mathcal{S}$ and C^∞; even for a causal s.o.f.e., it is not clear whether one can descend to the Schwartz spaces.

Let us sketch an abstract version of our approach: we start with a $\mathbb{Z}_2$-graded Banach space B and a strongly continuous group $(\mathrm{A}_t)_{t\in\mathbb{R}}$ of parity preserving operators; let $K : \operatorname{dom} K \to B$ denote the generator of this group. Also, let be given an entire superfunction $\Delta = \Delta[\Xi] \in \mathcal{M}^B(\mathrm{L}(B))$ the Taylor expansion Δ_0 of which in zero has lower degree ≥ 2 and satisfies $\Delta_0 \in \mathcal{P}(B, cU;\ B)$ for all $c > 0$ where U is the unit ball of B. Formally, the equation of interest is

$$\frac{d}{dt}\Xi' = K\Xi' + \Delta[\Xi']; \tag{3.10.2}$$

however, this makes sense only if Ξ' takes values in $\operatorname{dom} K$. Therefore we look for the integrated version

$$\Xi'(t) = \mathrm{A}_t\Xi'(0) + \int ds f(t,s)\mathrm{A}_{t-s}\Delta[\Xi'](s) \tag{3.10.3}$$

where $f(t,s)$ is as in (1.7.1).

For a connected subset $I \subseteq \mathbb{R}$, $I \ni 0$ with non-empty kernel, let $B(I) := C(I, B)$ equipped with the topology induced by the seminorms $\|\phi\|_{B([a,b])} := \max_{t\in[a,b]} \|\phi(t)\|$ where $a, b \in I$, $a < b$. If Z is any smf we call an element $\Xi' \in \mathcal{M}^{B(I)}(\mathrm{L}(B))$ a *solution family* iff (3.10.3) holds. (Of course, if Ξ' takes values in $D := \operatorname{dom} K$ (i. e. $\Xi' \in \mathcal{M}^{C(I,\operatorname{dom} K)}(\mathrm{L}(B))$ where $\operatorname{dom} K$ is equipped with the graph norm) then (3.10.3) is equivalent to the differentiated form (3.10.2)).

Now our approach generalizes to:

Corollary 3.10.2. *(i) There exists a formal solution* $\Xi^{\mathrm{sol}} = \Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{P}(B;\ B(\mathbb{R}))$ *of* (3.10.3), *with* $\Xi^{\mathrm{sol}}_{(\leq 1)}(t) = \mathrm{A}_t\Xi^{\mathrm{Cau}}$, *and*

$$\Xi^{\mathrm{sol}}_{(n)}(t) = \int ds f(t,s)\mathrm{A}_{t-s}\Delta[\Xi^{\mathrm{sol}}_{(\leq n-1)}]_{(n)}(s)$$

for $n \geq 2$.

(ii) For any $c > 0$ *there exists* $\theta > 0$ *such that* $\Xi^{\mathrm{sol}} \in \mathcal{P}(B, cU;\ B([-\theta,\theta]))$ *where* $U \subseteq B$ *is the unit ball.*

(iii) Suppose that for any solution family $\Xi' \in \mathcal{M}^{B(I)}(\mathrm{L}(B))$, *we have* $\sup_{t\in I} \|\phi(t)\| < \infty$. *Then the formal solution is the Taylor expansion at zero of a uniquely determined superfunction* $\Xi^{\mathrm{sol}}[\Xi^{\mathrm{Cau}}] \in \mathcal{M}^{B(\mathbb{R})}(\mathrm{L}(B))$ *which is the universal solution family. The image of the corresponding smf morphism* $\check{\Xi}^{\mathrm{sol}} : \mathrm{L}(B) \to \mathrm{L}(B(\mathbb{R}))$ *is a sub-supermanifold which is called the* solution supermanifold *of* (3.10.3) *or* (3.10.2). □

3.11. **Solutions with values in Grassmann algebras.** The most naive notion of a configuration in a classical field model with anticommuting fields arises by replacing the domain $\mathbb{R}$ for the real field components by a finite-dimensional Grassmann algebra $\Lambda_n = \mathbb{C}[\zeta_1, \ldots, \zeta_n]$ (we recall that, in accordance with our hermitian framework, only complex Grassmann algebras should be used). Here we consider only smooth configurations; thus, a Λ_n*-valued configuration* is a tuple $\xi = (\phi|\psi)$ with

$$\begin{aligned} \phi_i &\in C^\infty(\mathbb{R}^{d+1}, (\Lambda_n)_{\mathbf{0},\mathbb{R}}) \quad \text{for } i = 1, \ldots, N_{\mathbf{0}}, \\ \psi_j &\in C^\infty(\mathbb{R}^{d+1}, (\Lambda_n)_{\mathbf{1},\mathbb{R}}) \quad \text{for } j = 1, \ldots, N_{\mathbf{1}}. \end{aligned}$$

Now, comparing with [14, 1.11] we see that ξ encodes just a Z_n-family over $\mathbb{R}$ of quality $\mathcal{E}$ where Z_n is the $0|n$-dimensional smf, so that $\mathcal{O}(Z_n) = \Lambda_n$. Also, ξ is a solution family in our sense iff the field equations are satisfied in the plain sense. We now get an overview over all Λ_n-valued solutions:

Corollary 3.11.1. *Suppose that the s.o.f.e. is causal and complete, and let be given* Λ_n*-valued Cauchy data*

$$\xi^{\mathrm{Cau}} \in C^\infty(\mathbb{R}^d, (\Lambda_n \otimes V)_{\mathbf{0},\mathbb{R}}).$$

Then there exists a unique Λ_n*-valued solution* $\xi = (\phi|\psi)$ *with these Cauchy data. It is given by*

$$\xi = \left(\check{\Xi}^{\mathrm{sol}} \circ (\xi^{\mathrm{Cau}})^\vee\right)^\wedge = \Xi^{\mathrm{sol}}_{b(\phi^{\mathrm{Cau}})}[s(\xi^{\mathrm{Cau}})]$$

where $b(\cdot) : \Lambda_n \to \mathbb{C}$ denotes the body map, and $s(\cdot) = 1 - b(\cdot)$ the soul map. □

We now look for solutions in the infinite-dimensional Grassmann algebra Λ_∞ of supernumbers introduced by deWitt [4]:

$$\Lambda_\infty = \bigcup_{n>0} \Lambda_n = \varinjlim \Lambda_n. \tag{3.11.1}$$

Let $\mathbb{R}^\infty$ be the vector space of all number sequences $(a_i)_{i\geq 1}$, equipped with the product topology. The topological dual, $(\mathbb{R}^\infty)^*$, is algebraically generated by the projections π_i on the i-th component.

Let $Z_\infty := \mathrm{L}(\Pi\mathbb{R}^\infty)$; this is an infinite-dimensional smf the underlying manifold of which is a single point. (We recall that Π is just an odd formal symbol.) Now the elements $\zeta_i := e_i \circ \Pi^{-1}$ lie in the odd part of $(\Pi\mathbb{R}^\infty)^* \subseteq \mathcal{O}(Z_\infty)$; from the universal property of the Grassmann algebra we get an algebra homomorphism

$$\Lambda_\infty \to \mathcal{O}(Z_\infty),$$

and one shows that this is an isomorphism; thus, we can identify both sides.

Now, fixing some $k \geq 0$, and given an element $f \in \mathcal{O}^{C^\infty(\mathbb{R}^k)}(Z_\infty)$, we get a map

$$f' : \mathbb{R}^k \to \Lambda_\infty, \quad x \mapsto \delta_x \circ f \tag{3.11.2}$$

which has the property that for any bounded open $U \subseteq \mathbb{R}^k$ coincides with a C^∞ map $f'|_U : U \to \Lambda_n$ for sufficiently large n. In this way, we get an isomorphism

$$\mathcal{O}^{C^\infty(\mathbb{R}^k)}(Z_\infty) \xrightarrow{\cong} C^\infty(\mathbb{R}^k, \Lambda_\infty)$$

where we equip Λ_∞ with the locally convex inductive limit topology arising from (3.11.1).

Thus, a Λ_∞-valued smooth configuration $\xi \in (C^\infty(\mathbb{R}^{d+1}, \Lambda_\infty) \otimes V)_{\mathbf{0}}$ encodes the same information as an smf morphism $Z_\infty \to M^{\mathrm{cfg}}$, i. e. a Z_∞-valued point of M^{cfg}; analogously for the Cauchy data.

It follows that Cor. 3.11.1 holds also for $n = \infty$.

Remark . An element $f \in \mathcal{O}^{\mathcal{E}}(Z_\infty)$ lies in $\mathcal{O}^{\mathcal{E}_c}(Z_\infty)$ iff for each sequence $i_1 < \cdots < i_n$ of indices, the function $(\partial_{\zeta_{i_1}} \cdots \partial_{\zeta_{i_n}} f)^\sim \in C^\infty(\mathbb{R}^{d+1})$ has its support in some $\mathbb{V}_r$. The function (3.11.2) itself needs not to have this support property.

3.12. **Examples.** Here we consider only two characteristic examples; a systematic exploration of a large class of classical field theories will be given in the successor paper.

3.12.1. *The Φ^4 model.* Let us see how this model, considered in 1.2, fits into our model class: Here $(d, N_{\mathbf{0}}|N_{\mathbf{1}}, \tau, (L_i[\underline{\Xi}])) = (3, 2|0, (1,0), (L_i[\underline{\Phi}]))$ where L_1, L_2 are given by (1.2.2); thus $K = \begin{pmatrix} 0 & -1 \\ K' & 0 \end{pmatrix}$ with $K' = -\sum_{a=1}^3 \partial_a{}^2$ and $\Delta = (0, 4q\Phi_1^3)$. Now

$$A = \begin{pmatrix} \partial_t \mathcal{A} & \mathcal{A} \\ \partial_t{}^2 \mathcal{A} & \partial_t \mathcal{A} \end{pmatrix},$$

where $\mathcal{A}$ is given by (1.2.4), and the conditions (1.5.2), (1.5.3) as well as the causality condition (3.2.1) are easily checked. It is a classical result that this s.o.f.e. is complete. Since the kinetic operator is spatially second order but $\mathcal{A}$ improves spatial smoothness by one degree, the "smoothness loss" μ is one. (The smoothness condition would allow even a derivative coupling; however, such an interaction could not stem from a Lagrangian.)

3.12.2. *The Thirring model.* This is purely fermionic, with $d = 1$ and field contents given by a fermionic Dirac spinor field $\Psi^c = (\Psi_1^c, \Psi_2^c)^{\mathrm{T}}$; thus, the real field components are $(\Psi_i)_{i=1}^4 = (\mathrm{Re}\,\Psi_1^c, \mathrm{Im}\,\Psi_1^c, \mathrm{Re}\,\Psi_2^c, \mathrm{Im}\,\Psi_2^c)$. For $a = 0, 1$, set

$$j^a(\underline{\Psi}^c) := \overline{\underline{\Psi}^c} \gamma^a \underline{\Psi}^c;$$

where γ^0, γ^1 are the usual gamma matrices, and $\psi \mapsto \overline{\psi} := \overline{\psi}^{\mathrm{T}}\gamma^0$ denotes Dirac conjugation. We consider the Thirring model in its classical form, with Lagrangian

$$\mathcal{L}[\underline{\Psi}^{\mathrm{c}}] = \frac{\mathrm{i}}{2}\sum_{a=0}^{1}\left(\overline{\underline{\Psi}^{\mathrm{c}}}\gamma^a\partial_a\underline{\Psi}^{\mathrm{c}} - \partial_a\overline{\underline{\Psi}^{\mathrm{c}}}\gamma^a\underline{\Psi}^{\mathrm{c}}\right) - \frac{g}{2}\sum_{a=0}^{1} j_a(\underline{\Psi}^{\mathrm{c}})j^a(\underline{\Psi}^{\mathrm{c}})$$

where $g \in \mathbb{R}$.

The variational derivatives are

$$\frac{\delta}{\delta\overline{\underline{\Psi}^{\mathrm{c}}}_\alpha}\mathcal{L}[\underline{\Psi}^{\mathrm{c}}] = \big(\sum_a \gamma^a(\mathrm{i}\,\partial_a\underline{\Psi}^{\mathrm{c}} - g j_a(\underline{\Psi}^{\mathrm{c}})\underline{\Psi}^{\mathrm{c}})\big)_\alpha, \quad \alpha = 1, 2,$$

and its hermitian conjugate; thus, the "field equations" are equivalent to the vanishing of

$$L^{\mathrm{c}}[\underline{\Psi}] := \partial_t\underline{\Psi}^{\mathrm{c}} + (\gamma^0)^{-1}\gamma^1\partial_1\underline{\Psi}^{\mathrm{c}} + \mathrm{i}\,g(\gamma^0)^{-1}\sum_a \gamma^a j_a(\underline{\Psi}^{\mathrm{c}})\underline{\Psi}^{\mathrm{c}}.$$

It is easy to check that we get a causal s.o.f.e.

$$(d, N_{\mathbf{0}}|N_{\mathbf{1}}, \tau, (L_i[\underline{\Xi}])) := (1, 0|4, (0,0,0,0), (\mathrm{Re}\,L_1^{\mathrm{c}}[\underline{\Psi}], \mathrm{Im}\,L_1^{\mathrm{c}}[\underline{\Psi}], \mathrm{Re}\,L_2^{\mathrm{c}}[\underline{\Psi}], \mathrm{Im}\,L_2^{\mathrm{c}}[\underline{\Psi}])).$$

Since the underlying bosonic s.o.f.e. is "empty", this model (as well as its generalization, the Gross-Neveu model), is complete. It is well known that this model can be explicitly solved; using the method given e. g. in [1, 11.2.A] one gets a closed formula for Ψ^{sol}. We use complex notations, setting

$$\Psi^{\mathrm{c,Cau}} := (\Psi_1^{\mathrm{Cau}} + \mathrm{i}\,\Psi_2^{\mathrm{Cau}},\ \Psi_3^{\mathrm{Cau}} + \mathrm{i}\,\Psi_4^{\mathrm{Cau}})^{\mathrm{T}} \in \mathcal{O}^{\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau}}\otimes\mathbb{C}^2}(M^{\mathrm{Cau}})$$

and analogously for $\Psi^{\mathrm{c,sol}} \in \mathcal{O}^{\mathcal{E}_{\mathrm{c}}\otimes\mathbb{C}^2}(M^{\mathrm{Cau}})$. Also, set

$$\phi_2^{\mathrm{Cau}} := -\overline{\Psi^{\mathrm{c,Cau}}}\gamma^0\Psi^{\mathrm{c,Cau}}, \quad \phi_1^{\mathrm{Cau}}(x) := \int_0^x dx'\,\overline{\Psi^{\mathrm{c,Cau}}}(x')\gamma^1\Psi^{\mathrm{c,Cau}}(x').$$

Let $\Phi_1^{\mathrm{free}}[\Phi_1^{\mathrm{Cau}}, \Phi_2^{\mathrm{Cau}}]$ be the solution operator of the massless free scalar field:

$$\Phi_1^{\mathrm{free}}[\Phi_1^{\mathrm{Cau}}, \Phi_2^{\mathrm{Cau}}](t,y) = \int_{\mathbb{R}^3} dx\,\left(\partial_t\mathcal{A}(t, y-x)\Phi_1^{\mathrm{Cau}}(x) + \mathcal{A}(t, y-x)\Phi_2^{\mathrm{Cau}}(x)\right),$$

where $A(t,x)$ is the massless Pauli-Jordan exchange function given by its spatial Fourier transform as $\hat{A}(t,p) = (2\pi p^2)^{-1/2}\sin(\sqrt{p^2}t)$. Now the solution operator for the free Dirac field is given by

$$\Psi^{\mathrm{c,free}}[\Phi^{\mathrm{c,Cau}}](t,y) = \int_{\mathbb{R}^3} dx\sum_{a=0}^{1}(\gamma^a\partial_a\mathcal{A})(t, y-x)\Phi^{\mathrm{c,Cau}}(x).$$

Proposition 3.12.1. *We have*

$$\Psi^{\mathrm{c,sol}}[\Psi^{\mathrm{c,Cau}}] = \exp(\mathrm{i}\,g\Phi_1^{\mathrm{free}}[\phi_1^{\mathrm{Cau}}, \phi_2^{\mathrm{Cau}}])\ \Psi^{\mathrm{c,free}}[\exp(-\mathrm{i}\,g\phi_1^{\mathrm{Cau}})\Psi^{\mathrm{c,Cau}}].$$

□

All the smf's $M^{\mathrm{Cau}}, M^{\mathrm{cfg}}, M^{\mathrm{sol}}$ connected with this model have a single point P as underlying manifold; the whole information of $\check{\Xi}^{\mathrm{sol}} : M^{\mathrm{Cau}} \to M^{\mathrm{cfg}}$ lies in the homomorphism $\check{\Xi}^{\mathrm{sol}} : \mathcal{O}_{M^{\mathrm{cfg}}}(P) \to \mathcal{O}_{M^{\mathrm{Cau}}}(P)$ of infinite-dimensional Grassmann algebras.

It would be interesting to see how M^{sol} is connected with the tree approximation of the quantized model.

Appendix A. Table of spaces of configurations and Cauchy data

Quality	smooth (C^∞)	C^∞ with causal support	Sobolev ($k > \frac{d}{2}$)
Cauchy data for components	$\mathcal{E}^{\mathrm{Cau}} = C^\infty(\mathbb{R}^d)$	$\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau}} = \mathcal{D}(\mathbb{R}^d)$	$\mathcal{H}_k^{\mathrm{Cau}} = H_k(\mathbb{R}^d)$
Cauchy data	$\mathcal{E}^{\mathrm{Cau},V} = C^\infty(\mathbb{R}^d, V)$	$\mathcal{E}_{\mathrm{c}}^{\mathrm{Cau},V} = \mathcal{D}(\mathbb{R}^d, V)$	$\mathcal{H}_k^{\mathrm{Cau},V}$ (cf. (1.6.1))
Configurations for components	$\mathcal{E} = C^\infty(\mathbb{R}^{d+1})$	$\mathcal{E}_{\mathrm{c}}$ (cf. 3.1)	$\mathcal{H}_k^l(I)$ (cf. (1.6.3))
Configurations	$\mathcal{E}^V = C^\infty(\mathbb{R}^{d+1}, V)$	$\mathcal{E}_{\mathrm{c}}^V = \mathcal{E}_{\mathrm{c}} \otimes V$	$\mathcal{H}_k^{l,V}(I)$ (cf. (1.6.4))
Use	Variant of Main Thm.	Main Thm.	Technical

References

[1] Bogoliubov N N, Logunov A A, Oksak A I, Todorov I T: General principles of quantum field theory (in russian). "Nauka", Moskwa 1987

[2] Choquet-Bruhat Y, Christodoulou D: Existence of global solutions of the Yang-Mills, Higgs and spinor field equations in $3+1$ dimensions. Ann. de l'E.N.S., $4^{\text{ème}}$ série, Tome 14 (1981), p. 481–500

[3] Choquet-Bruhat Y: Classical supergravity with Weyl spinors. Proc. Einstein Found. Intern. Vol. 1, No. 1 (1983) 43-53

[4] DeWitt B: Supermanifolds. Cambridge University Press, Cambridge 1984

[5] Eardley D M, Moncrief V: The global existence of Yang-Mills-Higgs Fields in 4-dimensional Minkowski space. I. Local Existence and Smoothness Properties. II. Completion of the proof. Comm. Math. Phys. 83, 171–191 and 193–212 (1982)

[6] Ginibre J, Velo G: The Cauchy Problem for coupled Yang-Mills and scalar fields in the temporal gauge. Comm. Math. Phys. 82 (1982) 171-212

[7] Hörmander L: The Analysis of Linear Partial Differential Operators II. Differential Operators with Constant Coefficients. Grundl. d. math. Wiss. 256, Springer-Verlag 1971, Berlin-Heidelberg 1983

[8] Isenberg J, Bao D, Yasskin P B: Classical Supergravity. In: Mathematical Aspects of Superspace (ed.: H. J. Seifert et al). Nato Asi Series C: Math. and Phys. Sciences. Dordrecht 1984, 173–205

[9] Kostant B, Sternberg S: Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Annals of Physics 176, 49–113 (1987)

[10] Reed M, Simon B: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. Academic Press, New York, 1975

[11] Segal I: Symplectic Structures and the Quantization Problem for Wave Equations. Symposia Math. 14 (1974), 99-117

[12] Schmitt T: Supergeometry and hermitian conjugation. Journal of Geometry and Physics, Vol. 7, n. 2, 1990

[13] ——: Functionals of classical fields in quantum field theory. Reviews in Mathematical Physics, Vol. 7, No. 8 (1995), 1249-1301

[14] ——: Supergeometry and quantum field theory, or: What is a classical configuration? Preprint No. 419, Techn. Univ. Berlin, 1995

[15] ——: Infinitedimensional Supermanifolds I. Report 08/88 des Karl-Weierstraß-Instituts für Mathematik, Berlin 1988.
II, III. Mathematica Gottingensis. Schriftenreihe des SFBs Geometrie und Analysis, Heft 33, 34 (1990). Göttingen 1990

[16] Śniatycki J, Schwarz G: The existence and uniqueness of solutions of Yang-Mills equations with the bag boundary conditions. Comm. Math. Phys. 159 (1994), 593–604.

[17] Taylor M E: Pseudodifferential Operators. Princeton Math. Series, Princeton University Press, Princeton 1981

[18] Strichartz R S: Multipliers on fractional Sobolev spaces. J. Math. Mechanics 16 (1967), 1031 – 1060

Technische Universität Berlin
Fachbereich Mathematik, MA 7 – 2
Straße des 17. Juni 136
10623 Berlin
FR Germany

E-Mail address: schmitt@math.tu-berlin.de