Point interactions in a strip

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Abstract

We study the behavior of a quantum particle confined to a hard–wall strip of a constant width in which there is a finite number N of point perturbations. Constructing the resolvent of the corresponding Hamiltonian by means of Krein's formula, we analyze its spectral and scattering properties. The bound state–problem is analogous to that of point interactions in the plane: since a two–dimensional point interaction is never repulsive, there are m discrete eigenvalues, $1 \le m \le N$, the lowest of which is nondegenerate. On the other hand, due to the presence of the boundary the point interactions give rise to infinite series of resonances; if the coupling is weak they approach the thresholds of higher transverse modes. We derive also spectral and scattering properties for point perturbations in several related models: a cylindrical surface, both of a finite and infinite heigth, threaded by a magnetic flux, and a straight strip which supports a potential independent of the transverse coordinate. As for strips with an infinite number of point perturbations, we restrict ourselves to the situation when the latter are arranged periodically; we show that in distinction to the case of a point–perturbation array in the plane, the spectrum may exhibit any finite number of gaps. Finally, we study numerically conductance fluctuations in case of random point perturbations.

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1 Introduction

Point interactions were introduced in early days of quantum mechanics by R. Kronig, W.G. Penney, E. Fermi and others. It lasted several decades, however, before it became clear how they can be treated in a mathematically sound way and in what sense they correspond to the intuitive notion of a δ–shaped potential. This in turn made it possible to develop this method into a powerful tool for analyzing quantum systems in which the interaction is supported by a family of sets which are small on an appropriate scale and well separated.

A thorough and extensive discussion of the point interaction method can be found in the monograph [1]; references to some recent work are given, *e.g.*, in [3, 8]. This does not exhaust, however, the list of possible applications. In this paper we are going to study another system with point interactions, namely a quantum particle living in a strip of a fixed width d with the Dirichlet (*i.e.*, hard wall) boundary, in which there is a finite number of pointlike perturbations.

This problem has an obvious physical motivation. The strip is a natural model for quantum wires which are an object of intensive experimental and theoretical interest these days. Though the presently available technologies make it possible to fabricate mesoscopic structures of extremely pure semiconductor material in which the electron motion is predominantly ballistic, a real wire is always non–ideal, containing impurities in the crystallic lattice which affect its conductivity properties.

Even if one treats the one–electron problem within a suitable range of energy in the approximation of a particle with the effective mass whose motion is confined to a strip, the scattering problem in the presence of impurities is not easy to solve. The assumption that the impurities have a point character can turn the solution into a more manageable task; at the same time it is a reasonable approximation because a typical impurity consists of an alien atom in the lattice while the wire cross section includes at least several but mostly many atomic layers. Moreover, an experimental way to produce *artificial* impurities has been reported [21]; though the latter are not exactly pointlike, their existence stresses the need for better understanding to scattering in quantum waveguides.

Importance of point interactions as models of impurity scattering in a two–dimensional electron gas was realized long time ago [23], however, it seems to attract a new attention recently when several studies of point perturbations on strips, cylinders or tori has appeared — see, *e.g.*, [4, 9, 22] and references therein. In particular, P. Bagwell has pointed out the role played by quasibound states at excited transverse modes which are unstable due to intermode coupling. Let us remark that similar resonances appear generically if there is a locally attractive interaction in the wire (the latter may be of a various origin; it can be induced also in a purely geometrical way — see, *e.g.*, [6, 10, 18, 30, 31]). The most remarkable manifestation of the resonances are local deformations of the ideal steplike conductance pattern.

On the other hand, the point scatterers in the above mentioned papers are treated in a simplified way, usually with the model space restricted to a finite number of transverse modes. This does not allow us to introduce the interaction in a proper *local* way. In fact, the authors of Ref.[9] discuss the difference between the situation when the scatterers are attractive or repulsive. It is known, however, that — in distinction to one and three dimensions — a two–dimensional point interaction obtained as a limit of a family of scaled potentials is never repulsive; this fact follows from the logarithmic character of singularity of the corresponding resolvent kernel — *cf.* [2, 19] or [1, Sec.I.5].

This gives a motivation to analyze the problem of point perturbations in a strip more carefully; we follow the ideas which have yielded the one–center scattering matrix discussed in [15]. As in the case without the boundaries [1], the point perturbations are constructed as self–adjoint extensions of a symmetric operator obtained by restriction of the free Hamiltonian in the strip. Our main tool is the Krein's formula which allows us to write down the resolvent of the Hamiltonian; the difference from the free resolvent is a rank N operator, where N is the number of impurities. Since we are interested in local perturbations only, the number of parameters is again N ; they are naturally interpreted as the corresponding coupling constants.

The spectral problem is then reduced to finding eigenvalues of an $N\times N$ matrix. In distinction to the case of point interactions in a plane, however, the matrix elements are not elementary functions but sums of certain series. Nevertheless, one can say a lot about the spectrum. In particular, the mentioned fact about attractivity of two–dimensional point perturbations is not affected by the presence of the strip boundary. One can also derive the S–matrix which yields conductance modulations due to the impurities, as well as infinite series of resonances associated with each point perturbation. Another feature observed in scattering are turbulent patterns of the probability flow caused by the impurities, which have been noticed in Ref.[9] and in [6, 31] in a bent–guide context.

Furthermore, strips with point perturbations offer a useful test field for some properties dependent on the dimension of the configuration space, such as number of open gaps in a periodic situation, or existence of localization in case of random perturbations, because, roughly speaking, they are "halfway" between the one–dimensional and two–dimensional situation. We shall show that while a straight polymer in the plane has at most one gap [1, Sec.III.4], a "coated polymer", *i.e.*, a periodic array of point perturbations in a strip can exhibit any finite number of open gaps. This raises the question, of course, whether the Bethe–Sommerfeld conjecture can be proven in this situation.

Finally, if the perturbation positions or coupling constants are distributed at random, we can study statistical properties of observable quantities such as conductance of a wire with point perturbations. The explicit solvability of our model makes it suitable for comparison with predictions of random–matrix theory; this will be done in the concluding section.

2 A one–center perturbation in an infinite strip

Consider a straight planar strip $\Omega := I\!\!R\times[0,d]$ and suppose that the free motion in Ω is governed by the Dirichlet Laplacian [25, Sec.XIII.15]; since the boundary of Ω is smooth, it acts as $H_0\psi = -\Delta\psi$ on the domain of all ψ for which the *rhs* (in the sense of distributions) belongs to $L^2(\Omega)$ and which satisfy the boundary conditions

$$\psi(x,0) \,=\, \psi(x,d) \,=\, 0 \tag{2.1}$$

for all $x\in I\!\!R$. Unless stated otherwise, we put for simplicity $d=\pi$; the results for a general strip width are then obtained by a simple scaling transformation.

2.1 Boundary conditions

Our first goal is to construct a one–center singular perturbation to H_0 supported by a point $\vec a := (a,b)$ with $b\in(0,\pi)$. This can be done using the procedure mentioned in the introduction; since locally there is no difference between point interactions in the plane and in the strip, we can use the standard boundary conditions [1, Chap.I.5]. We introduce the regularized boundary values

$$L_0(\psi,\vec{a}) := \lim_{|\vec{x}-\vec{a}|\to 0} \frac{\psi(\vec{x})}{\ln|\vec{x}-\vec{a}|}\,, \qquad L_1(\psi,\vec{a}) := \lim_{|\vec{x}-\vec{a}|\to 0} [\psi(\vec{x}) - L_0(\psi,\vec{a})\,\ln|\vec{x}-\vec{a}|]$$

and require

$$L_1(\psi,\vec{a}) + 2\pi\alpha L_0(\psi,\vec{a}) = 0\,. \tag{2.2}$$

To any $\alpha \in I\!\!R$ we define the self–adjoint operator $H(\alpha,\vec{a})$ as

$$\begin{aligned} H(\alpha,\vec{a})\psi &:= -\Delta\psi\,, \\ D(H(\alpha,\vec{a})) &:= \Big\{\, \psi\,:\, -\Delta\psi \ \text{ is } L^2 \text{ on } \Omega\backslash\{\vec{a}\} \text{ and (2.1), (2.2) are satisfied} \Big\}\,; \end{aligned} \tag{2.3}$$

here again $-\Delta\psi$ is understood in the sense of distributions. The case $\alpha=\infty$, *i.e.* $L_0(\psi,\vec{a})=0$, corresponds to the free Hamiltonian H_0.

2.2 The resolvent

To find the resolvent of $H(\alpha,\vec{a})$ we start with that of H_0. We use the decomposition into transverse modes, $L^2(\Omega) = \bigoplus_{n=1}^{\infty} L^2(I\!\!R)\otimes\{\chi_n\}$, where $\chi_n(y) := \sqrt{\frac{2}{\pi}}\,\sin(ny)$; then the free Hamiltonian can be written as

$$H_0 = \bigoplus_{n=1}^{\infty} h_n\otimes I_n\,, \qquad h_n := -\frac{d^2}{dx^2} + n^2 \tag{2.4}$$

with $D(h_n) := AC^2(I\!\!R)$. It follows that the free resolvent is an integral operator with the kernel

$$G_0(\vec{x}_1,\vec{x}_2;z) \equiv (H_0-z)^{-1}(\vec{x}_1,\vec{x}_2) = \frac{i}{\pi}\sum_{n=1}^{\infty} \frac{e^{ik_n(z)|x_1-x_2|}}{k_n(z)}\,\sin(ny_1)\,\sin(ny_2)\,, \tag{2.5}$$

where $\vec{x}_j=(x_j,y_j)$ and $k_n(z) := \sqrt{z-n^2}$. The complex energy here belongs to the resolvent set of H_0, $z\in \mathbb{C}\setminus[1,\infty)$; the function $G_0(\cdot,\cdot;z)$ is defined and smooth everywhere outside the hyperplane $\vec{x}_1=\vec{x}_2$, but the sum on the *rhs* may not converge absolutely if the longitudinal coordinates coincide, $x_1=x_2$. Moreover, the *rhs* makes sense also for all non–integer $z>1$, where it gives the boundary value of the kernel at the cut; one has to choose properly the branch of the square root in $k_n(z)$. We note for further purposes that for real z below the lowest threshold, $z<1$, the kernel is strictly positive, $G_0(\vec{x}_1,\vec{x}_2;z)>0$ for all mutually different $\vec{x}_1,\vec{x}_2\in\Omega$ — *cf.* [25, App. to Sec.XIII.12].

The sought kernel of the full resolvent can be obtained by means of the Krein formula [1, App.A], which yields the Ansatz

$$(H(\alpha,\vec{a})-z)^{-1}(\vec{x}_1,\vec{x}_2) = G_0(\vec{x}_1,\vec{x}_2;z) + \lambda(\alpha,\vec{a};z)G_0(\vec{x}_1,\vec{a};z)G_0(\vec{a},\vec{x}_2;z)\,. \tag{2.6}$$

To determine the function λ we use the fact that, by definition, $(H(\alpha,\vec{a})-z)^{-1}$ maps into the domain of $H(\alpha,\vec{a})$. Writing $\psi := (H(\alpha,\vec{a})-z)^{-1}\varphi$, $\psi_0 := (H_0-z)^{-1}\varphi$ for an arbitrary $\varphi \in L^2(\Omega)$, we get from the last relation

$$\psi(x,y) \,=\, \psi_0(x,y) \,+\, \frac{i\lambda}{\pi} \sum_{n=1}^{\infty} \frac{e^{ik_n(z)|x-a|}}{k_n(z)} \sin(ny)\, \sin(nb)\, \psi_0(a,b)\,. \tag{2.7}$$

Since $\psi_0 \in D(H_0)$ is smooth at $\vec{x} = \vec{a}$, the generalized boundary values can be written as $L_j(\psi,\vec{a}) = \mathcal{L}_j(\vec{a})\psi_0(\vec{a})$. The singularity at the "diagonal" of the resolvent kernel for planar regions with a smooth boundary is well known [29, Chaps.11 and 14]. It is not difficult, however, to find it directly: we have

$$\begin{aligned}\mathcal{L}_0(\vec{a}) \,&=\, \lim_{|\vec{x}-\vec{a}|\to 0} \frac{1}{\ln|\vec{x}-\vec{a}|} \left\{ 1 \,+\, \frac{i\lambda}{\pi} \sum_{n=1}^{\infty} \frac{e^{ik_n|x-a|}}{k_n} \sin(ny)\, \sin(nb) \right\} \\ &= \frac{i\lambda}{\pi} \lim_{u\to 0} \frac{1}{\ln u} \sum_{n=1}^{\infty} \frac{e^{ik_n u}}{k_n} \sin^2(nb) \,=\, \frac{\lambda}{\pi} \lim_{u\to 0} \frac{1}{\ln u} \sum_{n=1}^{\infty} \frac{e^{-(n-z/2n)u}}{n} \sin^2(nb)\,,\end{aligned}$$

because the differences $k_n^{-1} - (in)^{-1}$ and $e^{ik_n+n-z/2n} - 1$ are $\mathcal{O}(n^{-3})$. The limit is easily computed giving $\mathcal{L}_0(\vec{a}) \,=\, -\lambda/2\pi$; this in turn yields

$$\mathcal{L}_1(\vec{a}) \,=\, 1 \,+\, \frac{i\lambda}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin^2(nb)}{k_n} \,-\, \frac{1}{2in} \right).$$

Using now the boundary conditions (2.2) we find that

$$\lambda(\alpha,\vec{a};z) \,=\, \Lambda(\alpha,\vec{a};z)^{-1} \tag{2.8}$$

with

$$\Lambda(\alpha,\vec{a};z) \,=\, \alpha \,+\, \frac{1}{\pi i} \sum_{n=1}^{\infty} \left(\frac{\sin^2(nb)}{k_n(z)} \,-\, \frac{1}{2in} \right) \,=:\, \alpha \,-\, \xi(\vec{a},z)\,; \tag{2.9}$$

for the sake of brevity we shall sometimes drop a part or all of the arguments in the following.

2.3 Scaling behavior

Although the modification of these results for a general strip is simple, it is worthy of a brief comment. The logarithmic factor in the definition of the regularized boundary values refers to a fixed length scale. If we choose the natural scale of our problem for it, we replace the factor by $\ln|\nu(\vec{x}-\vec{a})|$, where $\nu := \pi/d$. In that case $\mathcal{L}_0(\vec{a})$ does not change, while $\mathcal{L}_1(\vec{a})$ acquires in addition to the scaling also the additive factor $\frac{1}{2\pi} \ln\left(\frac{\pi}{d}\right)$. The formula (2.8) reads then $\lambda(\alpha,\vec{a};z) = (\alpha - \xi_d(\vec{a},z))^{-1}$ with

$$\xi_d(\vec{a},z) \,=\, \frac{i}{d} \sum_{n=1}^{\infty} \left(\frac{\sin^2(nb\nu)}{k_n(z)} \,-\, \frac{d}{2\pi i n} \right) \,+\, \frac{1}{2\pi} \ln\left(\frac{\pi}{d}\right), \tag{2.10}$$

where $k_n(z) := \sqrt{z-(\pi n/d)^2}$. Hence changing the strip width is equivalent to a shift in the coupling constant for a fixed d ; one has to replace α by $\alpha - \frac{1}{2\pi}\ln\left(\frac{\pi}{d}\right)$. In particular, making the strip thin with α preserved corresponds to choosing a large positive α in the rescaled problem.

2.4 The resolvent à la Zorbas

The resolvent kernel can be obtained even without the boundary conditions (2.2) if we employ an integral formula due to Zorbas [32, Thm.4.1]. According to it, the operator family in question can be parametrized by a real θ ; the function $\Lambda(\alpha,\vec a;z)$ is replaced by

$$\begin{aligned}\Lambda(\theta,\vec a;z) \;=\; (1+e^{i\theta})^{-1}\Big\{(i-z)\int_\Omega G_0(\vec x,\vec a;z)G_0(\vec x,\vec a;i)\,d\vec x \\ -\; e^{i\theta}(i+z)\int_\Omega G_0(\vec x,\vec a;z)G_0(\vec x,\vec a;-i)\,d\vec x\Big\}\,.\end{aligned}$$

Using the formula (2.5) together with the identities $i-z=k_n(i)^2-k_n(z)^2$, $i+z=k_n(z)^2-k_n(-i)^2$ the *rhs* can be easily computed; we get

$$\Lambda(\theta,\vec a;z) \;=\; \frac{1}{\pi i}\sum_{n=1}^{\infty}\left\{\frac{1}{k_n(z)} - i\mathrm{Im}\left(\frac{1}{k_n(i)}\right) + \frac{i\sin\theta}{1+\cos\theta}\mathrm{Re}\left(\frac{1}{k_n(i)}\right)\right\}\sin^2(nb)\,. \tag{2.11}$$

Since Re $(k_n(z)^{-1})$ as well as $k_n(z)-k_n(i)$ are $\mathcal{O}(n^{-3})$, the series converges and can be splitted into the θ–independent and θ–dependent parts. Comparing the two parametrizations, we find

$$\alpha(\theta) \;=\; F(b) + \frac{\sin\theta}{1+\cos\theta}\,G(b)\,, \tag{2.12}$$

where

$$F(b) \;:=\; \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\sqrt{\frac{\sqrt{n^4+1}+n^2}{2(n^4+1)}}\,\sin^2(nb) - \frac{1}{2n}\right)\,,$$

$$G(b) \;:=\; \frac{1}{\pi}\sum_{n=1}^{\infty}\sqrt{\frac{\sqrt{n^4+1}-n^2}{2(n^4+1)}}\,\sin^2(nb)\,;$$

the function α clearly maps $(-\pi,\pi)$ bijectively onto $I\!R$.

2.5 Spectral properties

As a rank–one perturbation in the resolvent, the point interaction does not change the essential spectrum, *i.e.*, we have $\sigma_{ess}(H(\alpha,\vec a)) = [1,\infty)$ with the multiplicity $2[\sqrt{E/n^2}]$ at a point E, where $[\cdot]$ denotes the integer part.

Figure 1: The function ξ for different values of b.

To determine the point spectrum, one has to find poles of the resolvent, *i.e.*, to solve the equation

$$\xi(\vec{a}, z) \;=\; \alpha \tag{2.13}$$

for $z \in I\!R$. First consider the case $z < 1$. It is useful to introduce $\kappa_n(z) := -ik_n(z) = \sqrt{n^2 - z}$; then the function $\xi(\vec{a}, \cdot)$ can be expressed as

$$\begin{aligned} \xi(\vec{a}, z) \;&=\; \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin^2(nb)}{\kappa_n(z)} \;-\; \frac{1}{2n} \right) \\ &=\; \xi(\vec{a}, z') \;+\; \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\kappa_n(z') - \kappa_n(z)}{\kappa_n(z)\kappa_n(z')} \sin^2(nb)\,. \end{aligned}$$

The sum at the right hand side converges because the coefficients at $\sin^2(nb)$ decay like $\mathcal{O}(n^{-3})$ as $n \to \infty$. In particular, one can compute easily

$$\xi(\vec{a}, 0) \;=\; \frac{1}{2\pi} \ln(2 \sin b)\;; \tag{2.14}$$

this quantity is positive close to the center of the strip, $b \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, and nonpositive otherwise. One has

$$\frac{\partial \xi}{\partial z} \;=\; \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin^2(nb)}{\kappa_n(z)^3} \;>\; 0\,,$$

so the function is monotonously growing in $(-\infty, 1)$; it diverges at both endpoints. We have

$$\xi(\vec{a}, z) \;=\; \frac{\sin^2 b}{\pi\sqrt{1-z}} \;+\; \mathcal{O}(1) \qquad \text{as} \qquad z \to 1-\,,$$

while on the negative side the definition formula shows that $\xi(\vec{a}, \cdot)$ diverges logarithmically. A natural conjecture about its asymptotics is

$$\xi(\vec{a}, z) \;=\; -\,\frac{1}{4\pi} \ln\left(-\frac{z}{4}\right) \;+\; \frac{1}{2\pi}\, \Psi(1) \;+\; \mathcal{O}(1) \qquad \text{as} \qquad z \to -\infty\,,$$

where $\Psi(1) = -\gamma = 0.57721$ (compare with [1, Sec.I.5]). Figure 1 suggests that it is indeed the case. The asymptotics is independent of b but not uniformly in a as the relation (2.14) shows.

Another property of $\xi(\vec{a}, \cdot)$ is its monotonicity across a halfstrip,

$$\xi(\vec{a}, z) \;>\; \xi(\vec{a}', z) \qquad \text{iff} \qquad \left| b - \frac{\pi}{2} \right| \;<\; \left| b' - \frac{\pi}{2} \right|\,.$$

To prove it, we employ the identity $\sin^2(nb) - \sin^2(nb') = \sin(n(b+b'))\sin(n(b-b'))$ for $0 < b' < b \le \frac{\pi}{2}$; it shows that

$$\xi(\vec{a}, z) - \xi(\vec{a}', z) \;=\; G_0(0, b+b'; 0, b-b'; z)\,,$$

Figure 2: Eigenvalue plots for different values of α, b.

Figure 3: Eigenfunctions for different values of b.

so the result follows from the above mentioned positivity of the free-resolvent kernel.

Properties of solutions to Eq.(2.13) follow easily from this discussion: to any coupling constant $\alpha \in I\!R$ there is *just one eigenvalue* $\epsilon(\alpha,\vec{a})$ in $(-\infty,1]$. The function $\epsilon(\cdot,\vec{a})$ is monotonously growing,

$$\epsilon(\alpha,\vec{a}) \,>\, \epsilon(\alpha',\vec{a}) \qquad \text{iff} \qquad \alpha>\alpha' \,, \tag{2.15}$$

and has the following asymptotic behavior,

$$\begin{aligned} \epsilon(\alpha,\vec{a}) \;&=\; 1-\left(\frac{\sin^2 b}{\pi\alpha}\right)^2 +\,\mathcal{O}(\alpha^{-3}) \qquad \text{as} \qquad \alpha\to\infty\,, \\ \epsilon(\alpha,\vec{a}) \;&\approx\; -4\,e^{2(\Psi(1)-2\pi\alpha)} \qquad \text{as} \qquad \alpha\to-\infty\,. \end{aligned} \tag{2.16}$$

Hence $H(\alpha,\vec{a})$ preserves the property of the point interaction in the plane, namely that it has always a bound state except in the free case; also the asymptotics for $\alpha\to-\infty$ is the same. On the other hand, proximity of the boundary pushes the bound–state energy up,

$$\epsilon(\alpha,\vec{a}) \,<\, \epsilon(\alpha,\vec{a}') \qquad \text{iff} \qquad \left|b-\frac{\pi}{2}\right| < \left|b'-\frac{\pi}{2}\right|\,. \tag{2.17}$$

The resolvent formula (2.6) provides us also with the (non–normalized) wavefunction of the bound state through the residue at the pole, $\psi(\cdot;\alpha,\vec{a})=G_0(\cdot,\vec{a};\epsilon(\alpha,\vec{a}))$, so we have

$$\psi(x;\alpha,\vec{a}) \;=\; \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\kappa_n(\epsilon)|x-a|}}{\kappa_n(\epsilon)}\, \sin(ny)\,\sin(nb)\,. \tag{2.18}$$

If α decreases, ψ becomes well localized and resembles the Hankel eigenfunction of the point interaction in the plane, while in the limit $\alpha\to\infty$ we have

$$\psi(x;\alpha,\vec{a}) \,\approx\, \alpha\,\frac{\sin y}{\sin b}\,e^{-|x-a|\sin^2 b/\pi\alpha} \,+\, \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{e^{-|x-a|\sqrt{n^2-1}}}{\sqrt{n^2-1}}\,\sin(ny)\,\sin(nb)\;;$$

the leading term here is the product of $\chi_1(y)$ with the eigenfuction of the *one–dimensional* attractive point interaction of the strength $-(2/\pi\alpha)\sin^2 b$; recall that in view of the preceding section this corresponds by rescaling to the limit $d\to 0$.

These results are illustrated on Figs. 2 and 3.

There are *no eigenvalues embedded in the continuous spectrum.* The function $\xi(\vec{a},\cdot)$ is defined everywhere in $(1,\infty)$ except at the thresholds, $z=m^2$, but it is now complex because $\kappa_n(z)$ is negative imaginary for the open channels. In particular,

$$\operatorname{Im}\xi(\vec{a},z) \;=\; \frac{1}{\pi}\sum_{n=1}^{[\sqrt{z}]}\frac{\sin^2(nb)}{k_n(z)} \;>\; 0 \tag{2.19}$$

for a non–integer $z>1$, so the function $a-\xi(\vec{a},\cdot)$ has no zeros there. The argument does not apply if $z=m^2$ because the corresponding $k_m(z)=0$ and none of the expressions at the *rhs* of (2.6) makes sense. However, one can check directly that the singularities cancel mutually. Around $z=m^2$, the free resolvent kernel behaves as

$$G(\vec{x}_1,\vec{x}_2;z) \;=\; \frac{i}{\pi}\,\frac{\sin(my_1)\sin(my_2)}{\sqrt{z-m^2}}\,\mathrm{Err}(z)\,,$$

where $\mathrm{Err}(z)$ stands as a shorthand for $1+\mathcal{O}(\sqrt{z-m^2})$, and

$$\Lambda(\alpha,\vec{a};z) \;=\; \tilde{\alpha} \;-\; \frac{i}{\pi}\,\frac{\sin^2(mb)}{\sqrt{z-m^2}} \;+\; \mathcal{O}(\sqrt{z-m^2})$$

with $\tilde{\alpha} := \alpha+\frac{1}{2\pi m}-\frac{i}{\pi}\sum_{n\neq m}\left(\frac{\sin^2(nb)}{k_n(m^2)}-\frac{1}{2in}\right)$. Hence the full–resolvent kernel behaves asymptotically as

$$\frac{i\tilde{\alpha}}{\pi}\,\frac{\sin(my_1)\sin(my_2)}{\sqrt{z-m^2}}\left(\tilde{\alpha}\;-\;\frac{i}{\pi}\,\frac{\sin^2(mb)}{\sqrt{z-m^2}}\right)^{-1}\mathrm{Err}(z)$$

and cannot have a pole at $z=m^2$. In conclusion, no real $z\geq 1$ is an eigenvalue of $H(\alpha,\vec{a})$. On the other hand, the resolvent has resonance zeros in the complex plane as we shall show in Section 2.7 below.

2.6 Scattering

The existence of the wave operators is easy to establish because the point interaction is a rank–one perturbation in the resolvent, and therefore Birman–Kuroda theorem applies [25, Sec.XI.3]. What is more interesting, however, is how the scattering matrix looks like. To find it we employ the relation (2.7) in combination with (2.8) changing slightly the notation: to any $\psi\in D(H(\alpha,\vec{a}))$ and a nonreal z there is a unique decomposition

$$\psi(\vec{x}) \;=\; \psi_z(\vec{x}) \;+\; \frac{1}{\alpha-\xi(\vec{a},z)}\,G_0(\vec{x},\vec{a};z)\psi_z(\vec{a})$$

with $\psi_z\in D(H_0)$ and $(H(\alpha,\vec{a})-z)\psi=(H_0-z)\psi_z$. If we choose

$$\psi_z^\varepsilon \;:=\; e^{ik_n(z)x-\varepsilon x^2}\chi_n(y) \tag{2.20}$$

for ψ_z, the corresponding $\psi=:\psi^\varepsilon$ belongs to $D(H(\alpha,\vec{a}))$ for all $\varepsilon>0$ and

$$((H(\alpha,\vec{a})-z)\psi^\varepsilon)(\vec{x}) \;=\; 2\varepsilon(2\varepsilon x^2-1-2ik_n(z)x)\psi_z^\varepsilon(\vec{x})\,.$$

Since the *rhs* converges in L^2 sense as z approaches the real line, $\psi^\varepsilon \in D(H(\alpha,\vec{a}))$ for $z \in [1,\infty)$ and the last relation holds again. Of course, ψ^ε ceases to be square integrable as $\varepsilon \to 0+$, but the pointwise limit exists and

$$\psi(\vec{x}) \;=\; e^{ik_n(z)x}\chi_n(y) \,+\, \frac{e^{ik_n(z)a}}{\alpha-\xi(\vec{a},z)}\, G_0(\vec{x},\vec{a};z)\chi_n(b)\,. \tag{2.21}$$

This function is locally square integrable, satisfies the appropriate boundary conditions and solves $(H(\alpha,\vec{a})-z)\psi=0$ as a differential equation. Substituting for G_0 we obtain easily the reflection and transmission amplitudes, $r_{nm}(z)$ and $t_{nm}(z)$, respectively, for the case when the incident wave corresponds to the n–th transverse mode and the scattered one appears in the m–th mode; they are given by the relation

$$(t_{nm}(z)-\delta_{nm})\, e^{i(k_m-k_n)a} \;=\; r_{nm}(z)\, e^{-i(k_n+k_m)a} \;=\; \frac{i}{\pi}\, \frac{\sin(nb)\sin(mb)}{k_m(z)(\alpha-\xi(\vec{a},z))}\,. \tag{2.22}$$

It is necessary to assume, of course, that both the involved channels are open, *i.e.*, $z>\max(n^2,m^2)$.

Since the point interaction gives rise to a rank–one perturbation in the resolvent, the Kato–Birman theory yields also the completeness of the wave operators, *i.e.*, unitarity of the S–matrix. In particular, the "diagonal" part of the unitarity condition can be expressed in terms of the transmission and reflection coefficients as

$$\sum_{m=1}^{[\sqrt{z}]} k_m(|t_{nm}|^2+|r_{nm}|^2) \;=\; k_n\,, \tag{2.23}$$

and has an obvious meaning of conservation of probability flow (taking into account different particle velocities in different channels). It is not difficult in the present case to check the identity (2.23) directly. Since $t_{nn}=1+r_{nn}e^{-2ik_na}$, it reduces to

$$k_n\,\mathrm{Re}\,\left(r_{nn}e^{-2ik_na}\right) \,+\, 2\sum_{m=1}^{[\sqrt{z}]} k_m|r_{nm}|^2 \;=\; 0\,;$$

substituting from (2.22) and multiplying both sides by $|\alpha-\xi(\vec{a},z)|^2$, we arrive at the relation

$$\mathrm{Im}\,\xi(\vec{a},z) \;=\; \frac{1}{\pi}\sum_{m=1}^{[\sqrt{z}]} \frac{\sin^2(mb)}{k_m(z)}\,,$$

which is nothing else than (2.19); recall that the closed channels do not contribute to the imaginary part of $\xi(\vec{a},z)$. In the same way one can check the remaining parts of the unitarity condition which read

$$\sum_{m=1}^{[\sqrt{z}]} k_m(t_{nm}\bar{t}_{sm}+r_{nm}\bar{r}_{sm}) \;=\; \delta_{ns}k_n$$

for $n \neq s$ and

$$\sum_{m=1}^{[\sqrt{z}]} k_m \left(t_{nm}\overline{r}_{sm}e^{2ik_m a} + r_{nm}\overline{t}_{sm}e^{-2ik_m a} \right) \;=\; 0\,,$$

since the "elementary block" of the S–matrix describing the transitions between the n–th and m–th channel is of the form

$$S_{nm} \;=\; \sqrt{\frac{k_m}{k_n}} \begin{pmatrix} t_{nm} & r_{nm} \\ \tilde{r}_{nm} & \tilde{t}_{nm} \end{pmatrix}\,, \tag{2.24}$$

where the tilded quantities in the second row are obtained by changing the sign of a, *i.e.*, $\tilde{r}_{nm} := r_{nm}e^{-2i(k_n+k_m)a}$ *etc.*

2.7 Resonances

Every square root in the definition of the channel momentum $k_n(z) := \sqrt{z-n^2}$ gives rise to a cut; hence the free resolvent kernel (2.5) as well as other quantities derived from it are in general multivalued with infinitely sheeted Riemann surfaces. In particular, the pole condition (2.13) may have solutions on the other (nonphysical) sheets which produce resonances.

For simplicity we introduce $q_n(z) := \sqrt{z-n^2}$ which takes values always in the upper complex halfplane or at the positive real halfline. On the $(N+1)$–th sheet we have then

$$k_n(z) \;:=\; \begin{cases} -q_n(z) & \dots & n \le N \\ \;\;q_n(z) & \dots & n > N \end{cases}$$

Denoting by $\xi_N(\vec{a},\cdot)$ the corresponding branch of the function $\xi(\vec{a},\cdot)$ we can write the pole condition (2.13) explicitly as

$$\begin{aligned} \alpha - \mathrm{Re}\,\xi_N(\vec{a},z) \;&=\; \alpha + \frac{1}{\pi}\sum_{n=1}^{N}\left(-\mathrm{Im}\left(\frac{1}{q_n(z)}\right)\sin^2(nb) + \frac{1}{2n}\right) \\ &\quad + \frac{1}{\pi}\sum_{n=N+1}^{\infty}\left(\mathrm{Im}\left(\frac{1}{q_n(z)}\right)\sin^2(nb) + \frac{1}{2n}\right) \;=\; 0\,, \\ -\,\mathrm{Im}\,\xi_N(\vec{a},z) \;&=\; \frac{1}{\pi}\sum_{n=1}^{N}\mathrm{Re}\left(\frac{1}{q_n(z)}\right)\sin^2(nb) \\ &\quad - \frac{1}{\pi}\sum_{n=N+1}^{\infty}\mathrm{Re}\left(\frac{1}{q_n(z)}\right)\sin^2(nb) \;=\; 0\,. \end{aligned} \tag{2.25}$$

For a weak coupling, there is generically one resonance pole close to each threshold, with exception of the lowest one; the resonance is absent if $(N+1)b/\pi$ is integer so that the incident wave has a node at the impurity and does not feel the perturbation. To show this more explicitly, let us rewrite $\xi_N(\vec{a},z) = \xi^N(\vec{a},z) + \tilde{\xi}^N(\vec{a},z)$, where the two terms sum contributions from the evanescent and propagating modes, $n \ge N+1$

and $1 \le n \le N$, respectively. Suppose we switch the latter on with the help of an additional parameter, *i.e.*, we look for solutions of the condition

$$F(z,\eta) \,:=\, \alpha - \xi^N(\vec a, z) - \eta\tilde\xi^N(\vec a, z) \,=\, 0\,. \tag{2.26}$$

If $\eta = 0$, we can repeat the argument of Sec. 2.5: we find that the function $\xi^N(\vec a,\cdot)$ is monotonously growing between $-\infty$ and ∞ when z runs over the interval $(-\infty,(N+1)^2)$; hence to a given α there is just one $z_N^0(\alpha)$ such that $F(z_N^0(\alpha),0) = 0$. Moreover, the leading behavior of $\xi^N(\vec a,\cdot)$ as $z \to (N+1)^2-$ is again given by a single term, so the "eigenvalues" behave as

$$z_N^0(\alpha) \,=\, (N+1)^2 - \left(\frac{\sin^2((N+1)b)}{\pi\alpha}\right)^2 + \mathcal{O}(\alpha^{-3}) \tag{2.27}$$

in the weak–coupling case, $\alpha \to \infty$.

For a nonzero η the condition (2.26) can be solved perturbatively by means of the implicit–function theorem. We have

$$\begin{aligned}
\left.\frac{\partial F}{\partial\eta}\right|_{(z_N^0(\alpha),0)} &= -\tilde\xi^N(z_N^0(\alpha)) \,=\, \frac{i}{\pi}\sum_{n=1}^{N}\left(\frac{\sin^2(nb)}{\sqrt{(N+1)^2-n^2}} + \frac{1}{2in}\right) + \mathcal{O}(\alpha^{-1})\,,\\
\left.\frac{\partial F}{\partial z}\right|_{(z_N^0(\alpha),0)} &= \frac{i}{2\pi}\sum_{n=N+1}^{\infty}\frac{\sin^2(nb)}{q_N(z_N^0(\alpha))^3} \,=\, \frac{\pi^3\alpha^3}{|\sin((N+1)b))|} + \mathcal{O}(\alpha^0)\,;
\end{aligned}$$

dividing these quantities we obtain $-(\partial z_N^\eta/\partial\eta)_{\eta=0}$. Moreover, the remainder term coming from $\partial^2 z_N^\eta/\partial\eta^2$ is $\mathcal{O}(\alpha^{-4})$, so for sufficiently large positive α we may use the expansion up to $\eta = 1$, obtaining an asymptotic formula for the resonance–pole position,

$$z_N(\alpha) \,=\, (N+1)^2 - 2i\,\frac{|\sin((N+1)b)|}{\pi^3\alpha^3}\sum_{n=1}^{N}\left(\frac{\sin^2(nb)}{\sqrt{(N+1)^2-n^2}} + \frac{1}{2in}\right) + \mathcal{O}(\alpha^{-4})\,; \tag{2.28}$$

the real shift appearing in (2.27) has been absorbed here into the error term. In particular, the resonance width behaves in the weak–coupling case as

$$\Gamma_N(\alpha) \,:=\, -2\,\mathrm{Im}\, z_N(\alpha) \,=\, 4\,\frac{|\sin((N+1)b)|}{\pi^3\alpha^3}\sum_{n=1}^{N}\frac{\sin^2(nb)}{\sqrt{(N+1)^2-n^2}} + \mathcal{O}(\alpha^{-4})\,. \tag{2.29}$$

If the coupling is not weak, resonance poles can be found from a numerical solution of the conditions (2.25); an example is shown on Figure 4. The pole trajectories can be followed for all values of α, however, only weak perturbations produce a substantial resonance scattering effect because the pole residues decrease rapidly with the coupling strength; this also is shown on Figure 4.

Figure 4: Pole trajectory in the z–plane.

3 A finite number of point interactions

Our next aim is to extend the above results to the situation when there is a finite number N of point interactions in the strip Ω. We suppose that their positions are $\vec{a}_j := (a_j, b_j)$ and the coupling constants α_j ; for the sake of brevity we denote $\vec{a} := \{\vec{a}_1, \dots, \vec{a}_N\}$ and $\alpha := \{\alpha_1, \cdots, \alpha_N\}$. The corresponding Hamiltonian $H(\alpha, \vec{a})$ is again given by (2.3) with the boundary conditions (2.2) replaced by

$$L_1(\psi, \vec{a}_j) \,+\, 2\pi\alpha_j L_0(\psi, \vec{a}_j) \,=\, 0\,, \qquad j \,=\, 1, \dots, N\,. \tag{3.1}$$

As in the one–center case, any of these point interactions may be switched off by means of the limit $\alpha_j \to \infty$.

3.1 The resolvent

We shall again start from derivation of the resolvent. Since the deficiency indices of the initial symmetric operator are now (N, N), a natural Ansatz is

$$(H(\alpha, \vec{a}) - z)^{-1}(\vec{x}_1, \vec{x}_2) \,=\, G_0(\vec{x}_1, \vec{x}_2; z) \,+\, \sum_{j,k=1}^{N} \lambda_{jk} G_0(\vec{x}_1, \vec{a}_j; z) G_0(\vec{a}_k, \vec{x}_2; z)\,, \tag{3.2}$$

where $G_0(\cdot, \cdot; z)$ is the free–resolvent kernel (2.5). Functions $\psi \in D(H(\alpha, \vec{a}))$ and $\psi_0 \in D(H_0)$ introduced as above are then related by

$$\psi(\vec{x}) \,=\, \psi_0(\vec{x}) \,+\, \sum_{j,k=1}^{N} \lambda_{jk} G_0(\vec{x}, \vec{a}_j; z)\, \psi_0(\vec{a}_k) \tag{3.3}$$

and one can compute easily the boundary values at the interaction support,

$$\begin{aligned} L_0(\psi, \vec{a}_m) \,&=\, -\sum_{j,k=1}^{N} \frac{\lambda_{jk}}{2\pi}\, \delta_{jm}\, \psi_0(\vec{a}_k)\,, \\ L_1(\psi, \vec{a}_m) \,&=\, -\,\psi_0(\vec{a}_m) \,+\, \sum_{j,k=1}^{N} \lambda_{jk} \left(\frac{i}{\pi}\, \delta_{jm} \sum_{n=1}^{\infty} \left(\frac{\sin^2(nb_j)}{k_n(z)} \,-\, \frac{1}{2in} \right) \right. \\ &\qquad \left. +\, (1 - \delta_{jm}) G_0(\vec{a}_j, \vec{a}_m; z) \right) \psi_0(\vec{a}_k)\,. \end{aligned}$$

Substituting this into (3.1) and using the fact that $\psi_0(\vec{a}_k)$ are independent quantities, we find

$$\lambda(\alpha, \vec{a}; z) \,=\, \Lambda(\alpha, \vec{a}; z)^{-1}\,, \tag{3.4}$$

where $\Lambda(\alpha,\vec{a};z)$ is now an $N\times N$ matrix given by

$$\begin{aligned}\Lambda_{jj} &= \alpha_j - \frac{i}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin^2(nb_j)}{k_n(z)} - \frac{1}{2in}\right),\\ \Lambda_{jm} &= -G_0(\vec{a}_j,\vec{a}_m;z) = -\frac{i}{\pi}\sum_{n=1}^{\infty}\frac{e^{ik_n(z)|a_j-a_m|}}{k_n(z)}\sin(nb_j)\sin(nb_m)\,,\quad j\neq m\,;\end{aligned} \tag{3.5}$$

we shall again drop the arguments occassionally if they are clear from the context. Notice that the scaling argument of the Section 2.3 can be easily adapted to the present situation; it shows that changing d is equivalent to the *simultaneous* shift of all the coupling constants α_j on $-\frac{1}{2\pi}\ln\left(\frac{\pi}{d}\right)$ in the rescaled strip.

3.2 The discrete spectrum

A finite number of point interactions represents a finite–rank perturbation in the resolvent, and therefore the conclusion of Sec.2.5 about the essential spectrum remains valid, $\sigma_{ess}(H(\alpha,\vec{a}))=[1,\infty)$. The discrete spectrum is again determined by poles of the resolvent coming from the coefficients λ_{jk} in (3.2); they are given by the condition

$$\det\Lambda(\alpha,\vec{a},z) = 0\,. \tag{3.6}$$

Comparing with the case $N=1$, it is now slightly more complicated to determine the eigenfunctions. One can adapt the argument from [1, Sec.II.1]. Suppose that $H:=H(\alpha,\vec{a})$ satisfies $H\varphi=z\varphi$ for some $z\in I\!R$. We pick an arbitrary $z'\in\rho(H)$; then in analogy with (3.3) there is a vector $\psi_0\in D(H_0)$ which allows us to express the eigenvector φ as

$$\varphi = \psi_0+\sum_{j=1}^{N} d_j G_0(\cdot,\vec{a}_j;z')\,, \tag{3.7}$$

where the coefficients are given by $d_j:=\sum_{k=1}^{N}(\Lambda(z')^{-1})_{jk}\psi_0(\vec{a}_k)$ and the relations

$$(H_0-z')\psi_0 = (H-z')\varphi = (z-z')\varphi$$

are valid. Applying $(H_0-z')^{-1}$ to the last identity we obtain

$$\psi_0 = (z-z')\left[(H_0-z')^{-1}\psi_0+\sum_{j=1}^{N} d_j(H_0-z')^{-1}G_0(\cdot,\vec{a}_j;z')\right],$$

and this in turn yields

$$(H_0-z)\psi_0 = (z-z')\sum_{j=1}^{N} d_j G_0(\cdot,\vec{a}_j;z')\,. \tag{3.8}$$

If $z < 1$, the resolvent $(H_0 - z)^{-1}$ exists and may be applied to both sides of the last relation giving

$$\psi_0 \, = \, \sum_{j=1}^{N} d_j \left(G_0(\cdot, \vec{a}_j; z) - G_0(\cdot, \vec{a}_j; z') \right) \; ; \tag{3.9}$$

we have employed here the first resolvent identity. Substituting into (3.7) we get an expression for the (non–normalized) eigenfunction

$$\varphi(\vec{x}) \, = \, \sum_{j=1}^{N} d_j G_0(\vec{x}, \vec{a}_j; z) \, . \tag{3.10}$$

To determine the coefficients, we notice that the relation (3.9) together with (3.5) gives

$$\psi_0(\vec{a}_j) \, = \, \sum_{m=1}^{N} \left(\Lambda(z')_{jm} - \Lambda(z)_{jm} \right) d_m \; ;$$

on the other hand, the above mentioned expression of d_j is equivalent to $\psi_0(\vec{a}_j) = \sum_{m=1}^{N} \Lambda(z')_{jm} d_m$. Hence

$$\sum_{m=1}^{N} \Lambda(z)_{jm} d_m \, = \, 0 \, , \tag{3.11}$$

i.e., $d := (d_1, \ldots, d_N)$ is an eigenvector of $\Lambda(\alpha, \vec{a}, z)$ corresponding to zero eigenvalue. Inverting the argument as in [1] one can check that any solution to (3.11) determines an eigenvector of $H(\alpha, \vec{a})$.

The next question concerns the existence of solutions to Eqs(3.6) and (3.11). For convenience we introduce again $\kappa_n = -ik_n(z) = \sqrt{n^2 - z}$ which is positive for $z < 1$. We have then

$$\Lambda_{jj} \, = \, \alpha_j \, - \, \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin^2(nb_j)}{\kappa_n(z)} \, - \, \frac{1}{2n} \right) \, ,$$

and an analogous expression for the nondiagonal elements. If $z \to -\infty$ the matrix behaves as

$$\Lambda(\alpha, \vec{a}, z) \, = \, \frac{1}{4\pi} \ln \left(-\frac{z}{4} \right) I \, + \, \mathcal{O}(z^0) \, ,$$

so all its eigenvalues tend to $+\infty$ as $z \to -\infty$. On the other hand, for $z \to 1-$ we have

$$\Lambda(\alpha, \vec{a}, z) \, = \, -\frac{1}{\pi\sqrt{1-z}} M_1 \, + \, \mathcal{O}(1) \, ,$$

where $M_1 := (\sin b_j \, \sin b_m)_{j,m=1}^{N}$. This matrix has, in particular, an eigenvector $(\sin b_1, \ldots, \sin b_N)$ corresponding to the *positive* eigenvalue $\sum_{j=1}^{N} \sin^2 b_j$, and therefore one of the eigenvalues of $\Lambda(\alpha, \vec{a}, z)$ tends to $-\infty$ as $z \to 1-$. The elements of $\Lambda(\alpha, \vec{a}, z)$ are continuous functions of z, hence the same is true for its eigenvalues. It follows that there is an eigenvalue which crosses zero, *i.e.*, $H(\alpha, \vec{a})$ has *at least one eigenvalue.*

Figure 5: Eigenfunctions for three point perturbations. (a) The ground state. (b) The second excited state.

In fact a stronger claim can be made concerning the eigenvalues of $\Lambda(\alpha,\vec a,z)$. By a straightforward differentiation we find

$$\frac{d}{dz}\,\Lambda(z)_{jm} \;=\; -\,\frac{1}{2\pi}\,\sum_{n=1}^{\infty} \frac{e^{-|a_j-a_m|\sqrt{n^2-z}}}{(n^2-z)^{3/2}}\, \left(1+|a_j-a_m|\sqrt{n^2-z}\right)\, \sin(nb_j)\sin(nb_m)\,.$$

The matrix function $\Lambda(\cdot)$ is monotonous if for any $c\in\mathbb{C}^N$ the quantity

$$\begin{aligned}\frac{d}{dz}\,(c,\Lambda(z)c) \;&=\; -\,\frac{1}{2\pi}\,\sum_{n=1}^{\infty}(n^2-z)^{-3/2} \sum_{j,m=1}^{N} \overline{c_j\sin(nb_j)}\, c_m\sin(nb_m) \\ &\quad\times e^{-|a_j-a_m|\sqrt{n^2-z}}\left(1+|a_j-a_m|\sqrt{n^2-z}\right)\end{aligned}$$

has a definite sign (is nonpositive in our case). The expression on the right hand side tells us that this is true provided the function $f:\ f(x)=e^{-\kappa|x|}(1+\kappa|x|)$ is of positive type for any $\kappa>0$. The last named property follows from the identity

$$(1+\kappa|x|)\,e^{-\kappa|x|} \;=\; \frac{2\kappa^3}{\pi}\int_{\mathbb{R}} \frac{e^{ipx}}{(p^2+\kappa^2)^2}\,dp\,,$$

because by Bochner's theorem [25, Sec.IX.2] a function is of positive type *iff* its Fourier image is positive. In fact, since the measure in the last integral is pointwise positive, $\frac{d}{dz}\Lambda(z)$ is even strictly positive; it means that all the eigenvalues of $\Lambda(\alpha,\vec a;z)$ are decreasing functions of z.

The most important consequence of this fact is the *nondegeneracy of the ground state.* To prove it one has to check that the lowest eigenvalue of $\Lambda(z)$ is simple for any $z\in(-\infty,1)$, which is equivalent to the claim that the matrix semigroup $\{\,e^{-t\Lambda(z)}\,:\ t\ge 0\,\}$ is positivity preserving [25, Sec.XIII.12]. The last property is ensured if all the nondiagonal elements of $\Lambda(z)$ are negative; we have $\Lambda(z)_{jm}=-G_0(\vec a_j,\vec a_m;z)$ by (3.5) so the desired result follows from the positivity of the free–resolvent kernel. The coefficients may be therefore chosen of the same sign for the ground state; in fact, as positive because $d_{j_0}=0$ would mean that the eigenfunction (3.10) is smooth at $\vec x=\vec a_{j_0}$ so the corresponding interaction is absent, $\alpha_{j_0}=\infty$.

3.3 Embedded eigenvalues

Let us now ask whether the continuous spectrum of $H:=H(\alpha,\vec a)$ may contain eigenvalues if $N>1$. Suppose that $H\varphi=z\varphi$ for some $z>1$. We employ again the expression (3.7) for the eigenvector and write ψ_0 as a series, $\psi_0(\vec x)=\sum_{n=1}^{\infty} g_n(x)\chi_n(y)$, with the coefficient functions $g_n\in L^2(\mathbb{R})$. Substituting this into (3.8) and using the fact that $\{\chi_n\}$ is an orthonormal basis in $L^2(0,d)$, we obtain the following system of equations,

$$-g_n''(x) - k_n(z)^2 g_n(x) \;=\; \frac{i}{2}\,(z-z')\,\sum_{j=1}^N d_j\chi_n(b_j)\,\frac{e^{ik_n(z')|x-a_j|}}{k_n(z')}$$

for $n=1,2,\dots$; the Fourier–Plancherel operator transforms it to

$$(p^2-z+n^2)\hat g_n(p) \;=\; \frac{z-z'}{2\pi}\,\sum_{j=1}^N d_j\chi_n(b_j)\,\frac{e^{-ipa_j}}{p^2-z'+n^2}\,. \tag{3.12}$$

If $g_n\in L^2$ the same is true for $\hat g_n$; this is impossible if

$$z \;>\; n^2 \tag{3.13}$$

and the *rhs* of (3.12) is nonzero at $\pm p_n$, where $p_n:=\sqrt{n^2-z}$, since $\hat g_n^2$ would have then a nonintegrable singularity. It is clear that it is the factor p^2-z+n^2 which matters, because z' belongs to $\rho(H)$ by assumption.

Hence the *rhs* of (3.12) has to be zero; we want to conclude that $\hat g_n=0$. If $N>1$ and a_j are not the same, it might happen that the *rhs* is not zero identically. However, we can choose a common phase factor to be put in front of the sum; then the square integrability requires

$$\sum_{j=1}^N d_j\chi_n(b_j)\,e^{\mp ip_n(a_j-a)} \;=\; 0$$

for an arbitrary a . If all the a_j are mutually different (mod $2\pi p_n^{-1}$) it follows that $d_j\chi_n(b_j)=0$ for each j . On the other hand, if some of them coincide we find $\sum_j d_j\chi_n(b_j)=0$ where the index runs through the values with the same longitudinal coordinate a_j , and therefore $\hat g_n=0$ again, *i.e.*, $\hat g_n$ may be nonzero at most if some a_j differ by multiples of $2\pi p_n^{-1}$.

Consider now an arbitrary $g\in L^2(I\!R)$ and n satisfying the condition (3.13). Using (3.7) and (2.5) we find

$$(g\chi_n,\varphi) \;=\; (\hat g,\hat g_n) \;+\; \frac{i}{2k_n(z')}\,\sum_{j=1}^N d_j\chi_n(b_j)(g,e^{ik_n(z')|\cdot-a_j|})\,,$$

where the inner product on the *rhs* refers to $L^2(I\!R)$ and $\hat g_n$ in the first term can be expressed by (3.12). If $d_j\chi_n(b_j)=0$ for each j , the right hand side is zero. In the exceptional case mentioned above we use the fact that the *lhs* is independent of z' . The explicit expression for d_j together with the asymptotic behavior of Λ show that $d_j\to 0$ as $z'\to-\infty$; the same is true for the inner product in the second term as well as for $(\hat g,\hat g_n)$; together we find $(g\chi_n,\varphi)=0$ again. We conclude that $z>1$ *cannot be an eigenvalue corresponding to an eigenvector from the subspace* $\bigoplus_{n=1}^{[\sqrt z]} L^2(I\!R)\otimes\{\chi_n\}$.

Figure 6: An embedded eigenvalue due to symmetry.

On the other hand, the condition (3.13) in the above argument is crucial; in the case $N > 1$ the operator $H(\alpha, \vec{a})$ *can have embedded eigenvalues* with eigenfunctions in the orthogonal complement of the mentioned subspace. As the simplest example, consider a pair of point perturbations with the *same* coupling constant α placed at $\vec{a}_1 := (0, b)$ and $\vec{a}_1 := (0, \pi - b)$. The eigenvalue problem can be divided into the part symmetric and antisymmetric with respect to the strip axis, the antisymmetric part being obtained by scaling of the single–center problem with $\vec{a} := (0, 2b)$ and coupling constant $\alpha - \frac{1}{2\pi} \ln 2$. The scaled eigenvalue tends to 4 as $\alpha \to \infty$, hence it is embedded in the continuum for all α large enough. This is illustrated on Figure 6. In the same way, one can construct other examples of embedded eigenvalues. Their common feature is the existence of a symmetry which prevents the (energetically allowed) decay of the eigenstate; a violation of the symmetry turns these eigenvalues into resonances. Recall also that embedded eigenvalues due to symmetric obstacles in *Neumann* waveguides have been treated recently in [11].

3.4 The limits of strong and weak coupling

If all the point interactions under consideration are strong, *i.e.*, having the α_j's large negative, one naturally expects the corresponding bound states to be strongly localized and weakly influenced both by the other perturbations and by the boundary. To show that this is indeed the case, let us write the matrix $\Lambda(\alpha, \vec{a}; z)$ in the form

$$\Lambda(z) = \left(\left(\alpha_j + \frac{1}{4\pi} \ln\left(-\frac{z}{4}\right)\right) \delta_{jk}\right) \left[I + \left(\left(\alpha_j + \frac{1}{4\pi} \ln\left(-\frac{z}{4}\right)\right) \delta_{jk}\right)^{-1} \tilde{\Lambda}(z)\right],$$

where $\tilde{\Lambda}(z)$ is the remainder matrix, which is independent of α and has a bounded norm as $z \to -\infty$. Given a finite energy interval $I \subset (-\infty, 1)$ one can always choose the α_j's large enough negative so that the two matrices in the above product are regular, and no eigenvalues are contained in I. Consequently, the roots of Eq.(2.13) in the strong–coupling limit are situated in the region where $\Lambda(z)$ is dominated by the diagonal part; then there are exactly N eigenvalues, including a possible degeneracy, and

$$\epsilon_j(\alpha, \vec{a}) \approx 4\, e^{-4\pi\alpha_j} \qquad \text{as} \qquad \max_{1 \le j \le N} \alpha_j \to -\infty\,. \tag{3.14}$$

One can conjecture that the asymptotics is in fact the same as in (2.16). The corresponding eigenfunctions coincide in the leading order with the strongly coupled one–center eigenfunctions of Section 2.5.

For a weak coupling the situation is different. We again restrict our attention to the case when *all* the interactions are weak, *i.e.*, the corresponding α_j's are large positive. We denote $A := \mathrm{diag}(\alpha_1, \dots, \alpha_N)$ and use the decomposition

$$\Lambda(z) = \left(A - \frac{1}{\pi\sqrt{1-z}}\, M_1\right) \left[I + \left(A - \frac{1}{\pi\sqrt{1-z}}\, M_1\right)^{-1} \tilde{\Lambda}(z)\right],$$

where $\tilde{\Lambda}(z)$ is a remainder independent of α, whose norm is bounded as $z \to 1-$. In the same way as above, to a given $z_0 < 1$ one can always choose the α_j's sufficiently large so that the two matrices are regular and no eigenvalues are contained in $(-\infty, z_0]$. In distinction to the previous case, however, $\Lambda(z)$ is asymptotically not a multiple of the unit matrix but rather a rank–one operator on $\mathbb{C}^N$ as we have seen in the previous section. Hence only one eigenvalue of $H(\alpha, \vec{a})$ can approach the threshold of the continuous spectrum; in combination with the above argument this means that if all the point interactions are weak enough, $H(\alpha, \vec{a})$ has *a single bound state.*

This is analogous to the behavior of interactions whose support has a nonzero measure. Recall that a two–dimensional Schrödinger operator with a weak potential V has always one bound state provided V is attractive in the mean and satisfies some decay restrictions [26], and the same is true in a tube, whatever dimension its cross section is, provided the potential is replaced by its projection on the lowest transverse mode [12].

To find the corresponding bound state in our case, we have to solve the spectral problem for $M_1 + \eta A$, where $\eta := -\pi\sqrt{1-z}$; we are interested in the eigenvalue that approaches $\sum_{j=1}^N \sin^2 b_j$ as $z \to 1-$. An elementary perturbative argument yields the expression

$$\sum_{j=1}^N \sin^2 b_j \,-\, \pi\sqrt{1-z}\, \frac{\sum_{j=1}^N \alpha_j \sin^2 b_j}{\sum_{j=1}^N \sin^2 b_j} \,+\, \mathcal{O}(\eta^2)\,;$$

putting it equal to zero, we obtain the energy of the weakly bound state,

$$\epsilon(\alpha, \vec{a}) \approx 1 - \left(\frac{\left(\sum_{j=1}^N \sin^2 b_j\right)^2}{\pi \sum_{j=1}^N \alpha_j \sin^2 b_j} \right)^2 \tag{3.15}$$

as $\min_{1\le j\le N} \alpha_j \to \infty$. Since the range of M_1 is spanned by $(\sin b_1, \dots, \sin b_N)$, the corresponding asymptotic expression for the eigenfunction is

$$\begin{aligned}\psi(x; \alpha, \vec{a}) \;\approx\; & \sin y\, \frac{\sum_{k=1}^N \alpha_k \sin^2 b_k}{\left(\sum_{k=1}^N \sin^2 b_k\right)^2} \sum_{j=1}^N e^{-\sqrt{1-\epsilon}|x-a_j|} \sin^2 b_j \\ & + \sum_{n=2}^\infty \sin(ny) \sum_{j=1}^N \frac{e^{-\sqrt{n^2-1}|x-a_j|}}{\sqrt{n^2-1}} \sin(nb_j) \sin b_j\,.\end{aligned}$$

The leading term is again a product of $\chi_1(y)$ with a linear combination of the eigenfunctions of one–dimensional point interactions placed at $a_j,\ j = 1, \dots, N$.

3.5 More about bound states

The spectral condition (3.6) makes it possible to derive also other properties of the discrete spectrum for the N–center Hamiltonian. We have seen that it has N bound states and just one bound state in the strong– and weak–coupling limits, respectively. One may ask, more generally, what are the regions in the space $I\!R^N$ of coupling constants, where the dimension of the discrete spectrum is $n = 1, 2, \dots, N$.

First consider the two–center case. As we already mentioned $\det \Lambda(\alpha, \vec{a}; z)$ tends to $+\infty$ as $z \to -\infty$. On the other hand, the behavior of this quantity as $z \to 1-$ depends on the parameters $\alpha_j,\ \vec{a}_j,\ j = 1, 2$; the limit may be either $+\infty$ or $-\infty$, *i.e.*, the equation $\det \Lambda(\alpha, \vec{a}; z) = 0$ can have one or two roots. The corresponding areas in the (α_1, α_2)–plane are divided by the line $P_2(\alpha_1, \alpha_2) = 0$, where

$$P_2(\alpha_1, \alpha_2) \;:=\; \alpha_1 \sin^2 b_2 \,+\, \alpha_2 \sin^2 b_1 \,+\, \frac{\sin^2 b_1 + \sin^2 b_2}{2\pi} \,-\, \frac{2}{\pi}\, |a_1 - a_2| \sin^2 b_1 \sin^2 b_2 \,;$$

there is only one bound state if $P_2(\alpha_2, \alpha_2) \ge 0$ and two in the opposite case. Inspecting the expression more closely we see that for a fixed $\vec{a}_1$ it represents a family of lines with the common intersection

$$\left[\frac{4|a_1 - a_2| \sin^2 b_1 - 1}{2\pi}, -\frac{1}{2\pi} \right].$$

Obviously, we can change the number of bound states for fixed coupling constants by changing the slope of the line (it depends on b_1, b_2 only) or shifting the line in the α_1–direction changing $|a_1 - a_2|$.

Similar conclusions can be made for a larger number of perturbations. Although parametric expressions of the hypersurfaces in $I\!R^N$ dividing regions with different number of bound states are easy to derive investigating $\lim_{z\to 1-} \det \Lambda(\alpha, \vec{a}; z)$, the formulas are somewhat lengthy and we do not write them down here. We restrict ourselves to the observation that

$$P_N(\alpha_1, \dots, \alpha_N) \;=\; P_{N-1}(\alpha_1, \dots, \alpha_{N-1})\, \alpha_N \,+\, Q_{N-1}(\alpha_1, \dots, \alpha_{N-1})$$

for some polynomial Q_{N-1}, which means that the dividing hypersurface for N perturbations is singular as a function of $\alpha_1, \dots, \alpha_{N-1}$ at those points for which $P_{N-1}(\alpha_1, \dots, \alpha_{N-1}) = 0$, *i.e.*, on the previous surface. Number of bound states is changed by one each time when the surface in the coupling–constant space is crossed. It means that having found the number of bound states for a given setting of $N-1$ centers, we can decide whether adding another center increases the number of bound states or not: it does if $P_N(\alpha_1, \dots, \alpha_N) < 0$, while in the opposite case it does not.

Another question concerns possible degeneracy of the discrete spectrum. For the sake of brevity, we write the matrix (3.5) as

$$\Lambda(\alpha, \vec{a}; z)_{jm} \;=\; \delta_{jm}(\alpha_j - \xi_j(z)) + (1 - \delta_{jm}) g_{jm}(z)$$

with $\xi_j(z) := \xi(\vec{a}_j, z)$ and $g_{jm}(z) := -G_0(\vec{a}_j, \vec{a}_m; z)$. Since $-g_{jm}(z) > 0$, the *maximum degeneracy* is $N-1$; in particular, the discrete spectrum is always simple for $N = 2$. This is not surprising; we know from Section 3.2 that the ground state is nondegenerate. On the other hand, degeneracies may occur. Consider the case $N = 3$ with $\vec{a}_{1,3} := \left(\pm a, \frac{\pi}{2}\right)$ and $\vec{a}_2 := (0, b)$, and fix an energy $z < 1$. We have $g_{12}(z) = g_{23}(z)$ for any $b \in (0, \pi)$. If $b = \frac{\pi}{2}$ this value is by (2.5) strictly greater than $g_{13}(z)$; on the other hand, $\lim_{b\to 0} g_{12}(z) = 0$, so there is a $b \in \left(0, \frac{\pi}{2}\right)$ for which all the three $g_{jm}(z)$ have the same value. Choosing now the coupling constants α_j in such a way that $\alpha_j - \xi(\vec{a}_j; z) = g_{jm}(z)$, $j = 1, 2, 3$, we find that z is an eigenvalue of multiplicity two.

It should be noted, however, that degenerated eigenvalues occur rather exceptionally. The matrix $\Lambda(\alpha, \vec{a}; z)$ has $3N$ real parameters, because one of the coordinates a_j may be chosen arbitrarily, while the number of different 2×2 submatrices is $\binom{N+1}{2}$. In particular, the number of the available parameters is always less than the number of matrices to be annulated for $N \geq 6$.

Suppose now that we fix $N-1$ coupling constants and let the remainig one run from $-\infty$ to ∞. The whole discrete spectrum moves up at that but in a peculiar way; if we do not hit a degeneracy point by a chance, we observe *avoided crossings* between the energy levels of the $(N-1)$–center Hamiltonian and the running eigenvalue corresponding to the chosen perturbation. If all the fixed α_j represent a strong coupling, so that they meet the graph of the chosen level at large negative energies where it is steep, and if the crossings are narrowly avoided, we reproduce in the present situation the *cascading phenomenon* studied in [17].

To explain the mechanism responsible for this behavior, consider the case of two perturbations with coupling constants $\alpha := \alpha_1$ and α_2. If one of the perturbations is absent, the other one has a single eigenvalue, $e_1(\alpha)$ and $e_2(\alpha) \equiv e_2$, respectively. Now we let α running and look how the eigenvalues influence each other, in particular, when $|\alpha|$ is large. One of the solutions to the condition (3.6) is

$$\epsilon_2(\alpha) = e_2 + \frac{c_2(e_2)}{\alpha_0 - \alpha} + \mathcal{O}(\alpha^{-2}), \tag{3.16}$$

where

$$c_j(z) := \frac{g_{12}(z)^2}{\xi_j'(z)} > 0, \qquad \alpha_0 := \xi_1(e_2) - \frac{2(g_{12}g_{12}')(e_2)}{\xi_2'(e_2)};$$

the first term on the *rhs* of the last expression corresponds to the crossing point, $e_1(\alpha) = e_2$, and the second one is the shift due to level interaction. In a similar way, the running eigenvalue changes to

$$\epsilon_1(\alpha) \approx e_1(\alpha) + \frac{g_{12}(e_1(\alpha))^2}{\xi_1'(e_1(\alpha))[\xi_2(e_1(\alpha)) - \alpha_2] - 2(g_{12}g_{12}')(e_1(\alpha))}.$$

Using asymptotic properties of ξ_j together with (2.16), we find that

$$\epsilon_1(\alpha) \approx e_1(\alpha) + \frac{4\pi}{\alpha} \left(g_{12}(e_1(\alpha))\right)^2 e^{2(\Psi(1) - 2\pi\alpha)} \tag{3.17}$$

Figure 7: The cascading effect for a cluster of eigenvalues

as $\alpha \to -\infty$; recall that $g_{12}(e_1(\cdot))$ itself is exponentially decaying. On the other hand, there is no reasonable asymptotics for $\alpha \to \infty$ because the cross terms $g_{12}(z)$ can no longer be regarded as a perturbation there; in fact, the perturbed eigenvalue may disappear in the continuum.

Nevertheless, if $2g_{12}g'_{12}$ remains small in a wide range of energies, there is an interval (α', α'') above the crossing point where the first term in the denominator dominates and we have

$$\epsilon_1(\alpha) \approx e_1(\alpha) + \frac{c_1(e_1(\alpha))}{\xi_2(e_1(\alpha)) - \tilde{\alpha}_0}$$

with $\tilde{\alpha}_0 := \alpha_2 + (2g_{12}g'_{12}/\xi'_1)(e_1(\alpha))$; the correction term is positive. We see therefore that away of the crossing region the two eigenvalues follow closely the two branches of the "decoupled" spectrum graph. It is also clear that the smaller are the terms g_{12} in the above formulas the closer come the two curves together; since the function $G_0(\vec{a}_1, \vec{a}_2; z)$ at a fixed energy decrease exponentially with the distance of the two points, the most profound cascading effect may be expected when the impurities are far apart. Moreover, if the perturbations produce a multiple eigenvalue or a cluster of almost identical simple eigenvalues, the cascading means that one eigenvalue leaves the cluster and one joins it; this is illustrated on Figure 7.

3.6 Scattering

The existence and completeness of wave operators follow again from the Kato–Birman theory. The argument of Section 2.6 can be easily modified: the generalized eigenfunction (2.21) at a non–integer $z > 1$ has to be replaced by

$$\psi(\vec{x}) = e^{ik_n(z)x}\chi_n(y) + \sum_{j,k=1}^{N} (\Lambda(z)^{-1})_{jk} G_0(\vec{x}, \vec{a}_j; z)\, e^{ik_n(z)a_k}\chi_n(b_k) \tag{3.18}$$

with the incident wave in the n–th channel. Consequently, the reflection and transmission amplitudes from the n–th to the m–th channel are

$$\begin{aligned} r_{nm}(z) &= \frac{i}{\pi} \sum_{j,k=1}^{N} (\Lambda(z)^{-1})_{jk} \frac{\sin(mb_j)\sin(nb_k)}{k_m(z)}\, e^{i(k_m a_j + k_n a_k)}\,, \\ t_{nm}(z) &= \delta_{nm} + \frac{i}{\pi} \sum_{j,k=1}^{N} (\Lambda(z)^{-1})_{jk} \frac{\sin(mb_j)\sin(nb_k)}{k_m(z)}\, e^{-i(k_m a_j - k_n a_k)}\,. \end{aligned} \tag{3.19}$$

The unitarity condition now reads

$$\sum_{m=1}^{[\sqrt{z}]} k_m (t_{nm}\bar{t}_{sm} + r_{nm}\bar{r}_{sm}) = \delta_{ns} k_n\,,$$

Figure 8: The conductance plot for a pair of point perturbations.

$$\sum_{m=1}^{[\sqrt{z}]} k_m \left(\tilde{t}_{nm}\overline{r}_{sm} + \tilde{r}_{nm}\overline{t}_{sm} \right) \;=\; 0\,, \tag{3.20}$$

because S_{nm} is given again by (2.24), where the tilded quantities are obtained by mirror transformation, *i.e.*, by changing each perturbation longitudinal coordinate a_j to $-a_j$.

The knowledge of the reflection and transmission coefficients makes it possible to express quantities of a direct physical interest, in the first place the conductance which is given by the Landauer formula

$$G(z) \;=\; \frac{2e^2}{h} \sum_{n,m=1}^{[\sqrt{z}]} \frac{k_m}{k_n} \left| t_{nm}(z) \right|^2 , \tag{3.21}$$

where $t_{nm}(z)$ are the coefficients (3.19). An example is shown on Figure 8; we see that the perturbations deform the ideal steplike shape. The most remarkable feature are the sharp resonance peaks and dips which approach the channel thresholds in the weak–coupling limits. The corresponding resonance poles can be found as in Section 2.7 but we shall not go into the details.

The fact that our model is explicitly solvable makes it a useful tool to study various methods in which the scattering matrix for a system of perturbations is constructed through a factorization from single–obstacle objects. In practice such techniques involve a restriction on the number of evanescent modes taken into account between subsequent scatterers. A detailed discussion of this situation in our framework is presented in another paper [16]; it shows that a particular caution is required in the resonance regions.

4 A cylindrical strip in axial magnetic field

Before proceeding further, let us show how the above reasoning can be modified for a different geometry: we shall suppose that the strip is coiled into the form of a cylinder and placed into a homogeneous magnetic field parallel to the cylinder axis. Here we consider the situation when the *longitudinal* coordinate is locked cyclically so the resulting cylinder is of a finite height; the case of an infinite cylinder which was discussed recently on a heuristic level in [22] will be treated in the next section.

For a finite–height cylinder the configuration space is compact and may be mapped onto the rectangle

$$\Omega \;:=\; [0, 2\pi) \,\times\, [0, d]$$

with the cylinder perimeter equal to 2π on a suitable length scale. The vector potential at the cylinder surface is $\vec{A} = \left(\frac{1}{2}B, 0\right)$, so

$$\phi \,:=\, \pi B \,=\, 2\pi A \tag{4.1}$$

is the magnetic flux through the cylinder.

4.1 The free Hamiltonian

The free problem can again be solved by separation of variables. It is convenient to include the magnetic field into boundary conditions: the unperturbed operator $H_0(\phi)$ acts then as Laplacian on Ω satisfying the condition (2.1) for all $x \in [0, 2\pi)$ together with

$$\psi(0+, y) \,=\, e^{-i\phi}\psi(2\pi-, y)\,, \qquad \frac{\partial\psi}{\partial x}(0+, y) \,=\, e^{-i\phi}\,\frac{\partial\psi}{\partial x}(2\pi-, y) \tag{4.2}$$

for all $y \in [0, d]$. Its spectrum is purely discrete (*i.e.*, finitely degenerated) and consists of the eigenvalues

$$\epsilon_{mn} \,:=\, (m+A)^2 + (\nu n)^2\,, \quad m \in \mathbb{Z}\,,\; n = 1, 2, \dots\,; \qquad \nu \,:=\, \frac{\pi}{d}\,. \tag{4.3}$$

In particular, if ν^2 is irrational the spectrum is simple for $2A \notin \mathbb{Z}$ and twice degenerated otherwise (with the exception of the ground state for $A \in \mathbb{Z}$). The corresponding eigenfunctions are of the form $\eta_m \otimes \chi_n$, where χ_n are the transverse–mode functions used above and

$$\eta_m(x) \,:=\, \frac{1}{\sqrt{2\pi}}\, e^{i(m+A)x}\,, \quad m \in \mathbb{Z}\,.$$

It is straightforward to write the unperturbed resolvent kernel

$$G_0(\vec{x}_1, \vec{x}_2; \phi, z) \,=\, \frac{1}{\pi d} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{i(m+A)(x_1-x_2)} \sin(\nu n y_1) \sin(\nu n y_2)}{\epsilon_{mn} - z}\,. \tag{4.4}$$

The *rhs* has to be handled with caution because the series is not absolutely convergent. Fortunately, one can get rid of the double summation by evaluating the inner series: we have

$$\sum_{m=-\infty}^{\infty} \frac{e^{imx}}{(m+A)^2 + \gamma} \,=\, \frac{\pi}{2i\sqrt{\gamma}} \left(\frac{e^{i(A-i\sqrt{\gamma})(\pi-x)}}{\sin(\pi(A-i\sqrt{\gamma}))} - \frac{e^{i(A+i\sqrt{\gamma})(\pi-x)}}{\sin(\pi(A+i\sqrt{\gamma}))} \right)$$

for $x \in [0, 2\pi)$ [24, 5.4.3.4], while for $x \in (-2\pi, 0)$ one has to replace $\pi - x$ by $-\pi - x$; this yields

$$G_0(\vec{x}_1, \vec{x}_2) \,=\, \sum_{n=1}^{\infty} \frac{\sinh((2\pi - |x_1-x_2|)\sqrt{\nu^2n^2-z}) + e^{2i\eta\pi A} \sinh(|x_1-x_2|\sqrt{\nu^2n^2-z})}{\cosh(2\pi\sqrt{\nu^2n^2-z}) - \cos(2\pi A)} \times \frac{\sin(\nu n y_1) \sin(\nu n y_2)}{d\sqrt{\nu^2n^2-z}}\,, \tag{4.5}$$

where $\eta := \mathrm{sgn}\,(x_1 - x_2)$.

4.2 The perturbed resolvent

As before we add now to the operator $H_0(\phi)$ a finite number of point perturbations with the coupling constants $\alpha_1,\dots,\alpha_N$ placed at $\vec{a}_1,\dots,\vec{a}_N \in \Omega$; the corresponding Hamiltonian $H(\alpha,\vec{a},\phi)$ is again determined by the boundary conditions (3.1). To find its resolvent we repeat the above argument writing a general vector from the domain of $H(\alpha,\vec{a},\phi)$ in the form (3.3). In view of (4.5), we have

$$\lim_{|x-a_j|\to 0} \frac{G_0(\vec{x},\vec{a}_j)}{\ln|x-a_j|} = \frac{1}{\pi} \lim_{y\to b_j} \frac{1}{\ln|y-b_j|} \sum_{n=1}^{\infty} \frac{\sin(\nu n y_1)\sin(\nu n y_2)}{n} \coth(\pi\nu n)\,.$$

Since $\coth(\pi\nu n)$ tends to one exponentially fast as $n\to\infty$, it may be neglected; Using further the identity

$$\sum_{n=1}^{\infty} \frac{\sin(\nu n y_1)\sin(\nu n y_2)}{n} = \frac{1}{4}\ln\left(\left(\frac{\sin\left(\nu\frac{y+b_j}{2}\right)}{\sin\left(\nu\frac{y-b_j}{2}\right)}\right)^2\right)$$

[24, 5.4.15.1] we find that $L_0(\psi,\vec{a}_j)$ is given by the same expression as in Section 3.1. In this way, the sought resolvent kernel is given by a formula analogous to (3.2) with

$$\lambda(\alpha,\vec{a},\phi;z) = \Lambda(\alpha,\vec{a},\phi;z)^{-1}\,, \tag{4.6}$$

where $\Lambda_{jr} = -G_0(\vec{a}_j,\vec{a}_r)$ for $j\neq r$, while the diagonal elements are given by

$$\Lambda_{jj} = \alpha_j - \lim_{|x-a_j|\to 0}\left(G_0(\vec{x},\vec{a}_j) + \frac{1}{2\pi}\ln|x-a_j|\right)\,.$$

The limit is easily evaluated using (4.5); we arrive at the expressions

$$\begin{aligned} \Lambda_{jj} &= \alpha_j - \sum_{n=1}^{\infty}\left(\frac{\sinh(2\pi\sqrt{\nu^2n^2-z})}{\cosh(2\pi\sqrt{\nu^2n^2-z})-\cos\phi}\,\frac{\sin^2(\nu n b_j)}{d\sqrt{\nu^2n^2-z}} - \frac{1}{2\pi n}\right)\,, \\ \Lambda_{jr} &= -G_0(\vec{a}_j,\vec{a}_r)\,, \qquad j\neq r\,; \end{aligned} \tag{4.7}$$

notice that the matrix $\Lambda(z)$ does not change if the flux ϕ is modified by an integer multiple of 2π .

4.3 Spectral properties

Since the point interactions represent a finite–rank perturbation to the resolvent, $\sigma_{ess}(H(\alpha,\vec{a},\phi)) = \sigma_{ess}(H_0(\phi)) = \emptyset$, *i.e.*, the spectrum of $H(\alpha,\vec{a},\phi)$ is again purely discrete.

Moreover, it is disjoint with $\mathcal{E} := \{\,\epsilon_{mn} :\ m\in\mathbb{Z},\ n=1,2,\dots\}$ provided there is at least one point perturbation present. If some ϵ_{mn} would be an eigenvalue of $H(\alpha,\vec{a},\phi)$, the corresponding eigenfunction had to satisfy the corresponding Helmholz equation everywhere outside the set $\{\vec{a}_1,\dots,\vec{a}_N\}$ being therefore a *finite* linear combination $\psi = \sum_{m,n} c_{mn}\eta_m \otimes \chi_n$. Such a function, however, is bounded in Ω and cannot have a logarithmic singularity, *i.e.*, $L_0(\psi, \vec{a}_j) = 0$ for any j .

A real number $z \notin \mathcal{E}$ therefore belongs to the resolvent set of $H(\alpha, \vec{a}, \phi)$, so we are allowed to use the argument of Section 3.3. The eigenvalues are given by the condition

$$\det \Lambda(\alpha, \vec{a}, \phi; z) \;=\; 0\,, \tag{4.8}$$

under which the linear system

$$\sum_{r=1}^{N} \Lambda(z)_{jr} d_r \;=\; 0$$

has a solution (or solutions); the corresponding (non–normalized) eigenfunctions are then

$$\varphi(\vec{x}) \;=\; \sum_{r=1}^{N} d_j G_0(\vec{x}, \vec{a}_j; \phi, z)\,. \tag{4.9}$$

5 Point perturbations on an infinite cylinder

If the cylinder heigth d of the previous section goes to infinity, the configuration space becomes again noncompact; one can map it on the strip

$$\Omega \;=\; [0, 2\pi) \times I\!R\,.$$

The results of Section 3 modify easily to the present situation.

5.1 The resolvent and scattering amplitudes

The free Hamiltonian is defined again by the cyclic boundary conditions (4.2), so the "transverse eigenfunctions" are $\eta_m\,:\; \eta_m(x) \;=\; (2\pi)^{-1/2} e^{i(m+A)x}$ corresponding to the eigenvalues $\epsilon_m \;:=\; (m+A)^2$, $m \in Z\!\!\!Z$. The free resolvent kernel is therefore

$$G_0(\vec{x}_1, \vec{x}_2; \phi, z) \;=\; \frac{i}{4\pi} \sum_{m \in Z\!\!\!Z} \frac{e^{i(m+A)(x_1-x_2)} e^{ik_m(z)|y_1-y_2|}}{k_m(z)}\,, \tag{5.1}$$

where

$$k_m(z) \;:=\; \sqrt{z-(m+A)^2}\,. \tag{5.2}$$

Introducing now N point perturbations as in Section 3, we find that the full resolvent kernel is given by the relations (3.2) and (3.4) with

$$\begin{aligned} \Lambda_{jj} \;&=\; \alpha_j - \frac{i}{4\pi} \left[\frac{1}{k_0(z)} + \sum_{m \neq 0} \left(\frac{1}{k_m(z)} + \frac{i}{|m|} \right) \right]\,, \\ \Lambda_{jk} \;&=\; -G_0(\vec{a}_j, \vec{a}_k)\,, \qquad j \neq k\,; \end{aligned} \tag{5.3}$$

Figure 9: A strip potential due to an external field.

It is convenient for computational purposes to rewrite the first of these relations in the form

$$\Lambda_{jj} \,=\, \tilde\alpha_j \,-\, \frac{i}{4\pi}\, \left\lbrack\, \frac{1}{k_0(z)} \,+\, \sum_{m\ne 0} \left(\frac{1}{k_m(z)} + \frac{i}{|m\!+\!A|} \right) \right\rbrack\,, \tag{5.4}$$

where α_j are renormalized coupling constants,

$$\tilde\alpha_j \,=\, \alpha_j \,+\, \frac{1}{4\pi}\, \sum_{m\ne 0} \left(\frac{1}{|A|} \,-\, \frac{1}{|m\!+\!A|} \right)\;;$$

we suppose, of course, that $A\notin\mathbb{Z}$.

¿From here one can find the discrete spectrum of $H(\alpha,\vec a,\phi)$ by virtue of the eqs. (4.8) and (4.9). Similarly, the reflection and transmission amplitudes are

$$\begin{aligned} r_{nm}(z) \;&=\; \frac{i}{4\pi}\, \sum_{j,k=1}^N (\Lambda(z)^{-1})_{jk} \frac{e^{-i(m+A)b_j}\, e^{i(n+A)b_k}}{k_m(z)}\, e^{i(k_m a_j + k_n a_k)}\,, \\ t_{nm}(z) \;&=\; \delta_{nm} \,+\, \frac{i}{4\pi}\, \sum_{j,k=1}^N (\Lambda(z)^{-1})_{jk} \frac{e^{-i(m+A)b_j}\, e^{i(n+A)b_k}}{k_m(z)}\, e^{-i(k_m a_j - k_n a_k)}\;; \end{aligned} \tag{5.5}$$

the elementary block S_{nm} of the S–matrix is given again by (2.24), where the tilded quantities are obtained by mirror transformation. The scattering is, of course, a one–dimensional problem with an additional degree of freedom corresponding to the transverse channels — it imposes no restriction on the size of the cylinder contrary to the claim made in [22].

6 An infinite strip in external field

Let us return now to the infinite strip and assume that, in addition to point perturbations, the particle interacts with a potential. Such a situation can occur, for instance, if the strip is cut not from a plane but from a curved surface (see Fig. 9) and exposed to a homogeneous electric field. We shall suppose that the potential depends only on the longitudinal variable; otherwise the problem would become much more complicated.

6.1 The unperturbed Hamiltonian

In the described situation the Hamiltonian free of point perturbations is therefore

$$H_V \,:=\, -\partial_x^2 - \partial_y^2 + V(x) \tag{6.1}$$

with the usual Dirichlet condition (2.1) at the strip boundary. Our assumptions about the potential are for the moment rather weak: we require only that $V \in L^1_{loc}(I\!R)$ and the operator

$$h_V \,:=\, -\partial_x^2 + V(x)$$

with the natural domain is *e.s.a.* Then H_V understood as the closure of $h_V \otimes (-\partial_y^2)$ is self–adjoint and its spectrum is

$$\sigma(H_V) \,=\, \{\, \lambda + n^2 \,:\, \lambda \in \sigma(h_V)\,,\; n = 1, 2, \dots \,\}\,. \tag{6.2}$$

In particular, the resolvent set $\rho(H_V) \supset (-\infty, V_- + 1)$, where $V_- := \inf_{x\in I\!R} V(x)$. Of course, the character of the spectrum depends strongly on that of $\sigma(H_V)$. If, for example, the potential vanishes at infinity, $\lim_{|x|\to\infty} V(x) = 0$, then h_V is below bounded with $\sigma_{ess}(h_V) = [0, \infty)$ by [25, Secs.X.2 and XIII.4], so

$$\sigma_{ess}(H_V) = [1, \infty)$$

as in the absence of the potential. In a similar way, if the potential grows at large distances (at least in the mean) then the spectrum of H_V is pure point *etc.*

By assumption, h_V is limit point at both $\pm\infty$, so it follows from the Weyl alternative that to any $z = k^2 \in \mathbb{C} \setminus I\!R$ there are functions $u \equiv u(\cdot, k)$ and $v \equiv v(\cdot, k)$, unique up to a multiplicative constant, which solve the equation $f'' + (k^2 - v)f = 0$ being square integrable at $\mp\infty$, respectively. They are pointwise continuous with respect to k; we use the same symbols for their extensions to the positive real axis in the k–plane. We denote $k_n(z) := \sqrt{z - n^2}$ as before and set

$$u_n \,:=\, u(\cdot, k_n(z))\,, \qquad v_n \,:=\, v(\cdot, k_n(z))\,.$$

The free resolvent kernel is then

$$G_V(\vec{x}_1, \vec{x}_2; z) \,=\, -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{u_n(x_<)v_n(x_>)}{W(u_n, v_n)} \sin(ny_1)\sin(y_2)\,, \tag{6.3}$$

where $x_<$, $x_>$ is the smaller and larger of x_1, x_2, respectively.

6.2 Point perturbations

The Hamiltonian $H(\alpha, \vec{a}, V)$ with point perturbations is defined again through the boundary conditions (3.1). To find its resolvent, we need the logarithmic singularity of the kernel (6.3). If the potential is smooth the latter is known to be the same as in the absence of the potential [29]. In fact the potential need not be regular; nevertheless an additional restriction is required because the property under consideration is local, and therefore could be affected by a local singularity. We shall assume that the potential is *locally bounded*, *i.e.*, bounded in any compact interval.

To find the singularity, we notice that what matters here is the behavior of the series at large n; however, these terms correspond to solutions which are deep in the classically forbidden region (*i.e.*, with the imaginary part of the *channel* momentum large positive) and therefore of the WKB–form,

$$u_n(x) = \kappa_n(x)^{-1/2}\, e^{\int_0^\infty \kappa_n(x')dx'}\, \mathrm{Err}_1(\kappa_n)\,, \;\; v_n(x) = \kappa_n(x)^{-1/2}\, e^{-\int_0^\infty \kappa_n(x')dx'}\, \mathrm{Err}_1(\kappa_n)\,,$$

where $\kappa_n(x) := \sqrt{n^2+V(x)-z}$ and $\mathrm{Err}_\alpha(\kappa)$ stands as a shorthand for $1+\mathcal{O}(\kappa^\alpha)$, so

$$\frac{u_n(x)v_n(x)}{W(u_n,v_n)} \;=\; -\,\frac{1}{2\kappa_n(x)}\,\mathrm{Err}_1(\kappa_n) \;=\; -\,\frac{1}{2n}\,\mathrm{Err}_1(n)$$

due to the local boundedness of the potential. Hence we obtain

$$\lim_{|\vec{x}-\vec{a}_j|\to 0} \frac{G_V(\vec{x},\vec{a}_j;z)}{\ln(\vec{x}-\vec{a}_j|} \;=\; -\,\frac{1}{2\pi}\,,$$

and the argument of Section 3.1 can be easily modified giving

$$(H(\alpha,\vec{a},V)–z)^{-1}(\vec{x}_1,\vec{x}_2) \;=\; G_V(\vec{x}_1,\vec{x}_2;z)+\sum_{j,k=1}^{N}(\Lambda(z)^{-1})_{jk}G_V(\vec{x}_1,\vec{a}_j;z)G_V(\vec{a}_k,\vec{x}_2;z)\,, \tag{6.4}$$

where $\Lambda(\alpha,\vec{a},V;z)$ is the following $N\times N$ matrix:

$$\begin{aligned} \Lambda_{jr} \;&=\; -\,G_V(\vec{a}_j,\vec{a}_r;z)\,, \qquad j\neq r\,, \\ \Lambda_{jj} \;&=\; \alpha_j \;+\; \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{u_n(a_j)v_n(a_j)}{W(u_n,v_n)}\,\sin^2(nb_j) \;+\; \frac{1}{4n}\right)\,. \end{aligned} \tag{6.5}$$

It becomes singular if the condition

$$\det\Lambda(\alpha,\vec{a},V;z) \;=\; 0 \tag{6.6}$$

is valid; the corresponding z are eigenvalues of $H(\alpha,\vec{a},V)$. If $z\notin\rho(H_V)$, the argument of Section 3.2 may be used: the non–normalized eigenfunctions are of the form

$$\varphi(x) \;=\; \sum_{j=1}^{N} d_j G_V(\vec{x},\vec{a}_j;z)\,, \tag{6.7}$$

where $d\in\mathbb{C}^N$ is the respective solution to $\sum_{k=1}^{N}\Lambda(z)_{jk}d_k = 0$.

6.3 Scattering

One has to assume that the underlying scattering problem for the potential V makes sense, the wave operators exist and are complete; since V is locally bounded by assumption, this is ensured, for instance, if

$$|V(x)| \;\leq\; C(1+|x|)^{-1-\varepsilon}\,.$$

for some positive $C,\,\varepsilon$ [25, Sec.XI.4]. Repeating the limiting–absorption argument of Secs.2.6 and 3.4, we get the generalized eigenfunction

$$\psi(\vec x) \;=\; e^{ik_n(z)x}\chi_n(y) \,+\, \sum_{j,k=1}^N (\Lambda(z)^{-1})_{jk} G_V(\vec x,\vec a_j;z)\, e^{ik_n(z)a_k}\chi_n(b_k)\,.$$

We substitute for G_V from (6.3) and consider the asymptotic behavior. To this end, we need the asymptotics of the elementary solutions which is given by

$$u_n(x) \;\approx\; \left\{ \begin{array}{lll} e^{-ik_n(z)x} & \dots & x\to -\infty \\ \overline{\tau}_n\, e^{-ik_n(z)x} + \overline{\rho}_n\, e^{ik_n(z)x} & \dots & x\to\infty \end{array} \right. \tag{6.8}$$

and an analogous formula with the interchange of signs for $v_n(x)$; here $\tau_n := \tau(k_n(z))$ and $\rho_n := \rho(k_n(z))$ are the standard (left–to–right) transmission and reflection amplitudes for the potential V at the momentum $k_n(z)$. This yields

$$\begin{aligned} r_{nm}(z) \;&=\; \frac{i}{\pi} \sum_{j,k=1}^N (\Lambda(z)^{-1})_{jk} \frac{\sin(mb_j)\sin(nb_k)}{k_m(z)\overline{\tau}(k_m(z))}\, u_m(a_j)\, e^{ik_n(z)a_k}\,, \\ t_{nm}(z) \;&=\; \delta_{nm} \,+\, \frac{i}{\pi} \sum_{j,k=1}^N (\Lambda(z)^{-1})_{jk} \frac{\sin(mb_j)\sin(nb_k)}{k_m(z)\overline{\tau}(k_m(z))}\, v_m(a_j)\, e^{ik_n(z)a_k}\,; \end{aligned} \tag{6.9}$$

the unitarity condition (3.20) remains the same.

7 Periodic perturbations

An extension of the results discussed above to an infinite number of perturbations is straightforward for the algebraic part; the matrices Λ become infinite–dimensional, *i.e.*, operators on the Hilbert space ℓ^2. However, since they are generally unbounded, the corresponding analysis is much harder. The problem should be treated along the same lines as in [1, Sec.III.1.1] but we are not going to do that; instead we concentrate on an important particular case of a strip in which the point perturbations are periodically arranged.

7.1 The Floquet–Bloch analysis

Suppose that the set of point perturbations $\{\alpha,\vec a\}\equiv\{[\alpha_j,\vec a_j]:\, j=1,2,\dots\}$ in Ω is countably infinite and has a periodic pattern with a period $\ell>0$. As usually we suppose that the width of the strip is $d=\pi$; the general case can be obtained by a simultaneous rescaling of the coupling constants.

The existence of the corresponding operator $H(\alpha,\vec a)$ can be established as in [1, Sec.III.1.1]. Following the standard procedure named Floquet by mathematicians and Bloch by physicists, we can find a unitary operator $U\,:\,L^2(\Omega)\to$

$L^2(\mathcal{B}, (\ell/2\pi)d\theta; L^2(\hat\Omega))$, where

$$\hat\Omega := [0,\ell] \times [0,\pi] , \qquad \mathcal{B} := \left[-\frac{\pi}{\ell}, \frac{\pi}{\ell} \right) \times [0,\pi] \, ; \tag{7.1}$$

the x–projections of these rectangles are the Wigner–Seitz cell of the underlying one–dimensional lattice, and the corresponding Brillouin zone, respectively — *cf.* [1, Sec.III.1.3]. By means of U , the operator $H(\alpha,\vec a)$ is unitarily equivalent to

$$U\, H(\alpha,\vec a)\, U^{-1} \;=\; \frac{\ell}{2\pi} \int_{|\theta\ell|\le\pi}^{\oplus} H(\alpha,\vec a;\theta)\, d\theta \,, \tag{7.2}$$

where $H(\alpha,\vec a;\theta)$ is the point–interaction Hamiltonian on $L^2(\hat\Omega)$, *i.e.*, the Laplacian satisfying the boundary conditions (2.2) for $x\in[0,\ell]$,

$$\psi(\ell-,y) \;=\; e^{i\theta\ell}\psi(0+,y) \,, \qquad \frac{\partial\psi}{\partial x}(\ell-,y) \;=\; e^{i\theta\ell}\, \frac{\partial\psi}{\partial x}(0+,y) \tag{7.3}$$

for $y\in[0,\pi]$ and (3.1); with an abuse of notation we shall use the symbol $(\alpha,\vec a)$ for the subset of the point interactions in $\hat\Omega$ and suppose that their number is N .

The operators $H(\alpha,\vec a;\theta)$ are easy to investigate because up to a change in notation and a simple rescaling they coincide with the magnetic cylinder Hamiltonians of Section 4. In particular, the "free" eigenvalues

$$\epsilon_{mn}(\theta) \;:=\; \left(\frac{2\pi m}{\ell} + \theta \right)^2 + n^2 \,, \quad m\in\mathbb{Z} \,,\; n=1,2,\dots \,, \tag{7.4}$$

correspond to the eigenfunctions $\eta_m^\theta \otimes \chi_n$, where χ_n are the usual transverse solutions and

$$\eta_m^\theta(x) \;:=\; \frac{1}{\sqrt{\ell}}\, e^{i(2\pi m+\theta\ell)x/\ell} \,, \quad m\in\mathbb{Z} \,.$$

Moreover, the free resolvent kernel is

$$\begin{aligned} G_0(\vec x_1,\vec x_2;\theta;z) \;=\; & \sum_{n=1}^{\infty} \frac{\sinh((\ell-|x_1-x_2|)\sqrt{n^2-z}) + e^{2i\eta\theta\ell}\sinh(|x_1-x_2|\sqrt{n^2-z})}{\cosh(\ell\sqrt{n^2-z}) - \cos(\theta\ell)} \\ & \times \frac{\sin(ny_1)\sin(ny_2)}{\pi\sqrt{n^2-z}} \,, \end{aligned} \tag{7.5}$$

where $\eta := \mathrm{sgn}\,(x_1-x_2)$, and the full kernel is expressed by a formula analogous to (3.2) with

$$\lambda(\alpha,\vec a,\theta;z) \;=\; \Lambda(\alpha,\vec a,\theta;z)^{-1} \,, \tag{7.6}$$

where $\Lambda_{jr} = -G_0(\vec a_j,\vec a_r)$ for $j\ne r$, while the diagonal elements are given by

$$\Lambda_{jj} \;=\; \alpha_j \;-\; \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sinh(\ell\sqrt{n^2-z})}{\cosh(\ell\sqrt{n^2-z}) - \cos\theta\ell}\, \frac{\sin^2 nb_j}{\sqrt{n^2-z}} \;-\; \frac{1}{2n} \right) . \tag{7.7}$$

We may also write the last formula as $\Lambda_{jj}(\alpha,\vec a,\theta;z) = \alpha_j - \xi(\vec a_j,\theta;z)$, where the function ξ is more explicitly given by

$$\begin{aligned}\xi(\vec a_j,\theta;z) \;&=\; \frac{1}{\pi}\sum_{n=1}^{n[z]}\left(\frac{\sin(\ell\sqrt{z-n^2})}{\cos(\ell\sqrt{z-n^2})-\cos\theta\ell}\,\frac{\sin^2 nb_j}{\sqrt{z-n^2}}\;-\;\frac{1}{2n}\right)\\ &+\;\frac{1}{\pi}\sum_{n=n[z]+1}^{\infty}\left(\frac{\sinh(\ell\sqrt{n^2-z})}{\cosh(\ell\sqrt{n^2-z})-\cos\theta\ell}\,\frac{\sin^2 nb_j}{\sqrt{n^2-z}}\;-\;\frac{1}{2n}\right)\;;\end{aligned} \tag{7.8}$$

we have denoted here $n[z] := \min\{0,[\sqrt{z}]\}$. It is defined everywhere except at

$$\mathcal{E}(\vec a,\theta) \;:=\; \{\,\epsilon_{mn}(\theta)\in\mathcal{E}(\theta)\;:\;\sin nb\neq 0\,\}\,, \tag{7.9}$$

where $\mathcal{E}(\theta)$ is the eigenvalue set (7.4); it is also easy to see that the function ξ is monotonously increasing between any pair of neighboring singularities.

To find the spectrum of the original operator, one has to solve first the spectral problem for $H(\alpha,\vec a;\theta)$ in analogy with the magnetic cylinder case: the eigenvalues are given by

$$\det\Lambda(\alpha,\vec a,\theta;z) \;=\; 0\,, \tag{7.10}$$

and the corresponding eigenfunctions are given by a relation similar to (4.9). Next one has to overlay the obtained eigenvalue sets for θ running through the x–projection of the Brillouin zone to get the spectrum of $H(\alpha,\vec a)$. The main question is, of course, whether we cover a single interval in this way as in the free case, or whether the interaction will open some gaps.

7.2 A single array of perturbations

Since it is not easy to draw general conclusions about the form of the spectrum, let us restrict to the simplest case where $N=1$, *i.e.*, the strip Ω contains a single periodic array of point perturbation of a strength α and spacing ℓ. The condition (7.10) then simplifies to

$$\xi(\vec a,\theta;z) \;=\; \alpha\,. \tag{7.11}$$

We have seen that the function on the left hand side is monotonously increasing between its singularities, *i.e.*, the points of $\mathcal{E}(\vec a,\theta)$. This means that for fixed $\alpha,\,\theta$ there is a sequence $\{\epsilon_r(\alpha,\vec a,\theta)\}_{r=0}^{\infty}$ arranged in the ascending order; each of them depends, in fact, only on the y–component b of the vector $\vec a$. The lowest one satisfies

$$\epsilon_0(\alpha,\vec a,\theta) \;<\; 1+\theta^2\,,$$

and between each two neighboring points of $\mathcal{E}(\vec a,\theta)$ there is just one of the other eigenvalues. It is also clear that any of $\epsilon_r(\alpha,\vec a,\theta)$ is continuous with respect to the parameters and $\epsilon_r(\cdot,\vec a,\theta)$ is increasing for fixed b and θ.

On the other hand, the eigenvalues need not be monotonous with respect to θ. The implicit–function theorem tells us, however, that

$$\frac{\partial\epsilon_r(\alpha,\vec a,\theta)}{\partial\theta} \;=\; -\;\frac{\partial\xi(\vec a,\theta;z)}{\partial\theta}\,\left(\frac{\partial\xi(\vec a,\theta;z)}{\partial z}\right)^{-1}\Bigg|_{(\epsilon_r,\theta)}$$

whenever the denominator is nonzero. Leaving aside the thresholds, $z=n^2$, and the points of $\mathcal{E}(\theta)$, a straightforward differentiation shows that $\xi(\vec a,\cdot;\cdot)$ is analytic in both variables. Since the function ξ is not identically zero, the derivative $\partial\epsilon_r(\alpha,\vec a,\theta)/\partial\theta$ may be zero at some points but never in an interval. This means that the spectrum of $H(\alpha,\vec a)$ is *absolutely continuous* — *cf.* [25, Sec.XIII.16].

Let us turn now to the question about the number of gaps. Below $z=1$ the spectrum may be estimated by means of extrema of the function ξ: we have

$$\xi_+(\vec a,z) \;:=\; \max_{|2\theta\ell|\le\pi}\xi(\vec a,\theta;z) \;=\; \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin^2 nb}{\sqrt{n^2-z}}\,\coth\left(\frac{\ell}{2}\sqrt{n^2-z}\right)\;-\;\frac{1}{2n}\right),$$

and a similar formula for the minimum, $\xi_-(\vec a,z)$, with coth replaced by tanh. Both functions are increasing and have the same logarithmic asymptotics as $\xi(\vec a,z)$ of Section 2.5 as $z\to-\infty$. On the other hand, $\xi_+(\vec a,\cdot)$ diverges as $z\to 1-$, while $\xi_-(\vec a,\cdot)$ has a finite limit. This shows that a gap exists provided

$$\begin{aligned}\alpha \;<\; \xi_-(\vec a,1-) \;&=\; \frac{\ell}{2\pi}\,\sin^2 b-\frac{1}{2\pi}\\ &+\;\frac{1}{\pi}\sum_{n=2}^{\infty}\left(\frac{\sin^2 nb}{\sqrt{n^2-1}}\,\tanh\left(\frac{\ell}{2}\sqrt{n^2-1}\right)\;-\;\frac{1}{2n}\right).\end{aligned} \tag{7.12}$$

The condition is satisfied for a strong enough coupling, or alternatively, for any fixed α and the spacing ℓ large enough.

This conclusion calls to mind the spectrum of a straight polymer in a plane described in [1, Sec.III.4] which has also one gap if the coupling is stronger than some critical value. However, for a "coated polymer", *i.e.*, our strip with an array of perturbations a much stronger result is valid; we are going to show that under a suitable choice of parameters it can have *any finite number* of gaps.

To this end, we consider $z\in(2,3)$ and $\ell\gg 1$, and rewrite the right hand side of the relation (7.8) as

$$\xi(\vec a,\theta;z) \;=\; \xi_0(\vec a,\theta;z)+\eta(\vec a,\theta;z)\,,$$

where

$$\xi_0(\vec a,\theta;z) \;:=\; \frac{\sin(\ell\sqrt{z-1})}{\cos(\ell\sqrt{z-1})-\cos\theta\ell}\,\frac{\sin^2 b}{\pi\sqrt{z-1}}$$

and $\eta(\vec a,\theta;z)$ is the rest. In the same way as above, one can show that $\eta(\vec a,\theta;\cdot)$ is monotonously increasing with a bounded derivative everywhere below the second threshold, in particular, in the chosen interval of energies. Moreover,

$$\eta(\vec{a},\theta;z)\;\le\;\eta_+(z)\;:=\;-\,\frac{1}{2\pi}\;+\;\frac{1}{\pi}\sum_{n=2}^{\infty}\left(\frac{\sin^2 nb}{\sqrt{n^2-z}}\,\coth\left(\frac{\ell}{2}\sqrt{n^2-z}\right)\;-\;\frac{1}{2n}\right)\,,$$

and the minimum, $\eta_-(z)$, is obtained when coth is replaced by tanh. These estimates become close if ℓ is large; using the inequality $\coth u-\tanh u<5\,e^{-2u}$ for $2u\ge 1$, we find

$$\eta_+(z)-\eta_-(z)\;<\;\frac{5}{\pi}\sum_{n=2}^{\infty}\frac{\sin^2 nb}{\sqrt{n^2-z}}\,e^{-\ell\sqrt{n^2-z}}\;<\;\frac{5}{\pi}\sum_{n=2}^{\infty}e^{-\ell(n-1)}$$

for $z\in(2,3)$, so

$$\eta_+(z)-\eta_-(z)\;<\;\frac{5}{\pi}\,\frac{e^{-\ell}}{1-e^{-\ell}}$$

and the allowed corridor shrinks exponentially with increasing ℓ. On the other hand, the function

$$g_\theta(u)\;:=\;\frac{\sin u}{\cos u-\cos\theta\ell}$$

is increasing between any two zeros of its denominator. In the intervals, where it is positive, it is estimated by the appropriate branch of $\tan\left(\frac{u}{2}+\pi m\right)$ from below; if it is negative, we have a similar estimate from above with tan replaced by $-\cot$. Hence independently of θ we have either

$$\xi_0(\vec{a},\theta;z)\;\ge\;\frac{\sin^2 b}{\pi\sqrt{z-1}}\;\tan\left(\frac{\pi}{2}\left\{\frac{\ell}{\pi}\sqrt{z-1}\right\}\right)$$

or

$$\xi_0(\vec{a},\theta;z)\;\le\;-\,\frac{\sin^2 b}{\pi\sqrt{z-1}}\;\cot\left(\frac{\pi}{2}\left\{\frac{\ell}{\pi}\sqrt{z-1}\right\}\right)\,,$$

where $\{\cdot\}$ denotes the fractional part. Putting the estimates together, we see that the oscillating part dominates, so for sufficiently large $|\alpha|$ there are gaps having $1+\left(\frac{\pi m}{\ell}\right)^2$ as one endpoint provided the latter lies in the chosen interval. In addition, $\tan u+\cot u\ge 2$, which means that the gap between the lower and the upper bound to $\xi(\vec{a},\theta;z)$ never closes for $z\in(2,3)$ if

$$\frac{5\,e^{-\ell}}{1-e^{-\ell}}\;<\;\sqrt{2}\,\sin^2 b\,.$$

Together we infer that *for any* $\alpha\in I\!R$, the Hamiltonian under consideration can have *an arbitrary finite number of gaps* in its spectrum provided ℓ is chosen large enough.

Recall that the number of gaps is a quantity strongly dependent on the dimension of a periodic system. For one–dimensional systems it is generically infinite [25, Sec.XIII.16], [1, Sec.III.2], while for higher dimensions the *Bethe–Sommerfeld conjecture* states this number is finite. This is known to be true for regular periodic potentials — see, *e.g.*, [27, 28]. If the potential is replaced by a point interaction, the one–dimensional situation suggests that the gap closing at high energies is slower. The Bethe–Sommerfeld conjecture is known to be valid for two– and three–dimensional lattices if each lattice cell contains a single point perturbation — *cf.* [1, Secs.III.1, III.4]. On the other hand, an example of a "multidimensional" periodic graph system has been presented recently [13, 14] which exhibits an infinite number of gaps — or none at all. In this connection it is interesting to ask whether a periodically perturbed strip considered here can exhibit an infinite number of gaps. However, we are not going to attack this problem in the present paper.

8 Random point perturbations

Again, we have no intention to develop a general theory which would modify the results of Sec.III.5 in [1] to the present situation. Our goal is more modest; we want to investigate numerically how the scattering changes in the above studied models if the set of point perturbations is chosen at random at an interval of a length L. We will thus investigate a family of random operators $H_\omega \equiv H(\alpha, \vec{a}_\omega, \phi)$, where $(a_{1,\omega}, a_{2,\omega}, \dots, a_{N,\omega})$ and $(b_{1,\omega}, b_{2,\omega}, \dots, b_{N,\omega})$ are independent random numbers, $a_{j,\omega} \in (0, L)$, $b_{j,\omega} \in (0, 2\pi)$ for all $j = 1, \dots, N$.

It is widely accepted that an irregular scattering on such a family of impurities leads to fluctuations of the corresponding S–matrix, and consequently, of the measured sample conductance. These mesoscopic fluctuations are generic in the sense that they do not depend on detailed properties of the sample under investigation. Our aim here is to find the conductance fluctuations for the present model.

The mentioned universality is usually manifested through relations to spectral properties of random–matrix ensembles. More specifically, the S–matrix eigenvalues of an irregular quantum scattering system are usually supposed to conform with those of a circular ensemble of random matrices. This correspondence has theoretically been demonstrated by Blumel and Smilansky for the first time using semiclassical arguments [7]. According to it, for systems with time–reversal symmetry S corresponds to the circular orthogonal ensemble (COE), while for systems without this symmetry its statistical properties refer to the circular unitary ensemble (CUE). However, as we have said, the use of the COE/CUE ensembles for description of the mesoscopic fluctuations was justified with the help of a semiclassical approximation [7, 5], which implicitly requires a large number of open channels. It is therefore interesting to investigate the situation, when only a few transverse modes contribute to the scattering, so the mentioned argument may not be applied.

The quantity to study is the sample conductance which is obtained (in the two–probe setting) from the transmission amplitudes (3.19) by the Landauer formula (3.21). A standard approach to conductance fluctuations consists of regarding the S–matrix as a random matrix of a dimension given by the number of open channels and averaging the corresponding conductance over the appropriate random ensemble;

Figure 10: var(G) for random perturbations *vs.* momentum

Figure 11: Conductance for random perturbations *vs.* momentum

this leads to [5]:

$$\langle G \rangle = \begin{cases} \frac{M}{2} - \frac{M}{4M+2} & \text{for COE} \\ \\ \frac{M}{2} & \text{for CUE,} \end{cases} \tag{8.1}$$

where $\langle G \rangle$ is the mean value and M is the number of open channels in the corresponding quantum wire. For $M \to \infty$ we get $\langle G \rangle_{\mathrm{CUE}} - \langle G \rangle_{\mathrm{COE}} = \frac{1}{4}$, *i.e.*, the averaged conductance is enhanced by $1/4$, if the time reversal symmetry is broken. For the variance var(G) the averaging over the circular ensembles leads to

$$\mathrm{var}(G) = \begin{cases} \frac{M(M+1)^2}{(2M+1)^2(2M+3)} & \text{for COE} \\ \\ \frac{M^2}{4(4M^2-1)} & \text{for CUE.} \end{cases} \tag{8.2}$$

This relation shows that, in the semiclassical case, the conductance variance is practically independent of M being equal to $1/8$ and $1/16$ for COE and CUE, respectively.

The model we are considering here offers an opportunity for testing these relations in a situation where the full solution to the quantum scattering problem is known and the semiclassical approximation is not required. To be specific, we will consider the dependence of the conductance on energy of the incomimg particle in the case when the number of impurities and their distribution inside the waveguide is fixed; we choose 25 impurities distributed at random inside a waveguide segment of length 25.

One expects that for a fixed number of impurities the universality regime described above can be valid only within a certain energy interval. For energies which are too low the localization effects will dominate the scattering scenario and the incoming wave will be nearly completely reflected. On the other hand for energies above a certain threshold direct processes will dominate and the major part of the incident wave function will be transmitted through the sample; the region intermediate between these two regimes is expected to be the universality "window".

Our numerical results support the above conjectured picture. In Figure 10 we plot var(G) as a function of the incident wave vector k in comparison with the random–matrix prediction (8.2). The universality window is localized approximately between $k = 3.5$ and $k = 5.5$. For smaller k the localization effects suppress the variance whereas for k above the window the variance depends significantly on k with maxima localized at integer values of k.

Figure 12: Level–spacing statistics for random–impurity scattering

The conductance G of the same randomly perturbed waveguide is plotted on the Figure 11. The prediction of (3.6) is again shown for the sake of comparison (the dashed line). It is clear that G coincides with the random–matrix prediction only inside the universality window; for smaller k it is suppressed by localization effects while for large k it is enhanced by direct transmission.

Finally, the level–spacing statistics of the corresponding S–matrix is shown on Figure 12. The first picture corresponds to k inside the universality window, while the other one is computed for a larger momentum value. The prediction of the random–matrix theory (Wigner distribution) and the Poisson distribution are plotted for the sake of comparison; a shift towards the latter for k above the window is clearly visible.

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Figure captions

Figure 1 The dependence of $\xi(\vec{a},z)$, where $\vec{a}=(0,b)$. The dotted, dash-dotted and dashed curves represent $b=\pi/2$, $b=\pi/6$ and $b=\pi/12$, respectively. The full line is the leading term of asymptotics, $\xi_{as}(\vec{a},z)=-\frac{1}{4\pi}\ln\left(-\frac{z}{4}\right)+\frac{1}{2\pi}\Psi(1)$.

Figure 2 Eigenvalue plots for different values of b, α is the coupling constant — *cf.* (2.13). The dashed, full and dash-dotted curves represent $b=\pi/2$, $b=\pi/6$ and $b=\pi/12$, respectively.

Figure 3 Unnormalized eigenfunctions for different values of b, $\alpha=1$. From top to bottom, they belong to $b=\pi/2$, $b=\pi/6$ and $b=\pi/12$, respectively.

Figure 4 Pole trajectory in the z–plane. The perturbation is located in the middle of strip, *i.e.* $b=\pi/2$. The resonance is on the third sheet of Riemann surface. The trajectory begins at the treshold $z=9$, which corresponds to $\alpha\to+\infty$ and runs through the lower part of the complex plane as α decreases to $-\infty$. In the inset is the dependence of the residue $\rho:=\xi'(\vec{a},z)$ depending on the coupling constant α.

Figure 5 Eigenfunctions for three point perturbations. The ground state, $E_0=-8.7051$ (bottom) and the second excited state, $E_2=-7.8199$ (top). The perturbation are placed at $(0,\pi/2)$, $(1,2\pi/3)$ and $(2,\pi/4)$; they have the same strength $\alpha=-0.15$.

Figure 6 An embedded eigenvalue due to symmetry. The ground and first excited energies of a strip with two perturbations placed symmetrically with respect to the axis of strip at $(0, \pi/3)$ and $(0, 2\pi/3)$. As α increases, the first excited state reaches the continuum and tends to 4 as $\alpha \to \infty$.

Figure 7 The cascading effect for a cluster of eigenvalues. Four perturbations placed at $(-1, 2\pi/3)$, $(0, \pi/2)$, $(1, \pi/3)$ and $(2, 2\pi/5)$; the first three have the same strength $\alpha_1 = \alpha_2 = \alpha_3 = -0.2$. If we change α_4 from $+\infty$ (very weak perturbation) to $-\infty$ (strong perturbation), there is a critical value at which the third excited states appears below the continuum. It becomes more bound as α_4 decreases, joins the triplet E_0, E_1, E_2 and the ground state energy E_0 parts the cluster and drops exponentially. The order of levels is always preserved.

Figure 8 The conductance plots for a pair of point perturbations. The plots show the dependence of conductance G on energy z. The perturbations are at $(0, \pi/3)$, $(0.25, 2\pi/3)$, having the same strength $\alpha = 0.2$ (the upper plot) and at $(0, 6\pi/11)$, $(10, 3\pi/11)$, $\alpha_1 = -0.25$, $\alpha_2 = -0.2$ (the lower plot).

Figure 9 A strip potential due to an external field.

Figure 10 $\mathrm{var}(G)$ for random perturbations *vs.* momentum.

Figure 11 Conductance for random perturbations *vs.* momentum.

Figure 12 Level–spacing statistics for random impurity–scattering.