Ψ-series and obstructions to integrability of periodically perturbed one degree of freedom Hamiltonians

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Abstract

A connection between the Ψ-series local expansion of the solution of a perturbed system of O.D.E.'s and the evaluation of the Mel'nikov vector with the method of residues has recently been found by Goriely and Tabor. By following an analogous procedure, we find a straightforward relation between the failure of the compatibility condition of the Painlevé test and the absence of an analytic integral for periodically perturbed Hamiltonians whose unperturbed part does not necessarily possess a homoclinic loop. We apply these results to a periodically perturbed unharmonic oscillator.

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1 Introduction

This work deals with the connection between the Ψ-series expression of the solution in complex time and a recently developed [1] real-time non-integrability criterion for one degree of freedom periodically perturbed Hamiltonian systems, whose integrable part is also Painlevé integrable. We follow the lines of reasoning of a recent paper by Goriely and Tabor [2], where they establish an analogous connection between Ψ-series of the solutions of perturbed O.D.E.'s and Mel'nikov's integral on the homoclinic loop of the unperturbed part, when this part possesses such a loop. On one hand, the non-integrability criterion developed in [1] involves an integral computed on the bounded solutions of the autonomous part of the Hamiltonian with a period that is in resonance with the period of the perturbation. This integral can be calculated by the method of residues. On the other hand, in the Ψ-series expansions for the solution of the perturbed system, certain coefficients of the logarithmic part are related to the same residues through the variational equations of the unperturbed system.

Painlevé and his associates, in their work on complex second-order O.D.E.'s, developed the so called α-method (e.g. [3]), in order to obtain nessecary conditions for which the only movable singularities of the solutions of such equations are poles. Then by direct computation, they found all such differential equations that possess the above property, which is termed the *Painlevé property.* Later, Ablowitz, Ramani and Segur [4] introduced a test which poses nessecary conditions for the Painlevé property in complex O.D.E.'s of any order. More references on Painlevé analysis and its connection to P.D.E.'s can be found in [2].

The solutions of the perturbed Hamiltonian can be expressed in series containing logarithmic terms, which belong to the class called Ψ-series and have found many applications in the theory of complex differential equations (e.g. [3,5]). They have also been employed in some recent works by several authors [6-10].

There is also a great deal of work done in the subject of non-integrability of Hamiltonian systems. It starts with the well known theorems of Poincaré ([11], Ch.5) and Bruns (e.g. [12]). Poincaré ([13], Ch.33) was the first to study the splitting of separatrices of hyperbolic periodic orbits. Mel'nikov [14] established his well-known theorem on the existence of transverse homoclinic intersections in the case of one degree of freedom non-autonomous systems. These transverse intersections have been used to prove the existence of chaotic Smale horseshoe dynamics (e.g. [15]). For the extension of Mel'nikov's homoclinic integral to the n degrees of freedom, see [16].

Ziglin [17] has proved a theorem relating certain properties of the monodromy group of the normal variational equations around a known isolated periodic solution of a Hamiltonian system to the existence of a second integral of motion. By applying Ziglin's theorem, Yoshida [18,19] has found algorithms for pinpointing non-integrable cases in homogeneous potentials of two or n degrees of freedom. These theorems of Ziglin and Yoshida are non-perturbative. There is also a perturbative theorem of Ziglin [20], that relates the non-vanishing of Mel'nikov's homoclinic integral to the multivaluedness of the integral of motion or the solution of the perturbed system.

Another integral which is evaluated on the periodic orbits of the unperturbed system is the *subharmonic* Mel'nikov function (e.g. [21], p.109). This function first apperared in ([11], Ch.3) for the case of n degrees of freedom autonomous Hamiltonians and in [14] for periodically perturbed one degree of freedom systems. Simple zeroes of this function correspond to bifurcations of the non-isolated periodic orbits of the unperturbed system to the perturbed one. The non-vanishing of this integral has been linked to the non-integrability of the perturbed system both in the autonomous n degrees of freedom case [22,23] and the periodically perturbed one [1]. In the autonomous case of 2 degrees of freedom, this integral supplies a non-integrability criterion which is equivalent to the non-integrability theorem of Poincaré. The non-vanishing of this integral has also been related to the infinite Riemann sheets of the solution of the perturbed system [24].

2 Laurent and Ψ-series

We consider the one degree of freedom Hamiltonian

$$H = H_0(q,p) + \varepsilon H_1(q,p,t). \tag{1}$$

The perturbation H_1 is a polynomial in q, p and periodic with respect to time with period T_1. The equations of motion are

$$\dot{x} = f(x) \tag{2}$$

where

$$x = \begin{pmatrix} q \\ p \end{pmatrix}$$

and

$$f(x) = \begin{pmatrix} \frac{\partial H_0}{\partial p} + \varepsilon \frac{\partial H_1}{\partial p} \\ -\frac{\partial H_0}{\partial q} - \varepsilon \frac{\partial H_1}{\partial q} \end{pmatrix} = \Omega D H$$

where Ω is the standard symplectic structure

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{pmatrix}.$$

Another assumption is that the unperturbed part, which is integrable, has the Painlevé property, i.e. the only movable singularities in its solution are poles. We also suppose that its solution can be expanded in formal Laurent series

$$x = (t - t_*)^p \sum_{k=0}^{\infty} a_k (t - t_*)^k \tag{3}$$

around the poles, where t_* is the position of the pole, which contains one more arbitrary constant, in addition to the arbitrary t_*.

The Painlevé test [4] plays an important role in the following analysis. In order to apply this test, one finds first the dominant behaviour of the solution around any existing pole, i.e.

$$x \sim \alpha(t - t_*)^p,$$

where

$$\alpha = a_0 \in C^2 \setminus \{0\}, \quad p \in Z^2.$$

By balancing the dominant terms, the unperturbed part of (2) takes the form

$$\dot{x} = f(x) = \hat{f}(x) + \check{f}(x) \tag{4}$$

where $\hat{f}$ is the part of f that exhibits the dominant behaviour $(t - t_*)^{p-1}$, while the behaviour of $\check{f}$ is $(t-t_*)^{p+\bar{p}-1}$, $\bar{p} \in Z^2 \setminus \{0\}$. There may be different integer vectors p that balance equation (4) and all of them have to be taken into account. Non-integer (rational or irrational) powers p imply that the system, although integrable, does not comply with the requirements of the Painlevé property. Since the unperturbed equations are Painlevé integrable by assumption, they pass the first step of the test with integer p as dominant behaviour for all balances.

The next step is to determine the resonances, that is the coefficients of the Laurent series, where the arbitrary constant enters in the expansion. The resonances are determined by the recursive procedure which determines the coefficients a_j in the Laurent expansion, by the equation

$$(R - jI)a_j = -P_j(\alpha, a_1, ..., a_{j-1}) \tag{5}$$

where

$$R = D\hat{f}(\alpha) - \text{diag}p$$

and P_j is a polynomial vector which depends only on $\alpha, ..., a_{j-1}$.

The position of the resonances in the Laurent series (3) is determined by the eigenvalues r of the matrix R, since for these eigenvalues the determinant of the matrix in the left-hand side of equation (5) becomes zero and therefore the corresponding solutions a_r involve arbitrary constants. In order for the a_r to exist, P_r must be orthogonal to the eigenvector $\bar{\beta}_r$ of R^T of the eigenvalue r, i.e.

$$\bar{\beta}_r^T P_r = 0. \tag{6}$$

The eigenvalue $r_1 = -1$ always appears, since it corresponds to the arbitrariness of the position of the pole. If the unperturbed system possesses the Painlevé property, the other eigenvalue is a positive integer and the compatibility condition (6) is satisfied.

In the Painlevé integrable case, the assumptions that the system has the Painlevé property and that there exist formal Laurent solutions which contain the arbitrary constants, that can be checked directly by the Painlevé test, suffice for the convergence of these Laurent series in a domain around the pole. The above mentioned assumptions also suffice for for the non-existence of a natural boundary and fixed essential singularities.

We are also going to deal with periodic orbits of the unperturbed system, with period T_2 (in real time), that are in resonance with the external period T_1, i.e.

$$T = nT_1 = mT_2 \tag{7}$$

where $m, n \in Z \setminus \{0\}$ and are relative primes. The *frequency* of the unperturbed orbits is

$$\omega = \frac{dH_0}{dJ} = \frac{2\pi}{T_2}$$

where J is the action variable of H_0. We assume that the unperturbed system is *non-degenerate*, i.e. that it holds

$$\frac{d^2H_0}{dJ^2} \neq 0,$$

in an open domain of phase space. As a consequence, the period T_2 of the solutions varies continuously in this domain, and in this case a dense set of periodic orbits of H_0 satisfying the resonance condition (7) exists. We also assume that these periodic solutions possess, in complex time, a second, complex period T_c, such that $\mathrm{Im}(T_c) \neq 0$.

The last assumption concerns the perturbation H_1. We need the term DH_1 to be non-dominant, i.e. the dominant behaviour of this term should be $(t - t_*)^{p+\tilde{p}-1}$ with $\tilde{p} \in C^2 \setminus \{0\}$.

If one tries to find the Laurent expansion of the perturbed solution

$$x' = \sum_{j=0}^{\infty} a'_j (t - t_*)^{p+j},$$

the non-dominant behaviour of the perturbation leads to $a'_0 = \alpha$ while the corresponding equation (5) for the perturbed system takes the form

$$(R - jI)a'_j = -P'_j(\alpha, a'_1, ..., a'_{j-1}) \tag{8}$$

where the matrix $R = D\hat{f}(\alpha) - \mathrm{diag}p$ is the same as in (5). Therefore the position $r \neq -1$ of the arbitrary coefficient remains the same, i.e. is eigenvalue of the same matrix, but now in general the compatibility condition (6) is not satisfied, i.e.

$$\bar{\beta}_r^T P_r' \neq 0, \tag{9}$$

since the perturbed system is expected not only to violate the Painlevé property, but also to be non-integrable. Finally, the last assumption, in conjunction to the above mentioned consequences, allows us to use, instead of the Laurent series, an expansion containing logarithmic terms which is termed logarithmic Ψ-series.

Proposition 1 [2]: The solutions of the perturbed system can be expanded in the formal Ψ-series

$$x' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} a_{ijk}(t-t_*)^{p+i}\varepsilon^j Z^k \tag{10}$$

or, alternatively

$$x' = \sum_{i=0}^{\infty}\sum_{j=0}^{i} s_{ij}\varepsilon^i Z^j \tag{11}$$

where $Z = \log(t-t_*)$ and s_{ij} are Laurent series that converge in a complex domain around the pole of the unperturbed system.

For a proof of this proposition see [2].

3 An obstruction to integrability and its evaluation with residues

In [1] we have proved the following

Proposition 2 [1]: A one degree of freedom perturbed Hamiltonian of the form

$$H = H_0(x) + \varepsilon H_1(x,t)$$

where the perturbation is periodic in time with period T_1 and H_0 is non-degenerate, does not possess an integral of motion, analytic in ε in an open interval around zero if, in an open domain of the phase space, one can find a dense or a key set of periodic orbits (invariant circles) of H_0 with period T_2 for which the resonance relation (7) holds, such that

$$I = \int_{t_0}^{t_0+T} \frac{\partial H_1}{\partial t}(x(t-t_0),t)dt \neq 0 \tag{12}$$

for at least one t_0 for each orbit.

Let $[H_0, H_1] = (DH_0)^T \Omega DH_1$ be the standard Poisson bracket. Then the above obstruction can also be expressed as

$$I = \int_{t_0}^{t_0+T} [H_0, H_1] dt \neq 0 \tag{13}$$

for this set of resonant periodic orbits of H_0, since

$$\int_{t_0}^{t_0+T} \frac{dH_1}{dt} dt = \int_{t_0}^{t_0+T} [H_1, H_0] dt + \int_{t_0}^{t_0+T} \frac{\partial H_1}{\partial t} dt \equiv 0. \tag{14}$$

The above calculations are performed in real time and the above criterion guarantees that no real-analytic and single-valued integral of the perturbed system exists in the domain where the above mentioned set of periodic orbits is dense, or in an open domain determined by a key set of them.

We may now put without loss of generality $T_1 = 2\pi$ by a scale transformation in time, so that the resonance relation (7) becomes

$$T = 2\pi n = mT_2.$$

Let us now evaluate the integral (13) by the method of residues. First we evaluate the Poisson bracket $[H_0, H_1]$ along the periodic solutions of the unperturbed part. Since H_0 is autonomous, x depends on time only through $\tau = t - t_0$, while H_1 depends on time both explicitly and through the unperturbed solution $x(\tau)$, i.e.

$$[H_0, H_1](x(t - t_0), t) = [H_0, H_1](x(\tau), \tau + t_0).$$

Since the Poisson bracket is, as shown above, a periodic function with period 2π with respect to its second argument, we can expand it in Fourier series as

$$[H_0, H_1] = \sum_{k=-\infty}^{\infty} g^{(k)}(x(\tau)) e^{ik(\tau + t_0)}. \tag{15}$$

Then the integral (13) becomes

$$I = \sum_{k=-\infty}^{\infty} I^{(k)} e^{ikt_0}$$

where

$$I^{(k)} = \int_0^T g^{(k)} e^{ik\tau} d\tau$$

i.e. I is periodic in t_0 with period 2π and the integrals $I^{(k)}$ are the coefficients of the Fourier expansion of I with respect to t_0. These integrals are going to be determined, for $k \neq 0$, with the help of the following path integrals.

$$S^{(k)} = \oint_{\gamma} g^{(k)}(x(z))e^{ikz}dz$$

where γ is the path in the complex plane depicted in figure 1 and can be decomposed as $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$.

The relationship between $I^{(k)}$ and $S^{(k)}$ is the following

$$S^{(k)} = I^{(k)} + S_2^{(k)} + S_3^{(k)} + S_4^{(k)} \tag{16}$$

where

Figure 1: The path γ in the complex τ-plane

$$S_i^{(k)} = \int_{\gamma_i} g^{(k)}(x(z))e^{ikz}dz\ , \quad i = 2, 3, 4.$$

It can be easily shown that $S_2^{(k)} = -S_4^{(k)}$ for every $k \in Z \setminus \{0\}$, therefore equation (16) becomes

$$S^{(k)} = I^{(k)} + S_3^{(k)}.$$

It can also be easily shown that

$$S_3^{(k)} = -e^{ikT_c} I^{(k)},$$

therefore

$$I^{(k)} = \frac{1}{1 - e^{ikT_c}} S^{(k)}. \tag{17}$$

The value of the path integral $S^{(k)}$ is given by

$$S^{(k)} = 2\pi i \sum_{t_*} \mathrm{res}\{g^{(k)}(x(t))e^{ikt}\} \tag{18}$$

where the summation is performed over the residues of the function $g^{(k)}(x(\tau))$ on all the poles of the solution of the unperturbed Hamiltonian, $x(\tau)$, that are included in the previous parallelogram.

The above procedure is not valid for $k = 0$ but, since the function $\partial H_1/\partial t$ does not have a zeroth order Fourier coefficient in an expansion in $\tau + t_0$ with respect to its explicit dependence on time, analogous to (15), and since equality (14) is valid, i.e. $I^{(k)}$ is also the k-th Fourier coefficient of

$$\int_0^T \frac{\partial H_1}{\partial t} d\tau,$$

it holds that $I^{(0)} = 0$.

In the previous analysis we have assumed that no pole lies on the period lines of the solution $x(t)$ of the unperturbed system. If a number of poles lie on the period lines γ_2, γ_4, then the path taken in order to perform the integration can be selected in a convenient way around the poles, such that the previous results are still valid. There are no poles on the period lines γ_1, γ_3 since we have assumed that H_0 is analytic in its arguments and therefore so is the real solution $x(t)$.

4 Variational equations of the unperturbed system and the Laurent expansion of their solutions

By expanding around a periodic solution $x_0(t)$ of system (2), we obtain for the first variation ξ the linear system of variational equations

$$\dot{\xi} = Df(x_0(t))\xi \tag{19}$$

where Df is the Jacobian of the right-hand side of (2), evaluated along the periodic solution $x_0(t)$. In the case of a Hamiltonian system with Hamiltonian H_0, the variational equations have the form

$$\dot{\xi} = \Omega D^2 H_0 \xi \tag{20}$$

where D^2H_0 is the Hessian of H_0, evaluated along the solution $x_0(t)$. It is known ([11], § 64) that $\Omega DH_0(x_0(t))$ is a periodic solution of (20).

In the following we are also going to use the adjoint variational equations (e.g. [25], p. 87) of (19),

$$\dot{\xi}^T = -\xi^T Df(x_0(t)) \tag{21}$$

which in the Hamiltonian case become

$$\dot{\xi}^T = -\xi^T \Omega D^2 H_0(x_0(t)). \tag{22}$$

It is easy to show that, if $\xi = \Omega y$ is a solution of (20), then y^T is a solution of (22). Let $Y, \bar{Y} \in GL(2, C(t))$ be fundamental solutions of (19) and (21) respectively, i.e.

$$\dot{Y} = DfY \tag{23}$$

and

$$\dot{\bar{Y}} = -\bar{Y}Df. \tag{24}$$

If Y is a solution of (23), then it can be shown (e.g. [2]) that Y^{-1} is is a solution of (24).

As a next step, we are going to use the following proposition, establishing the fact that the solutions of (19) and (21) can be expanded in Laurent series around the poles of the solution $x_0(t)$ and providing their dominant behaviour.

Proposition 3 [2]: The system of variational equations (19) around a periodic solution of the unperturbed system has a fundamental local solution Y, such that each column $y^{(i)}$ has a convergent local Laurent expansion in a punctured disk $B(t_*, y^{(i)})$ around each pole t_* of the solution $x_0(t)$. The form of the column $y^{(i)}$ around t_* is

$$y^{(i)}(t) = (t - t_*)^{p+r_i}(\beta_{r_i} + \sum_{j=1}^{\infty} b_j^{(i)}(t - t_*)^j)$$

where β_{r_i} is the eigenvector of R, corresponding to the eigenvalue r_i. The adjoint variational equations (21) have also a fundamental local solution $\bar{Y}$, whose rows $\bar{y}^{(i)}$ have the form

$$\bar{y}^{(i)}(t) = (t - t_*)^{-p-r_i}(\bar{\beta}_{r_i} + \sum_{j=1}^{\infty} \bar{b}_j^{(i)}(t - t_*)^j)$$

in a neighbourhood $B(t_*, \bar{y}^{(i)})$ around t_*, where $\bar{\beta}_{r_i}$ is the eigenvector of R^T, of eigenvalue r_i. The matrices $Y, \bar{Y}$ are defined in $B(t_*, Y) = \bigcap_{i=1}^{2} B(t_*, y^{(i)})$ and $B(t_*, \bar{Y}) = \bigcap_{i=1}^{2} B(t_*, \bar{y}^{(i)})$ respectively. In the open domain $B(t_*, Y) \bigcap B(t_*, \bar{Y})$ we have $\bar{Y} = Y^{-1}$.

For a proof, see [2].

For the Hamiltonian case at hand, we will prove the following

Proposition 4: DH_0, evaluated at x_0, is a solution of the adjoint variational equations with dominant behaviour $-p-r$ where r is the other (in addition to -1) eigenvalue of R . Moreover, the coefficient at the dominant behaviour is an eigenvector of R^T with eigenvalue r and therefore its local expansion can be selected as a row of the fundamental solution $\bar{Y}$.

Proof: We will show first that the local expansion of the known solution of (20), ΩDH_0, can be selected as a column, e.g. $y^{(1)}$, of Y. Since the leading behaviour of $x_0(t)$ is

$$x_{0i} \sim \alpha_i (t-t_*)^{p_i},$$

the leading behaviour of ΩDH_0 is $p-1$, since

$$\Omega DH_0(x_0)_i = \dot{x}_{0i} \sim \alpha_i p_i (t-t_*)^{p_i-1}.$$

By inserting now ΩDH_0 in the variational equations, we obtain

$$\alpha_i p_i (p_i - 1) \sim \sum_j \frac{\partial \hat{f}_i}{\partial x_j}(\alpha)\alpha_j p_j$$

or

$$R\alpha p = -\alpha p, \tag{25}$$

i.e. -1 is an eigenvalue of R and the coefficient of ΩDH_0 at the dominant behaviour $p-1$, i.e. $\alpha p = (\alpha_1 p_1, \alpha_2 p_2)$ is the corresponding eigenvector.

By inserting ΩDH_0 in (20) and transposing we obtain

$$\frac{d}{dt}(DH_0)^T = -(DH_0)^T \Omega D^2 H_0,$$

i.e. $DH_0(x_0)^T$ is a solution of the adjoint variational equations (22). Let q be its dominant behaviour. From the identity

$$DH_0 = \Omega^{-1}(\Omega DH_0)$$

we see that

$$q_1 = p_2 - 1,\ q_2 = p_1 - 1. \tag{26}$$

Since $\Omega D^2 H_0$ is traceless, the trace of R equals $-(p_1+p_2)$. Let r be the eigenvalue of R, different from -1. Then $r-1=-p_1-p_2$ and the leading behaviour of DH_0 is also

$$q_1 = -p_1 - r,\ q_2 = -p_2 - r. \tag{27}$$

Next we will show that the vector of the coefficients of the dominant behaviour of DH_0, which is $\Omega^{-1}\alpha p = (-\alpha_2 p_2,\ \alpha_1 p_1)$, is an eigenvector of R^T of eigenvalue r. The matrix R is given by

$$R = \Omega D^2 H_0(\alpha) - \mathrm{diag} p,$$

while, as seen from (26) and (27), it holds

$$\mathrm{diag} p - I = -\mathrm{diag} p - rI,$$

so that

$$(R^T - rI)\Omega^{-1}\alpha p = \Omega(R+I)\alpha p = 0$$

and this completes the proof of proposition 4. □

The conclusion of the above analysis, which we will use later, is that the adjoint variational equations (22) have a local fundamental solution $\bar{Y}$, which can always be selected as

$$\bar{Y} = \begin{pmatrix} \bar{y}^{(1)T} \\ (DH_0)^T \end{pmatrix} \in GL(2, C(t)). \tag{28}$$

The dominant behaviour of $\bar{y}^{(1)}$ is, according to proposition 3, $-p+1$.

5 Relation of the compatibility condition to the non-integrability of the perturbed system

By truncating the formal logarithmic expansion of the solution of the perturbed system around a pole t_* of the unperturbed one to order ε, equation (11) yields

$$x' = s_{00} + \varepsilon(s_{10} + s_{11}Z) + O(\varepsilon^2)$$

On the other hand, if we consider the solution x' as a perturbation of the unperturbed solution x, i.e.

$$x' = x + \varepsilon x_1 + O(\varepsilon^2)$$

and differentiate with respect to time, we get up to order ε

$$\dot{x}_1 = \Omega D^2 H_0 x_1 + \Omega D H_1 \tag{29}$$

where D^2H_0 and DH_1 are evaluated on the unperturbed solution x. Since $x_1 = s_{10} + s_{11}Z$, we have

$$\dot{s}_{10} + \dot{s}_{11}Z + s_{11}(t - t_*)^{-1} = Dfs_{10} + Dfs_{11}Z + \Omega DH_1$$

where $Df = \Omega D^2H_0$. Therefore, since the polynomial and the logarithmic functions are linearly independent, we arrive at the following system

$$\dot{s}_{10} = Dfs_{10} + \Omega DH_1 - s_{11}(t - t_*)^{-1} \tag{30}$$

$$\dot{s}_{11} = Dfs_{11} \tag{31}$$

Proposition 5 [2]: The solution of the system (30), (31) can be written as

$$s_{10} = YK_0, \quad s_{11} = YK_1 \tag{32}$$

where

$$K_0 = \int^t \bar{Y}(s)\Omega DH_1(x(s), s)ds - K_1\log(t - t_*) \tag{33}$$

$$K_1 = \operatorname*{res}_{t_*}\{\bar{Y}(t)\Omega DH_1(x(t), t)\} \tag{34}$$

and t_* is the pole around which we expand the solution $x = s_{00}$ of the unperturbed and x' of the perturbed system.

The proof, which can be found in [2], involves the general solution (32) of (30) and (31), while (34) is obtained by demanding s_{10} to be a Laurent series, according to proposition 1.

Now, if we select the matrix $\bar{Y}$ as in (28), for the components of K_1 we have

$$K_{11} = \operatorname*{res}_{t_*}(\bar{y}^{(1)T}\Omega DH_1) \tag{35}$$

and

$$K_{12} = \operatorname*{res}_{t_*}\{[H_0, H_1]\}, \tag{36}$$

while from (32) we obtain $K_1 = \bar{Y}s_{11}$, i.e.

$$K_{11} = \bar{y}^{(1)T}s_{11} \tag{37}$$

and

$$K_{12} = DH_0^Ts_{11}. \tag{38}$$

According to equations (10) and (11) of proposition 1,

$$s_{11} = \sum_{i=0}^{\infty} a_{i11}(t-t_*)^{i+p} \tag{39}$$

and, according to proposition 4,

$$\begin{aligned} \bar{y}^{(1)} &= (t-t_*)^{-p+1}(\bar{\beta}_{-1} + \textstyle\sum_{j=1}^{\infty} b_j^{(1)}(t-t_*)^j), \\ DH_0 &= (t-t_*)^{-p-r}(\bar{\beta}_r + \textstyle\sum_{j=1}^{\infty} b_j^{(2)}(t-t_*)^j) \end{aligned} \tag{40}$$

where $\bar{\beta}_{-1}$ and $\bar{\beta}_r$ are eigenvectors of R^T of eigenvalues -1 and r respectively. By inserting (39), (40) in (37), (38) and taking into account that $K_1 = \text{const.}$, we obtain

$$K_{11} = 0, \tag{41}$$

$$K_{12} = \bar{\beta}_r^T a_{r11} \overset{\text{def}}{=} c_r. \tag{42}$$

Considering now, following [2], the logarithmic expansion of x' up to $p+r$ and expanding with respect to ε we obtain

$$x' = \sum_{i=1}^{r} \left\{ a_i|_{\varepsilon=0} + \varepsilon \left.\frac{\partial a_i}{\partial \varepsilon}\right|_{\varepsilon=0} \right\} (t-t_*)^{p+i} + \left\{ b_0|_{\varepsilon=0} + \varepsilon \left.\frac{\partial b_0}{\partial \varepsilon}\right|_{\varepsilon=0} \right\} (t-t_*)^{p+r} Z + \text{h.o.t.}$$

Since there are no logarithmic terms in the unperturbed expansion, it holds $b_0|_{\varepsilon=0} = 0$ and the coefficient of the term $\varepsilon \log(t-t_*)$ at $p+r$ is

$$\left.\frac{\partial b_0}{\partial \varepsilon}\right|_{\varepsilon=0},$$

which equals a_{r11}, according to (10).

We have already mentioned that due to the non-dominant behaviour of the perturbation, the matrix R is the same as in the unperturbed system. Introducing now the logarithmic term, equation (8) becomes

$$(R - rI)a'_r = -P'_r + b_0$$

and, in order for a'_r to exist, the right-hand side must be orthogonal to the eigenvector $\bar{\beta}_r$, i.e.

$$\bar{\beta}_r^T P'_r = \bar{\beta}_r^T b_0 \overset{\text{def}}{=} C_r, \tag{43}$$

therefore, according to (42), we have

$$K_{12} = \bar{\beta}_r^T a_{r11} = \bar{\beta}_r^T \left.\frac{\partial b_0}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial C_r}{\partial \varepsilon}\right|_{\varepsilon=0} = c_r. \tag{44}$$

On the other hand, inserting into (36) the expansion (15) of the Poisson bracket we get

$$K_{12} = \sum_{-\infty}^{\infty} \operatorname*{res}_{t_*} \{g^{(k)}(x(\tau)) e^{ik\tau}\} e^{ikt_0}. \tag{45}$$

and considering the Fourier expansion of c_r

$$c_r = \sum_{-\infty}^{\infty} c_r^{(k)} e^{ikt_0}$$

we obtain that

$$c_r^{(k)} = \operatorname*{res}_{t_*} \left\{ g^{(k)}(x(\tau)) e^{ik\tau} \right\}.$$

Taking now into account equations (17) and (18), we get

$$I^{(k)} = \frac{2\pi i}{1 - e^{ikT_c}} \sum_{t_*} c_r^{(k)} \tag{46}$$

and the integral I in the obstruction (12) or (13) equals to

$$I = 2\pi i \sum_{k=-\infty}^{\infty} \frac{\sigma_r^{(k)}}{1 - e^{ikT_c}} e^{ikt_0}, \quad k \neq 0 \tag{47}$$

where

$$\sigma_r^{(k)} = \sum_{t_*} c_r^{(k)} \tag{48}$$

is the sum of the Fourier coefficients $c_r^{(k)}$ of the terms of order ε of the compatibility condition (9) over all poles of the solution $x(\tau)$ of the unperturbed system inside the parallelogram depicted in figure 1. Since e^{ikt_0} are linearly independent functions, from (47) we see that if, for at least one k, the sums $\sigma_r^{(k)}$ are different from zero for the invariant circle of H_0 obeying the resonance condition (7), then $I \neq 0$ on this circle. By combining this result with proposition 2, we have proved the following

Proposition 6: If at least one $\sigma_r^{(k)}$ is different from zero on a dense or a key set of resonant invariant circles of H_0, the perturbed Hamiltonian does not possess an analytic integral of motion for $\varepsilon \neq 0$ in an open domain around zero.

6 Application to a periodically perturbed unharmonic oscillator

We will apply the above results to the Hamiltonian

$$H = H_0 + \varepsilon H_1 = \frac{1}{2} p_x^2 + \frac{1}{4} x^4 + \varepsilon x \cos t, \tag{49}$$

which corresponds to the perturbed unharmonic oscillator

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{p}_x &= -x^3 - \varepsilon \cos t. \end{aligned} \tag{50}$$

The solution of the unperturbed system is (e.g. [26], p. 207)

$$x(\tau) = \lambda \mathrm{cn}(\lambda\tau, 1/\sqrt{2}), \quad \tau = t - t_0 \tag{51}$$

where λ and t_0 are arbitrary constants. The periods of $x(\tau)$ are (e.g. [27], p. 914)

$$T_2 = \frac{4K}{\lambda}, \quad T_c = \frac{2(K + iK')}{\lambda}$$

where K and K' are respectively the complete and the complementary complete elliptic integral of the first kind with modulus $k = 1/\sqrt{2}$.

The resonance condition (7) takes the form

$$\frac{4K}{\lambda} = \frac{2\pi n}{m} \tag{52}$$

where m, n are relative prime positive integers. For the invariant circle of H_0 defined by (52), the path of integration γ is the parallelogram shown in figure 1, in which $2m$ simple poles of the solution (51) are included, at the points

$$\begin{aligned} t_*^s &= \tfrac{1}{\lambda}(2(2s+1)K + iK'), \\ \bar{t}_*^s &= \tfrac{1}{\lambda}(4sK + iK') \end{aligned} \tag{53}$$

for $s = 0, 1, ..., m-1$, with residues

$$i/k = i\sqrt{2} \quad \text{and} \quad -i/k = -i\sqrt{2}$$

respectively.

Applying the standard Painlevé test to (50), we obtain for the leading behaviour the following results

$$p = (-1,\ -2), \quad \alpha = \pm(i\sqrt{2},\ -i\sqrt{2})$$

where the plus sign corresponds to the expansion around t_*^s and the minus sign to the one around $\bar{t}_*^s$. The matrix R is

$$R = \Omega D^2 H_0(\alpha) - \mathrm{diag}p = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$$

and the eigenvalues are -1 and $r=4$. By expanding DH_0 we obtain the normalized eigenvectors

$$\bar{\beta}_4 = \mp i\sqrt{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The vector P_4' on the other hand is

$$P_4' = \begin{pmatrix} 0 \\ -\varepsilon\sin(t_* + t_0) \end{pmatrix},$$

so that

$$c_4 = \pm i\sqrt{2}\sin(t_* + t_0).$$

The corresponding Fourier coefficients $c_r^{(k)}$ are non-zero only for $k=\pm 1$ and are given by

$$c_4^{(-1)}(t_*^s) = -\tfrac{1}{\sqrt{2}}e^{-it_*^s}, \quad c_4^{(1)}(t_*^s) = \tfrac{1}{\sqrt{2}}e^{it_*^s},$$

$$c_4^{(-1)}(\bar{t}_*^s) = \tfrac{1}{\sqrt{2}}e^{-it_*^s}, \quad c_4^{(1)}(\bar{t}_*^s) = -\tfrac{1}{\sqrt{2}}e^{it_*^s}.$$

We apply (46) for $k=\pm 1$ and distinguish the following cases:
a) $m \neq 1$ and b) $m=1,\ n=2j$
In these cases $I^{(-1)} = I^{(1)} = 0$ so that $I=0$.
c) $m=1,\ n=2j+1$
In this case (46) yields

$$I^{(-1)} = -\frac{2\pi i}{K}\frac{q^{-(j+1/2)}}{1+q^{-(2j+1)}},$$

$$I^{(1)} = \frac{2\pi i}{K}\frac{q^{(j+1/2)}}{1+q^{(2j+1)}}$$

where q is the elliptic nome, which in this case is $q=e^{-\pi}$, and we obtain

$$I = -\frac{4\pi}{K}\frac{q^{(j+1/2)}}{1+q^{(2j+1)}}\sin t_0. \tag{54}$$

This integral is not identically zero, so the obstruction of proposition 2 is satisfied on the set of invariant circles of H_0 defined by

$$\lambda = \frac{2K}{\pi(2j+1)}, \quad j=0,1,...,$$

which is given from (52) for the above values of m and n. These circles, for $j\to\infty$ accumulate on the equilibrium $x=p_x=0$ and they form a key set in an open neighbourhood of it. According to proposition 2, the system does not possess an analytic integral of motion for $\varepsilon\neq 0$. This result is valid, however, for all ε, since the above analysis remains unaltered if we transform $\varepsilon\to C\varepsilon$ with arbitrary C.

7 Conclusions

The main result of this paper is included in proposition 6, which relates in a straightforward way the compatibility condition of the Painlevé test to the non-integrability of a periodically perturbed nearly integrable Hamiltonian of one degree of freedom. In obtaining this result, we followed the main steps of the method developed by Goriely and Tabor in [2], where they established such a relation for the homoclinic Mel'nikov vector of a perturbed system of O.D.E.'s whose integrable part possesses a homoclinic loop. Our result connects the Ψ-series expansion of the perturbed solution directly to the non-existence of an analytic integral, irrespectively of whether the unperturbed system possesses such a loop or not.

We have shown that if the sum $\sigma_r^{(k)}$ of the compatibility conditions at order ε over all poles of the local expansions for a dense or a key set of periodic orbits of H_0 is different from zero, then the perturbed Hamiltonian cannot possess an analytic integral for $\varepsilon \neq 0$. This does not mean, however, that the introduction of logarithmic terms in the local expansion around each pole at order ε leads to the non-existence of an analytic integral, since $c_r^{(k)}$ may be in general different from zero, but their sums may eventually vanish. In this case, the obstruction to integrability of proposition 2 is not satisfied and one cannot prove non-integrability. This is exactly the case for most of the invariant circles of the integrable part in the example of the preceeding section, although (12) is satisfied in a key set of them.

The present analysis combines also the singularity structure of the perturbed solutions to the subharmonic bifurcations of periodic orbits, since the integral I, as commented in [1], is actually the subharmonic Mel'nikov integral, whose simple zeroes define the points of continuation of periodic orbits of the unperturbed to the perturbed system. The possible existence of such a relation was already pointed out in [2]. As we mentioned in [1], the obstruction (12) poses a weaker condition than the existence of simple zeroes of I.

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