GLOBAL SPECIFICATIONS AND NON–QUASILOCALITY OF PROJECTIONS OF GIBBS MEASURES

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Abstract: We study the question of whether the quasilocality (continuity, almost-Markovianness) property of Gibbs measures remains valid under a projection on a sub–σ–algebra. Our method is based on the construction of global specifications, whose projections yield local specifications for the projected measures. For Gibbs measures compatible with monotonicity–preserving local specifications, we show that the set of configurations where quasilocality is lost is an event of the tail field. This set is shown to be empty whenever a strong-uniqueness property is satisfied, and of measure zero when the original specification admits a single Gibbs measure. Moreover, we provide a criterion for non-quasilocality (based on a quantity related to the surface-tension). We apply these results to projections of the extremal measures of the Ising model. In particular, our non-quasilocality criterion allows us to extend and make more complete previous studies of projections to a sublattice of one less dimension (Schonmann example).

Running Head: Specifications and non–quasilocality

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1 Introduction

We study random fields X_i indexed by the elements i of a countable set $\mathcal{L}$, with values in $\{-1,1\}$. We set $\Omega := \{-1,1\}^{\mathcal{L}}$ and $\mathcal{F} := \sigma\{X_i,\, i \in \mathcal{L}\}$. Let $S \subset \mathcal{L}$; we set $\mathcal{F}_S := \sigma\{X_j,\, j \in S\}$, $S^c := \mathcal{L}\backslash S$ and denote by ω_S the restriction of ω to S. The symbol $|M|$ will denote the cardinality of $M \subset \mathcal{L}$ and $1_F(\,\cdot\,)$ the characteristic function of $F \in \mathcal{F}$. We always suppose that the conditional probability

$$\mathbb{E}(1_F \,|\, X_j = \omega(j),\, j \in S^c)\ ,\quad F \in \mathcal{F} \text{ and } S \subset \mathcal{L}\,, \tag{1.1}$$

is given by a probability kernel γ_S on $(\Omega, \mathcal{F})$,

$$\mathbb{E}(1_F \,|\, X_j = \omega(j),\, j \in S^c) = \gamma_S(F|\omega)\,. \tag{1.2}$$

An interesting case is when $\mathcal{L}$ is the set of vertices of a simple graph $(\mathcal{L}, V)$. The graph structure defines a notion of adjacency, two elements i and j of $\mathcal{L}$ are adjacent if and only if they are connected by an edge $e \in V$ of the graph. For each set $S \subset \mathcal{L}$ the boundary ∂S of S is the set

$$\partial S := \{j \in \mathcal{L} : j \in S^c, j \text{ adjacent to some } i \in S\}. \tag{1.3}$$

If for all finite subsets $\Lambda \subset \mathcal{L}$, all $\mathcal{F}_\Lambda$–measurable sets F, $\gamma_\Lambda(F|\omega)$ is $\mathcal{F}_{\partial\Lambda}$–measurable, then the random field is called a *Markov random field*. The importance of such random fields in applied sciences (e.g. neural networks, statistical mechanics) comes from the fact that the *local* laws governing the system are modelled by specifying the probability kernels γ_Λ for all finite $\Lambda \subset \mathcal{L}$. The collection $\Gamma := \{\gamma_\Lambda,\, \Lambda \text{ finite}\}$ is called a *local specification* (see definition 2.2). The main problem is then to describe the set $\mathcal{G}(\Gamma)$ of all random fields *compatible* with a given specification Γ, i.e. such that for all finite Λ

$$\mathbb{E}(1_F \,|\, X_j = \omega(j),\, j \in \Lambda^c) = \gamma_\Lambda(F|\omega)\,. \tag{1.4}$$

A fundamental aspect of this problem is that $\mathcal{G}(\Gamma)$ may contain several elements. (In this paper we always have $\mathcal{G}(\Gamma) \neq \emptyset$). It is therefore possible to have *different global* behaviours compatible with given local laws. This problem is often called the DLR–problem, because Dobrushin (1968), Lanford and Ruelle (1969) proposed the formulation of statistical mechanics of infinite systems precisely in those terms. Usually one does not study a single local specification Γ, but a model, i.e. a family $\Gamma^{\underline{t}}$ of local specifications indexed by parameters $\underline{t}$. A famous example is the Ising model, where the parameters are interpreted as the temperature and the external magnetic field. The set $\mathcal{G}(\Gamma^{\underline{t}})$ of all random fields compatible with $\Gamma^{\underline{t}}$ depends now on $\underline{t}$; one says that there is a *phase transition* at $\underline{t}$ if $\mathcal{G}(\Gamma^{\underline{t}})$ contains more than one element.

An obvious generalization of the above framework is to replace the condition that the conditional probability in (1.4) is $\mathcal{F}_{\partial\Lambda}$–measurable by the weaker condition that it is $\mathcal{F}_{W_\Lambda}$–measurable, where $W_\Lambda \supset \partial\Lambda$ is some given finite set. More generally, the weakest natural condition is to require that the function $\omega \mapsto \gamma_\Lambda(F|\omega)$ be a continuous function on Ω (with product topology). The continuity requirement, also called *quasilocality* or *almost–Markov property* in Sullivan (1973)[a], means that given a positive ε there exists a *finite* set $\Lambda_1 \subset \mathcal{L}\backslash\Lambda$ such that

$$|\gamma_\Lambda(F|\omega) - \gamma_\Lambda(F|\omega')| \leq \varepsilon\,, \tag{1.5}$$

whenever $\omega(j) = \omega'(j)$ for all $j \in \Lambda_1$. Georgii (1988) is the standard reference on the subject.

Let $\{X_i,\, i \in \mathcal{L}\}$ be a random field (described by a probability measure μ) compatible with a local specification $\Gamma = \{\gamma_\Lambda\}$, which is quasilocal. There are various natural situations where one is interested only in the subprocess $\{X_i,\, i \in T\}$, T being some infinite subset of $\mathcal{L}$. (In a problem of transmission of information with a random source we may have access only to the transmitted messages.) We also suppose that T^c is an infinite set. The subprocess is of course described by the projection μ_T of the measure μ on the σ–algebra $\mathcal{F}_T$. It was noticed in Griffiths Pearce (1979), and clarified in Israel (1981), that the quasilocality property may not be valid for the subprocess $\{X_i,\, i \in T\}$. The same observation has been made later on by Schonmann (Schonmann (1989)) for a different choice of T, but without making the connection with the earlier works Griffiths Pearce (1979) and Israel (1981). In the comprehensive work van Enter Fernández Sokal (1993) these problems are analyzed in depth in the original context of Griffiths Pearce (1979), namely the renormalization group transformations. (Here the above transformation is called a decimation transformation; many other transformations of the random fields are also analyzed.) In all known examples the lack of quasilocality means that there exists η such that *no* version of the conditional probability,

$$\mathbb{E}_{\mu_T}(1_{\{X_i=\eta(i)\}}\,|\,X_j = \eta(j),\, j \in T\backslash\{i\})\,,\; i \in T\,, \tag{1.6}$$

is continuous at η. Compared with (1.5) this reveals an instability of the system by a dependence of (1.6) on the values of the subprocess "at infinity", i.e. outside any finite subset $\Lambda_1 \subset T$. This phenomenon may be traced back to the phenomenon of phase transition on the "hidden part" of the process. For any $\eta \in \Omega$ we define a local specification $\Gamma^\eta_{T^c}$ on T^c by setting for finite $\Lambda \subset T^c$

$$\gamma^{T^c}_{\Lambda,\eta}(F|\omega) := \gamma_\Lambda(F|\omega_{T^c}\eta_T)\,. \tag{1.7}$$

We denote by $\mathcal{G}(\Gamma^\eta_{T^c})$ the set of random fields $\{X_j,\, j \in T^c\}$ compatible with $\Gamma^\eta_{T^c}$; the role of the parameters $\underline{t}$ mentioned before is played here by η. In all known examples the lack of quasilocality at η of (1.6) is established by proving that $\mathcal{G}(\Gamma^\eta_{T^c})$ contains more than one element, that is, there is a phase transition for the system on T for such a choice of η.

In this paper we study this type of problems for a class of random fields defined by monotonicity–preserving local specifications Γ. We say that $\omega \leq \eta$ if $\omega(i) \leq \eta(i)$

[a] For the general case $\Omega = \Omega_0^{\mathcal{L}}$ with Ω_0 compact, the requirement of continuity is in fact slightly different from quasilocality or almost-Markovianness. If Ω_0 is finite, as it is here, the three qualifiers become synonymous. We shall mostly use here the word "quasilocal", except in some instances where we shall use instead "continuous", to avoid confusion with notions like "local specifications" (see below).

for all $i \in \mathcal{L}$, and a function f is increasing if $\omega \leq \eta$ implies that $f(\omega) \leq f(\eta)$. The monotonicity–preserving condition reads

$$f(\cdot) \text{ increasing} \Longrightarrow \gamma_\Lambda(f|\cdot) \text{ increasing}. \tag{1.8}$$

We also suppose that the local specifications are *Gibbs specifications* in the sense of definition 2.6. Denote by $+$, respectively by $-$ the element η such that $\eta(i) = 1$ for all i, respectively $\eta(i) = -1$ for all i. There are two probability measures on $(\Omega, \mathcal{F})$, denoted by μ^+ and μ^-,

$$\mu^+(\cdot) := \lim_{\Lambda \uparrow \mathcal{L}} \gamma_\Lambda(\cdot|+) \quad , \quad \mu^-(\cdot) := \lim_{\Lambda \uparrow \mathcal{L}} \gamma_\Lambda(\cdot|-)\,, \tag{1.9}$$

so that the corresponding random fields belong to $\mathcal{G}(\Gamma)$ and any probability measure μ corresponding to a random field in $\mathcal{G}(\Gamma)$ is such that for any increasing function f,

$$\mu^-(f) \leq \mu(f) \leq \mu^+(f)\,. \tag{1.10}$$

In particular $|\mathcal{G}(\Gamma)| = 1$ if and only if $\mu^+ = \mu^-$. This class includes the Ising model and we apply our general results to that case at the end of our paper, recovering and extending earlier results. Our strategy is to examine first another question, which is of interest in itself, whether for a random field in $\mathcal{G}(\Gamma)$ we can extend the local specification to a *global specification* such that the random field remains compatible. A global specification is a family of probability kernels $\{\gamma_S\}$ indexed by *all* subsets $S \subset \mathcal{L}$ and satisfyfing the axioms of a local specification. Contrary to the local specification there is at most one random field compatible with a global specification. In the case of a Markov random field the existence of a compatible global specification is equivalent to the validity of the global Markov property of the random field. [See Albeverio and Zegarlinski (1992) for a review on this last property.] We prove the existence of a global specification in two cases: when the local specification is monotonicity–preserving and the random field corresponds to the probability measure μ^+ or μ^-, or when the local specification satisfies a strong uniqueness condition (definition 3.3). Then we study subprocesses of the random field described by μ^+. Using the global specification we can define a local specification $Q_T^+ = \{q_\Lambda^+\,; \Lambda \text{ finite} \subset T\}$ for the subprocess $\{X_i,\, i \in T\}$. Let Ω_q^+ be the set of continuity points of Q_T^+ (see definition 2.5). In Propositions 4.1 and 4.2 we prove

- The set Ω_q^+ is measurable with respect to the tail–field σ–algebra

$$\mathcal{T}_T^\infty := \bigcap_{\substack{\Lambda \subset T \\ \Lambda \text{ finite}}} \mathcal{F}_{T\backslash\Lambda}\,. \tag{1.11}$$

- The set Ω_q of all η such that $|\mathcal{G}(\Gamma_{T^c}^\eta)| = 1$ (no phase transition for η) is a subset of Ω_q^+.

- If $|\mathcal{G}(\Gamma_{T^c}^-)| = 1$, then $\Omega_q^+ = \Omega_q$, i.e. ω is a continuity point if and only if $|\mathcal{G}(\Gamma_{T^c}^\omega)| = 1$.

- If $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$, then no point of discontinuity of Q_T^+ can be removed by modifying the local specification Q_T^+ on a set of μ_T–measure zero.

Similar results hold for the measure μ^-. The results concerning the set of continuity points of the local specification Q_T^+ are summarized in the

Theorem *Let Γ be a Gibbs specification which is monotonicity–preserving. Let $T \subset \mathcal{L}$ so that $|T| = \infty$ and $|T^c| = \infty$. Let Ω_q^+ be the set of continuity points of the local specification $Q_T^+ = \{q_\Lambda^+\}_{\Lambda \subset T}$ and Ω_q^- the set of continuity points of $Q_T^- = \{q_\Lambda^-\}_{\Lambda \subset T}$.*

1. *If $|\mathcal{G}(\Gamma)| = 1$, i.e. $\mu^+ = \mu^-$, then Ω_q^+ has μ^+–measure one.*

Assume that $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$ and $|\mathcal{G}(\Gamma_{T^c}^-)| = 1$. Then the set

$$\Omega_q = \{\omega : \ |\mathcal{G}(\Gamma_{T^c}^\omega)| = 1\} , \tag{1.12}$$

$\Omega_q \subset \{-1,1\}^T$, has the following properties:

2. *Ω_q is $\mathcal{T}_T^\infty$–measurable and hence it is dense in $\{-1,1\}^T$,*
3. *$\Omega_q^+ = \Omega_q^- = \Omega_q$,*
4. *$\Omega_q = \{\omega \in \Omega : q_\Lambda^+(\cdot|\omega) = q_\Lambda^-(\cdot|\omega) \ \ \forall \text{ finite } \Lambda \subset T\}$,*
5. *$\mu^+(\Omega_q) = 1$ if and only if $\mu^+ \in \mathcal{G}(Q_T^-)$; $\mu^-(\Omega_q) = 1$ if and only if $\mu^- \in \mathcal{G}(Q_T^+)$.*

In section 5 we establish a criterion proving non–quasilocality for Q_T^+. The method is inspired by Sullivan (1973) and Kozlov (1974). It is based on an interesting estimate of the relative entropy for two random fields compatible with a local specification.

The plan of the paper is as follows. In section 2 we set the general notation and give the main definitions. All results are formulated in terms of measures μ rather than random fields. In section 3 we treat the question of the existence of a global specification. In section 4 we define and study a local specification for the projected measure. This section contains the main theorem. A criterion for absence of quasilocality is established in section 5. Finally in section 6 we apply our results to the Ising model.

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2 Notation, local specification, quasilocality

2.1 General notation

We find it useful to adopt the following convention: Λ or Λ_1,..., always denote sets of finite cardinality. The expression $\lim_\Lambda a_\Lambda$ is the limit of the net $\{a_\Lambda; \{\Lambda\}, \subset\}$. If we consider a subnet $\{a_\Lambda; \{\Lambda\}_{\Lambda\subset S}, \subset\}$, where S is an infinite subset of $\mathcal{L}$, its limit is written $\lim_{\Lambda\uparrow S} a_\Lambda$. Let $E = \{-1, +1\}$ (with the discrete topology) and $\Omega := E^{\mathcal{L}}$ with the product topology and product σ–algebra; the elements of Ω are functions $\omega : \mathcal{L} \to E$, $i \mapsto \omega(i)$; they are called *configurations.* ω_M is the restriction of ω to a subset $M \subset \mathcal{L}$; two configurations play a special rôle, $\omega(i) \equiv 1$ and $\omega(i) \equiv -1$ which are denoted by $+$ and $-$. Let $\Lambda \subset \mathcal{L}$, $\eta \in \Omega$ and $\omega \in \Omega$; we define $\omega^\eta_\Lambda \in \Omega$ by

$$\omega^\eta_\Lambda(k) := \begin{cases} \omega(k) & k \in \Lambda \\ \eta(k) & k \notin \Lambda\,. \end{cases} \tag{2.1}$$

For example ω^+_Λ is the configuration equal to ω in Λ and equal to 1 outside Λ. The value at ω of the evaluation-map X_i, $i \in \mathcal{L}$, is $X_i(\omega) := \omega(i)$; $\mathcal{F}_M = \sigma\{X_i; i \in M\}$ is the σ–algebra generated by the X_i's, $i \in M$; when $M = \mathcal{L}$ we set $\mathcal{F} := \mathcal{F}_{\mathcal{L}}$. Let S be any subset of $\mathcal{L}$; the tail-field σ– algebra on S is

$$\mathcal{T}^\infty_S := \bigcap_{\Lambda\subset S} \mathcal{F}_{S\backslash\Lambda}\,. \tag{2.2}$$

If S is not finite, then $\mathcal{T}^\infty_S \neq \{\Omega, \emptyset\}$. When $S = \mathcal{L}$ we set $\mathcal{T}^\infty := \mathcal{T}^\infty_{\mathcal{L}}$. We say that two configurations ω and ω' are *almost equal,* $\omega \sim \omega'$, iff $\omega(k) = \omega'(k)$ for all but a finite number of k. The relation $\sim$ is an equivalence relation. Given $\omega \in \Omega$, its equivalence class

$$\tau_\omega := \{\omega' :\ \omega' \sim \omega\} \tag{2.3}$$

is a (countable) $\mathcal{T}^\infty$-measurable set. Conversely, if A is a $\mathcal{T}^\infty$–measurable set and $\omega \in A$, then $\{\omega' :\ \omega' \sim \omega\} \subset A$, and hence

$$A = \bigcup_{\omega\in A} \tau_\omega\,. \tag{2.4}$$

The family of all subsets $C(\omega, \Lambda) := \{\omega' :\ \omega'_\Lambda = \omega_\Lambda\}$, $\Lambda \subset \mathcal{L}$, $\omega \in \Omega$, forms a base of open neighbourhoods of ω. For any ω the set τ_ω is dense in Ω, and therefore all non-empty $\mathcal{T}^\infty$–measurable sets are dense.

All $\mathcal{F}_\Lambda$–measurable functions are continuous since $\Lambda \subset \mathcal{L}$ is finite. They are called *local* or Λ*–local.* The set of local functions is dense in the set of all continuous functions with the sup-norm topology. The only $\mathcal{T}^\infty$–measurable and continuous functions are the constant functions. Indeed, let ω and η belong to Ω. By definition $\omega^\eta_\Lambda \sim \eta$ and $\lim_\Lambda \omega^\eta_\Lambda = \omega$. If f is $\mathcal{T}^\infty$–measurable then $f(\omega^\eta_\Lambda) = f(\eta)$; if f is continuous then $\lim_\Lambda f(\omega^\eta_\Lambda) = f(\omega)$. Hence, f is constant if it is $\mathcal{T}^\infty$–measurable and continuous.

We introduce an order on Ω by defining

$$\omega \leq \eta \ \text{ iff } \ \omega(k) \leq \eta(k) \ \ \forall k\,. \tag{2.5}$$

A function is *increasing* iff $f(\omega) \leq f(\eta)$ whenever $\omega \leq \eta$.

Definition 2.1 *A function f is* right–continuous *at ω if*

$$\lim_{\Lambda} f(\omega_{\Lambda}^{+}) = f(\omega) \ . \tag{2.6}$$

A function f is left–continuous *at ω if*

$$\lim_{\Lambda} f(\omega_{\Lambda}^{-}) = f(\omega) \ . \tag{2.7}$$

We introduce a (weak) topology on the set of probability measures on $(\Omega, \mathcal{F})$. A sequence of probability measures μ_n *converges to* μ iff for all continuous functions

$$\lim_{n} \mu_n(f) = \mu(f) \ . \tag{2.8}$$

In our case it is sufficient to verify (2.8) for the local functions and even only for the non-negative increasing local functions.

Most of our results are valid for a countable set $\mathcal{L}$. On the other hand most of the examples studied in the literature are defined on $\mathcal{L} = \mathbb{Z}^d$, $d \geq 1$. In such a case there is a natural action of $\mathbb{Z}^d$ on $\mathcal{L}$ which lifts to Ω, $\tau_a(\omega)(k) := \omega(k-a)$. A function f is $\mathbb{Z}^d$*–invariant* if for all $a \in \mathbb{Z}^d$

$$f \circ \tau_a = f \ . \tag{2.9}$$

A measure μ is $\mathbb{Z}^d$*–invariant* if for all $a \in \mathbb{Z}^d$ and all bounded functions f

$$\mu(f \circ \tau_a) = \mu(f) \ . \tag{2.10}$$

2.2 Local specification and quasilocality

We recall the definitions of the main concepts which we study in this paper. A good reference is Georgii (1988). Notice however that definition 2.4, which is central for us, is not the conventional one.

Definition 2.2 *A* local specification *Γ on $\mathcal{L}$ is a family of probability kernels $\Gamma = \{\gamma_\Lambda \, , \, \Lambda \subset \mathcal{L}\}$ on $(\Omega, \mathcal{F})$, such that*

(s_1) $\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$ for all $\omega \in \Omega$;

(s_2) $\gamma_\Lambda(F|\cdot)$ is $\mathcal{F}_{\Lambda^c}$–measurable for all $F \in \mathcal{F}$;

(s_3) $\gamma_\Lambda(F|\omega) = 1_F(\omega)$ if $F \in \mathcal{F}_{\Lambda^c}$;

(s_4) $\gamma_{\Lambda_2}\gamma_{\Lambda_1} = \gamma_{\Lambda_2}$ if $\Lambda_1 \subset \Lambda_2$.

Remark: We consider $(E, \mathcal{F})$ as the product space $(E^\Lambda \times E^{\Lambda^c}, \mathcal{F}_\Lambda \otimes \mathcal{F}_{\Lambda^c})$. Properties (s_1) and (s_2) imply that, for fixed ω, the probability measure $\gamma_\Lambda(\cdot|\omega)$ is the product measure on $(E^\Lambda \times E^{\Lambda^c}, \mathcal{F}_\Lambda \otimes \mathcal{F}_{\Lambda^c})$,

$$\mu_{\Lambda,\omega} \otimes \delta_{\omega_{\Lambda^c}} \ , \tag{2.11}$$

where $\mu_{\Lambda,\omega}$ is the restriction of $\gamma_\Lambda(\cdot|\omega)$ to $\mathcal{F}_\Lambda$ and $\delta_{\omega_{\Lambda^c}}$ is the Dirac mass at ω_{Λ^c}. Indeed, we claim that if $F = F_1 \times F_2 \in \mathcal{F}_\Lambda \otimes \mathcal{F}_{\Lambda^c}$ then

$$\gamma_\Lambda(F|\omega) = \gamma_\Lambda(F_1|\omega)\, 1_{F_2}(\omega) \ . \tag{2.12}$$

To prove (2.12) it is sufficient to consider $F_2 \ni \omega$; in that case we have for any $F_1 \in \mathcal{F}_\Lambda$

$$\gamma_\Lambda(F_1 \times F_2|\omega) \leq \gamma_\Lambda(F_1|\omega)1_{F_2}(\omega) \ , \tag{2.13}$$

and, therefore, identity (2.12) follows from

$$0 = (\gamma_\Lambda(F_1|\omega)1_{F_2}(\omega) - \gamma_\Lambda(F_1 \times F_2|\omega)) + (\gamma_\Lambda(F_1^c|\omega)1_{F_2}(\omega) - \gamma_\Lambda(F_1^c \times F_2|\omega)) \ . \tag{2.14}$$

Definition 2.3 *Let Γ be a local specification. A probability measure is* Γ–compatible *if for all $F \in \mathcal{F}$ and all $\Lambda \subset \mathcal{L}$*

$$\mathbb{E}_\mu(1_F|\mathcal{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(F|\omega) \ \ \mu - a.s. \ . \tag{2.15}$$

The set of all Γ–compatible probability measures is a convex set $\mathcal{G}(\Gamma)$ which may be empty; each $\mu \in \mathcal{G}(\Gamma)$ has a unique extremal decomposition; the extremal elements of $\mathcal{G}(\Gamma)$ are those probability measures $\mu \in \mathcal{G}(\Gamma)$ which satisfy a zero-one law on $\mathcal{T}^\infty$, $\mu(F) = 0$ or $\mu(F) = 1$ for all $F \in \mathcal{T}^\infty$ (see Theorems 7.26 and 7.7 in Georgii (1988)).

Remark: If μ is a given probability measure, then the condition of Γ being a local specification such that μ is compatible with it is a stronger requirement than being a system of proper regular conditional probabilities for all sub–σ–algebra $\mathcal{F}_\Lambda$, $\Lambda \subset \mathcal{L}$. Indeed, for conditional probabilities, the identity

$$\gamma_{\Lambda_2}\gamma_{\Lambda_1} = \gamma_{\Lambda_2} \tag{2.16}$$

with $\Lambda_1 \subset \Lambda_2$ is in general valid only μ–almost surely. Goldstein (1978) and Sokal (1981) give conditions which ensure the existence of a local specification for a given measure. In particular, in our setting, every system of conditional probabilities can be extended to a local specification (in a measure-dependent fashion), for instance by using Definition 7 in Goldstein (1978).

Definition 2.4 *Let $f : \Omega \to \mathbb{R}$.*

(i) f is quasilocal at ω *if for any $\varepsilon > 0$ there exists Λ_ε such that*

$$\sup_{\substack{\theta: \\ \theta_{\Lambda_\varepsilon} = \omega_{\Lambda_\varepsilon}}} |f(\omega) - f(\theta)| \leq \varepsilon \ . \tag{2.17}$$

(ii) f is quasilocal *if it is quasilocal at every $\omega \in \Omega$.*

Our point of view is that the *pointwise* notion of quasilocality is a useful concept. Quasilocality is equivalent to continuity if we choose the discrete topology on E as in our case. We shall therefore use both terminologies in the paper. Recently Definition 2.4 has been introduced independently by Grimmett (1995) in a similar context; see also Lörinczi (1994). The standard definition of quasilocality is the following one: a function is quasilocal iff for any $\varepsilon > 0$ there exists Λ_ε such that

$$\sup_{\substack{\theta\,,\,\omega:\\ \theta_{\Lambda_\varepsilon}=\omega_{\Lambda_\varepsilon}}} |f(\omega) - f(\theta)| \leq \varepsilon \,. \tag{2.18}$$

In our terminology (2.18) corresponds to uniform quasilocality on Ω. In general it is a stronger notion than *(ii)*. However, in the context of this paper (2.18) coincides with *(ii)* because Ω is compact.

Definition 2.5 *A specification* Γ *is* quasilocal at ω *if the functions* $\omega \mapsto \gamma_\Lambda(f|\omega)$ *are quasilocal at* ω *for each finite* Λ *and each bounded local function* f. *It is* quasilocal *if it is quasilocal at every* ω.

Gibbs specifications are the most studied type of local specifications. For our purposes we define them as follows. Let $A \subset \Lambda \subset \mathcal{L}$, with $|A| < \infty$, and let $\sigma_A \in E^A$; we set

$$1_{\sigma_A}(\omega) := \begin{cases} 1 & \text{if } \omega_A = \sigma_A \\ 0 & \text{otherwise} \end{cases} \,, \tag{2.19}$$

$$\gamma_\Lambda(\sigma_A|\omega) := \int \gamma_\Lambda(d\eta|\omega) 1_{\sigma_A}(\eta) \,, \tag{2.20}$$

$$\gamma_\Lambda(\sigma|\omega) := \int \gamma_\Lambda(d\eta|\omega) 1_{\sigma_\Lambda}(\eta) \,. \tag{2.21}$$

Definition 2.6 *A local specification is a* Gibbs specification *if it satisfies*

(G_1) Let A, $|A| < \infty$; there exists a constant $c_1(A) > 0$, so that for all $\Lambda \supset A$, σ and ω

$$\inf_\omega \gamma_\Lambda(\sigma_A|\omega) \geq c_1(A) \,. \tag{2.22}$$

(G_2) Let B, $|B| < \infty$; there exist $d_1(B) > 0$ and $d_2(B) < \infty$ such that

$$d_1(B)\gamma_\Lambda(\sigma|\omega) \leq \gamma_\Lambda(\sigma|\omega') \leq d_2(B)\gamma_\Lambda(\sigma|\omega) \tag{2.23}$$

for all σ, all $\Lambda \subset B$, and all ω, ω' such that $\omega_{B^c} = \omega'_{B^c}$.

(G_3) For any σ and any Λ the function $\omega \mapsto \gamma_\Lambda(\sigma|\omega)$ is quasilocal.

Usually Gibbs specifications are defined via the notion of an absolutely summable potential. In our context the two definitions are equivalent (Georgii (1988) Corollary 2.31). If Γ is a Gibbs specification, then the set of Γ–compatible measures $\mathcal{G}(\Gamma)$ is non–empty (Ω is compact and the local specification is quasilocal).

In the case $\mathcal{L} = \mathbb{Z}^d$ it is natural to consider $\mathbb{Z}^d$–invariant local specifications.

Definition 2.7 *A local specification* Γ *is* $\mathbb{Z}^d$–invariant *if for all $a \in \mathbb{Z}^d$, all Λ, ω, and bounded functions f,*

$$\gamma_\Lambda(f \circ \tau_a|\omega) = \gamma_{\Lambda+a}(f|\tau_a(\omega)) \,. \tag{2.24}$$

If a local specification is defined through an absolutely summable potential, then it is necessarily continuous. If the potential is $\mathbb{Z}^d$–invariant, then the local specification is also $\mathbb{Z}^d$–invariant. It is an open question, whether a continuous $\mathbb{Z}^d$–invariant local specification can always be defined by a $\mathbb{Z}^d$*–invariant absolutely summable* potential (see van Enter Fernández Sokal (1993) remark p.935). This is one reason why in this paper we avoid the use of potentials.

2.3 Monotonicity–preserving local specification

All local specifications which we consider here are monotonicity–preserving specifications. This property is our main technical tool.

Definition 2.8 *Let Γ be a local specification. Γ is* monotonicity–preserving *if for all bounded increasing functions f the function*

$$\omega \mapsto \gamma_\Lambda(f|\omega) := \int \gamma_\Lambda(d\eta|\omega) f(\eta) \tag{2.25}$$

is increasing.

In the literature monotonicity–preserving local specifications are sometimes called *attractive* because of the interpretation of some examples of such local specifications.

Proposition 2.1 *Let Γ be a local specification which is monotonicity–preserving. Then*

(i) For any increasing bounded function f the nets $\{\gamma_\Lambda(f|+);\ \{\Lambda\}, \subset\}$, resp. $\{\gamma_\Lambda(f|-);\ \{\Lambda\}, \subset)\}$, are monotone decreasing, resp. increasing.

(ii) The nets $\{\gamma_\Lambda(\cdot|\omega)\}$ with $\omega = +$ or $\omega = -$ converge to probability measures

$$\mu^+(\cdot) := \lim_\Lambda \gamma_\Lambda(\cdot|+) \quad , \quad \mu^-(\cdot) := \lim_\Lambda \gamma_\Lambda(\cdot|-)\,. \tag{2.26}$$

If furthermore $\mathcal{L} = \mathbb{Z}^d$ and the local specification is $\mathbb{Z}^d$–invariant, then the measures μ^+ and μ^- are $\mathbb{Z}^d$–invariant.

(iii) For any $\mu \in \mathcal{G}(\Gamma)$ and any bounded increasing function f

$$\mu^-(f) \le \mu(f) \le \mu^+(f)\,. \tag{2.27}$$

(iv) If furthermore Γ is quasilocal, then μ^+ and μ^- are Γ–compatible, moreover they are extremal elements of $\mathcal{G}(\Gamma)$. Furthermore $|\mathcal{G}(\Gamma)| = 1$ iff $\mu^+ = \mu^-$.

Proof : The proof is standard. Let f be an increasing function; if $\Lambda_1 \subset \Lambda_2$, then

$$\begin{aligned} \gamma_{\Lambda_2}(f|+) &= \int \gamma_{\Lambda_2}(d\eta|+)\gamma_{\Lambda_1}(f|\eta) \\ &\le \int \gamma_{\Lambda_2}(d\eta|+)\gamma_{\Lambda_1}(f|+) \\ &= \gamma_{\Lambda_1}(f|+)\,. \end{aligned} \tag{2.28}$$

The existence of the limits follows now easily. $\mathbb{Z}^d$–invariance follows from monotonicity. If $\mu \in \mathcal{G}(\Gamma)$, then by the backward martingale convergence theorem

$$\limsup_\Lambda \gamma_\Lambda(f|\omega) \tag{2.29}$$

is a version of $\mathbb{E}_\mu(f|\mathcal{T}^\infty)$. Since Γ is monotonicity–preserving

$$\mu^-(f) \le \limsup_\Lambda \gamma_\Lambda(f|\omega) \le \mu^+(f)\,; \tag{2.30}$$

therefore

$$\mu^-(f) \le \mu(f) \le \mu^+(f)\,. \tag{2.31}$$

If Γ is quasilocal, then μ^+ and μ^- are also Γ–compatible, hence $\mathcal{G}(\Gamma) \neq \emptyset$. Indeed, for continuous f and any Λ_1

$$\mu^+(f) = \lim_{\Lambda} \gamma_\Lambda(f|+) = \lim_{\Lambda} \int \gamma_\Lambda(d\omega|+)\gamma_{\Lambda_1}(f|\omega) = \mu^+(\gamma_{\Lambda_1}(f|\cdot))\,. \tag{2.32}$$

The extremality of μ^+ and μ^- follows from (2.31). This also shows that $|\mathcal{G}(\Gamma)| = 1$ iff $\mu^+ = \mu^-$. □

By the definition of μ^+ in (2.26) we have

$$\lim_{\Lambda} \gamma_\Lambda(f|+) = \mu^+(f) \tag{2.33}$$

for any continuous function f. When Γ is monotonicity–preserving a similar property also holds for monotone right–continuous functions. This allows us to give a variant of (iv) in Proposition 2.1.

Lemma 2.1 *Assume that Γ is monotonicity–preserving and let f be a monotone function. If f is right–continuous, i.e. $\lim_\Lambda f(\omega_\Lambda^+) = f(\omega)$ for all ω, then*

$$\mu^+(f) = \lim_{\Lambda} \gamma_\Lambda(f|+)\,. \tag{2.34}$$

Similarly if f is left–continuous, i.e. $\lim_\Lambda f(\omega_\Lambda^-) = f(\omega)$ for all ω, then

$$\mu^-(f) = \lim_{\Lambda} \gamma_\Lambda(f|-)\,. \tag{2.35}$$

Proof : It is sufficient to prove the lemma for monotone increasing functions. Let $M \subset \Lambda \subset N$, $|N| < \infty$; we set (for this proof)

$$f_\Lambda^+(\omega) := f(\omega_\Lambda^+)\,, \tag{2.36}$$

and suppose that f is increasing. Then $f_\Lambda^+ \le f_M^+$ and by Proposition 2.1

$$\gamma_N(f_\Lambda^+|+) \le \gamma_\Lambda(f_\Lambda^+|+) = \gamma_\Lambda(f|+) \le \gamma_\Lambda(f_M^+|+)\,. \tag{2.37}$$

Since f_Λ^+ is local we can take the limit over N; we get (see Proposition 2.1)

$$\mu^+(f_\Lambda^+) = \inf_N \gamma_N(f_\Lambda^+|+) \le \gamma_\Lambda(f|+) \le \gamma_\Lambda(f_M^+|+)\,. \tag{2.38}$$

By the monotone convergence theorem we have

$$\mu^+(f) \le \liminf_{\Lambda} \gamma_\Lambda(f|+) \le \limsup_{\Lambda} \gamma_\Lambda(f|+) \le \mu^+(f_M^+)\,; \tag{2.39}$$

finally by taking the limit over M,

$$\mu^+(f) \le \liminf_{\Lambda} \gamma_\Lambda(f|+) \le \limsup_{\Lambda} \gamma_\Lambda(f|+) \le \mu^+(f)\,. \tag{2.40}$$

A similar proof holds for the second part of the lemma. □

Definition 2.9 *A local specification Γ is* right–continuous *if $\omega \mapsto \gamma_\Lambda(f|\omega)$ is right–continuous for all Λ, all local bounded functions f and all ω.*

Corollary 2.1 *Let Γ be a monotonicity–preserving, right–continuous local specification. Then μ^+ defined by (2.26) is Γ–compatible.*

Proof : The same proof (see (2.32)) as above holds; indeed $\gamma_{\Lambda_1}(f|\omega)$ is right–continuous by hypothesis and the last equality in (2.32) follows now from Lemma 2.1. □

Remark: Recall that quasilocality of a local specification Γ is equivalent in our setting to the Feller property, that is, $f \mapsto \gamma_\Lambda(f|\cdot)$ maps the (bounded) continuous functions into the (bounded) continuous functions. If Γ is a monotonicity–preserving right–continuous local specification the natural space of functions which is mapped into itself by the mapping $f \mapsto \gamma_\Lambda(f|\cdot)$ is the space of monotone right–continuous functions. Indeed, let f be a monotone increasing right–continuous function. For any finite set N the function

$$\omega \mapsto f_N(\omega) := f(\omega_N^+) \tag{2.41}$$

is a local function, $f_N(\omega) \geq f(\omega)$ and $\lim_N f_N(\omega) = f(\omega)$. Let Λ_1 be fixed; we have

$$\lim_\Lambda \gamma_{\Lambda_1}(f|\omega_\Lambda^+) \geq \gamma_{\Lambda_1}(f|\omega) \; . \tag{2.42}$$

On the other hand for any fixed finite N we have by right–continuity

$$\lim_\Lambda \gamma_{\Lambda_1}(f|\omega_\Lambda^+) \leq \lim_\Lambda \gamma_{\Lambda_1}(f_N|\omega_\Lambda^+) = \gamma_{\Lambda_1}(f_N|\omega) \; . \tag{2.43}$$

Taking now the limit over N and using the monotone convergence theorem we get

$$\lim_\Lambda \gamma_{\Lambda_1}(f|\omega_\Lambda^+) \leq \gamma_{\Lambda_1}(f|\omega) \; . \tag{2.44}$$

This proves that $\gamma_{\Lambda_1}(f|\cdot)$ is (monotone and) right continuous.

3 Global specification

In the whole section we consider a countable infinite set $\mathcal{L}$.

Definition 3.1 *A* global specification Γ *on $\mathcal{L}$ is a family of probability kernels* $\Gamma = \{\gamma_S\,,\, S \subset \mathcal{L}\}$ *on $(\Omega, \mathcal{F})$, S any subset of $\mathcal{L}$, such that*

(s_1) $\gamma_S(\cdot|\omega)$ *is a probability measure on $(\Omega, \mathcal{F})$ for all $\omega \in \Omega$;*

(s_2) $\gamma_S(F|\cdot)$ *is $\mathcal{F}_{S^c}$–measurable for all $F \in \mathcal{F}$;*

(s_3) $\gamma_S(F|\omega) = 1_F(\omega)$ *if $F \in \mathcal{F}_{S^c}$;*

(s_4) $\gamma_{S_2}\gamma_{S_1} = \gamma_{S_2}$ *if $S_1 \subset S_2$.*

We remark that the difference between a global and a local specification (definition 2.2) is that for the former condition *(s_4)* is required for *any* subset S of $\mathcal{L}$, while for the latter it only holds for *finite* subsets of $\mathcal{L}$.

Definition 3.2 *Let Γ be a global specification. A probability measure μ is Γ–*compatible*, if for all $F \in \mathcal{F}$ and all $S \subset \mathcal{L}$*

$$\mathbb{E}_\mu(1_F|\mathcal{F}_{S^c})(\omega) = \gamma_S(F|\omega) \;\; \mu - a.s.\,. \tag{3.1}$$

In the definition of a global specification it is understood that $\mathcal{F}_{\mathcal{L}^c} = \mathcal{F}_\emptyset = \{\emptyset, \Omega\}$. There is at most one probability measure μ compatible with a global specification Γ, and if there is one, then for any ω

$$\mu(f) = \gamma_{\mathcal{L}}(f|\omega)\,. \tag{3.2}$$

The main question in this section is: *given a local specification Γ and a probability measure μ compatible with Γ, can we extend Γ into a global specification Γ^μ so that μ is compatible with Γ^μ?* We answer this question in two cases: (i) Γ is monotonicity–preserving, right–continuous, and $\mu = \mu^+$ of Proposition 2.1; (ii) Γ verifies the strong–uniqueness property.

3.1 Monotonicity–preserving and right–continuous case

Let $\Gamma = \{\gamma_\Lambda\}$ be a local specification which is monotonicity–preserving and right–continuous. We construct a global specification $\Gamma^+ = \{\gamma_S^+\}$ for μ^+ of Proposition 2.1. It will be evident that there is a similar construction of a global specification $\Gamma^- = \{\gamma_S^-\}$ for the measure μ^- defined by (2.26), if the local specification $\Gamma = \{\Lambda\}$ is monotonicity–preserving and left–continuous.

The idea of the construction is simple and not new, see Goldstein (1980), Föllmer (1980). For Λ finite we set

$$\gamma_\Lambda^+ := \gamma_\Lambda\,. \tag{3.3}$$

Let S be an infinite subset of $\mathcal{L}$; given $\omega \in \Omega$ we first define a local specification Γ_S^ω on S, that is a family of probabibility kernels on $(E^S, \mathcal{F}_S)$ indexed by all finite subsets of S.

$$\Gamma_S^\omega := \{\gamma_{\Lambda,\omega}^S\,,\, \Lambda \subset S\}\,, \tag{3.4}$$

where the probability kernel $\gamma^S_{\Lambda,\omega}$ is defined on $\mathcal{F}_S \times E^S$ by

$$\gamma^S_{\Lambda,\omega}(\,\cdot\,|\eta) := \gamma_\Lambda(\,\cdot\,|\eta^\omega_S)\ . \tag{3.5}$$

We have

(s_1) $\gamma^S_{\Lambda,\omega}(\cdot|\eta_S)$ is a probability measure on $(E^S, \mathcal{F}_S)$ for all $\eta_S \in E^S$;

(s_2) $\gamma^S_{\Lambda,\omega}(F|\cdot)$ is $\mathcal{F}_{S\setminus\Lambda}$–measurable for all $F \in \mathcal{F}_S$;

(s_3) $\gamma^S_{\Lambda,\omega}(F|\eta_S) = 1_F(\eta^\omega_S)$ if $F \in \mathcal{F}_{S\setminus\Lambda}$;

(s_4) $\gamma^S_{\Lambda_2,\omega}\gamma^S_{\Lambda_1,\omega} = \gamma^S_{\Lambda_2,\omega}$ if $\Lambda_1 \subset \Lambda_2 \subset S$.

This local specification is again monotonicity–preserving and right–continuous. The set of probability measures on $(E^S, \mathcal{F}_S)$ compatible with Γ^ω_S is denoted by $\mathcal{G}(\Gamma^\omega_S)$. The crucial observation is that Γ^+ is formed by measures in $\mathcal{G}(\Gamma^\omega_S)$. Indeed, suppose that Γ^+ exists; the compatibility condition (s_4) implies for all $\Lambda \subset S$ and any ω,

$$\int \gamma^+_S(d\eta|\omega)\gamma^+_\Lambda(f|\eta) = \gamma^+_S(f|\omega)\ . \tag{3.6}$$

The product property of the measure $\gamma^+_S(d\eta|\omega)$ on $\mathcal{F} = \mathcal{F}_S \otimes \mathcal{F}_{S^c}$ implies

$$\begin{aligned} \gamma^+_S(f|\omega) &= \int \gamma^+_S(d\eta|\omega)\gamma^+_\Lambda(f|\eta) \\ &= \int \gamma^+_S(d\eta|\omega)\gamma^+_\Lambda(f|\eta^\omega_S) \\ &= \int \gamma^+_S(d\eta|\omega)\gamma^S_{\Lambda,\omega}(f|\eta) \end{aligned} \tag{3.7}$$

since $\gamma^+_\Lambda = \gamma_\Lambda$ and (3.5) holds. Therefore $\gamma^+_S(d\eta|\omega) \in \mathcal{G}(\Gamma^\omega_S)$. In order to define a global specification we must choose for each ω and each S an element of $\mathcal{G}(\Gamma^\omega_S)$. It is not clear that we can make a choice compatible with property (s_2), whenever there are several elements in $\mathcal{G}(\Gamma^\omega_S)$. In our case there is a canonical choice since for any fixed $\omega \in \Omega$ the net $\{\gamma_\Lambda(g|\omega^+_{S^c})\,,\,\{\Lambda\}_{\Lambda\subset S}, \subset\}$ converges to a probability measure $\mu^+_{S,\omega}$ which is compatible with Γ^ω_S (Proposition 2.1 and Corollary 2.1). Since $\mathcal{F} = \mathcal{F}_S \otimes \mathcal{F}_{S^c}$, we define a probability measure on $(\Omega, \mathcal{F})$ for any $\omega \in \Omega$ by

$$\gamma^+_S(d\eta|\omega) := \mu^+_{S,\omega}(d\eta_S) \otimes \delta_{\omega_{S^c}}(d\eta_{S^c})\ . \tag{3.8}$$

Because

$$\gamma^+_S(d\eta|\omega) = \lim_{\Lambda\uparrow S}\gamma_\Lambda(d\eta|\omega^+_{S^c})\ , \tag{3.9}$$

the family of probability kernels $\Gamma^+ := \{\gamma^+_S\}_{S\subset\mathcal{L}}$ satisfies (s_1), (s_2) and (s_3). It is clearly monotonicity–preserving. We now prove that it is right–continuous (Lemma 3.1) and that it satisfies (s_4) (Lemma 3.2). Since $\mu^+ = \gamma^+_{\mathcal{L}}$, this last property also implies that μ^+ is compatible with Γ^+.

Lemma 3.1 *Let Γ be a local specification which is monotonicity–preserving and right–continuous. Let $S \subset \mathcal{L}$, $|S| = \infty$, and let g be a monotone right–continuous function. Then $\omega \mapsto \gamma^+_S(g|\omega)$ is right–continuous, that is*

$$\lim_\Lambda \gamma^+_S(g|\omega^+_\Lambda) = \gamma^+_S(g|\omega)\ . \tag{3.10}$$

Proof : By the remark at the end of section 2 it is sufficient to consider the case of a local non-negative monotone increasing function g. Therefore

$$\gamma_S^+(g|\omega_\Lambda^+) \geq \gamma_S^+(g|\omega) \;, \tag{3.11}$$

and

$$\lim_{\Lambda\uparrow S^c} \gamma_S^+(g|\omega_\Lambda^+) \geq \inf_{\Lambda\subset S^c} \gamma_S^+(g|\omega_\Lambda^+) \geq \gamma_S^+(g|\omega) \;. \tag{3.12}$$

Let $\Lambda_1 \subset S$. Since the local specification Γ is right–continuous, the function $\omega \mapsto \gamma_{\Lambda_1}(g|\omega)$ is right–continuous, as well as the function

$$\omega \mapsto \gamma_{\Lambda_1}(g|\omega_{S^c}^+) \tag{3.13}$$

which is the composition of $\omega \mapsto \gamma_{\Lambda_1}(g|\omega)$ and of the continuous map $\omega \mapsto \omega_{S^c}^+$. In particular

$$\gamma_{\Lambda_1}(g|\omega) = \lim_{\Lambda\uparrow S^c} \gamma_{\Lambda_1}(g|\omega_\Lambda^+) \;. \tag{3.14}$$

Since g is increasing, $\Lambda \subset S^c$ and $\Lambda_1 \subset S$,

$$\gamma_{\Lambda_1}(g|\omega_\Lambda^+) \geq \gamma_S^+(g|\omega_\Lambda^+) \;; \tag{3.15}$$

consequently

$$\begin{aligned} \gamma_{\Lambda_1}(g|\omega) &= \lim_{\Lambda\uparrow S^c} \gamma_{\Lambda_1}(g|\omega_\Lambda^+) \\ &\geq \lim_{\Lambda\uparrow S^c} \gamma_S^+(g|\omega_\Lambda^+) \\ &\geq \inf_{\Lambda\subset S^c} \gamma_S^+(g|\omega_\Lambda^+) \;. \end{aligned} \tag{3.16}$$

Taking now the limit $\Lambda_1 \uparrow S$, we get

$$\gamma_S^+(g|\omega) \geq \inf_{\Lambda\subset S^c} \gamma_S^+(g|\omega_\Lambda^+) \;. \tag{3.17}$$

□

Lemma 3.2 *Let Γ be a local specification which is monotonicity–preserving and right–continuous. Let $S_1 \subset S_2$ be two infinite subsets of $\mathcal{L}$. Then*

$$\gamma_{S_2}^+\gamma_{S_1}^+ = \gamma_{S_2}^+ \;. \tag{3.18}$$

Proof : It is sufficient to prove

$$\int \gamma_{S_2}^+(d\eta|\omega)\gamma_{S_1}^+(h|\eta) = \gamma_{S_2}^+(h|\omega) \tag{3.19}$$

for h a Λ_1–local function, $\Lambda_1 \subset S_2$. Using (3.8) formula (3.19) becomes

$$\int \mu_{S_2,\omega}^+(d\eta_{S_2})\gamma_{S_1}^+(h|\eta_{S_2}^\omega) = \int \mu_{S_2,\omega}^+(d\eta_{S_2})h(\eta_{S_2}) \;. \tag{3.20}$$

Since ω is fixed and we integrate over the space $(E^{S_2}, \mathcal{F}_{S_2})$, all configurations below are configurations of E^{S_2} and are extended by ω outside S_2 if necessary; in order to simplify the notation we omit the index S_2 and write η instead of η_{S_2} or $\eta_{S_2}^\omega$, $\mu^+(d\eta)$

instead of $\mu^+_{S_2,\omega}(d\eta_{S_2})$, $\gamma^+_{S_1}(\,\cdot\,|\eta)$ instead of $\gamma^+_{S_1}(\,\cdot\,|\eta^\omega_{S_2})$ and so on. Equation (3.19) is true if

$$\mathbb{E}_{\mu^+}(h|\mathcal{F}_{S_2\backslash S_1})(\eta) = \gamma^+_{S_1}(h|\eta) \quad \mu^+ - a.s. \,. \tag{3.21}$$

Therefore it is sufficient to prove that

$$\mathbb{E}_{\mu^+}(g\,f) = \mathbb{E}_{\mu^+}(\gamma^+_{S_1}(g|\cdot)\,f) \tag{3.22}$$

for g a non–negative increasing Λ_1–local function with $\Lambda_1 \subset S_1$, and f a non–negative increasing Λ_2–local function with $\Lambda_2 \subset S_2\backslash S_1$. By Lemma 3.1 $\eta \mapsto \gamma^+_{S_1}(g|\eta)$ is right–continuous; using Lemma 2.1 we have for fixed $\Lambda' \subset S_2$

$$\begin{aligned} \mathbb{E}_{\mu^+}(\gamma^+_{S_1}(g|\cdot)\,f) &= \lim_{\Lambda''\uparrow S_2} \gamma_{\Lambda''}(\gamma^+_{S_1}(g|\cdot)\,f|+) \\ &\leq \gamma_{\Lambda'}(\gamma^+_{S_1}(g|\cdot)\,f|+) \\ &\leq \int \gamma_{\Lambda'}(d\eta|+)\gamma_\Lambda(g|+^\eta_{S_1})\,f(\eta)\,, \end{aligned} \tag{3.23}$$

since for any $\Lambda \subset S_1$

$$\gamma^+_{S_1}(g|\eta) \leq \gamma_\Lambda(g|+^\eta_{S_1})\,. \tag{3.24}$$

We choose Λ' so that $\Lambda' \cap S_1 = \Lambda$ and Λ large enough so that g is Λ–local; then

$$\begin{aligned} \int \gamma_{\Lambda'}(d\eta|+)\gamma_\Lambda(g|+^\eta_{S_1})\,f(\eta) &= \int \gamma_{\Lambda'}(d\eta|+)\gamma_\Lambda(g|\eta)\,f(\eta) \\ &= \int \gamma_{\Lambda'}(d\eta|+)g(\eta)\,f(\eta)\,. \end{aligned} \tag{3.25}$$

Therefore

$$\mathbb{E}_{\mu^+}(\gamma^+_{S_1}(g|\cdot)\,f) \leq \mathbb{E}_{\mu^+}(f\,g)\,. \tag{3.26}$$

On the other hand, if $M \subset S_1 \cap \Lambda'$,

$$\begin{aligned} \mathbb{E}_{\mu^+}(f\,g) &= \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+)f(\eta)\,g(\eta) \\ &= \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+)f(\eta)\,\gamma_M(g|\eta) \\ &\leq \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+)f(\eta)\,\gamma_M(g|+^\eta_{S_1}) \\ &= \int \mu^+(d\eta)f(\eta)\,\gamma_M(g|+^\eta_{S_1})\,. \end{aligned} \tag{3.27}$$

By the monotone convergence theorem

$$\mathbb{E}_{\mu^+}(f\,g) \leq \lim_{M\uparrow S_1} \int \mu^+(d\eta)f(\eta)\,\gamma_M(g|+^\eta_{S_1}) = \mathbb{E}_{\mu^+}(f\,\gamma^+_{S_1}(g|\,\cdot))\,. \tag{3.28}$$

□

We summarize the results of this section in the

Proposition 3.1 *Let Γ be a local right–continuous and monotonicity–preserving specification on $\mathcal{L}$. Then Γ can be extended into a global specification Γ^+ on $\mathcal{L}$ so that μ^+ is compatible with Γ^+. The global specification is right–continuous and monotonicity–preserving.*

The global specification Γ^+ inherits the right–continuity of the local specification Γ. The main question of the paper can be so formulated: is the global specification Γ^+ quasilocal, if it is the case for the local specification. In general *this is not true* as the results of section 5 show.

3.2 Strong–uniqueness case

In connection with the last remark of the preceding section the results of Proposition 3.2 below show that there is a case for which a global specification inherits the quasilocality of the local specification. These results are due essentially to Föllmer (1980); see also Theorem 8.23 in Georgii (1988).

Definition 3.3 *Let Γ be a local specification on $\mathcal{L}$. Γ has the* strong uniqueness property *if for any ω and any infinite subset $S \subset \mathcal{L}$, $|\mathcal{G}(\Gamma_S^\omega)| = 1$ (see (3.4) and (3.5)).*

Dobrushin's uniqueness condition implies the strong uniqueness property (see Georgii (1988) Chapter 8, in particular Theorem 8.23).

Proposition 3.2 *Let Γ be a continuous local specification on $\mathcal{L}$, which has the strong uniqueness property. Then Γ can be extended into a continuous global specification.*

Proof : We extend Γ into a global specification, still denoted by Γ. Let $S \subset \mathcal{L}$, $|S| = \infty$; we set

$$\Gamma_S(d\eta|\omega) := \mu_{S,\omega}(d\eta_S) \otimes \delta_{\omega_{S^c}}(d\eta_{S^c}) \ , \tag{3.29}$$

where $\mu_{S,\omega}(d\eta_S)$ is the unique measure in $\mathcal{G}(\Gamma_S^\omega)$. Clearly properties (s_1) and (s_3) are verified. We prove that $\omega \mapsto \Gamma_S(f|\omega)$ is continuous for any continuous function f. Let ω_n be a sequence in Ω converging to ω. The sequence of measures $\{\Gamma_S(\,\cdot\,|\omega_n)\}_n$ has an accumulation point ν. We now make the following claim:
Assume that there is a sequence of configurations $\omega_n \to \omega$ and a sequence of probability measures $\Gamma_S(\,\cdot\,|\omega_n) \in \mathcal{G}(\Gamma_S^{\omega_n})$ converging to some probability measure ν. Then, if Γ is a continuous local specification, $\nu \in \mathcal{G}(\Gamma_S^\omega)$.
Indeed, for f continuous $\gamma_\Lambda(f|\,\cdot\,)$ is continuous by hypothesis, and so for any η

$$\lim_n \gamma_\Lambda(f|\eta_S^{\omega_n}) = \gamma_\Lambda(f|\eta_S^\omega) \ . \tag{3.30}$$

By compactness the convergence in (3.30) is uniform, and therefore if $\Lambda \subset S$ we have

$$\begin{aligned} \int \nu(d\eta) f(\eta) &= \lim_n \int \Gamma_S(d\eta|\omega_n) f(\eta) && (3.31)\\ &= \lim_n \int \Gamma_S(d\eta|\omega_n) \gamma_\Lambda(f|\eta) \\ &= \lim_n \int \Gamma_S(d\eta|\omega_n) \gamma_\Lambda(f|\eta_S^\omega) && (3.32)\\ &= \int \nu(d\eta) \gamma_\Lambda(f|\eta_S^\omega) \ . \end{aligned}$$

This proves the claim.

By uniqueness $\nu \in \mathcal{G}(\Gamma_S^\omega)$ has a single element, namely $\Gamma_S(d\eta|\omega)$. Hence

$$\nu(d\eta) = \Gamma_S(d\eta|\omega) \ . \tag{3.33}$$

This proves that $\{\Gamma_S(\,\cdot\,|\omega_n)\}_n$ converges to $\Gamma_S(d\eta|\omega)$ and that $\omega \mapsto \Gamma_S(f|\omega)$ is continuous for continuous f, in particular for any local function. Hence $\Gamma_S(F|\,\cdot\,)$,

$F \in \mathcal{F}$, is $\mathcal{F}$–measurable. By uniqueness $\Gamma_S(F|\omega) = \Gamma_S(F|\omega')$ for all $\omega' = \omega$ with $\omega'_{S^c} = \omega_{S^c}$. Consequently $\Gamma_S(F|\cdot)$ is $\mathcal{F}_{S^c}$–measurable. This proves property (s_2). The compatibility property (s_4) is proved as in Föllmer (1980). □

Remark: Proposition 3.2 admits variants, see e.g. Theorem 8.23 in Georgii (1988). The approach of Georgii is different from the approach of Föllmer (1980), which we follow here.

4 Local specifications for projections of μ^+

Let $\mathcal{L}$ be an infinite countable set, T any infinite subset of $\mathcal{L}$, such that $|T^c| = \infty$ to avoid trivial cases. In the whole section we assume that Γ is a local Gibbs specification on $\mathcal{L}$, which is monotonicity–preserving. We use the global specification Γ^+ [see (3.8) and (3.9)] constructed in the previous section to define a local specification $Q_T^+ = \{q_\Lambda^+ \,, \Lambda \subset T\}$ on T: the probability kernel q_Λ^+ is defined for any $\mathcal{F}_T$–measurable function f by

$$\int q_\Lambda^+(d\eta|\omega) f(\eta) := \int \gamma_{T^c\cup\Lambda}^+(d\theta|\omega) f(\theta)\,. \tag{4.1}$$

The local specification Q_T^+ is monotonicity–preserving and satisfies properties (G_1) and (G_2). In particular the first part of Proposition 2.1 holds for Q_T^+. As in (2.21) we set

$$q_\Lambda^+(\sigma|\omega) := \int q_\Lambda^+(d\eta|\omega) 1_{\sigma_\Lambda}(\eta)\,. \tag{4.2}$$

Our first result is that the projection of μ on $(E^T, \mathcal{F}_T)$, denoted by μ_T^+, is Q_T^+–compatible and is an extremal element of $\mathcal{G}(Q_T^+)$ (Lemma 4.1). Quasilocality properties of Q_T^+ are then studied via properties of the set of Gibbs measures $\mathcal{G}(\Gamma_{T^c}^\omega)$ compatible with the local specification $\Gamma_{T^c}^\omega = \{\gamma_{\Lambda,\omega}^{T^c} \,, \Lambda \subset T^c\}$ [see (3.4) and (3.5)]. We prove in Lemma 4.3 that $|\mathcal{G}(\Gamma_{T^c}^\omega)| = 1$ implies $|\mathcal{G}(\Gamma_{T^c}^{\omega'})| = 1$ for $\omega' \sim \omega$, and in Lemma 4.4 that $|\mathcal{G}(\Gamma_{T^c}^\omega)| = 1$ implies $|\mathcal{G}(\Gamma_{T^c\cup\Lambda}^\omega)| = 1$ for $\Lambda \subset T$. ¿From these results it follows that $|\mathcal{G}(\Gamma_{T^c}^\omega)| = 1$ implies continuity of Q_T^+ at ω and that the set Ω_q^+ of continuity points of Q_T^+ is in the tail-field

$$\mathcal{T}_T^\infty = \bigcap_{\Lambda\subset T} \mathcal{F}_{T\setminus\Lambda}. \tag{4.3}$$

If furthermore $|\mathcal{G}(\Gamma_{t^c}^-)| = 1$, then $\Omega_q^+ = \Omega_q$, where Ω_q is the set of points ω where $|\mathcal{G}(\Gamma_{T^c}^\omega)| = 1$ (no phase transition). If $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$, then any discontinuity point of Q_T^+ *cannot be removed* by modifying the specification on a set of μ^+–measure zero. (Propositions 4.1 and 4.2). The results about the set of continuity points of the local specification Q_T^+ (or Q_T^-) are summarized in Theorem 4.1.

Lemma 4.1 *The projection of μ^+ onto $\mathcal{F}_T$ defines a probability measure μ_T^+ on $(E^T, \mathcal{F}_T)$ which is the limit of the net $(q_\Lambda^+(\cdot|+)\,, \Lambda \subset T)$. The measure μ_T^+ is Q_T^+–compatible. It is an extremal element of $\mathcal{G}(Q_T^+)$.*

Proof : Let f be a bounded increasing local function in T; since f is local, by definition of q^+_Λ we have

$$q^+_\Lambda(f|+) = \lim_{\Lambda_1 \uparrow T^c \cup \Lambda} \gamma_{\Lambda_1}(f|+) \,. \tag{4.4}$$

If $\Lambda' \cap T = \Lambda$, then

$$\gamma_{\Lambda'}(f|+) \geq \lim_{\Lambda_1 \uparrow T^c \cup \Lambda} \gamma_{\Lambda_1}(f|+) \,, \tag{4.5}$$

and hence

$$\mu^+(f) = \lim_{\Lambda'} \gamma_{\Lambda'}(f|+) \geq \lim_{\Lambda \uparrow T} q^+_\Lambda(f|+) \,. \tag{4.6}$$

On the other hand, for any Λ_1

$$\mu^+(f) \leq \gamma_{\Lambda_1}(f|+) \,. \tag{4.7}$$

Hence

$$\mu^+(f) \leq q^+_\Lambda(f|+) \tag{4.8}$$

for each Λ. From (4.6)–(4.8) μ^+ is the limit of the net $(q^+_\Lambda(\cdot|+) \,,\, \Lambda \subset T)$. The measure μ^+_T is Q^+_T–compatible by Proposition 3.1. It is extremal by Proposition 2.1. □

The results concerning quasilocality of Q^+_T are based on the following

Lemma 4.2 *Let $S \subset \mathcal{L}$ and $\Lambda \subset S^c$ and Γ be a local Gibbs specification on $\mathcal{L}$, which is monotonicity–preserving. For any ω, any ω', $\omega' \sim \omega$, and any positive $\mathcal{F}_S$–measurable function f*

$$d_1(\Lambda \cup \Lambda')\gamma^+_S(f|\omega') \leq \gamma^+_{S\cup\Lambda}(f|\omega) \leq d_2(\Lambda \cup \Lambda')\gamma^+_S(f|\omega') \tag{4.9}$$

and

$$d_1(\Lambda \cup \Lambda')\gamma^-_S(f|\omega') \leq \gamma^-_{S\cup\Lambda}(f|\omega) \leq d_2(\Lambda \cup \Lambda')\gamma^-_{T^c}(f|\omega') \,; \tag{4.10}$$

Λ' is the subset of S^c where ω and ω' are different; the constants d_1, d_2 are those appearing in condition (G_2).

Proof : We compute

$$\begin{aligned}
\gamma^+_{S\cup\Lambda}(f|\omega) &= \int \gamma^+_{S\cup\Lambda}(d\eta|\omega) f(\eta) \\
&= \int \gamma^+_{S\cup\Lambda}(d\eta|\omega) \int \gamma^+_S(d\theta|\eta) f(\theta) \\
&= \int \gamma^+_{S\cup\Lambda}(d\eta|\omega) \int \gamma^+_S(d\theta|\eta_\Lambda \omega_{\Lambda^c}) f(\theta_S \eta_\Lambda \omega_{S^c\setminus\Lambda}) \\
&= \sum_{\eta_\Lambda} \gamma^+_{S\cup\Lambda}(1_{\eta_\Lambda}|\omega) \int \gamma^+_S(d\theta|\eta_\Lambda \omega_{\Lambda^c}) f(\theta_S \eta_\Lambda \omega_{S^c\setminus\Lambda}) \\
&= \sum_{\eta_\Lambda} \gamma^+_{S\cup\Lambda}(1_{\eta_\Lambda}|\omega) \gamma^+_S(f|\eta_\Lambda \omega_{\Lambda^c}) \,.
\end{aligned} \tag{4.11}$$

Since f is $\mathcal{F}_S$–measurable it does not depend on $\eta_\Lambda \omega_{S^c\setminus\Lambda}$; hence using property (G_2) of the local specification Γ we have

$$d_1(\Lambda \cup \Lambda')\gamma^+_S(f|\omega') \leq \gamma^+_S(f|\eta_\Lambda \omega_{\Lambda^c}) \leq d_2(\Lambda \cup \Lambda')\gamma^+_S(f|\omega') \,. \tag{4.12}$$

Therefore (4.11) and (4.12) imply

$$d_1(\Lambda \cup \Lambda')\gamma^+_S(f|\omega') \leq \gamma^+_{S\cup\Lambda}(f|\omega) \leq d_2(\Lambda \cup \Lambda')\gamma^+_S(f|\omega') \,. \tag{4.13}$$

□

Lemma 4.3 *If $|\mathcal{G}(\Gamma^{\omega}_{T^c})| = 1$, i.e. if $\gamma^+_{T^c}(\cdot|\omega) = \gamma^-_{T^c}(\cdot|\omega)$, then the same is true for any ω', $\omega' \sim \omega$. The set $\{\omega : \ |\mathcal{G}(\Gamma^{\omega}_{T^c})| = 1\}$ belongs to the tail-field*

$$\mathcal{T}^{\infty}_{T} = \bigcap_{\Lambda \subset T} \mathcal{F}_{T\setminus\Lambda} \ . \tag{4.14}$$

Proof : Let ω', $\omega' \sim \omega$; we prove that $|\mathcal{G}(\Gamma^{\omega'}_{T^c})| = 1$. By Lemma 4.2 there exist constants b_1 and b_2 such that for all positive $\mathcal{F}_{T^c}$–measurable functions f

$$b_1\gamma^+_{T^c}(f|\omega') \le \gamma^+_{T^c}(f|\omega) \le b_2\gamma^+_{T^c}(f|\omega') \tag{4.15}$$

and

$$b_1\gamma^-_{T^c}(f|\omega') \le \gamma^-_{T^c}(f|\omega) \le b_2\gamma^-_{T^c}(f|\omega') \ . \tag{4.16}$$

Therefore, if $\gamma^+_{T^c}(\cdot|\omega) = \gamma^-_{T^c}(\cdot|\omega)$, then the two measures $\gamma^-_{T^c}(\cdot|\omega')$ and $\gamma^+_{T^c}(\cdot|\omega')$ are equivalent. Since they are extremal elements of $\mathcal{G}(\Gamma^{\omega'}_{T^c})$ they satisfy a zero-one law on the tail-field σ–algebra

$$\mathcal{T}^{\infty}_{T^c} = \bigcap_{\Lambda \subset T^c} \mathcal{F}_{T^c\setminus\Lambda} \ , \tag{4.17}$$

and consequently they coincide on this σ–algebra. Therefore they are equal, $|\mathcal{G}(\Gamma^{\omega'}_{T^c})| = 1$ (Proposition 2.1) and the characteristic function of the set $\{\omega : \ |\mathcal{G}(\Gamma^{\omega}_{T^c})| = 1\}$ is $\mathcal{F}_{T\setminus\Lambda}$–measurable for any $\Lambda \subset T$. □

Lemma 4.4 *Let $\Lambda \subset T$. Then there is an affine bijection between $\mathcal{G}(\Gamma^{\omega}_{T^c\cup\Lambda})$ and $\mathcal{G}(\Gamma^{\omega}_{T^c})$. In particular $|\mathcal{G}(\Gamma^{\omega}_{T^c})| = 1$ if and only if $|\mathcal{G}(\Gamma^{\omega'}_{T^c\cup\Lambda})| = 1$ when $\omega' \sim \omega$.*

Proof : This follows from Lemma 4.2. For details see Theorem 7.33 in Georgii (1988). □

We come to the study of the quasilocality of the local specification Q^+_T. Quasilocality or continuity of $q^+_{\Lambda}(\sigma|\cdot)$ at ω means that

$$\lim_{\Lambda'\uparrow T} q^+_{\Lambda}(\sigma|\omega^+_{\Lambda'}) = \lim_{\Lambda'\uparrow T} q^+_{\Lambda}(\sigma|\omega^-_{\Lambda'}) \ . \tag{4.18}$$

¿From this expression and Lemma 4.4 is not hard to conclude that $|\mathcal{G}(\Gamma^{\omega}_{T^c})| = 1$ implies the quasilocality of Q^+_T at ω (the proof is spelled out in Proposition 4.1 below).

Lemma 4.5 *Let $\Lambda_1 \subset \Lambda_2 \subset T$. If the function $q^+_{\Lambda_2}(\sigma|\cdot)$ is continuous at ω for any σ, then for any σ the function $q^+_{\Lambda_1}(\sigma|\cdot)$ is continuous at any ω' such that $\omega'_{\Lambda_2^c} = \omega_{\Lambda_2^c}$.*

Proof : Let f be an increasing Λ_1–local function. We have

$$q^+_{\Lambda_2}(f|\omega^+_{\Lambda}) - q^+_{\Lambda_2}(f|\omega^-_{\Lambda}) = \tag{4.19}$$

$$\int q^+_{\Lambda_2}(d\eta|\omega)[q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^+_{\Lambda})_{\Lambda\setminus\Lambda_2}) - q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^-_{\Lambda})_{\Lambda\setminus\Lambda_2})] \ . \tag{4.20}$$

By monotonicity the function between square brackets in (4.19) is non–negative, and since (G_1) holds $q^+_{\Lambda_2}(d\eta|\omega)$ is a strictly positive measure. The continuity of $q^+_{\Lambda_2}(f|\cdot)$ implies

$$\begin{aligned} 0 &= \lim_{\Lambda\uparrow T}\left(q^+_{\Lambda_2}(f|\omega^+_{\Lambda}) - q^+_{\Lambda_2}(f|\omega^-_{\Lambda})\right) & (4.21)\\ &= \int q^+_{\Lambda_2}(d\eta|\omega)\lim_{\Lambda\uparrow T}[q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^+_{\Lambda})_{\Lambda\setminus\Lambda_2}) - q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^-_{\Lambda})_{\Lambda\setminus\Lambda_2})] \ ; \end{aligned}$$

hence for any η_{Λ_2} and any increasing Λ_1–local function f

$$\lim_{\Lambda\uparrow T} q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^+_\Lambda)_{\Lambda\setminus\Lambda_2}) = \lim_{\Lambda\uparrow T} q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^-_\Lambda)_{\Lambda\setminus\Lambda_2})\,. \tag{4.22}$$

This proves the continuity of $q^+_{\Lambda_1}(\sigma|\cdot)$ for any σ and any ω' such that $\omega'_{\Lambda_2^c} = \omega_{\Lambda_2^c}$. $\square$

Proposition 4.1 *Let Γ be a local specification on $\mathcal{L}$ which is Gibbs and monotonicity–preserving.*

1. *Then the set of the continuity points of the local specification Q^+_T,*

$$\Omega^+_q := \{\omega :\ q^+_\Lambda(\sigma|\cdot) \text{ is continuous at } \omega \text{ for all } \sigma, \Lambda\subset T\}\,, \tag{4.23}$$

is in the tail-field

$$\mathcal{T}^\infty_T = \bigcap_{\Lambda\subset T} \mathcal{F}_{T\setminus\Lambda}\,. \tag{4.24}$$

2. *Let Ω_q be the set*

$$\Omega_q := \{\omega :\ |\mathcal{G}(\Gamma^\omega_{T^c})| = 1\} \tag{4.25}$$

and Q^-_T be the local specification for the projection of the measure μ^-. Then

$$\Omega_q = \{\omega :\ q^+_\Lambda(\sigma|\omega) = q^-_\Lambda(\sigma|\omega)\ ,\ \forall\sigma\ ,\ \forall\Lambda\subset T\} \tag{4.26}$$

and

$$\Omega_q \subset \Omega^+_q\,. \tag{4.27}$$

3. *If $|\mathcal{G}(\Gamma^-_{T^c})| = 1$, then*

$$\Omega_q = \Omega^+_q\,. \tag{4.28}$$

Proof : 1. Let $\omega\in\Omega^+_q$. Lemma 4.5 implies that $\omega'\in\Omega^+_q$ if ω' differs from ω only on a finite subset $\Lambda'\subset T$. Hence Ω^+_q is $\mathcal{T}^\infty_T$–measurable.

2. Let f be an increasing local function and $\Lambda\subset T$. Since f is local it is continuous, and we can apply Lemma 3.1, getting

$$q^+_\Lambda(f|\omega) = \lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^+_{\Lambda'}) \ge \lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^-_{\Lambda'}) \ge \lim_{\Lambda'\uparrow T} q^-_\Lambda(f|\omega^-_{\Lambda'}) = q^-_\Lambda(f|\omega)\,. \tag{4.29}$$

By definition of Q^+_T and Q^-_T we have

$$q^+_\Lambda(\cdot|\omega) = \gamma^+_{\Lambda\cup T^c}(\cdot|\omega)\ \ ,\ \ q^-_\Lambda(\cdot|\omega) = \gamma^-_{\Lambda\cup T^c}(\cdot|\omega)\,. \tag{4.30}$$

If $\omega\in\Omega_q$ then Lemma 4.4 implies that

$$q^+_\Lambda(\sigma|\omega) = q^-_\Lambda(\sigma|\omega)\ ,\ \forall\sigma\ ,\ \forall\Lambda\subset T\,. \tag{4.31}$$

Conversely if (4.31) holds, then (4.29) implies that $\omega\in\Omega_q$ and claim (4.26) is proven. Let $\omega\in\Omega_q$; from (4.30) and Lemma 4.4 we have

$$q^+_\Lambda(\cdot|\omega) = \gamma^+_{\Lambda\cup T^c}(\cdot|\omega) = \gamma^-_{\Lambda\cup T^c}(\cdot|\omega) = q^-_\Lambda(\cdot|\omega)\,. \tag{4.32}$$

Combined with (4.29) we get

$$\lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^+_{\Lambda'}) = \lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^-_{\Lambda'})\,, \tag{4.33}$$

that is, $\omega \in \Omega_q^+$.

3. If $|\mathcal{G}(\Gamma^-_{T^c})| = 1$, then for any $\Lambda \subset T$ and any $\omega \sim -$,

$$\gamma^+_{T^c \cup \Lambda}(\cdot|\omega) = \gamma^-_{T^c \cup \Lambda}(\cdot|\omega)\,; \tag{4.34}$$

Therefore

$$q^+_\Lambda(\cdot|\omega^-_{\Lambda'}) = q^-_\Lambda(\cdot|\omega^-_{\Lambda'})\,, \tag{4.35}$$

and (4.29) can be replaced by

$$q^+_\Lambda(f|\omega) = \lim_{\Lambda' \uparrow T} q^+_\Lambda(f|\omega^+_{\Lambda'}) \geq \lim_{\Lambda' \uparrow T} q^+_\Lambda(f|\omega^-_{\Lambda'}) = \lim_{\Lambda' \uparrow T} q^-_\Lambda(f|\omega^-_{\Lambda'}) = q^-_\Lambda(f|\omega)\,. \tag{4.36}$$

Let $\omega \in \Omega_q^+$; (4.30), (4.36) and Lemma 4.4 imply that $\omega \in \Omega_q$ follows. □

Remark: If for all $j \in \mathcal{L}$ and all σ the functions $q_j^+(\sigma|\cdot)$ are continuous, then the same is true for the functions $q^+_\Lambda(\sigma|\cdot)$ (see e.g. (5.4) below). The local specification Q_T^+ is therefore quasilocal.

In the next proposition we give a sufficient condition so that the discontinuities of Q_T^+ cannot be removed by changing the local specification on a set of μ^+–measure zero.

Proposition 4.2 *Let Γ be a local specification on $\mathcal{L}$ which is monotonicity–preserving and quasilocal. Let $\varepsilon > 0$ and $\omega \in \Omega$, such that*

$$\lim_{\Lambda'} |q_j^+(\sigma|\omega^+_{\Lambda'}) - q_j^+(\sigma|\omega^-_{\Lambda'})| \geq \varepsilon\,. \tag{4.37}$$

If $\eta \mapsto q_j^+(\sigma|\eta)$ is continuous at any $\eta \sim +$,

$$\lim_{\Lambda \uparrow T} q_j^+(\sigma|\eta^-_\Lambda) = q_j^+(\sigma|\eta)\,, \tag{4.38}$$

then for any neighbourhood of ω, $V_\Lambda = \{\omega' :\ \omega'_\Lambda = \omega_\Lambda\}$, we can find two neighbourhoods, $V^+_{\Lambda,M}$ and $V^-_{\Lambda,M}$, $\Lambda \subset M$, $|M| < \infty$,

$$V^+_{\Lambda,M} = \{\omega' :\ \omega'_\Lambda = \omega_\Lambda\,,\ \omega'_{M\setminus\Lambda} = +\}\,, \tag{4.39}$$

$$V^-_{\Lambda,M} = \{\omega' :\ \omega'_\Lambda = \omega_\Lambda\,,\ \omega'_{M\setminus\Lambda} = -\}\,, \tag{4.40}$$

which have the following property: for any $\alpha \in V^+_{\Lambda,M}$ and $\theta \in V^-_{\Lambda,M}$,

$$\lim_{\Lambda'} |q_j^+(\sigma|\alpha^+_{\Lambda'}) - q_j^+(\sigma|\theta^-_{\Lambda'})| \geq \frac{\varepsilon}{2}\,. \tag{4.41}$$

Proof : By hypothesis $q_j^+(\sigma|\cdot)$ is continuous at every $\omega \sim +$,

$$\lim_{M \uparrow T} q_j^+(\sigma|(\omega^+_\Lambda)^-_M) = q_j^+(\sigma|\omega^+_\Lambda)\,. \tag{4.42}$$

Lemma 3.1 implies

$$\lim_{M \uparrow T} q_j^+(\sigma|(\omega^-_\Lambda)^+_M) = q_j^+(\sigma|\omega^-_\Lambda)\,. \tag{4.43}$$

We choose $M \supset \Lambda$ so that

$$|q_j^+(\sigma|(\omega_\Lambda^+)_M^-) - q_j^+(\sigma|\omega_\Lambda^+)| \le \frac{\varepsilon}{4} \tag{4.44}$$

and

$$|q_j^+(\sigma|(\omega_\Lambda^-)_M^+) - q_j^+(\sigma|\omega_\Lambda^-)| \le \frac{\varepsilon}{4}\,. \tag{4.45}$$

By monotonicity, if $\alpha \in V_{\Lambda,M}^+$ and $\theta \in V_{\Lambda,M}^-$, then

$$\begin{aligned}(q_j^+(1|\alpha) - q_j^+(1|\theta)) &\ge (q_j^+(1|\alpha_M^-) - q_j^+(1|\theta_M^+)) \\ &\ge (q_j^+(1|\omega_\Lambda^+) - q_j^+(1|\omega_\Lambda^-)) - \frac{\varepsilon}{2} \\ &\ge \frac{\varepsilon}{2}\,.\end{aligned} \tag{4.46}$$

Similar inequalities hold if $\sigma(j) = -1$. □

Remark: If $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$, then $|\mathcal{G}(\Gamma_{T^c}^\omega)| = 1$ for any $\omega \sim +$ (Lemma 4.3); therefore $\omega \in \Omega_q^+$ and Proposition 4.2 applies.

Theorem 4.1 *Let Γ be a Gibbs specification which is monotonicity–preserving. Let $T \subset \mathcal{L}$ so that $|T| = \infty$ and $|T^c| = \infty$. Let Ω_q^+ be the set of continuity points of the local specification $Q_T^+ = \{q_\Lambda^+\}_{\Lambda\subset T}$ and Ω_q^- the set of continuity points of $Q_T^- = \{q_\Lambda^-\}_{\Lambda\subset T}$.*

1. *If $|\mathcal{G}(\Gamma)| = 1$, i.e. $\mu^+ = \mu^-$, then Ω_q^+ has μ^+–measure one.*

Assume that $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$ and $|\mathcal{G}(\Gamma_{T^c}^-)| = 1$. Then there exists a dense subset $\Omega_q \subset \{-1,1\}^T$ with the following properties.

2. *Ω_q is $\mathcal{T}_T^\infty$–measurable, $\Omega_q = \{\omega : |\mathcal{G}(\Gamma_{T^c}^\omega)| = 1\}$, and $\Omega_q^+ = \Omega_q^- = \Omega_q$.*
3. *$\mu^+(\Omega_q) = 1$ if and only if $\mu^+ \in \mathcal{G}(Q_T^-)$; $\mu^-(\Omega_q) = 1$ if and only if $\mu^- \in \mathcal{G}(Q_T^+)$. Furthermore*

$$\Omega_q = \{\omega \in \Omega : q_\Lambda^+(\cdot|\omega) = q_\Lambda^-(\cdot|\omega) \;\; \forall \text{ finite } \Lambda \subset T\}\,. \tag{4.47}$$

Proof : 1. Let $\Lambda \subset T$; let f be an increasing local function in $T^c \cup \Lambda$. We have

$$\mu^+(f) - \mu^-(f) \;=\; \int \mu^+(d\omega)\gamma_{T^c\cup\Lambda}^+(f|\omega) - \int \mu^-(d\omega)\gamma_{T^c\cup\Lambda}^-(f|\omega)\,, \tag{4.48}$$

which, if $\mu^+ = \mu^-$, yields

$$0 \;=\; \int \mu^+(d\omega)\Big[\gamma_{T^c\cup\Lambda}^+(f|\omega) - \gamma_{T^c\cup\Lambda}^-(f|\omega)\Big]\,. \tag{4.49}$$

Since the square bracket is non-negative (monotonicity) it must be zero μ^+-a.e.,

$$\gamma_{T^c\cup\Lambda}^+(f|\omega) = \gamma_{T^c\cup\Lambda}^-(f|\omega) \;\; \mu^+ - a.e. \tag{4.50}$$

We conclude using 2. of Proposition 4.1.

2. follows from Proposition 4.1.

3. Let $\mu^+ \in \mathcal{G}(Q_T^-)$; then for any increasing local function f

$$0 = \mu^+(f) - \mu^+(f) \;=\; \int \mu^+(d\omega)[q_\Lambda^+(f|\omega) - q_\Lambda^-(f|\omega)] \,. \tag{4.51}$$

Since $[q_\Lambda^+(f|\omega) - q_\Lambda^-(f|\omega)]$ is non–negative, we have

$$q_\Lambda^+(f|\omega) - q_\Lambda^-(f|\omega) = 0 \;\; \mu^+ - \text{a.s.}. \tag{4.52}$$

Conversely, if $\mu^+(\Omega_q) = 1$, then for any increasing local function f

$$\begin{aligned} \int \mu^+(d\omega) q_\Lambda^+(f|\omega) &= \int_{\Omega_q} \mu^+(d\omega) q_\Lambda^+(f|\omega) \\ &= \int_{\Omega_q} \mu^+(d\omega) q_\Lambda^-(f|\omega) \,, \\ &= \int \mu^+(d\omega) q_\Lambda^-(f|\omega) \,, \end{aligned} \tag{4.53}$$

The second part of 3. follows from Proposition 4.1. □

5 A criterion for non–quasilocality

Let $\Gamma = \{\gamma_\Lambda \,, \, \Lambda \subset \mathcal{L}\}$ be a local specification defined on $\mathcal{L}$, which is monotonicity–preserving and satisfies (G_1). We establish in this section our main criterion for non–quasilocality of a local specification, Corollary 5.1. This is done by estimating the relative entropy

$$\frac{1}{|\Lambda|} H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) := \frac{1}{|\Lambda|} \sum_{\sigma_\Lambda} \gamma_\Lambda(\sigma|+) \log \frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(\sigma|-)} \,. \tag{5.1}$$

We do this in subsection 5.1. The method is inspired by Sullivan (1973) and Kozlov (1974).

5.1 Estimates of the relative entropy

We define on the set $\mathcal{L}$ a total order denoted by $\geq$. Given any $\sigma \in \Omega$ and $j \in \mathcal{L}$, we define a new element ${}_j\sigma \in \Omega$ by

$${}_j\sigma(k) := \begin{cases} - & \text{if } k < j \\ \sigma(k) & \text{if } k \geq j \end{cases} \,. \tag{5.2}$$

We write the quotient in the right–hand side of (5.1) as

$$\frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(\sigma|-)} = \frac{\gamma_\Lambda(-|+)}{\gamma_\Lambda(-|-)} \cdot \frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(-|+)} \cdot \frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(\sigma|-)} \,. \tag{5.3}$$

Using the identity

$$\frac{\gamma_\Lambda(\sigma_{\Lambda_1}\eta_{\Lambda_2}|\omega)}{\gamma_\Lambda(\tau_{\Lambda_1}\eta_{\Lambda_2}|\omega)} = \frac{\gamma_{\Lambda_1}(\sigma_{\Lambda_1}|\eta_{\Lambda_2}\omega_{\Lambda_2^c})}{\gamma_{\Lambda_1}(\tau_{\Lambda_1}|\eta_{\Lambda_2}\omega_{\Lambda_2^c})} \,, \tag{5.4}$$

where $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, we have

$$\frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(-|+)} = \prod_{j\in\Lambda} \frac{\gamma_j(\sigma|_j\sigma_\Lambda^+)}{\gamma_j(-|_j\sigma_\Lambda^+)} \tag{5.5}$$

and

$$\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(\sigma|-)} = \prod_{j\in\Lambda} \frac{\gamma_j(-|_j\sigma_\Lambda^-)}{\gamma_j(\sigma|_j\sigma_\Lambda^-)} \,. \tag{5.6}$$

Using (5.2), (5.5) and (5.6) we can write (5.1) as

$$\frac{1}{|\Lambda|} H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) = \tag{5.7}$$

$$\frac{1}{|\Lambda|}\log\frac{\gamma_\Lambda(-|+)}{\gamma_\Lambda(-|-)} + \frac{1}{|\Lambda|}\sum_{\sigma_\Lambda}\gamma_\Lambda(\sigma|+)\log\prod_{j\in\Lambda}\frac{\gamma_j(\sigma|_j\sigma_\Lambda^+)}{\gamma_j(-|_j\sigma_\Lambda^+)}\frac{\gamma_j(-|_j\sigma_\Lambda^-)}{\gamma_j(\sigma|_j\sigma_\Lambda^-)} \,.$$

Let us define

$$f_{j,\Lambda}(\eta) := \log\frac{\gamma_j(+|\eta_\Lambda^+)}{\gamma_j(+|\eta_\Lambda^-)}\frac{\gamma_j(-|\eta_\Lambda^-)}{\gamma_j(-|\eta_\Lambda^+)} \,. \tag{5.8}$$

The j^{th} factor of the product in (5.7) is equal to one if $\sigma(j) = -$; it is larger than one if $\sigma(j) = +$ by monotonicity; thus

$$0 \;\le\; \log\frac{\gamma_j(\sigma|_j\sigma_\Lambda^+)}{\gamma_j(-|_j\sigma_\Lambda^+)}\frac{\gamma_j(-|_j\sigma_\Lambda^-)}{\gamma_j(\sigma|_j\sigma_\Lambda^-)} \;\le\; f_{j,\Lambda}(\sigma)\,. \tag{5.9}$$

A similar expression can be derived using ${}^j\sigma$ instead of ${}_j\sigma$, where

$${}^j\sigma(k) := \begin{cases} + & \text{if } k < j \\ \sigma(k) & \text{if } k \ge j \end{cases} ; \tag{5.10}$$

we get

$$\frac{1}{|\Lambda|} H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) = \tag{5.11}$$

$$\frac{1}{|\Lambda|}\log\frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} + \frac{1}{|\Lambda|}\sum_{\sigma_\Lambda}\gamma_\Lambda(\sigma|+)\log\prod_{j\in\Lambda}\frac{\gamma_j(\sigma|^j\sigma_\Lambda^+)}{\gamma_j(+|^j\sigma_\Lambda^+)}\frac{\gamma_j(+|^j\sigma_\Lambda^-)}{\gamma_j(\sigma|^j\sigma_\Lambda^-)} \,.$$

By monotonicity

$$\log\frac{\gamma_j(\sigma|^j\sigma_\Lambda^+)}{\gamma_j(+|^j\sigma_\Lambda^+)}\frac{\gamma_j(+|^j\sigma_\Lambda^-)}{\gamma_j(\sigma|^j\sigma_\Lambda^-)} \;\le\; 0\,. \tag{5.12}$$

Lemma 5.1 *Let the local specification $\Gamma = \{\gamma_\Lambda\,,\ \Lambda \subset \mathcal{L}\}$ on $\mathcal{L}$ satisfy (G_1) and be monotonicity–preserving. Then $\lim_\Lambda f_{j,\Lambda} = f_j$ exists, and*

$$\frac{1}{|\Lambda|}\log\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \le \frac{1}{|\Lambda|}\sum_{j\in\Lambda}\int\gamma_\Lambda(d\sigma|+)f_{j,\Lambda}({}_j\sigma)\,, \tag{5.13}$$

$$\frac{1}{|\Lambda|} H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) \le \frac{1}{|\Lambda|}\log\frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)}\,. \tag{5.14}$$

Let $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$; then

$$\log\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \le \log\frac{\gamma_{\Lambda_1}(-|-)}{\gamma_{\Lambda_1}(-|+)} + \log\frac{\gamma_{\Lambda_2}(-|-)}{\gamma_{\Lambda_2}(-|+)}\,, \tag{5.15}$$

and

$$\log\frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} \le \log\frac{\gamma_{\Lambda_1}(+|+)}{\gamma_{\Lambda_1}(+|-)} + \log\frac{\gamma_{\Lambda_2}(+|+)}{\gamma_{\Lambda_2}(+|-)}\,. \tag{5.16}$$

Proof : The existence of $\lim_\Lambda f_{j,\Lambda}$ follows by monotonicity. Since the relative entropy is non-negative

$$\frac{1}{\Lambda}\log\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \le \frac{1}{\Lambda}\sum_{j\in\Lambda}\int \gamma_\Lambda(d\sigma|+) f_{j,\Lambda}({}_j\sigma)\ . \tag{5.17}$$

Starting from (5.11), and observing that the second term of the right–hand side of (5.11) is non-positive (see (5.12)), we get (5.14).
Let $\Lambda = \Lambda_1 \cup \Lambda_2$, so that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let χ_j be the characteristic function of the set $\{\eta:\ \eta(j) = +\}$, and

$$\chi_\Lambda := \prod_{j\in\Lambda}\chi_j\ . \tag{5.18}$$

Then, since χ_Λ is increasing,

$$\begin{aligned}
\gamma_\Lambda(\chi_\Lambda|+) &= \int \gamma_\Lambda(d\eta|+)\chi_{\Lambda_1}(\eta)\chi_{\Lambda_2}(\eta) \\
&= \int \gamma_\Lambda(d\eta|+)\gamma_{\Lambda_1}(\chi_{\Lambda_1}|\eta)\chi_{\Lambda_2}(\eta) \\
&\le \gamma_{\Lambda_1}(\chi_{\Lambda_1}|+)\gamma_\Lambda(\chi_{\Lambda_2}|+) \\
&\le \gamma_{\Lambda_1}(\chi_{\Lambda_1}|+)\gamma_{\Lambda_2}(\chi_{\Lambda_2}|+)\ .
\end{aligned} \tag{5.19}$$

Similarly,

$$\gamma_\Lambda(\chi_\Lambda|-) \ge \gamma_{\Lambda_1}(\chi_{\Lambda_1}|-)\gamma_{\Lambda_2}(\chi_{\Lambda_2}|-)\ . \tag{5.20}$$

Therefore

$$\frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} \le \frac{\gamma_{\Lambda_1}(+|+)}{\gamma_{\Lambda_1}(+|-)}\frac{\gamma_{\Lambda_2}(+|+)}{\gamma_{\Lambda_2}(+|-)}\ ; \tag{5.21}$$

we can prove analogously that

$$\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \le \frac{\gamma_{\Lambda_1}(-|-)}{\gamma_{\Lambda_1}(-|+)}\frac{\gamma_{\Lambda_2}(-|-)}{\gamma_{\Lambda_2}(-|+)}\ . \tag{5.22}$$

□

5.2 Criterion for non–quasilocality

The main property of the function f_j is that it is non-negative; it is equal to zero at η iff $\gamma_j(\sigma|\cdot)$ is continuous at η for any σ. Our criterion for non–quasilocality is stated for $\mathbb{Z}^d$–invariant local specifications, but it can be generalized to other situations with suitable modifications.

Proposition 5.1 *Let $\mathcal{L} = \mathbb{Z}^d$ and Γ be a $\mathbb{Z}^d$–invariant, monotonicity–preserving local specification satisfying (G_1). Then*

$$\lim_{\Lambda_n}\frac{1}{|\Lambda_n|}\log\frac{\gamma_{\Lambda_n}(-|-)}{\gamma_{\Lambda_n}(-|+)} \le \int \mu^+(d\sigma) f_j({}_j\sigma)\ , \tag{5.23}$$

where $\{\Lambda_n\}$ is a sequence tending to $\mathbb{Z}^d$ in the sense of Fisher, e.g. $\{\Lambda_n\}$ is a sequence of increasing cubes with $|\Lambda_n| \to \infty$; j is any lattice point.

Proof : For η fixed the function $\Lambda \mapsto f_{j,\Lambda}(\eta)$ is decreasing in Λ. ¿From (5.13) of Lemma 5.1, if $\Lambda_n \supset \Lambda_m$,

$$\lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \log \frac{\gamma_{\Lambda_n}(-|-)}{\gamma_{\Lambda_n}(-|+)} \leq \lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \int \gamma_{\Lambda_n}(d\sigma|+) f_{j,\Lambda_n}({}_j\sigma) \tag{5.24}$$
$$\leq \lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \int \gamma_{\Lambda_n}(d\sigma|+) f_{j,\Lambda_m}({}_j\sigma) \; .$$

The function $f_{j,\Lambda}$ can be decomposed into

$$f_{j,\Lambda}(\eta) = a_{j,\Lambda}(\eta) + b_{j,\Lambda}(\eta) \tag{5.25}$$

with

$$a_{j,\Lambda}(\eta) := \log \frac{\gamma_j(+|\eta_\Lambda^+)}{\gamma_j(-|\eta_\Lambda^+)} \;\;,\;\; b_{j,\Lambda}(\eta) := \log \frac{\gamma_j(-|\eta_\Lambda^-)}{\gamma_j(+|\eta_\Lambda^-)} \; . \tag{5.26}$$

By monotonicity the function $a_{j,\Lambda}(\cdot)$ is increasing, and the function $b_{j,\Lambda}(\cdot)$ decreasing. Therefore

$$\int \gamma_{\Lambda_n}(d\sigma|+) a_{j,\Lambda_m}({}_j\sigma) \tag{5.27}$$

is decreasing as a function of Λ_n, and

$$\int \gamma_{\Lambda_n}(d\sigma|+) b_{j,\Lambda_m}({}_j\sigma) \tag{5.28}$$

is increasing as a function of Λ_n. Given any $\varepsilon > 0$, we can find a cube Λ_ε containing the origin 0, such that if $j + \Lambda_\varepsilon \subset \Lambda_n$, then

$$\left| \int \gamma_{\Lambda_\varepsilon}(d\sigma|+) a_{j,\Lambda_m}({}_j\sigma) - \int \mu^+(d\sigma) a_{j,\Lambda_m}({}_j\sigma) \right| \leq \varepsilon \tag{5.29}$$

$$\left| \int \gamma_{\Lambda_\varepsilon}(d\sigma|+) b_{j,\Lambda_m}({}_j\sigma) - \int \mu^+(d\sigma) b_{j,\Lambda_m}({}_j\sigma) \right| \leq \varepsilon \; . \tag{5.30}$$

Using the $\mathbb{Z}^d$–invariance of μ^+ and (5.29) and (5.30),

$$\lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \int \gamma_{\Lambda_n}(d\sigma|+) f_{j,\Lambda_m}({}_j\sigma) = \int \mu^+(d\sigma) f_{j,\Lambda_m}({}_j\sigma) \; . \tag{5.31}$$

We can take now the limit $\Lambda_m \uparrow \mathbb{Z}^d$. □

Corollary 5.1 *Let $\mathcal{L} = \mathbb{Z}^d$ and Γ be a $\mathbb{Z}^d$–invariant, monotonicity–preserving local specification satisfying (G_1). If*

$$\lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \log \frac{\gamma_{\Lambda_n}(-|-)}{\gamma_{\Lambda_n}(-|+)} > 0 \; , \tag{5.32}$$

then the local specification Γ cannot be continuous (quasilocal) everywhere, hence it is not Gibbs.

6 Ising model

Our basic example is the Ising model on $\mathbb{Z}^d$, $d \geq 2$. Let $\langle i,j \rangle$ denote a pair of nearest neighbour points i and j in $\mathbb{Z}^d$. For any $\Lambda \subset \mathbb{Z}^d$, we define a function $I_\Lambda(\omega)$ on Ω,

$$I_\Lambda(\omega) := \sum_{\langle i,j \rangle \cap \Lambda \neq \emptyset} X_i(\omega) X_j(\omega) + h \sum_{i \in \Lambda} X_i(\omega) \ . \tag{6.1}$$

We define a local specification $\Gamma(\beta)$ on $\mathbb{Z}^d$, $\beta > 0$, by the Boltzmann-Gibbs formula

$$\gamma_\Lambda(\sigma|\eta) := \frac{\exp\left(\beta I_\Lambda(\sigma_\Lambda \eta_{\Lambda^c})\right)}{\sum_{\omega_\Lambda} \exp\left(\beta I_\Lambda(\omega_\Lambda \eta_{\Lambda^c})\right)} \ . \tag{6.2}$$

It is a Gibbs specification which is $\mathbb{Z}^d$–invariant and monotonicity–preserving. When $h = 0$ it is also invariant under the symmetry $\omega \mapsto \overline{\omega}$, where $\overline{\omega}(k) := -\omega(k)$. It is well-known that there exists $\beta_c(d)$ such that for any $\beta \leq \beta_c(d)$ there is a unique probability measure which is $\Gamma(\beta)$–compatible, and for any $\beta > \beta_c(d)$ the measures μ^+ and μ^- are different. By Lemma 5.1 we can define the quantity

$$\zeta_T := \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^+_{\Lambda_n}(-|-)}{q^+_{\Lambda_n}(-|+)} \geq 0 \ , \tag{6.3}$$

where $\{\Lambda_n\}$ is an increasing sequence of cubes in T such that $|\Lambda_n| \to \infty$.

We consider Schonmann's example with $T \cong \mathbb{Z}^{d-1}$ and $h = 0$ and verify the hypothesis of Theorem 4.1. We first recall the following result , Lemma 3.5 in Fröhlich Pfister (1987b).

Proposition 6.1 *For the d–dimensional Ising model, $d \geq 2$, if $T := \mathbb{Z}^{d-1}$, then $|\mathcal{G}(\Gamma^+_{T^c})| = 1$ and $|\mathcal{G}(\Gamma^-_{T^c})| = 1$ for any β.*

As a consequence of Proposition 6.1 and the spin-flip symmetry of the random field, we can write

$$\begin{aligned} \zeta_T &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^+_{\Lambda_n}(-|-)}{q^+_{\Lambda_n}(-|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^-_{\Lambda_n}(-|-)}{q^+_{\Lambda_n}(-|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^+_{\Lambda_n}(+|+)}{q^+_{\Lambda_n}(-|+)} \ . \end{aligned} \tag{6.4}$$

For $\beta > \beta_c$ large enough one can show directly, and easily, by a perturbative argument that ζ_T is strictly positive. However, using results from Fröhlich Pfister (1987b) we can prove

Corollary 6.1 *Let $d \geq 2$, $h = 0$ and $T \cong \mathbb{Z}^{d-1}$. Then:*

1. *For the Ising model on $\mathbb{Z}^d$ the local specification Q^+_T is continuous whenever Dobrushin's strong uniqueness condition holds.*

2. *Q^+_T is quasilocal μ^+–a.s. if $\beta \leq \beta_c(d)$.*

3. *$\zeta_T > 0$ if and only if $\beta > \beta_c(d)$. Therefore when $\beta > \beta_c(d)$ Q_T^+ is not quasilocal everywhere, hence not Gibbs.*

Remark: By Proposition 4.2, the discontinuities of Q_T^+ cannot be removed by changing the local specification on a set of μ^+–measure zero. That is, item [3.] of the corollary implies that for $\beta > \beta_c(d)$ the measure μ^+ is not compatible with *any* quasilocal specification.

Proof : By (6.4)

$$\begin{aligned} \zeta_T &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^+_{\Lambda_n}(+|+)}{q^+_{\Lambda_n}(-|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \lim_{\Lambda' \uparrow T^c \cup \Lambda_n} \log \frac{\gamma_{\Lambda'}(+_{\Lambda_n}|+)}{\gamma_{\Lambda'}(-_{\Lambda_n}|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \lim_{\Lambda' \uparrow T^c \cup \Lambda_n} \log \frac{Z^+_{\Lambda'}(+_{\Lambda_n})}{Z^+_{\Lambda'}(-_{\Lambda_n})} . \end{aligned} \tag{6.5}$$

In this last formula $Z^+_{\Lambda'}(+_{\Lambda_n})$ is the partition function of the Ising model in the box Λ', with + boundary condition, and such that all spins at $i \in \Lambda_n$ are equal to +. By monotonicity in Λ' and in Λ_n, the limit is also equal to

$$\lim_{\Lambda'' \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda_n|} \log \frac{Z^+_{\Lambda''}(+_{\Lambda_n})}{Z^+_{\Lambda''}(-_{\Lambda_n})} . \tag{6.6}$$

Here Λ'' is a cube centered at the origin, such that $\Lambda'' \cap T = \Lambda_n$. ζ_T is two times the surface tension as a consequence of d) p.54 in Fröhlich Pfister (1987b). The surface tension is strictly positive iff $\beta > \beta_c(d)$ (Lebowitz Pfister (1981)). □

Remarks :

(i) Questions concerning the size of Ω_q^+ are also considered in Maes Vande Velde (1992) and Lörinczi (1993).

(ii) In Lörinczi Vande Velde (1994) the authors indicate that one recovers quasilocality everywhere in the case $d = 2$, if one chooses instead of $T = \mathbb{Z}^1$ a subgroup T' of $\mathbb{Z}^1$ with a lattice–spacing large enough. However, the specifications Q_T^+ and Q_T^- so obtained are *different*. That is, the projected measures μ_T^+ and μ_T^- can *not* be simultaneously compatible with the same continuous specification, and, moreover, any non-trivial convex combination of them has conditional probabilities that are *everywhere discontinuous* (van Enter Lörinczi (1996)).

(iii) It is an *open* question whether we have $\mu^+(\Omega_q) = 1$ or $\mu^+(\Omega_q) = 0$.

(iv) In the region $h \neq 0$, it is known that the projected measure is quasilocal in $d = 2$ (Lörinczi (1995a–b)), while the expected existence of layering transitions would imply non-quasilocality for low field and low temperature in $d \geq 3$ (see the discussion of Lörinczi (1995a–b)).

Finally, we consider Griffiths–Pearce–Israel's example for $d \geq 3$. Here T is a d–dimensional subgroup of $\mathbb{Z}^d$. Let us fix β sufficiently large. There exists $h(\beta) > 0$ such that for any h, $0 \leq h \leq h(\beta)$, the measure μ_T^+ cannot be consistent with any local specification on T, which is quasilocal everywhere. In particular the measure μ_T^+ on $(T, \mathcal{F}_T)$ is not a Gibbs measure (van Enter Fernández Sokal (1993)). There are two cases to consider.

(I) For $h > 0$ $\mu^+ = \mu^-$ so that we have almost–sure quasilocality. It has been proved in Martinelli Olivieri (1993) that one recovers quasilocality everywhere if one chooses instead of T a subgroup T' of T of the same dimension, but with a lattice–spacing of order $O(1/h)$.

(II) For $h = 0$ we have $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$ and $|\mathcal{G}(\Gamma_{T^c}^-)| = 1$ (on T^c we have an Ising model with a magnetic field when $\omega_T = +$ or $\omega_T = -$). A simple estimation shows that $\log \frac{q_{\Lambda_n}^+(-|-)}{q_{\Lambda_n}^+(-|+)}$ is of the order of the length of the boundary of Λ_n when Λ_n is a square; therefore $\zeta_T = 0$. Thus, this example is of different nature than Schonmann's. Our criterion about non–quasilocality does not apply.

References

Albeverio S., Zegarlinski B. (1992). Global Markov property in quantum field theory and statistical mechanics, in *Ideas and Methods in Quantum and Statistical Physics*, eds. S. Albeverio *et al*, 331–369, Cambridge University Press.

Dobrushin R.L. (1968). Gibbsian random fields for lattice systems with pairwise interactions, *Funct. Anal. Appl.* **3**, 22–28

van Enter A.C.D., Fernández R., Sokal A.D. (1993). Regularity Properties and Pathologies of Position-Space Renormalization-Group Transformations: Scope and Limitations of Gibbsian Theory, *J. Stat. Phys.* **72**, 879–1167

van Enter A.C.D., Lörinczi J. (1996). Robustness of non-Gibbsian property: some examples, *J. Phys. A* **29**, 2465–2473

Föllmer H. (1980). On the Global Markov Property, in *Quantum Fields –algebras, processes* , ed. L. Streit, 293–302, Springer Wien, New-York

Fröhlich J., Pfister C.–E. (1987a). Semi-Infinite Ising Model I: Thermodynamic Functions and Phase Diagram in Absence of Magnetic Field, *Commun. Math. Phys.* **109** 493–523

Fröhlich J., Pfister C.–E. (1987b). Semi-Infinite Ising Model II: the Wetting and Layering Transitions, *Commun. Math. Phys.* **112** 51–74

Georgii H.–O. (1988). Gibbs Measures and Phase Transitions, de Gruyter, Berlin, New-York

Goldstein S. (1978). A Note on Specifications, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **46**, 45–51

Goldstein S. (1980). Remarks on the Global Markov Property, *Commun. Math. Phys.* **74**, 223–234

Griffiths R.B., Pearce P.A. (1979). Mathematical properties of renormalization–group transformations, *J. Stat. Phys.* **20**, 499–545

Grimmett G. (1995). The Stochastic Random–Cluster Process, and the Uniqueness of Random–Cluster Measures *Ann. Prob.* **23**, 1461–1510

Israel R.B. (1981). Banach algebras and Kadanoff transformations, in *Random Fields (Esztergom, 1979)*, eds J. Fritz, J.L. Lebowitz and D. Szász, North–Holand, Amsterdam, Vol. II, 593–608

Kozlov O.K. (1974). Gibbs description of a system of random variables, *Probl. Inform. Transmiss.* **10**, 258–265

Lanford O.E., Ruelle D. (1969). Observables at infinity and states with short range correlations in statistical mechanics *Commun. Math. Phys.* **13**, 194–215

Lebowitz J.L., Pfister C.–E. (1981). Surface tension and phase coexistence, *Phys. Rev. Letters* **46**, 1031–1033

Lörinczi J. (1994). Some results on the projected two-dimensional Ising model, in "On Three Levels" Ed. by M. Fannes, C. Maes, A. Verbeure, Plenum Press New-York, London, 373–380

Lörinczi J. (1995a). *On Limits of the Gibbsian Formalism in Thermodynamics*, PhD. thesis, University of Groningen.

Lörinczi J. (1995b). Quasilocality of projected Gibbs measures through analyticity techniques, *Helv. Phys. Acta* **68**, 605-626

Lörinczi J., Vande Velde K. (1994). A note on the projection of Gibbs measures, *J. Stat. Phys.* **77**, 881–887

Maes C., Vande Velde K. (1992). Defining Relative Energies For The Projected Ising Measure, *Helv. Phys. Acta* **65** 1055–1068

Martinelli F., Olivieri E. (1993). Some remarks on pathologies of the renormalization–group transformations for the Ising model, *J. Stat. Phys.* **72**, 1169–1177

Sokal A.D. (1981). Existence of Compatible Families of Proper Regular Conditional Probabilities, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **56**, 537–548

Schonmann R.H. (1989). Projections of Gibbs Measures May Be Non-Gibbsian, *Commun. Math. Phys.* **124**, 1–7

Sullivan W.G. (1973). Potentials for almost Markovian Random Fields, *Commun. Math. Phys.* **33**, 61–74

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