DISTRIBUTION OF OVERLAP PROFILES IN THE ONE-DIMENSIONAL KAC-HOPFIELD MODEL

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Abstract: We study a one-dimensional version of the Hopfield model with long, but finite range interactions below the critical temperature. In the thermodynamic limit we obtain large deviation estimates for the distribution of the "local" overlaps, the range of the interaction, γ^{-1}, being the large parameter. We show in particular that the local overlaps in a typical Gibbs configuration are constant and equal to one of the mean-field equilibrium values on a scale $o(\gamma^{-2})$. We also give estimates on the size of typical "jumps". i.e. the regions where transitions from one equilibrium value to another take place. Contrary to the situation in the ferromagnetic Kac-model, the structure of the profiles is found to be governed by the quenched disorder rather than by entropy.

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1.Introduction

Models of statistical mechanics where particles (or spins) interact through potentials $J_\gamma(r) \equiv \gamma^d J(\gamma r)$, $r \in I\!R^d$, with J some function that either has bounded support or is rapidly decreasing were introduced by Kac et al. [KUH] in 1963 as links between short-range, microscopic models and mean field theories such as the van der Waals theory of the liquid-gas transition. The main success of these models can be seen in that they explain, through the Lebowitz-Penrose theorem, the origin of the Maxwell rule that has to be invoked in an ad hoc way to overcome the problem of the non-convexity of the thermodynamic functions arising in mean-field theories.

Recently, there has been renewed interest in this model in the context of attempting to obtain a precise description of equilibrium configurations [COP] and their temporal evolution [DOPT] in magnetic systems at low temperatures. In [COP] large deviation techniques were used to describe precisely the profiles of local magnetization in a one dimensional Ising model with Kac potential in infinite volume in the limit $\gamma \downarrow 0$. It turned out that this apparently simple system exhibits a surprisingly rich structure when considered at appropriate scales and it appears that the Kac-type models can still offer an interesting test ground for the study of low-temperature phenomena. The purpose of the present paper is to extend such an analysis to a class of models with *random interactions*.

Spin systems where spins at sites i and j interact through a random coupling J_{ij} whose mean value is zero (or close to zero) are commonly termed *spin glasses*. The prototype models are the *Sherrington-Kirkpatrick model* (SK-model) [SK] where the lattice is the completely connected graph on N vertices and the couplings J_{ij} are i.i.d. centered gaussian variables with variance $N^{-1/2}$, and the *Edwards-Anderson model* [EA], defined on the lattice $\mathbb{Z}^d$ and with J_{ij} i.i.d. centered random variables with variance 1 if i and j are nearest neighbors in the lattice, whereas $J_{ij} \equiv 0$ otherwise.

These systems are notoriously difficult to analyse and little is known on a firm basis about their low temperature properties. The situation is somewhat better in the case of the *mean-field* SK-model, for which there is at least a rather elaborate picture based on the so-called *replica-method* (for a review see [MPV]) which is quite commonly accepted, although almost no results exist that are mathematically rigorous. Exceptions concern the high-temperature phase [ALR, FZ, CN, T1] and some self-averaging properties of the thermodynamic quantities [PS, BGP3]. For short-range models (the Edwards-Anderson model [EA] the situation is much worse, and there exist conflicting theories on such fundamental questions as the upper and lower critical dimension and the number of low temperature phases, all of which are more or less supported by heuristic arguments (see e.g. [FH, BF, vE, NS]), and the interpretation of numerical simulations on finite systems (for a recent analysis and a critical assessment of the situation see [MPR]).

The difficulties with the SK-model have soon prompted the proposal of simplified models for spin-glasses in which the statistics of the random couplings was changed while some of the features are conserved. The *Mattis-model* [Ma] where $J_{ij} \equiv \epsilon_i \epsilon_j$ with ϵ_i independent symmetric Bernoulli variables was realized to be trivially equivalent to a ferromagnet and lacking the essential feature of *frustration*; Luttinger [Lu] amended this by setting $J_{ij} \equiv \xi_i^1 \xi_j^1 + \xi_i^2 \xi_j^2$ while Figotin and Pastur [FP1, FP2] proposed and analysed a generalization of this interaction with an arbitrary fixed number of summands and more general distribution of the random variables ξ_i^μ. While these models could be solved exactly, they lacked essential features expected for real spin glasses and thus did not become very popular until they were again proposed in a quite different context by Hopfield [Ho] as models for autoassociative memory. Hopfield also considered the number of summands, M, to be a function of the size, N, of the graph ('network') and observed numerically a drastic change of behaviour of the system as the ratio $\alpha \equiv M/N$ exceeded a certain threshold. This was confirmed by Amit et al. [AGS] through a theoretical analysis using the replica trick. Indeed, the Hopfield model can be seen as a family of models depending on the different growth rate of $M(N)$ that mediates between simple ferromagnets and the SK spin-glass.

The Hopfield model offers the advantage to be more amenable to a mathematically rigorous analysis then the SK-model, at least as long as $M(N)$ does not grow too fast with N. By now we have a fairly complete understanding of the structure of the low temperature Gibbs states [BGP1, BGP3, BG4] in the case where $\lim_{N\uparrow\infty} M/N \leq \alpha_0$, for α_0 sufficiently small. It is thus interesting to take advantage of this situation in order to get some insight into the relation between finite dimensional spin-glasses and the corresponding mean field models by studying the finite dimensional version of the Hopfield model with a Kac-type interaction. It should be noted that such a model had already been considered by Figotin and Pastur [FP3] in 1982 in the case of bounded M. In a recent paper [BGP2] we have proven the analogue of the classical Lebowitz-Penrose theorem for this model, i.e. we have proven the convergence of the thermodynamic functions to the convex hulls of those of the mean-field model as $\gamma \downarrow 0$ under the condition that $\lim_{\gamma\downarrow 0} M(\gamma)|\ln\gamma|/\gamma = 0$. In the present paper we turn to the more detailed analysis of the Gibbs states of the Kac-Hopfield model and consider, as a first step, the one dimensional case along the lines of [COP].

Let us start by defining our model in a precise way and by fixing our notations. Let $(\Omega, \mathcal{F}, I\!P)$ be an abstract probability space. Let $\xi \equiv \{\xi_i^\mu\}_{i\in \mathbb{Z}, \mu\in I\!N}$ be a two-parameter family of independent, identically distributed random variables on this space such that $I\!P(\xi_i^\mu = 1) = I\!P(\xi_i^\mu = -1) = \frac{1}{2}$. (the precise form of the distribution of ξ_i^μ is not really essential and far more general distributions can be considered). We denote by σ a function $\sigma : \mathbb{Z} \to \{-1, 1\}$ and call σ_i, $i \in \mathbb{Z}$ the spin at site i. We denote by $\mathcal{S}$ the space of all such functions, equipped with the product topology of the discrete topology in $\{-1,1\}$. We choose the function $J_\gamma(i-j) \equiv \gamma J(\gamma|i-j|)$, and

$$J(x) = \begin{cases} 1, & \text{if } |x| \leq 1/2 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

(Note that other choices for the function $J(x)$ are possible. They must satisfy the conditions $J(x) \geq 0$, $\int dx J(x) = 1$, and must decay rapidly to zero on a scale of order unity. For example, the original choice of Kac was $J(x) = e^{-|x|}$. For us, the choice of the characteristic function is particularly convenient).

The interaction between two spins at sites i and j will be chosen for given $\omega \in \Omega$, as

$$-\frac{1}{2}\sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.2}$$

and the *formal* Hamiltonian will be

$$H_\gamma[\omega](\sigma) = -\frac{1}{2}\sum_{(i,j)\in \mathbb{Z}\times\mathbb{Z}}\sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.3}$$

As usual, to make mathematically meaningful statements, we have to consider restrictions of this quantity to *finite volumes*. We will do this in a particular way which requires some prior discussion. Note that the parameter γ introduces a natural length scale γ^{-1} into our model which is the distance over which spins interact directly. We will be interested later in the behaviour of the system on that and larger scales and will refer to it as the *macroscopic scale*, whereas the sites i of the underlying lattice $\mathbb{Z}$ are referred to as the microscopic scale. In the course of our analysis we will have to introduce two more intermediate, *mesoscopic scales*, as shall be explained later. We find it convenient to measure distances and to define finite volumes in the macroscopic rather than the microscopic scale, as this allows to deal with volumes that actually do not change with γ. Although this will require some slightly unconventional looking definitions, we are convinced the reader will come to appreciate the advantages of our conventions later on. Let thus $\Lambda = [\lambda_-, \lambda_+] \subset \mathbb{R}$ be an interval on the real line. Thus for points $i \in \mathbb{Z}$ referring to sites on the microscopic scale we will write

$$i \in \Lambda \quad \textit{iff} \quad \lambda_- \leq \gamma i \leq \lambda_+ \tag{1.4}$$

Note that we will stick very strictly to the convention that the letters i, j, k *always* refer to microscopic sites. The Hamiltonian corresponding to a volume Λ (with free boundary conditions) can then be written as

$$H_{\gamma,\Lambda}[\omega](\sigma) = -\frac{1}{2}\sum_{(i,j)\in \Lambda\times\Lambda}\sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega] J_\gamma(i-j)\sigma_i\sigma_j \tag{1.5}$$

We shall also write in the same spirit $\mathcal{S}_\Lambda \equiv \times_{i\in\Lambda}\{-1,1\}$ and denote its elements by σ_Λ. The *interaction* between the spins in Λ and those outside Λ will be written as

$$W_{\gamma,\Lambda}[\omega](\sigma_\Lambda,\sigma_{\Lambda^c}) = -\sum_{i\in\Lambda}\sum_{j\in\Lambda^c}\sum_{\mu=1}^{M(\gamma)} \xi_i^\mu[\omega]\xi_j^\mu[\omega]J_\gamma(i-j)\sigma_i\sigma_j \tag{1.6}$$

The finite volume Gibbs measure for such a volume Λ with fixed external configuration σ_{Λ^c} (the 'local specification') is then defined by assigning to each $\sigma_\Lambda \in \mathcal{S}_\Lambda$ the mass

$$\mathcal{G}_{\beta,\gamma,\Lambda}^{\sigma_{\Lambda^c}}[\omega](\sigma_\Lambda) \equiv \frac{1}{Z_{\beta,\gamma,\Lambda}^{\sigma_{\Lambda^c}}[\omega]} e^{-\beta[H_{\gamma,\Lambda}[\omega](\sigma_\Lambda)+W_{\gamma,\Lambda}[\omega](\sigma_\Lambda,\sigma_{\Lambda^c})]} \tag{1.7}$$

where $Z_{\beta,\gamma,\Lambda}^{\sigma_{\Lambda^c}}[\omega]$ is a normalizing factor usually called *partition function*. We will also denote by

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](\sigma_\Lambda) \equiv \frac{1}{Z_{\beta,\gamma,\Lambda}[\omega]} e^{-\beta H_{\gamma,\Lambda}[\omega](\sigma_\Lambda)} \tag{1.8}$$

the Gibbs measure with free boundary conditions. It is crucial to keep in mind that we are always interested in taking the infinite volume limit $\Lambda \uparrow I\!R$ first for fixed γ and to study the asymptotic of the result as $\gamma \downarrow 0$ (this is sometimes referred to as the 'Lebowitz-Penrose limit').

In [BGP2] we have studied the distribution of the global 'overlaps' $m_\Lambda^\mu(\sigma) \equiv \frac{\gamma}{|\Lambda|}\sum_{i\in\Lambda}\xi_i^\mu\sigma_i$ under the Gibbs measure (1.7). Here we are going into more detail in that we want to analyse the distribution of *local overlaps*. To do this we will actually have to introduce *two* intermediate *mesoscopic* length scales, $1 \ll \ell(\gamma) \ll L(\gamma) \ll \gamma^{-1}$. Note that both $\ell(\gamma)$ and $L(\gamma)$ will tend to infinity as $\gamma \downarrow 0$ while $\ell(\gamma)/L(\gamma)$ as well as $\gamma L(\gamma)$ tend to zero. We will assume that ℓ, L and γ^{-1} are integer multiples of each other. Further conditions on this scales will be imposed later. To simplify notations, the dependence on γ of ℓ and L will not be made explicit in the sequel. We now divide the real line into boxes of length $\gamma\ell$ and γL, respectively, with the first box, called 0 being centered at the origin. The boxes of length $\gamma\ell$ will be called x, y, or z, and labelled by the integers. That is, the box x is the interval of length $\gamma\ell$ centered at the point $\gamma\ell x$. No confusion should arise from the fact that we use the symbol x as denoting both the box *and* its label, since again x, y, z are used *exclusively* for this type of boxes. In the same way, the letters r, s, t are reserved for the boxes of length γL, centered at the points $\gamma L \mathbb{Z}$, and finally we reserve u, v, w for boxes of length one centered at the integers. With these conventions, it makes sense to write e.g. $i \in x$ shorthand for $\ell x - \ell/2 \leq i \leq \ell x + \ell/2$, etc.[1]In this spirit we define the $M(\gamma)$ dimensional vector $m_\ell(x,\sigma)$ and $m_L(r,\sigma)$ whose μ-th components are

$$m_\ell^\mu(x,\sigma) \equiv \frac{1}{\ell}\sum_{i\in x}\xi_i^\mu\sigma_i \tag{1.9}$$

[1] On a technical level we will in fact have to use even more auxiliary intermediate scales, but as in [COP] we will try to keep this under the carpet as far as possible.

and

$$m_L^\mu(r,\sigma) \equiv \frac{1}{L}\sum_{i\in r}\xi_i^\mu\sigma_i \tag{1.10}$$

respectively. Note that we have, for instance, that

$$m_L^\mu(r,\sigma) = \frac{\ell}{L}\sum_{x\in r} m_\ell^\mu(x,\sigma) \tag{1.11}$$

We will also have to be able to indicate the box on some larger scale containing a specified box on the smaller scale. Here we write simply, e.g., $r(x)$ for the unique box of length L that contains the box x of length ℓ. Expressions like $x(i)$, $u(y)$ or $s(k)$ have corresponding meanings.

Remark: It easy to connect from our notation to the continuum notation used in [COP]. For instance, (1.9) can be rewritten as

$$m_\ell(x,u) = \frac{1}{\gamma\ell}\gamma\sum_{i\in x}\xi_i^\mu\sigma_i \tag{1.12}$$

where $\gamma\sum_{i\in x}$ can be interpreted as a Riemann sum; the same occurs in all other expressions.

The rôle of the different scales will be the following. We will be interested in the typical profiles of the overlaps on the scale L, i.e. the typical $m_L(r,\sigma)$ as a function of r; we will control these functions within volumes on the macroscopic scale γ^{-1}. The smaller mesoscopic scale ℓ enters only in an auxiliary way. Namely, we will use a block-spin approximation of the Hamiltonian with blocks of that size. We will see that it is quite crucial to use a much smaller scale for that approximation than the scale on which we want to control the local overlaps. This was noted already in [COP].

We want to study the probability distribution induced by the Gibbs measure on the functions $m_L(r)$ through the map defined by (1.10). The corresponding measure space is for fixed γ simply the discrete space $\{-1,-1+2/L,\dots,1-2/L,1\}^{M(\gamma)\times\mathbb{Z}}$, which should be equipped with the product topology. Since this topology is quite non-uniform with respect to γ (note that both L and M tend to infinity as $\gamma\downarrow 0$), this is, however, not well adapted to take the limit $\gamma\downarrow 0$. Thus we replace the discrete topology on $\{-1,-1+2/L,\dots,1-2/L,1\}^{M(\gamma)}$ by the Euclidean ℓ_2-topology (which remains meaningful in the limit) and the product topology corresponding to $\mathbb{Z}$ is replaced by the weak local L_2 topology w.r.t. the measure $\gamma L\sum_{r\in\cdot}$; that is to say, a family of profiles $m_L^n(r)$ converges to the profile $m_L(r)$, *iff* for all finite $R\in\mathbb{R}$, $\gamma L\sum_{r\in[-R,R]}\|m_L^n(r)-m_L(r)\|_2\downarrow 0$ as $n\uparrow\infty$. While for all finite γ this topology is completely equivalent to the product topology of the discrete topology, the point here is that it is meaningful to ask for *uniform* convergence with respect to the parameter γ. We will denote this space by $\mathcal{T}_\gamma$, or simply $\mathcal{T}$ and call it the space of profiles (on scale L).

Before presenting our results, it may be useful to discuss in a somewhat informal way the heuristic expectations based on the the work of [COP] and the results known from [BGP1, BG-P3, BG4]. In [COP] it was shown that the typical magnetization profiles are such that almost everywhere, $m_L(r,\sigma)$ is very close to one of the two equilibrium values of the mean field model, $\pm a(\beta)$; moreover, the profile is essentially constant over macroscopic distances of the order $e^{\gamma^{-1}}$. The distances between jumps are actually independent exponentially distributed random variables. Heuristically, this picture is not too difficult to understand. First, one approximates the Hamiltonian by a block-spin version by replacing the interaction potential by a function that is constant over blocks of length L. Ignoring the error term, the resulting model depends on σ only through the variables $m_L(r,\sigma)$. In fact, at each block r there is a little mean-field model and these mean field models interact through a ferromagnetic interaction of the form $J_{\gamma L}(r-s)(m_L(r)-m_L(s))^2$. This interaction can only bias a given block to choose between the two possible equilibrium values, but never prevent it from taking on an equilibrium value over a longer interval. Moreover, it tends to align the blocks. To jump from one equilibrium into the other costs in fact an energy of the order of γ^{-1}, so that the probability that this happens in a given unit interval is of the order $e^{-\gamma^{-1}}$. This explains why the entropy can force this to happen only on distances of the order of the inverse of this value. Finally, the Markovian character of a one-dimensional model leaves only a Poisson-distribution as a candidate for the distribution of the jumps. The main difficulty in turning these arguments into rigorous proofs lies in the control of the error terms.

It is crucial for the above picture that there is a complete symmetry between the two equilibrium states of the mean field model. As we have shown in [BGP2], the Kac-Hopfield model can be approximated by a blocked model just the same, and in [BGP1] we have shown that the mean field Hopfield model has its equilibrium states sharply concentrated at the $2M$ points $\pm a(\beta)e^\mu$, where e^μ is the μ-th standard unit vector. Thus we can again expect the overlap profiles to be over long distances constant close to one of these values. What is different here, however, is that due to the disorder the different equilibrium positions are not entirely equivalent. We have shown in [BGP3] that the fluctuations are only of the order of the square root of the volume, but since they are independent from block to block, they can add up over a long distance and effectively enforce jumps to different equilibrium positions at distances that are much shorter than those between entropic jumps. In fact, within the blocked approximation, it is not hard to estimate that the typical distance over which the profiles remain constant should be of the order γ^{-1} on the macroscopic scale (i.e. γ^{-2} on the microscopic scale). Using a concentration of measure estimates in a form developed by M. Talagrand [T2], we extent these estimates to the full model. Our main results on the typical profiles can then be summarized (in a slightly informal way) as follows:

Assume that $\lim_{\gamma\downarrow 0}\gamma M(\gamma)=0$. *Then there is a scale* $L\ll\gamma^{-1}$ *such that with* $I\!P$*-probability tending to one (as* $\gamma\downarrow 0$*) the following holds:*

(i) In any given macroscopic finite volume in any configuration that is "typical" with respect to the infinite volume Gibbs measure, for "most" blocks r, $m_L(r,\sigma)$ *is very close to one of the*

values $\pm a(\beta)e^{\mu}$ (we will say that $m_L(u,\sigma)$ is "close to equilibrium").

(ii) In any macroscopic volume Δ that is small compared to γ^{-1}, in a typical configuration, there is at most one connected subset J (called a "jump") with $|J| \sim \frac{1}{\gamma L}$ on which m_L is not close to equilibrium. Moreover, if such a jump occurs, then there exist (s_1,μ_1) and (s_2,μ_2), such that for all $u \in \Delta$ to the left of J, $m_L(u,\sigma) \sim s_1 a(\beta)e^{\mu_1}$ and for all $u \in \Delta$ to the right of J, $m_L(u,\sigma) \sim s_2 a(\beta)e^{\mu_2}$

The precise statement of these facts will require more notation and is thus postponed to Section 6 where it will be stated as Theorem 6.15. That section contains also the large deviation estimates that are behind these results. We should mention that we have no result that would prove the existence of a "jump" in a sufficiently large region. We discuss this problem in Section 7 in some more detail.

We also remark that the condition $\lim_{\gamma\downarrow 0} \gamma M(\gamma) = 0$ will be imposed thoughout the paper. It could be replaced with $\limsup_{\gamma\downarrow 0} \gamma M(\gamma) \leq \alpha_c(\beta)$ for some strictly positive $\alpha_c(\beta)$ for all $\beta > 1$. However, an actual estimate of this constant would be outrageously tedious and does not really appear, in our view, to be worth the trouble.

The remainder of the paper is organized in the following way. The next two sections provide some technical tools that will be needed throughout. Section 2 introduces the mesoscopic approximation of the Hamilitonian and corresponding error estimates. Section 3 contains large deviation estimates for the standard Hopfield model that are needed to analyse the mesoscopic approximation introduced before. Here we make use of some fundamental results from [BGP2] and [BG3] but present them in a somewhat different form. In Section 4 we begin the actual analysis of typical profiles. Here we show that for events that are local, we can express their probabilities in terms of a finite volume measure with random boundary conditions (see Corollary 4.2). In Section 5 we derive estimates on the random fluctuations of the free energies corresponding to these measures. In Section 6 we make use of these estimates to show that local events can be analysed using the mesoscopic approximation introduced in Section 2. This section is divided into three parts. Section 6.1 contains an analysis of measures with free boundary condition in macroscopic volumes of order $o\left(\gamma^{-1}\right)$. It is shown that they are asymptotically concentrated on constant profiles (see Theorem 6.1). This result is already quite instructive, and technically rather easy. In Sections 6.2 and 6.3 the measures with non-zero boundary conditions are studied. In Section 6.2 the case where the boundary conditions are the same on both sides of the box. It is shown that here, too, the profiles are typically constant and take the value favored by the boundary conditions (see Theorem 6.9). In Section 6.3 the case with different boundary conditions is treated. Here we show that the typical profile has exactly one "jump" and is constant otherwise (see Theorem 6.14). The results of Sections 4 and 6 are then combined to yield Theorem 6.15 which gives a precise statement the result announced above. In Section 7 we discuss some of the open points of our analysis. In particular we argue, that typical profiles are non-constant on a sufficiently large scale and that their precise form is entirely disorder determined (up to the global sign). We also formulate some conjectures for the model in dimensions greater than one. In Appendix A we give a proof of a technical estimate on the minimal energy associated to profiles that contain "jumps" between different equilibrium positions that is needed in Section 6.

2. Block-spin approximations

While mean-field models are characterized by the fact that the Hamiltonian is a function of global averages of the spin variables, in Kac-models the Hamiltonian is "close", but not identical to a function of "local" averages. In this section we make this statement precise by introducing the block version of the Hamiltonian and deriving the necessary estimates on the error terms. We define

$$H_{\gamma,\Lambda}(\sigma_\Lambda) = \gamma^{-1} E^{\ell}_{\gamma,\Lambda}(m_\ell(\sigma)) + \Delta H^{\ell}_{\gamma,\Lambda}(\sigma_\Lambda) \tag{2.1}$$

and

$$W_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\Lambda^c}) = \gamma^{-1} E^{\ell,L}_{\gamma,\Lambda}(m_\ell(\sigma), m_L(\sigma)) + \Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\Lambda^c}) \tag{2.2}$$

where

$$E^{\ell}_{\gamma,\Lambda}(m) \equiv -\frac{1}{2}\gamma\ell \sum_{(x,y)\in\Lambda\times\Lambda} J_{\gamma\ell}(x-y)(m(x), m(y)) \tag{2.3}$$

and

$$E^{\ell,L}_{\gamma,\Lambda}(m, \tilde m) \equiv -\gamma\ell L \sum_{x\in\Lambda}\sum_{r\in\Lambda^c} J_\gamma(\ell x - Lr)(m(x), \tilde m(r)) \tag{2.4}$$

For our purposes, we only need to consider volumes Λ of the form $\Lambda = [\lambda^-, \lambda^+]$ with $|\Lambda| > 1$. For such volumes we set $\partial\Lambda \equiv \partial^-\Lambda \cup \partial^+\Lambda$, $\partial^-\Lambda \equiv [\lambda^- - \frac{1}{2}, \lambda^-)$, and $\partial^+\Lambda \equiv (\lambda^+, \lambda^+ + \frac{1}{2}]$. Thus, obviously, $W_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\Lambda^c}) = W_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial\Lambda})$ and $\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\Lambda^c}) = \Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial\Lambda})$.

Lemma 2.1: *For all $\delta > 0$*

i)

$$I\!P\left[\sup_{\sigma\in\mathcal{S}_\Lambda} \frac{\gamma}{|\Lambda|}|\Delta H_\Lambda(\sigma)| \geq \gamma\ell(\gamma)8\sqrt{2}(\log 2 + \delta) + 2\sqrt{2}\gamma M(\gamma)\right] \leq 16 e^{-\delta\frac{|\Lambda|}{\gamma}} \tag{2.5}$$

ii)

$$I\!P\left[\sup_{\sigma\in\mathcal{S}_{\Lambda\cup\partial\Lambda}} \gamma|\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial\Lambda})| > (4\gamma L(\gamma)(\log 2 + \delta) + \gamma M(\gamma))\left(1 + \frac{\ell}{L}\right)^{\frac{1}{2}}\right] \leq 8 e^{-\frac{\delta}{\gamma}} \tag{2.6}$$

Proof: We will give the proof of (ii) only; the proof of (i) is similar and can be found in [BGP2]. Since $|\Lambda| > 1$, the spins inside $\partial^-\Lambda$ do not interact with those inside $\partial^+\Lambda$ and $\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial\Lambda})$ can be written as

$$\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial\Lambda}) = \Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial^-\Lambda}) + \Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial^+\Lambda}) \tag{2.7}$$

where

$$\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda, \sigma_{\partial^\pm\Lambda}) = -\sum_{x\in\Lambda}\sum_{r\in\partial^\pm\Lambda}\sum_{i\in x}\sum_{j\in r}[J_\gamma(i-j) - J_\gamma(\ell x - Lr)](\xi_i, \xi_j)\sigma_i\sigma_j \tag{2.8}$$

Both terms (2.7) being treated similarly, we will only consider $\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})$. First notice that since

$$J_\gamma(i-j)-J_\gamma(\ell x-Lr)=\gamma\left[1\!\!\mathrm{I}_{\{|i-j|\le(2\gamma)^{-1}\}}1\!\!\mathrm{I}_{\{|\ell x-Lr|>(2\gamma)^{-1}\}}-1\!\!\mathrm{I}_{\{|i-j|>(2\gamma)^{-1}\}}1\!\!\mathrm{I}_{\{|\ell x-Lr|\le(2\gamma)^{-1}\}}\right] \tag{2.9}$$

we can write $\Delta W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})=\gamma\left[\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})-\Delta^2 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})\right]$ with

$$\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})=-\sum_{x\in\Lambda}\sum_{r\in\partial^+\Lambda}\sum_{i\in x}\sum_{j\in r}1\!\!\mathrm{I}_{\{|i-j|\le(2\gamma)^{-1}\}}1\!\!\mathrm{I}_{\{|\ell x-Lr|>(2\gamma)^{-1}\}}(\xi_i,\xi_j)\sigma_i\sigma_j \tag{2.10}$$

and

$$\Delta^2 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})=-\sum_{x\in\Lambda}\sum_{r\in\partial^+\Lambda}\sum_{i\in x}\sum_{j\in r}1\!\!\mathrm{I}_{\{|i-j|>(2\gamma)^{-1}\}}1\!\!\mathrm{I}_{\{|\ell x-Lr|\le(2\gamma)^{-1}\}}(\xi_i,\xi_j)\sigma_i\sigma_j \tag{2.11}$$

Again, both terms $\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})$ and $\Delta^2 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})$ can be treated in the same way so that we only present an estimate of the former. Using the identity

$$1\!\!\mathrm{I}_{\{|i-j|\le(2\gamma)^{-1}\}}1\!\!\mathrm{I}_{\{|\ell x-Lr|>(2\gamma)^{-1}\}}=1\!\!\mathrm{I}_{\{|i-j|\le(2\gamma)^{-1}\}}1\!\!\mathrm{I}_{\{(2\gamma)^{-1}<|\ell x-Lr|\le(2\gamma)^{-1}+(\ell+L)/2\}} \tag{2.12}$$

and setting

$$g^\mu_{\gamma,\Lambda}(r)=\sum_{\substack{x\in\Lambda:\\(2\gamma)^{-1}<|\ell x-Lr|\le(2\gamma)^{-1}+(\ell+L)/2}}\sum_{i\in x}\sum_{j\in r}1\!\!\mathrm{I}_{\{|i-j|\le(2\gamma)^{-1}\}}\xi^\mu_i\xi^\mu_j\sigma_i\sigma_j \tag{2.13}$$

we have

$$\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})=-\sum_{\mu=1}^{M}\sum_{r\in\partial^+\Lambda}g^\mu_{\gamma,\Lambda}(r) \tag{2.14}$$

Note that the right hand side of (2.14) is a sum of independent random variables since for any two distinct r_1, r_2 in $\partial^+\Lambda$, the sets $\{x\in\Lambda:(2\gamma)^{-1}<|\ell x-Lr_1|\le(2\gamma)^{-1}+(\ell+L)/2\}$ and $\{x\in\Lambda:(2\gamma)^{-1}<|\ell x-Lr_2|\le(2\gamma)^{-1}+(\ell+L)/2\}$ are disjoint. Therefore,

$$I\!\!P\left[\sup_{\sigma\in\mathcal{S}_{\Lambda\cup\partial^+\Lambda}}\gamma^2|\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+})|>\frac{\epsilon}{4}\right]\le 2^{(\gamma^{-1}+1)}I\!\!P\left[\sum_{\mu=1}^{M}\sum_{r\in\partial^+\Lambda}g^\mu_{\gamma,\Lambda}(r)>\gamma^{-2}\frac{\epsilon}{4}\right] \tag{2.15}$$

where the probability in the right hand side is independent of the chosen spin configuration $\sigma_{\Lambda\cup\partial^+\Lambda}$. For convenience we will choose the configuration whose spins are all one's. Using the exponential Markov inequality together with the independence, we get

$$I\!\!P\left[\sup_{\sigma\in\mathcal{S}_{\Lambda\cup\partial^+\Lambda}}\gamma^2|\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})|>\frac{\epsilon}{4}\right]\le 2^{(\gamma^{-1}+1)}\inf_{t\ge0}e^{-t\gamma^{-2}\frac{\epsilon}{4}}\left[\prod_{r\in\partial^+\Lambda}I\!\!E e^{tg^1_{\gamma,\Lambda}(r)}\right]^M \tag{2.16}$$

Thus we have to estimate the Laplace-transform of $g^1_{\gamma,\Lambda}(r)$ for any $r \in \partial^+\Lambda$. We write

$$
\mathbb{E} e^{t g^1_{\gamma,\Lambda}(r)} = \mathbb{E} \exp\left\{ t \sum_{j\in r} \xi^1_j \sum_{\substack{x\in\Lambda: \\ (2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell+L)/2}} \sum_{i\in x} \mathbb{1}_{\{|i-j|\le(2\gamma)^{-1}\}} \xi^1_i \right\} \tag{2.17}
$$

Note that all the ξ^1_j with $j \in r$ are independent of the ξ^1_i with $i \in x$ for x satisfying $(2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell + L)/2$, and that, conditioned on these latter variables, the variables $\xi^1_j \sum_{x\in\Lambda} \mathbb{1}_{\{(2\gamma)^{-1}<|\ell x - Lr|\le(2\gamma)^{-1}+(\ell+L)/2\}} \mathbb{1}_{\{|i-j|\le(2\gamma)^{-1}\}} \xi^1_i$ are independent. If we denote by $\mathbb{E}_j$ the expectation w.r.t. ξ^1_j, this allows us to write

$$
\begin{aligned}
\mathbb{E} e^{t g^1_{\gamma,\Lambda}(r)} &= \mathbb{E} \prod_{j\in r} \mathbb{E}_j \exp\left\{ t \xi^1_j \sum_{\substack{x\in\Lambda: \\ (2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell+L)/2}} \sum_{i\in x} \mathbb{1}_{\{|i-j|\le(2\gamma)^{-1}\}} \xi^1_i \right\} \\
&\le \mathbb{E} \prod_{j\in r} \exp\left\{ \frac{t^2}{2} \left(\sum_{\substack{x\in\Lambda: \\ (2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell+L)/2}} \sum_{i\in x} \mathbb{1}_{\{|i-j|\le(2\gamma)^{-1}\}} \xi^1_i \right)^2 \right\}
\end{aligned} \tag{2.18}
$$

where we have used that $\ln\cosh x \le \frac{1}{2}x^2$. Using the Hölder-inequality on the last line, we arrive at

$$
\mathbb{E} e^{t g^1_{\gamma,\Lambda}(r)} \le \prod_{j\in r} \left[\mathbb{E} \exp\left\{ \frac{Lt^2}{2} \left(\sum_{\substack{x\in\Lambda: \\ (2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell+L)/2}} \sum_{i\in x} \mathbb{1}_{\{|i-j|\le(2\gamma)^{-1}\}} \xi^1_i \right)^2 \right\} \right]^{\frac{1}{L}} \tag{2.19}
$$

Now

$$
\begin{aligned}
&\mathbb{E} \exp\left\{ \frac{Lt^2}{2} \left(\sum_{\substack{x\in\Lambda: \\ (2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell+L)/2}} \sum_{i\in x} \mathbb{1}_{\{|i-j|\le(2\gamma)^{-1}\}} \xi^1_i \right)^2 \right\} \\
\le &\mathbb{E} \exp\left\{ \frac{Lt^2}{2} \left(\sum_{\substack{x\in\Lambda: \\ (2\gamma)^{-1} < |\ell x - Lr| \le (2\gamma)^{-1} + (\ell+L)/2}} \sum_{i\in x} \xi^1_i \right)^2 \right\} \\
\le &\frac{1}{\sqrt{1 - t^2 L(L+\ell)/2}}
\end{aligned} \tag{2.20}
$$

where we have used the Khintchine inequality and the fact that, for all $r \in \partial^+\Lambda$,

$$
\sum_{x\in\Lambda} \sum_{i\in x} \mathbb{1}_{\{(2\gamma)^{-1}<|\ell x - Lr|\le(2\gamma)^{-1}+(\ell+L)/2\}} \le \frac{L+\ell}{2} \tag{2.21}
$$

Since for $0 \le x \le 1/2$, $1/\sqrt{1-x} \le e^x$, for $t^2 \le \frac{1}{\ell(L+\ell)}$, we finally get, collecting (2.18)-(2.20),

$$
\mathbb{E} e^{t g^1_{\gamma,\Lambda}(r)} \le e^{t^2 \frac{L(L+\ell)}{2}} \tag{2.22}
$$

Therefore, since $\sharp\{r \in \partial^+\Lambda\} \leq (2\gamma L)^{-1}$, choosing $t = \frac{1}{\sqrt{L(L+\ell)}}$ in (2.22) yields

$$I\!P\left[\sup_{\sigma\in\mathcal{S}_{\Lambda\cup\partial^+\Lambda}} \gamma^2|\Delta^1 W^{\ell,L}_{\gamma,\Lambda}(\sigma_\Lambda,\sigma_{\partial^+\Lambda})| > \frac{\epsilon}{4}\right] \leq 2^{2/\gamma+1}\exp\left\{-\frac{1}{\gamma}\left[\frac{\epsilon}{4\gamma\sqrt{L(L+\ell)}}\right]\right\}\exp\left\{\frac{M}{4\gamma L}\right\} \tag{2.23}$$

Choosing ϵ in 2.6 as $\epsilon = 4\gamma\sqrt{L(L+\ell)}\left(\log 2 + \frac{M(\gamma)}{4\ell(\gamma)} + \delta\right)$ for some $\delta > 0$, gives (2.6). $\diamondsuit$

3. Some large deviation estimates for the Hopfield model

In the preceeding chapter we have introduced the block-approximation for the Hamiltonian of the Kac-Hopfield model. To make use of these, we need some large deviation results for the standard Hopfield model. They are essentially contained in [BGP1] and [BGP2], but we present them here in a slightly different way that focuses on our actual needs. We set $\frac{M}{N} \equiv \alpha$ throughout this section.

Recall that we have to consider the quantities

$$Z_{N,\beta,\rho}(m) \equiv 2^{-N} \sum_{\sigma\in\mathcal{S}_N} e^{\frac{\beta N}{2}\|m_N[\omega](\sigma)\|_2^2} \mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} \tag{3.1}$$

We set $f_{N,\beta,\rho}(m) \equiv -\frac{1}{\beta N}\ln Z_{N,\beta,\rho}(m)$. In this paper we are mostly interested in the localization of the minima of the functions $f_{N,\beta,\rho}(m)$. Thus we will only need the following estimates:

Lemma 3.1: *Define the random function*

$$\Phi_{N,\beta}(m) \equiv \frac{1}{2}\|m\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N} \ln\cosh\left(\beta(\xi_i, m)\right) \tag{3.2}$$

Then

$$f_{N,\beta,\rho}(m) \geq \Phi_{N,\beta}(m) - \tfrac{1}{2}\rho^2 \tag{3.3}$$

and for $\rho \geq \sqrt{2\alpha}$, if m^* is a critical point of $\Phi_{N,\beta}(m)$,

$$f_{N,\beta,\rho}(m^*) \leq \Phi_{N,\beta}(m^*) + \tfrac{\ln 2}{\beta N} \tag{3.4}$$

Proof: To prove Lemma 3.1, we define probability measures $\tilde{I\!P}$ on $\{-1,1\}^N$ through their expectation $\tilde{I\!E}_\sigma$, given by

$$\tilde{I\!E}_\sigma(\cdot) \equiv \frac{I\!E_\sigma e^{\beta N(m,m_N(\sigma))}(\cdot)}{I\!E_\sigma e^{\beta N(m,m_N(\sigma))}} \tag{3.5}$$

We have obviously that

$$\begin{aligned} Z_{N,\beta,\rho}(m) &= \tilde{I\!E}_\sigma e^{\frac{\beta N}{2}\|m_N(\sigma)\|_2^2-\beta N(m,m_N(\sigma))} \mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} I\!E_\sigma e^{\beta N(m,m_N(\sigma))} \\ &= e^{-\frac{\beta N}{2}\|m\|_2^2} \tilde{I\!E}_\sigma e^{\frac{\beta N}{2}\|m_N(\sigma)-m\|_2^2} \mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} I\!E_\sigma e^{\beta N(m,m_N(\sigma))} \\ &= e^{\beta N\left(-\frac{1}{2}\|m\|_2^2+\frac{1}{\beta N}\sum_{i=1}^N \ln\cosh\beta(\xi_i,m)\right)} \tilde{I\!E}_\sigma e^{\frac{\beta N}{2}\|m_N(\sigma)-m\|_2^2} \mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} \end{aligned} \tag{3.6}$$

But

$$\mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} \leq e^{\frac{\beta N}{2}\|m_N(\sigma)-m\|_2^2} \mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} \leq e^{\frac{\beta N}{2}\rho^2} \mathbb{I}_{\{\|m_N(\sigma)-m\|_2\leq\rho\}} \tag{3.7}$$

so that we get on the one hand

$$Z_{N,\beta\rho}(m) \leq e^{-\beta N\left[\Phi_{N,\beta}(m)-\frac{1}{2}\rho^2\right]} \tag{3.8}$$

which yields (3.3), and on the other hand

$$Z_{N,\beta\rho}(m) \geq e^{-\beta N\Phi_{N,\beta}(m)}\tilde{I\!P}\left[\|m_N(\sigma)-m\|_2 \leq \rho\right] \tag{3.9}$$

But, using Chebychev's inequality, we have that

$$\tilde{I\!P}\left[\|m_N(\sigma)-m\|_2 \leq \rho\right] \geq 1-\frac{1}{\rho^2}\tilde{I\!E}_\sigma\|m_N(\sigma)-m\|_2^2 \tag{3.10}$$

and

$$\begin{aligned}
&\tilde{I\!E}\|m_N(\sigma)-m\|_2^2 = \\
&\frac{I\!E_\sigma \prod_{i=1}^N e^{\beta(m,\xi_i\sigma_i)}\sum_\nu\left(N^{-2}\sum_{j,k}\xi_j^\nu\xi_k^\nu\sigma_j\sigma_k - 2m^\nu N^{-1}\sum_j \mu_j^\nu\sigma_j + (m^\nu)^2\right)}{\prod_{i=1}^N\cosh\beta(\xi_i,m)} \\
&=\frac{1}{N^2}\sum_\nu\sum_j 1 + \frac{1}{N^2}\sum_\nu\sum_{j\neq k}\tanh(\beta(m,\xi_j))\tanh(\beta(m,\xi_k))\xi_j^\nu\xi_k^\nu \\
&-\frac{2}{N}\sum_j\sum_\nu m^\nu\tanh(\beta(m,\xi_j))\xi_j^\nu + \sum_\nu (m^\nu)^2 \\
&=\frac{M}{N}-\sum_\nu\frac{1}{N}\sum_i\tanh^2(\beta(m,\xi_i)) + \sum_\nu\left(\frac{1}{N}\sum_i\xi_i^\nu\tanh(\beta(m,\xi_i))-m^\nu\right)^2
\end{aligned} \tag{3.11}$$

IF m^* is a critical point of Φ,

$$m^* = \frac{1}{N}\sum_i \xi_i\tanh(\beta(m^*,\xi_i)) \tag{3.12}$$

and so the last terms in (3.11) vanish and we remain with

$$\tilde{I\!E}\|m_N(\sigma)-m\|_2^2 \leq \frac{M}{N}\left(1-\frac{1}{N}\sum_i\tanh^2(\beta(\xi_i,m))\right) \leq \alpha \tag{3.13}$$

from which (3.4) follows immediately.$\diamondsuit$

Given the upper and lower bounds in terms of Φ, it remains to show that this function takes its absolute minima near the points $m^{(\mu,s)} \equiv sa(\beta)e^\mu$ only. This was done in [BGP1] and, with sharper estimates in [BG3]. We recall this result in a form suitable for our purposes. We denote by $a(\beta)$ the positive solution of the equation $a=\tanh(\beta a)$.

Proposition 3.2: *Assume that $\sqrt{\alpha}/a(\beta)^2$ is sufficiently small. Then there exists a set $\Omega_4(N)\subset\Omega$ with $I\!P(\Omega_4(N))\geq 1-e^{-cM}$ such that for all $\omega\in\Omega_4$, for all $m\in I\!R^M$*

$$\Phi_{N,\beta}[\omega](m)-\Phi_{N,\beta}[\omega](m^{(\mu,s)}) \geq \epsilon(m) \tag{3.14}$$

where ϵ is a non random function that satisfies

$$\epsilon(m) = \begin{cases} 0, & \textit{if } \inf_{\mu,s} \|m - m^{(\mu,s)}\|_2 \le c_1\sqrt{\alpha}/a(\beta) \\ ca(\beta)^2 \inf_{\mu,s} \|m - m^{(\mu,s)}\|_2^2, & \textit{if } c_1\sqrt{\alpha}/a(\beta) \le \inf_{\mu,s} \|m - m^{(\mu,s)}\|_2 \le c_2 a(\beta) \\ cc_2 a(\beta)^4, & \textit{if } \inf_{\mu,s} \|m - m^{(\mu,s)}\|_2 \ge c_2 a(\beta) \end{cases} \tag{3.15}$$

where c, c_1, c_2 are finite positive constants.

Proof: By some trivial changes of notations this follows from the estimates in Section 3 of [BG3], in particular Theorem 3.1 and Lemma 3.9. $\diamondsuit$

4. Local effective measures

In Section 2 we have seen that the Kac-Hopfield Hamiltonian can be approximated by a block-spin Hamiltonian up to errors that are essentially proportional to $\gamma\ell$ times the volume. This means of course that we cannot use this approximation throughout the entire volume Λ if we are interested in controlling local observables, as the errors would grow without bounds in the thermodynamic limit. A clever way to solve this difficulty was given in [COP] for the ferromagnetic Kac-model. The crucial point is that if one is interested in local observables in a box V, it is possible to show that with large probability (w.r.t. the Gibbs measure) not far away from this volume, there are intervals of macroscopic length 1 where the mesoscopic magnetization profiles are very close to one of the "equilibrium" values of the mean-field model. This knowledge allows to effectively decouple the system inside and outside this region, with the outside acting only as a "boundary condition". Due to the randomness of the interaction, an additional difficulty presents itself in terms of the randomness of the effective boundary conditions. This makes it necessary to perform this analysis on two separate length scales: in this section we consider a rather large volume (which we will see later can be chosen of order $o(\gamma^{-1})$ (on the macroscopic scale); in Section 6 these measures will be further analyzed by localizing them to much smaller boxes.

To begin, we imitate [COP] by defining variables η that serve as a decomposition of the configuration space through

$$\eta(u,\sigma) \equiv \eta_{\zeta,L}(u,\sigma) = \begin{cases} se^{\mu} & \text{if } \forall_{r\in u} \|m^{(\mu,s)} - m_L(r,\sigma)\|_2 \leq \zeta \\ 0 & \text{if } \forall_{\mu,s} \exists_{r\in u} : \|m^{(\mu,s)} - m_L(r,\sigma)\|_2 > \zeta \end{cases} \tag{4.1}$$

(This definition is unequivocal if ζ is chosen small enough i.e. $\zeta < \sqrt{2}a(\beta)$). For a given configuration σ, η determines whether a unit interval is close to equilibrium on the scale L. For a given volume $V \equiv [v_-, v_+] \subset \Lambda$, with $|V| > 1$, we set

$$\tau^+ = \begin{cases} \inf\{u \geq v_+ : \eta(u,\sigma) \neq 0\} \\ \infty \ \text{ if no such } u \ \text{ exists} \end{cases} \tag{4.2}$$

and

$$\tau^- = \begin{cases} \sup\{u \leq v_- : \eta(u,\sigma) \neq 0\} \\ -\infty \ \text{ if no such } u \ \text{ exists} \end{cases} \tag{4.3}$$

for a given configuration σ, $\tau^{\pm}$ indicates the position of the first unit interval to the right, respec. the left, of V where the configurations σ is close to equilibrium.

Let us introduce the indices $\mu^+, \mu^-, s^+, s^-, w_+, w_-$ where $\mu^{\pm} \in \{1, \ldots, M(\gamma)\}$, $s^{\pm} \in \{-1, 1\}$ and $w_+ \in [v_+, \infty)$, $w_- \in (-\infty, v_-]$. In the sequel, if not otherwise specified, all sums and unions over these indices run over the above sets. With these notations we define a partition of the configuration space $\mathcal{S}$ whose atoms are given by

$$\mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm}) \equiv \left\{\sigma \in \mathcal{S} : \tau^{\pm} = w_{\pm}, \eta(\tau^{\pm}, \sigma) = s^{\pm}e^{\mu^{\pm}}\right\} \tag{4.4}$$

and we denote by

$$\mathcal{S}_R = \bigcup_{\substack{\mu^\pm, s^\pm, w_\pm \\ 0\le \pm(w_\pm - v_\pm)\le R}} \mathcal{A}(\mu^\pm, s^\pm, w_\pm) \tag{4.5}$$

Notice that

$$\mathcal{S}_R^c = A^+(R) \cup A^-(R) \tag{4.6}$$

where

$$A^+(R) \equiv \left\{\sigma \in \mathcal{S} : \tau^+ > v_+ + R\right\} = \left\{\sigma \in \mathcal{S} : \forall_{v_+ \le w \le v_+ + R}\ \eta(w,\sigma) = 0\right\} \tag{4.7}$$

and

$$A^-(R) \equiv \left\{\sigma \in \mathcal{S} : \tau^- < v_- - R\right\} = \left\{\sigma \in \mathcal{S} : \forall_{v_- - R \le w \le v_-}\ \eta(w,\sigma) = 0\right\} \tag{4.8}$$

Before stating the main results of this chapter we need some further notations. For given indices $\mu^\pm, s^\pm, w_\pm$ we write $\Delta \equiv [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$ and we set

$$\widehat{\mathcal{A}}(\mu^\pm, s^\pm, w_\pm) \equiv \left\{\sigma \in \mathcal{S} : \eta(w_\pm, \sigma) = s^\pm e^{\mu^\pm}\right\} \tag{4.9}$$

We define the Gibbs measure on Δ with *mesoscopic boundary conditions* $m^{(\mu^\pm, s^\pm)}$ as the measure that assigns, to each $\sigma_\Delta \in \mathcal{S}_\Delta$, the mass,

$$\mathcal{G}^{\mu^\pm, s^\pm}_{\beta,\gamma,\Delta}[\omega](\sigma_\Delta) = \frac{1}{Z^{\mu^\pm, s^\pm}_{\beta,\gamma,\Delta}[\omega]} e^{-\beta\left\{H_{\gamma,\Delta}[\omega](\sigma_\Delta) + W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm, s^\pm)})\right\}} \tag{4.10}$$

where $Z^{\mu^\pm, s^\pm}_{\beta,\gamma,\Delta}[\omega]$ is the corresponding normalization factor and

$$W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm, s^\pm)}) \equiv -\sum_{i\in\Delta} s^- a(\beta)\xi_i^{\mu^-}\sigma_i \sum_{j\in\partial^-\Delta} J_\gamma(i-j) - \sum_{i\in\Delta} s^+ a(\beta)\xi_i^{\mu^+}\sigma_i \sum_{j\in\partial^+\Delta} J_\gamma(i-j) \tag{4.11}$$

Proposition 4.1: *Let F be a cylinder event with base contained in $[v_-, v_+]$. Then*

i) There exists a positive constant c such that, for all integer R, there exists Ω_R with $I\!P(\Omega_R) \ge 1 - Re^{-c\gamma^{-1}}$ such that for all $\mu^\pm, s^\pm, w_\pm, v_+ \le w_+ \le v_+ + R, v_- - R \le w \le v_-$ and $\omega \in \Omega_R$ for all $\Lambda \supset [v_- - R, v_+ + R]$

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\Big(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)\Big) \le \mathcal{G}^{\mu^\pm, s^\pm}_{\beta,\gamma,\Delta}[\omega]\,(F)\,\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, w_\pm)\right) e^{8\beta\gamma^{-1}(\zeta + 2\gamma L)} \tag{4.12}$$

and for any $u_+ \ge v_+$, $u_- \le v_-$,

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\Big(F \cap \widehat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\Big) \ge \mathcal{G}^{\mu^\pm, s^\pm}_{\beta,\gamma,[u_-,u_+]}[\omega]\,(F)\,\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right) e^{-8\beta\gamma^{-1}(\zeta + 2\gamma L)} \tag{4.13}$$

ii) There exist a positive constant c' such that for all integer R, there exists Ω_R with $I\!P(\Omega_R) \geq 1 - \gamma^{-1} R e^{-c'M}$ and there exist finite positive constants c_1 and c_2 such that if $\zeta\epsilon(\zeta)\gamma L > 2c_1\sqrt{\frac{M}{\ell}}$, then for all $\omega \in \Omega_R$ and $\Lambda \supset [v_- - R, v_+ + R]$

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F \cap S_R^c) \leq \exp\left(-\beta L R c_2 \zeta \epsilon(\zeta)\right) \tag{4.14}$$

Corollary 4.2: *Let F be a cylinder event with base contained in $[v_-, v_+]$. Then there exist a positive constant c' such that for all integer R, there exists Ω_R with $I\!P(\Omega_R) \geq 1 - \gamma^{-1} R e^{-c'M}$ and there exist finite positive constants c_1 and c_2 such that if $\zeta\epsilon(\zeta)\gamma L > 2c_1\sqrt{\frac{M}{\ell}}$, then for all $\omega \in \Omega_R$ and $\Lambda \supset [v_- - R, v_+ + R]$*

$$\begin{aligned}\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F) \leq & \sum_{\substack{\mu^\pm, s^\pm \\ -R < w_- \leq v_- \\ v_+ \leq w_+ < R}} \mathcal{G}^{\mu^\pm, s^\pm}_{\beta,\gamma,\Delta}[\omega]\,(F)\, \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, w_\pm)\right) e^{8\beta\gamma^{-1}(\zeta + 2\gamma L)} \\ & + \exp\left(-\beta L R c_2 \zeta \epsilon(\zeta)\right)\end{aligned} \tag{4.15}$$

and there exist $u_\pm$ with $\pm(u_\pm - v_\pm) \leq R$ such that for all $\Lambda \supset [v_- - R, v_+ + R]$

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F) \geq \sum_{\mu^\pm, s^\pm} \mathcal{G}^{\mu^\pm, s^\pm}_{\beta,\gamma,[u_-,u_+]}[\omega]\,(F)\, \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right) e^{-8\beta\gamma^{-1}(\zeta + 2\gamma L)} \tag{4.16}$$

and there exists $(\mu^\pm, s^\pm)$ such that

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right) \geq \frac{1}{8R^2M^2} \tag{4.17}$$

Remark: Corollary 4.2 tells us that in order to estimate the probability of some local event in V with respect to the infinite volume Gibbs measure we only need to control finite volume Gibbs measures in volumes $|\Delta|$ with all possible boundary conditions corresponding to one of the mean field equilibrium states. This analysis will be performed in Section 6. On the other hand, it appears quite hopeless to get a more precise information than (4.27) on the terms $\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right)$ appearing in both bounds. This is, after some thought, not surprising, but reflects the fact that the exact shape of typical profiles depends strongly on the disorder and only some of their properties on relatively short scales can be effectively controlled. In particular, it is clear that we cannot hope to get something like a full large deviation principle (in the sense of the results of [COP] in the deterministic case) for the infinite volume Gibbs measures.

Proof: The first assertion of Corollary 4.2 is obvious from (4.12) and (4.14). To prove the second, we need to show that

$$\sup_{\mu^\pm, s^\pm} \sup_{\pm(u_\pm - v_\pm) \leq R} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\widehat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right) \geq \frac{1}{8R^2M^2} \tag{4.18}$$

But from (4.14) we see that

$$\begin{aligned}
\frac{1}{2} &\leq 1-\exp\left(-\beta L R c_2 \zeta \epsilon(\zeta)\right) \leq 1-\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](S_R^c) \\
&\leq \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\tau_+ \leq v_+ + R, \tau_- \geq v_- - R\right) \\
&\leq \sum_{\pm(u_\pm - v_\pm)\leq R} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\tau_- = u_-, \tau_+ = u_+\right) \\
&\leq \sum_{\pm(u_\pm - v_\pm)\leq R} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\eta(u_-,\sigma)\neq 0, \eta(u_+,\sigma)\neq 0\right) \\
&\leq \sum_{\pm(u_\pm - v_\pm)\leq R} \sum_{\mu^\pm, s^\pm} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\eta(u_-,s) = s^- e^{\mu^-}, \eta(u_+,s) = s^+ e^{\mu^+}\right) \\
&\leq 4R^2M^2 \sup_{\pm(u_\pm - v_\pm)\leq R} \sup_{\mu^\pm, s^\pm} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\hat{\mathcal{A}}(\mu^\pm, s^\pm, u_\pm)\right)
\end{aligned} \tag{4.19}$$

which gives (4.18). $\diamondsuit$

In order to prove Proposition 4.1, we need the following lemmata.

Lemma 4.3: *For any finite subset $\Gamma \subset \mathbb{Z}$ we denote by $A(\Gamma)$ the $M\times M$-matrix with elements*

$$A_{\mu,\nu}(\Gamma) = \frac{1}{|\Gamma|}\sum_{i\in\Gamma} \xi_i^\mu \xi_i^\nu \tag{4.20}$$

and let B be the $N\times N$-matrix with entries

$$B_{i,j} = \frac{1}{N}\sum_{\mu=1}^{M} \xi_i^\mu \xi_j^\mu \tag{4.21}$$

Set $N=|\Gamma|$ and assume that $M \geq N^{1/6}$. Then,

(i)

$$\mathbb{E}\|A(\Gamma)-\mathbb{1}\| \leq \sqrt{\tfrac{M}{N}}\left(2+\sqrt{\tfrac{M}{N}}\right) + C\tfrac{\ln N}{N^{1/6}} \tag{4.22}$$

and

(ii) There exists a universal constant $K<\infty$ such that for all $0\leq\delta\leq 1$.

$$\mathbb{P}\left[\left|\|B(\Gamma)\| - \mathbb{E}\|B(\Gamma)\|\right| > \delta\right] \leq K\exp\left(-N\frac{\delta^2}{2K}\right) \tag{4.23}$$

In particular,

$$\mathbb{P}\left[\|A(\Gamma)\| \geq \left(1+\sqrt{\tfrac{M}{N}}\right)^2(1+\delta)\right] \leq K\exp\left(-N\frac{\delta^2}{2K}\right) \tag{4.24}$$

Proof: For the proof of this Lemma, see [BG3], Section 2. Somewhat weaker estimates were previously obtained in [Ge,ST,BG1,BGP1]. $\diamondsuit$

Lemma 4.4: *Let $\{X_i(n), i \geq 1\}$ be independent random variables with $X_i(n) \geq 0$, satisfying, for any $z \geq 0$,*

$$\mathbb{P}\left[X_i(n) \geq (1+z)a_n\right] \leq c_n e^{-zb_n} \tag{4.25}$$

where a_n, b_n, c_n are strictly positive parameters satisfying $b_n \uparrow \infty$ and $(\ln c_n)/b_n \downarrow 0$ as $n \uparrow \infty$. Then,

$$\mathbb{E}(X_i(n)) \leq a_n \left(1 + \frac{\ln c_n}{b_n}\right) \tag{4.26}$$

and, for all $\epsilon > 0$ and n sufficiently large,

$$\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{K} X_i(n) \geq (1+z+\epsilon)a_n\right] \leq e^{-zb_n(1-\eta)K} \tag{4.27}$$

where $\eta \equiv \eta(\epsilon, b_n, c_n) \downarrow 0$ as $n \uparrow \infty$.

Proof: Setting $Y_i(n) \equiv X_i(n)/a_n$, we have,

$$\mathbb{E}(Y_i(n)) = \mathbb{E}\int_0^\infty 1\!\!1_{\{y \leq Y_i(n)\}} dy = \int_0^\infty \mathbb{P}(Y_i(n) \geq y) dy \tag{4.28}$$

Thus, for any $x \geq 0$,

$$\mathbb{E}(Y_i(n)) \leq 1 + x + \int_{1+x}^\infty \mathbb{P}(Y_i(n) \geq y) dy \tag{4.29}$$

Performing the change of variable $y = 1 + z$ and making use of (4.15) yields

$$\mathbb{E}(Y_i(n)) \leq 1 + x + c_n \int_x^\infty e^{-b_n z} dz = 1 + x + \frac{c_n}{b_n} e^{-xb_n} \tag{4.30}$$

Now, choosing $x = (\ln c_n)/b_n$ minimizes the r.h.s. of (4.30) and gives (4.26). To prove (4.24) we first use that, by the exponential Markov inequality, for any $t > 0$,

$$\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{K} Y_i(n) \geq 1+z+\epsilon\right] \leq e^{-Kt(1+z+\epsilon)} \prod_{i=1}^{K} \mathbb{E}e^{tY_i(n)} \tag{4.31}$$

To estimate the Laplace transform of $Y_i(n)$, we write that,

$$\mathbb{E}e^{tY_i(n)} = \mathbb{E}(1 + \int_0^\infty te^{ty} 1\!\!1_{\{y \leq Y_i(n)\}} dy) = 1 + \int_0^\infty te^{ty} \mathbb{P}(Y_i(n) \geq y) dy \tag{4.32}$$

and, for any $x \geq 0$,

$$\begin{aligned}
\mathbb{E}e^{tY_i(n)} &= 1 + \int_0^{1+x} te^{ty} \mathbb{P}(Y_i(n) \geq y) dy + \int_{1+x}^\infty te^{ty} \mathbb{P}(Y_i(n) \geq y) dy \\
&\leq e^{t(1+x)} + \int_{1+x}^\infty te^{ty} \mathbb{P}(Y_i(n) \geq y) dy \\
&\leq e^{t(1+x)} + c_n te^t \int_x^\infty e^{-z(b_n - t)} dz
\end{aligned} \tag{4.33}$$

where we used (4.25) in the last line after having performed the change of variable $y = 1 + z$. Choosing $t = b_n(1-\eta)$ for some $0 < \eta \leq 1$, we get

$$
\begin{aligned}
I\!E e^{tY_i(n)} &\leq e^{b_n(1-\eta)(1+x)} \left[1 + c_n \frac{1-\eta}{\eta} e^{-xb_n}\right] \\
&\leq \exp\left(b_n(1-\eta)(1+x) + c_n \frac{1-\eta}{\eta} e^{-xb_n}\right)
\end{aligned}
\tag{4.34}
$$

and finally, inserting (4.34) in (4.31) yields

$$
I\!P\left[\frac{1}{K}\sum_{i=1}^{K} Y_i(n) \geq 1 + z + \epsilon\right] \leq e^{-zb_n(1-\eta)K} \exp\left(-(1-\eta)K\left[b_n(\epsilon - x) - \frac{c_n}{\eta} e^{-xb_n}\right]\right) \tag{4.35}
$$

For n large enough, choosing $x = \epsilon/2$, one can always choose $\eta \equiv \eta(\epsilon, b_n, c_n)$ such that the last exponential in (4.35) is less than 1 and $\eta(\epsilon, b_n, c_n) \downarrow 0$ as $n \uparrow \infty$. $\diamondsuit$

Lemma 4.5: *There exists a positive constant c such that, for all integer R, there exists Ω_R with $I\!P(\Omega_R) \geq 1 - R\gamma^{-1}e^{-c\gamma^{-1}}$ such that for all $\mu^\pm, s^\pm, w_\pm, v_+ \leq w_+ \leq v_+ + R, v_- - R \leq w_- \leq v_-$ and $\omega \in \Omega_R$*

(i)

$$
\sup_{\sigma:\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}} \left|\gamma^{-1}E^{1,L}_{\gamma,\Delta}[\omega](\sigma_\Delta, m_L(\sigma_{\partial\Delta})) - W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)})\right| \leq \zeta\gamma^{-1}(1+\sqrt{2\gamma M(\gamma)})\sqrt{2} \tag{4.36}
$$

and

(ii)

$$
\sup_\sigma |W_{\gamma,\Delta}[\omega](\sigma_\Delta, \sigma_{\partial\Delta})| \leq \gamma^{-1}4(1+\sqrt{M/\ell})^2 \tag{4.37}
$$

where $\Delta = [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$.

Proof: We first prove (i). We set

$$
W_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^\pm,s^\pm)}) = W^+_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^+,s^+)}) + W^-_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^-,s^-)}) \tag{4.38}
$$

where

$$
W^-_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^-,s^-)}) \equiv -L\sum_{i\in\Delta} s^- a(\beta)\xi_i^{\mu^-}\sigma_i \sum_{r\in\partial^-\Delta} J_\gamma(i - Lr) \tag{4.39}
$$

and

$$
W^+_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^+,s^+)}) \equiv -L\sum_{i\in\Delta} s^+ a(\beta)\xi_i^{\mu^+}\sigma_i \sum_{r\in\partial^+\Delta} J_\gamma(i - Lr) \tag{4.40}
$$

We will consider only the terms corresponding to the interaction with the right part of Δ, the other ones being similar. We have

$$\begin{aligned}
&\left|\gamma^{-1}E^{1,L}_{\gamma,\Delta}[\omega](\sigma_\Delta, m_L(\sigma_{\partial^+\Delta})) - W^+_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^+,s^+)})\right| \mathbb{1}_{\{\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}} \\
&\le L\left|\sum_{i\in\Delta}\sum_{r\in\partial^+\Delta} J_\gamma(i-Lr)\sigma_i\left(\xi_i,\left[m_L(r,\sigma_{\partial^+\Delta}) - m^{(\mu^+,s^+)}\right]\right)\right| \mathbb{1}_{\{\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}} \\
&\le L\sum_{r\in\partial^+\Delta}\left\|\sum_{i\in\Delta} J_\gamma(i-Lr)\xi_i\sigma_i\right\|_2 \left\|m_L(r,\sigma_{\partial^+\Delta}) - m^{(\mu^+,s^+)}\right\|_2 \mathbb{1}_{\{\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}} \\
&\le \zeta L\sum_{r\in\partial^+\Delta}\left\|\sum_{i\in\Delta} J_\gamma(i-Lr)\xi_i\sigma_i\right\|_2 \equiv T^+(\sigma)
\end{aligned} \tag{4.41}$$

$T^-(\sigma)$ is defined in an analogous way. Recalling the definition (4.21) we have

$$\begin{aligned}
T^+(\sigma) =& \zeta L\sum_{r\in\partial^+\Delta}\left(\sum_{i\in[w_+-1,w_+-\frac{1}{2}]}\sum_{j\in[w_+-1,w_+-\frac{1}{2}]}(\xi_i,\xi_j)\sigma_i\sigma_j J_\gamma(i-Lr)J_\gamma(j-Lr)\right)^{\frac{1}{2}} \\
\le& \zeta L\sum_{r\in\partial^+\Delta}\left(\gamma^{-1}\|B\|\sum_{i\in[w_+-1,w_+-\frac{1}{2}]}(\sigma_i J_\gamma(i-Lr))^2\right)^{\frac{1}{2}} \\
\le& \zeta L\sum_{r\in\partial^+\Delta}\|B\|^{\frac{1}{2}} \\
\le& \zeta(2\gamma)^{-1}\|B\|^{\frac{1}{2}}
\end{aligned} \tag{4.42}$$

where we have used in the last inequality that $\#\{r\in\partial^+\Delta\} = (2\gamma L)^{-1}$. Thus, by Lemma 4.3, for all $\epsilon>0$,

$$I\!P\left[\sup_{\sigma\in\mathcal{S}} T^+(\sigma) \ge \zeta(2\gamma)^{-1}(1+\sqrt{2\gamma M})\sqrt{1+\epsilon}\right] \le 2K\gamma^{-1}\exp\left(-\frac{\epsilon}{2K\gamma}\right) \tag{4.43}$$

from which (i) follows.

We turn to the proof of (ii). Using (2.2) we have, for all $\epsilon>0$,

$$\begin{aligned}
&I\!P\left[\sup_{\sigma\in\mathcal{S}}|W_{\gamma,\Delta}[\omega](\sigma_\Delta,\sigma_{\partial\Delta})| \ge 4\epsilon^2\right] \\
\le& I\!P\left[\sup_{\sigma\in\mathcal{S}}\left|\gamma^{-1}E^{\ell,\ell}_{\gamma,\Delta}(m_\ell(\sigma_\Delta), m_\ell(\sigma_{\partial\Delta}))\right| \ge 2\epsilon^2\right] + I\!P\left[\sup_{\sigma\in\mathcal{S}}\left|\Delta W^{\ell,\ell}_{\gamma,\Delta}(\sigma_\Delta,\sigma_{\partial\Delta})\right| \ge 2\epsilon^2\right]
\end{aligned} \tag{4.44}$$

Let us consider the first probability in the r.h.s. of (4.44). By definition,

$$E^{\ell,\ell}_{\gamma,\Delta}(m_\ell(\sigma_\Delta), m_\ell(\sigma_{\partial\Delta})) = \gamma\ell\sum_{x\in\Delta}\sum_{y\in\partial\Delta} J_{\gamma\ell}(x-y)(m_\ell(x,\sigma_\Delta), m_\ell(y,\sigma_{\partial\Delta})) \tag{4.45}$$

Now

$$(m_\ell(x,\sigma_\Delta), m_\ell(y,\sigma_{\partial\Delta})) \le \|m_\ell(x,\sigma_\Delta)\|_2\,\|m_\ell(y,\sigma_{\partial\Delta})\|_2 \le \|B(x)\|^{\frac{1}{2}}\|B(y)\|^{\frac{1}{2}} \tag{4.46}$$

where $B(x)$ is the $\ell \times \ell$-matrix $B(x) = \{B(x)_{i,j}\}_{i \in x, j \in x}$ with $B(x)_{i,j} = \frac{1}{\ell}\sum_{\mu=1}^{M} \xi_i^\mu \xi_j^\mu$. Thus

$$
\begin{aligned}
\left|E^{\ell,\ell}_{\gamma,\Delta}(m_\ell(\sigma_\Delta), m_\ell(\sigma_{\partial\Delta}))\right| &\leq (\gamma\ell)^2 \sum_{x\in\Delta}\sum_{y\in\partial\Delta} \mathbb{1}_{\{|\ell x - \ell y| \leq (2\gamma)^{-1}\}} \|B(x)\|^{\frac{1}{2}} \|B(y)\|^{\frac{1}{2}} \\
&\leq \left(\gamma\ell \sum_{x\in[w_+-1, w_+-\frac{1}{2}]} \|B(x)\|^{\frac{1}{2}}\right)\left(\gamma\ell \sum_{y\in[w_+-\frac{1}{2}, w_++1]} \|B(y)\|^{\frac{1}{2}}\right) \\
&+ \left(\gamma\ell \sum_{x\in[w_-+\frac{1}{2}, w_-+1]} \|B(x)\|^{\frac{1}{2}}\right)\left(\gamma\ell \sum_{y\in[w_-, w_-+\frac{1}{2}]} \|B(y)\|^{\frac{1}{2}}\right) \\
&\equiv T_1 T_2 + T_3 T_4
\end{aligned}
\tag{4.47}
$$

and,

$$
I\!P\left[\sup_{\sigma\in\mathcal{S}}\left|E^{\ell,\ell}_{\gamma,\Delta}(m_\ell(\sigma_\Delta), m_\ell(\sigma_{\partial\Delta}))\right| \geq 2\epsilon^2\right] \leq \sum_{k=1}^{4} I\!P(T_k \geq \epsilon) \tag{4.48}
$$

where the last equality in (4.47) defines the quantities T_k. All four probabilities on the right hand side of (4.48) will be bounded in the same way. Let us consider $I\!P(T_1 \geq \epsilon)$. Note that $\left\{\|B(x)\|^{\frac{1}{2}}\right\}_{x\in[w_+-1, w_+-\frac{1}{2}]}$ are independent random variables. It follows from Lemma 4.3 that, for all $\tilde{\epsilon} > 0$,

$$
I\!P\left[\|B(x)\|^{\frac{1}{2}} > \left(1 + \sqrt{M/\ell}\right)(1+\tilde{\epsilon})\right] \leq 4K\ell \exp\left(-\frac{\tilde{\epsilon}\ell}{K}\right) \tag{4.49}
$$

and by Lemma 4.4, we get that for large enough ℓ,

$$
I\!P\left[T_1 \geq \frac{1}{2}(1 + \sqrt{M/\ell})(1+\tilde{\epsilon})\right] \leq K \exp\left(-\frac{\tilde{\epsilon}}{2K\gamma}\right) \tag{4.50}
$$

Therefore, choosing $\epsilon \equiv \frac{1}{2}(1 + \sqrt{M/\ell})(1+\tilde{\epsilon})$ in (4.44), (4.48) yields

$$
I\!P\left[\sup_{\sigma\in\mathcal{S}}\left|\gamma^{-1}E^{\ell,\ell}_{\gamma,\Delta}(m_\ell(\sigma_\Delta), m_\ell(\sigma_{\partial\Delta}))\right| \geq (2\gamma)^{-1}(1 + \sqrt{M/\ell})^2(1+\tilde{\epsilon})^2\right] \leq 4K \exp\left(-\frac{\tilde{\epsilon}}{2K\gamma}\right) \tag{4.51}
$$

Choosing $\tilde{\epsilon} = 1$ and using Lemma 2.1 to bound the second term in (4.44) we get (4.37) which concludes the proof of Lemma 4.5. $\diamondsuit$

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1 part i): Setting $\Delta^c \equiv \Lambda \setminus \Delta$, some simple manipulations allow us to write

$$
\begin{aligned}
&\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm)) \le \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F\cap\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm))\\
&=\frac{1}{Z_{\beta,\gamma,\Lambda}[\omega]}I\!E_{\sigma_\Delta}\Bigg[e^{-\beta\left[H_{\gamma,\Delta}[\omega](\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]}\\
&\times I\!E_{\sigma_{\Delta^c}}e^{-\beta\left[H_{\gamma,\Delta^c}[\omega](\sigma_{\Delta^c})+\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]\right]}1\!\!1_{\{\sigma\in F\cap\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}}\Bigg]\\
&=I\!E_{\sigma_\Delta}\Bigg[\frac{1}{Z^{\mu^\pm,s^\pm}_{\beta,\gamma,\Delta}[\omega]}e^{-\beta\left[H_{\gamma,\Delta}[\omega](\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]}\\
&\times I\!E_{\sigma_{\Delta^c}}I\!E_{\tilde\sigma_\Delta}\frac{1}{Z_{\beta,\gamma,\Lambda}[\omega]}e^{-\beta\left[H_{\gamma,\Delta^c}[\omega](\sigma_{\Delta^c})+H_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta)+W_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta,\sigma_{\Delta^c})\right]}1\!\!1_{\{\sigma\in F\cap\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}}\\
&\quad\times e^{-\beta\left[\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]+\left[W_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta,m^{(\mu^\pm,s^\pm)})-W_{\gamma,\Delta}[\omega](\tilde\sigma_\Delta,\sigma_{\Delta^c})\right]\right]}\Bigg]\\
&=I\!E_{\sigma_\Delta}\Bigg[\mathcal{G}^{\mu^\pm,s^\pm}_{\beta,\gamma,\Delta}[\omega](\sigma_\Delta)1\!\!1_{\{\sigma\in F\}}I\!E_{\bar\sigma_\Lambda}\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](\bar\sigma_\Lambda)1\!\!1_{\{\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)\}}\\
&\quad\times e^{-\beta\left[\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]+\left[W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\Delta^c})\right]\right]}\Bigg]
\end{aligned}
\tag{4.52}
$$

Now, if $\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)$

$$
\begin{aligned}
&\left|\left[W_{\gamma,\Delta}[\omega](\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right]+\left[W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\Delta^c})\right]\right|\\
\le&2\sup_{\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)}\left|W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\Delta^c})-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right|\\
\le&2\sup_{\bar\sigma\in\widehat{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm)}\left|\gamma^{-1}E^{1,L}_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m_L(\bar\sigma_{\partial\Delta}))-W_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,m^{(\mu^\pm,s^\pm)})\right|\\
&+2\sup_{\bar\sigma\in\mathcal{S}}\left|\Delta W^{1,L}_{\gamma,\Delta}[\omega](\bar\sigma_\Delta,\bar\sigma_{\partial\Delta})\right|
\end{aligned}
\tag{4.53}
$$

Finally, by Lemma 4.5 and Lemma 2.1, the supremum over $\mu^\pm,s^\pm$ and $w_\pm$, $v_+\le w_+\le v_++R$ $v_--R\le w_-\le v_-$, of the last line of (4.53) is bounded from above by $8\gamma^{-1}(\zeta+2\gamma L)$ with a $I\!P_\xi$-probability, greater than $1-4\gamma^{-1}R\exp(-c\gamma^{-1})$ for some positive constant c. Thus from (4.52) and (4.53) follow both (4.12) and (4.13). $\diamondsuit$

Proof of Proposition 4.1 part ii): Using (4.6) the l.h.s. of (4.14) is bounded from above by $\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](A^+(R))+\mathcal{G}_{\beta,\gamma,\Lambda}[\omega](A^-(R))$. We estimate the first term, the second one being similar. Since the spin configuration are away from the equilibria on a length R, we can decouple the interaction between this part and the rest of the volume Λ, by making a rough estimate of those interaction terms. The fact that we are out of equilibrium will give terms proportional to R that will be dominant if R is chosen large enough. More precisely, calling $\Delta_R\equiv[v_+,v_++R]$, we have, for all fixed R

$$\begin{aligned}\mathcal{G}_{\beta,\gamma,\Lambda}\left(A^+(R)\right) &= \frac{1}{Z_{\beta,\gamma,\Lambda}} I\!E_{\sigma_\Lambda}\left[e^{-\beta H_{\gamma,\Lambda\setminus\Delta_R}(\sigma_{\Lambda\setminus\Delta_R})} e^{-\beta\left[H_{\gamma,\Delta_R}(\sigma_{\Delta_R})+W_{\gamma,\Delta_R}(\sigma_{\Delta_R},\sigma_{\Lambda\setminus\Delta_R})\right]} 1\!\!1_{\{\sigma\in A^+(R)\}}\right] \\ &\le e^{4c\gamma^{-1}} \frac{1}{Z_{\beta,\gamma,\Delta_R}} I\!E_{\sigma_{\Delta_R}}\left[e^{-\beta H_{\gamma,\Delta_R}(\sigma_{\Delta_R})} 1\!\!1_{\{\sigma\in A^+(R)\}}\right]\end{aligned} \tag{4.54}$$

with a $I\!P_\xi$-probability greater than $1-4\gamma^{-1}e^{-c\gamma^{-1}}$ for some positive constant c, where we have used Lemma 4.5 to bound the interaction between Δ_R and $\Lambda\setminus\Delta_R$. To estimate the last term in (4.54), we express it in terms of block spin variables on the scale ℓ. Using (2.5) we get

$$\mathcal{G}_{\beta,\gamma,\Delta_R}\left(A^+(R)\right) \le e^{2c\gamma^{-1}|\Delta_R|(4\gamma\ell+\gamma M)} \frac{I\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))} 1\!\!1_{\{\sigma\in A^+(R)\}}}{I\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))}} \tag{4.55}$$

with a $I\!P_\xi$-probability greater than $1-e^{-c\gamma^{-1}|\Delta_R|}$

We derive first a lower bound on the denominator which will be given effectively by restricting the configurations to be in the neighborhood of a constant profile near one of the equilibrium positions $sa(\beta)e^\mu$. We will choose without lost of generality to be $s=1,\mu=1$. To make this precise, we define for any $\rho>0$ the balls

$$\mathcal{B}_\rho^{(\mu,s)} \equiv \left\{ m\in I\!R^M \,\middle|\, \|m-m^{(\mu,s)}\|_2\le\rho \right\} \tag{4.56}$$

We will moreover write

$$\mathcal{B}_\rho \equiv \bigcup_{(\mu,s)\in\{1,\dots,M\}\times\{-1,1\}} \mathcal{B}_\rho^{(\mu,s)} \tag{4.57}$$

Obviously,

$$I\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))} \ge I\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))} \prod_{x\in\Delta_R} 1\!\!1_{\{m_\ell(x,\sigma)\in\mathcal{B}_\rho^{(1,1)}\}} \tag{4.58}$$

It can easily be shown that, on the set $\{m_\ell(x,\sigma)\in\mathcal{B}_\rho, \forall x\in\Delta_R\}$,

$$-\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma)) \ge \frac{\ell}{2}\sum_{x\in\Delta_R}\left(\|m_\ell(x,\sigma)\|_2^2-4\rho^2\right) \tag{4.59}$$

from which (4.58) yields

$$\begin{aligned} I\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))} &\ge e^{-4\beta\gamma^{-1}|\Delta_R|\rho^2} \prod_{x\in\Delta_R} I\!E_{\sigma_x} e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)\|_2^2} 1\!\!1_{\{m_\ell(x,\sigma)\,\in\,\mathcal{B}_\rho^{(1,1)}\}} \\ &= e^{-4\beta\gamma^{-1}|\Delta_R|\rho^2} \prod_{x\in\Delta_R} Z_{x,\beta,\rho}\left(a(\beta)e^1\right)\end{aligned} \tag{4.60}$$

provided that ρ is sufficiently large so that $\mathcal{B}_\rho^{(1,1)}$ contains the lowest minimum of Φ in the neighborhood of $a(\beta)e^1$, which is the case if $\rho\ge c\sqrt{\frac{M}{\ell}}$, for some finite constant c with a $I\!P_\xi$-probability $\ge 1-e^{-cM}$.

Next we derive an upper bound for the numerator of the ratio in (4.55). Using the inequality $ab \leq \frac{1}{2}(a^2+b^2)$ we get

$$-\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma)) \leq \frac{\ell}{2}\sum_{x\in\Delta_R}\|m_\ell(x,\sigma)\|_2^2 \tag{4.61}$$

and whence

$$I\!\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1}E^{\ell}_{\gamma,\Delta_R}(m_\ell(\sigma))}\mathbb{1}_{\{\sigma\in A^+(R))\}} \leq I\!\!E_{\sigma_{\Delta_R}} e^{\frac{\beta\ell}{2}\sum_{x\in\Delta_R}\|m_\ell(x,\sigma)\|_2^2}\mathbb{1}_{\{\sigma\in A^+(R)\}} \tag{4.62}$$

Let us now recall that, by definition,

$$A^+(R) = \left\{\sigma\in\mathcal{S}\,\middle|\, \forall_{u\in\Delta_R}\exists_{r\in u}: \inf_{\mu,s}\|m^{(\mu,s)} - m_L(r,\sigma)\|_2 > \zeta\right\} \tag{4.63}$$

Using that $m_L(r,\sigma) = \frac{\ell}{L}\sum_{x\in r} m_\ell(x,\sigma)$ we have

$$\|m^{(\mu,s)} - m_L(r,\sigma)\|_2 \leq \frac{\ell}{L}\sum_{x\in r}\|m^{(\mu,s)} - m_\ell(x,\sigma)\|_2 \tag{4.64}$$

so that

$$A^+(R) \subset \left\{\sigma\in\mathcal{S}\,\middle|\, \forall_{u\in\Delta_R}\exists_{r\in u}: \inf_{\mu,s}\frac{\ell}{L}\sum_{x\in r}\|m^{(\mu,s)} - m_\ell(x,\sigma)\|_2 > \zeta\right\} \tag{4.65}$$

We will use the following fact

Lemma 4.6: *Let $\{X_k, k=1,2,\dots,K\}$ be a sequence of real numbers satisfying $0\leq X_k\leq c$ for some $c<\infty$. Let $\zeta<c$ and assume that*

$$\frac{1}{K}\sum_{k=1}^{K} X_k > \zeta \tag{4.66}$$

For $0\leq\delta\leq\zeta$, define the set $V_{\delta,\zeta}\equiv\{k|X_k\leq\delta\zeta\}$. Then

$$|\{1\leq k\leq K : X_k > \delta\zeta\}| \geq K\frac{\zeta(1-\delta)}{c-\delta\zeta} \tag{4.67}$$

Proof : Set $V^c_{\delta,\zeta}\equiv\{1,\dots,K\}\setminus V_{\delta,\zeta}$. Then

$$\frac{1}{K}\sum_{k=1}^{K}X_k \leq \frac{1}{K}\sum_{k\in V_{\delta,\zeta}}X_k + \frac{1}{K}\sum_{k\in V^c_{\delta,\zeta}}X_k \leq \frac{1}{K}c|V_{\delta,\zeta}| + \frac{1}{K}\delta\zeta|V^c_{\delta,\zeta}| = \frac{1}{K}(c-\delta\zeta)|V_{\delta,\zeta}| + \delta\zeta \tag{4.68}$$

which, together with (4.67) implies the bound (4.68) $\diamondsuit$

Let us denote by $\mathcal{V}_{\delta,\zeta}(r)$ the set of all subsets $S\subset\{x\in r\}$ with cardinality $\frac{L}{\ell}\frac{\zeta(1-\delta)}{2-\delta\zeta}$, respectively volume

$$|S| \geq \gamma L\frac{\zeta(1-\delta)}{2-\delta\zeta} \tag{4.69}$$

Then, since $\|m^{(\mu,s)} - m_\ell(x,\sigma)\|_2 < 2$, Lemma 4.7 implies

$$A^+(R) \subset \left\{ \sigma \in \mathcal{S} \,\middle|\, \forall_{u\in\Delta_R} \exists_{r\in u} \exists_{S\in\mathcal{V}_{\delta,\zeta}(r)} : \forall_{x\in S}\,,\, m_\ell(x,\sigma) \in \mathcal{B}^c_{\delta\zeta} \right\} \tag{4.70}$$

Therefore

$$\begin{aligned} &I\!\!E_{\sigma_{\Delta_R}} e^{-\beta\gamma^{-1} E^\ell_{\gamma,\Delta_R}(m_\ell(\sigma))} 1\!\!1_{\{\sigma\in A^+(R)\}} \\ &\leq \prod_{u\in\Delta_R} I\!\!E_{\sigma_u} e^{\frac{\beta\ell}{2}\sum_{x\in u}\|m_\ell(x,\sigma)\|_2^2} 1\!\!1_{\{\exists_{r\in u}\exists_{S\in\mathcal{V}_{\delta,\zeta}(r)} : \forall x\in S\,,\, m_\ell(x,\sigma)\in\mathcal{B}^c_{\delta\zeta}\}} \\ &\leq \prod_{u\in\Delta_R} \sum_{r\in u} \sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} I\!\!E_{\sigma_u} e^{\frac{\beta\ell}{2}\sum_{x\in u}\|m_\ell(x,\sigma)\|_2^2} \prod_{x\in S} 1\!\!1_{\{m_\ell(x,\sigma)\in\mathcal{B}^c_{\delta\zeta}\}} \end{aligned} \tag{4.71}$$

Inserting this and (4.60) into (4.55) we have

$$\begin{aligned} &\mathcal{G}_{\beta,\gamma,\Delta_R}[\omega]\left(A^+(R)\right) \\ &\leq e^{\gamma^{-1}|\Delta_R|(16\gamma\ell+4\gamma M+4\beta\rho^2)} \prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} \prod_{x\in u\setminus S} \frac{Z_{x,\beta}}{Z_{x,\beta,\rho}(a(\beta)e^1)} \prod_{x\in S} \frac{Z^c_{x,\beta,\delta\zeta}}{Z_{x,\beta,\rho}(a(\beta)e^1)} \\ &\equiv e^{\gamma^{-1}|\Delta_R|(16\gamma\ell+4\gamma M+4\beta\rho^2)} \prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} T_S^{(1)} T_S^{(2)} \end{aligned} \tag{4.72}$$

where we have defined

$$Z^c_{x,\beta,\delta\zeta} \equiv I\!\!E_{\sigma_x} e^{\frac{\beta\ell}{2}\|m_\ell(x,\sigma)\|_2^2} 1\!\!1_{\{m_\ell(x,\sigma)\in\mathcal{B}^c_{\delta\zeta}\}} \tag{4.73}$$

It follows from Proposition 2.3 of [BGP1] that

$$Z_{x,\beta} \leq \exp\left(-\beta\ell\left[\phi(a(\beta)) - c\sqrt{\tfrac{M}{\ell}}\right]\right) \tag{4.74}$$

so that using Lemma 3.1 we get that

$$T_S^{(1)} \leq \prod_{x\in u\setminus S} \exp\left(+\beta\ell c\sqrt{\tfrac{M}{\ell}}\right) \leq e^{+\beta\gamma^{-1}c\sqrt{\frac{M}{\ell}}} \tag{4.75}$$

with a $I\!\!P_\xi$-probability $\geq 1-(\gamma\ell)^{-1}e^{-cM}$ On the other hand, to bound $Z^c_{x,\beta,\delta\zeta}$, we proceed as in [BG2] and first note that $\|m_\ell(x,\sigma)\|_2^2 \leq 2$ for all σ. Next, we introduce the lattice $\mathcal{W}_{\ell,M}$ with spacing $1/\sqrt{\ell}$ in $I\!\!R^M$ and we denote by $\mathcal{W}_{\ell,M}(2)$ the intersection of this lattice with the ball of radius 2 in $I\!\!R^M$. We have

$$|\mathcal{W}_{\ell,M}(2)| \leq \exp\left(M\ln\left(\frac{2\ell}{M}\right)\right) \tag{4.76}$$

Now, we may cover the ball of radius 2 in $I\!\!R^M$ with balls of radii $\hat\rho \equiv \sqrt{M/\ell}$ centered at the points of $\mathcal{W}_{\ell,M}(2)$. Supposing that $\delta\zeta > \hat\rho$ this yields,

$$\begin{aligned} Z^c_{x,\beta,\delta\zeta} &\leq \sum_{m\in\mathcal{W}_{\ell,M}(2)} 1\!\!1_{\{m\in\mathcal{B}^c_{\delta\zeta-\hat\rho}\}} Z_{x,\beta,\hat\rho}(m)[\omega] \\ &\leq \sum_{m\in\mathcal{W}_{\ell,M}(2)} 1\!\!1_{\{m\in\mathcal{B}^c_{\delta\zeta-\hat\rho}\}} \exp\left(-\beta\ell\left(\Phi_{x,\beta}(m)[\omega] - \frac{1}{2}\hat\rho^2\right)\right) \end{aligned} \tag{4.77}$$

Let us now assume that $\delta\zeta - \hat\rho$ satisfies the hypothesis of Proposition 3.2, then

$$Z^c_{x,\beta,\delta\zeta} \le \exp\left(-\beta\ell\left(\phi(a(\beta)) + \epsilon(\delta\zeta-\hat\rho) - 4(\delta\zeta-\hat\rho)\sqrt{\tfrac{M}{\ell}} - \tfrac{1}{2}\hat\rho^2 - \tfrac{M}{\beta\ell}\ln\left(\tfrac{2\ell}{M}\right)\right)\right) \tag{4.78}$$

with a $I\!P_\xi$-probability $\ge 1 - e^{-cM}$, where $\epsilon(\cdot)$ is the function defined in Proposition 3.2. We will assume that $\delta\zeta \gg \sqrt{\frac{M}{\ell}}$. Thus

$$\frac{Z^c_{x,\beta,\delta\zeta}}{Z_{x,\beta,\rho}(a(\beta)e^1)} \le \exp\left(-\beta\ell\left[\epsilon\left(\delta\zeta-\hat\rho\right) - c\delta\zeta\sqrt{\tfrac{M}{\ell}}\right]\right) \tag{4.79}$$

with a $I\!P_\xi$-probability $\ge 1 - e^{-c'M}$. Thus the product $T_S^{(1)}T_S^{(2)}$ defined in (4.72) is bounded by

$$T_S^{(1)}T_S^{(2)} \le \exp\left(\beta\gamma^{-1}c\left[\sqrt{\tfrac{M}{\ell}} - \epsilon(\zeta)|S|\right]\right) \tag{4.80}$$

with a $I\!P_\xi$-probability $\ge 1 - (\gamma\ell)^{-1}|S|e^{-c'M}$. Hence

$$\begin{aligned}
&\prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} T_S^{(1)}T_S^{(2)} \\
&\le \prod_{u\in\Delta_R}\sum_{r\in u}\sum_{S\in\mathcal{V}_{\delta,\zeta}(r)} \exp\left(-\beta\gamma^{-1}c\left[|S|\epsilon(\zeta) - \sqrt{\tfrac{M}{\ell}}\right]\right) \\
&\le \exp\left(-\beta\gamma^{-1}|\Delta_R|\left[\gamma L\zeta c\epsilon(\zeta) - \gamma|\ln(\gamma L)| - \gamma L\frac{\ln 2}{\ell} - c\sqrt{\tfrac{M}{\ell}}\right]\right)
\end{aligned} \tag{4.81}$$

with a $I\!P_\xi$-probability $\ge 1 - (\gamma)^{-1}Re^{-c'M}$ and finally, inserting (4.81) in (4.72) we arrive at

$$\mathcal{G}_{\beta,\gamma,\Delta_R}[\omega]\ \left(A^+(R)\right) \le \exp\left(-\beta\gamma^{-1}R\left[\gamma Lc\zeta\epsilon(\zeta) - c'\left(\sqrt{\tfrac{M}{\ell}} + 8\gamma\ell + 2\rho^2\right)\right]\right) \tag{4.82}$$

with a $I\!P_\xi$-probability $\ge 1 - (\gamma\ell)^{-1}Re^{-c'\ell}$, where we have used the fact that $M \ll \ell$. $\diamondsuit$

5. Self averaging properties of the free energy

In this chapter we study the self averaging properties of the free energy of the Hopfield-Kac model with *mesoscopic* boundary conditions.

We denote the partition function on the volume Δ with boundary condition $s^- a(\beta)e^{\mu^-}$ on the left of Δ and $s^+ a(\beta)e^{\mu^+}$ on the right of Δ by

$$Z_\Delta^{(\mu^\pm,s^\pm)} \equiv I\!\!E_{\sigma_\Delta}\left[e^{-\beta\left(H_{\gamma,\Delta}(\sigma)+W_{\gamma,\Delta,\partial^-\Delta}(\sigma_\Delta|m^{(\mu^-,s^-)})+W_{\gamma,\Delta,\partial^+\Delta}(\sigma_\Delta)|m^{(\mu^+,s^+)})\right)}\right] \tag{5.1}$$

and the corresponding free energy

$$f_\Delta^{(\mu^\pm,s^\pm)} \equiv f_\Delta = -\frac{\gamma}{\beta|\Delta|}\ln Z_\Delta^{(\mu^\pm,s^\pm)} \tag{5.2}$$

To include the case of free boundary conditions, we set $m^{(0,0)} \equiv 0$.

We are interested in the behavior of the fluctuations of $f_\Delta^{(\mu^\pm,s^\pm)}$ around it mean value. We will use the Theorem 6.6 of Talagrand [T2] that we state for the convenience of the reader. We denote by $I\!\!M X$ a median of the random variable X. Recall that a number x is called the median of a random variable X if both $I\!\!P[X \geq x] \geq \frac{1}{2}$ and $I\!\!P[X \leq x] \geq \frac{1}{2}$.

Theorem 5.1: [T2]*Consider a real valued function f defined on $[-1,+1]^N$. We assume that, for each real number a the set $\{f \leq a\}$ is convex. Consider a convex set $B \subset [-1,+1]^N$, and assume that for all $x,y \in B$, $|f(x)-f(y)| \leq k\|x-y\|_2$ for some positive k. Let X denote a random vector with i.i.d. components $\{X_i\}_{1\leq i\leq N}$ taking values in $[-1,+1]$. Then for all $t>0$,*

$$I\!\!P\left[|f(X) - I\!\!M f(X)| \geq t\right] \leq 4b + \frac{4}{1-2b}\exp\left(-\frac{t^2}{16k^2}\right) \tag{5.3}$$

where $b \equiv I\!\!P\left[X \notin B\right]$ and we assume that $b < \frac{1}{2}$.

The main result of this chapter is the following proposition:

Proposition 5.2:. *If $\gamma\ell$, M/ℓ and γM are small enough, then for all $t>0$, there exists a universal numerical constant K such that*

$$\begin{aligned} &I\!\!P\left[\left|f_\Delta^{(\mu^\pm,s^\pm)} - I\!\!E f_\Delta^{(\mu^\pm,s^\pm)}\right| \geq t + K\sqrt{\gamma^{-1}|\Delta|}\right] \\ &\leq K\exp\left(-\frac{\gamma^{-1}}{8}|\Delta|(\sqrt{1+t^2}-1)\right) \end{aligned} \tag{5.4}$$

Proof: Note first that the set $\{f_\Delta \leq a\}$ is convex. This follows from the fact that the Hamiltonian $H_{\gamma,\Delta}$ is a convex function of the variable ξ. The main difficulty that remains is to establish that

f_Δ is a Lipshitz function of the independent random variables ξ with a constant k that is small with large probability. To prove the Lipshitz continuity of f_Δ it is obviously enough to prove the corresponding bounds for $H_{\gamma,\Delta}(\sigma)$ and $W_{\gamma,\Delta,\partial^\pm\Delta}(\sigma_\Delta|m^{(\mu^\pm,s^\pm)})$.

Let us first prove that $H_{\gamma,\Delta}(\sigma)$ is Lipshitz in the random variable ξ. Let us write $\xi\equiv\xi[\omega]$ and $\hat\xi\equiv\xi[\omega']$. Denoting by $\xi^\mu\sigma$ the coordinatewise product of the two vectors ξ^μ and σ and $J_\gamma(i-j)$ the symmetric $\gamma^{-1}|\Delta|\times\gamma^{-1}|\Delta|$ matrix with i,j entries, we have

$$
|H_{\gamma,\Delta}[\omega](\sigma)-H_{\gamma,\Delta}[\omega'](\sigma)|=\left|\sum_{\mu=1}^{M}\left(\left[\xi^\mu\sigma-\hat\xi^\mu\sigma\right],J_\gamma\left[\xi^\mu\sigma+\hat\xi^\mu\sigma\right]\right)\right| \tag{5.5}
$$

Since J_γ is a symmetric and positive definite matrix, its square root $J_\gamma^{1/2}$ exists. Thus using the Schwarz inequality we may write

$$
\begin{aligned}
&\left|\sum_{\mu=1}^{M}\left([\xi^\mu\sigma-\hat\xi^\mu\sigma],J_\gamma[\xi^\mu\sigma+\hat\xi^\mu\sigma]\right)\right|\leq\\
&\sum_{\mu=1}\|J_\gamma^{1/2}[\xi^\mu\sigma-\hat\xi^\mu\sigma]\|_2\|J_\gamma^{1/2}[\xi^\mu\sigma+\hat\xi^\mu\sigma]\|_2\\
&\leq\mathcal{J}^+\mathcal{J}^-
\end{aligned} \tag{5.6}
$$

where

$$
\mathcal{J}^+\equiv\left(\sum_{\mu=1}^{M}([\xi^\mu\sigma+\hat\xi^\mu\sigma],J_\gamma[\xi^\mu\sigma+\hat\xi^\mu\sigma])\right)^{1/2} \tag{5.7}
$$

and

$$
\mathcal{J}^-\equiv\left(\sum_{\mu=1}^{M}([\xi^\mu\sigma-\hat\xi^\mu\sigma],J_\gamma[\xi^\mu\sigma-\hat\xi^\mu\sigma])\right)^{1/2}\leq\|\xi-\hat\xi\|_2 \tag{5.8}
$$

The last inequality in (5.8) follows since $\|J_\gamma\|\leq 1$.

On the other hand, by convexity

$$
\begin{aligned}
\left(\mathcal{J}^+\right)^2&\leq 2\sum_{\mu=1}^{M}(\xi^\mu\sigma J_\gamma\xi^\mu\sigma)+2\sum_{\mu=1}^{M}(\hat\xi^\mu\sigma J_\gamma\hat\xi^\mu\sigma)\\
&=2H_{\gamma,\Delta}[\omega](\sigma)+2H_{\gamma,\Delta}[\omega'](\sigma)
\end{aligned} \tag{5.9}
$$

Collecting, we get

$$
|H_{\gamma,\Delta}[\omega](\sigma)-H_{\gamma,\Delta}[\omega'](\sigma)|\leq\sqrt{2}\|\xi-\hat\xi\|_2\left(H_{\gamma,\Delta}[\omega](\sigma)+H_{\gamma,\Delta}[\omega'](\sigma)\right)^{1/2} \tag{5.10}
$$

This means that as in [T2], we are in a situation where the upper bound for the Lipshitz norm of $H_{\gamma,\Delta}[\omega](\sigma)$ is not uniformly bounded. However the estimates of Section 2, allow us to give reasonable estimates on the probability distribution of this Lipshitz norm. Recalling (2.5) we have

$$
I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta}|\Delta H_{\gamma,\Delta}(\sigma)|\geq\gamma^{-1}|\Delta|(16(1+c))\gamma\ell+4\gamma M)\right]\leq 16e^{-c\gamma^{-1}|\Delta|} \tag{5.11}
$$

Therefore, using (2.1) we get

$$
\begin{aligned}
&I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta}|H_{\gamma,\Delta}(\sigma)|\geq\gamma^{-1}|\Delta|(C+(16(1+c))\gamma\ell+4\gamma M)\right]\\
&\leq 16e^{-C\gamma^{-1}|\Delta|}+I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta}|\gamma^{-1}E^{\ell}_{\gamma,\Delta}(m_\ell(\sigma))|\geq C\gamma^{-1}\Delta\right]
\end{aligned}
\tag{5.12}
$$

To estimate this last probability, we notice that by convexity

$$
2(m_\ell(x,\sigma),m_\ell(y,\sigma))\leq\|m_\ell(x,\sigma)\|_2^2+\|m_\ell(y,\sigma)\|_2^2
\tag{5.13}
$$

Therefore

$$
\begin{aligned}
|\gamma^{-1}E^{\ell}_{\gamma,\Delta}(m_\ell(\sigma))| &= 1/2\left|\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)(m_\ell(x,\sigma),m_\ell(y,\sigma))\right|\\
&\leq \ell/2\sum_{x\in\Delta}\|m_\ell(x,\sigma)\|_2^2
\end{aligned}
\tag{5.14}
$$

Now we have

$$
\begin{aligned}
&I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta}\ell\sum_{x\in\Delta}\|m_\ell(x,\sigma)\|_2^2\geq 2C\gamma^{-1}|\Delta|\right]\\
&\leq 2^{\gamma^{-1}|\Delta|}I\!P\left[\ell\sum_{x\in\Delta}\|m_\ell(x,\sigma)\|_2^2\geq 2C\gamma^{-1}|\Delta|\right]\\
&\leq 2^{\gamma^{-1}|\Delta|}\inf_{0\leq t<1/2}e^{-2C\gamma^{-1}|\Delta|t}\prod_{x\in\Delta}\prod_{\mu=1}^{M}I\!E e^{t\ell\left(\frac{1}{\ell}\sum_{i\in x}\xi_i^\mu\sigma_i\right)^2}
\end{aligned}
\tag{5.15}
$$

Using the well known inequality [BG1]

$$
I\!E\exp\left(t\ell\left(\frac{1}{\ell}\sum_{i\in x}\xi_i^\mu\sigma_i\right)^2\right)\leq\frac{1}{\sqrt{1-2t}}
\tag{5.16}
$$

and choosing $t=1/4$, the r.h.s of (5.15) is bounded from above by

$$
\exp\left(-\gamma^{-1}|\Delta|\left(\frac{C}{2}-(1+M/2\ell)\ln 2\right)\right)
\tag{5.17}
$$

Collecting, we get

$$
I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta}\ell\sum_{x\in\Delta}\|m_\ell(x,\sigma)\|_2^2\geq\gamma^{-1}|\Delta|2\left(c+(1+M/2\ell)\ln 2\right)\right]\leq e^{-c\gamma^{-1}|\Delta|}
\tag{5.18}
$$

which implies, if $\gamma\ell$, γM and M/ℓ are small enough, that

$$
I\!P\left[\sup_{\sigma\in\mathcal{S}_\Delta}|H_{\gamma,\Delta}(\sigma)|\geq\gamma^{-1}|\Delta|(4c+1)\right]\leq 17e^{-c\gamma^{-1}|\Delta|}
\tag{5.19}
$$

which is the estimate we wanted.

To treat the boundary terms, *c.f* (4.12), let us call $W^-_{\gamma,\Delta}[\omega]$ (respectively $W^+_{\gamma,\Delta}[\omega]$) the terms corresponding to interactions with the left (respectively right) part of the boundary $\partial\Delta$. We estimate first the Lipshitz norm of $W^-_{\gamma,\Delta}[\omega]$, the one of $W^+_{\gamma,\Delta}[\omega]$ being completely identical.

$$
\begin{aligned}
&|W^-_{\gamma,\Delta}[\omega](\sigma_\Delta, m^{(\mu^-,s^-)}) - W^-_{\gamma,\Delta}[\omega'](\sigma_\Delta, m^{(\mu^-,s^-)})| \\
&\le a(\beta)\left|\sum_{i\in\Delta}\sigma_i(\xi_i^{\mu^-} - \hat{\xi}_i^{\mu^-})\left(\sum_{j\in\partial^-\Delta} J_\gamma(i-j)\right)\right| \\
&\le a(\beta)\left(\sum_{i\in\Delta}(\xi_i^{\mu^-} - \hat{\xi}_i^{\mu^-})^2\right)^{1/2}\left(\sum_{i\in\Delta}\left(\sum_{j\in\delta^-\Delta} J_\gamma(i-j)\right)^2\right)^{1/2} \\
&\le \gamma^{1/2}a(\beta)\|\xi-\hat{\xi}\|_2^2 \\
&\le \gamma^{1/2}\|\xi-\hat{\xi}\|_2^2
\end{aligned}
\tag{5.20}
$$

where we have used the Schwarz inequality and

$$
\sum_{i\in\Delta}\left(\sum_{j\in\partial^-\Delta} J_\gamma(i-j)\right)^2 \le \gamma^{-1} \tag{5.21}
$$

Therefore if we denote by

$$
\Omega_B \equiv \left\{\xi\in[-1,+1]^{\gamma^{-1}\Delta M} \big| \sup_{\sigma\in\mathcal{S}_\Delta}|H_{\gamma,\Delta}(\sigma)| \le \gamma^{-1}|\Delta|(4c+1)\right\} \tag{5.22}
$$

Using (5.3), (5.19), (5.20) and some easy computations, we get

$$
I\!\!P\left[\left|f_\Delta^{(\mu^\pm,s^\pm)} - I\!\!M f_\Delta^{(\mu^\pm,s^\pm)}\right| \ge t\gamma^{-1}|\Delta|\right] \le 68e^{-c\gamma^{-1}|\Delta|} + 68e^{-\frac{t^2}{16(4c+2)}\gamma^{-1}|\Delta|} \tag{5.23}
$$

Choosing c such that $c = \frac{t^2}{16(4c+2)}$ we get

$$
I\!\!P\left[\left|f_\Delta^{(\mu^\pm,s^\pm)} - I\!\!M f_\Delta^{(\mu^\pm,s^\pm)}\right| \ge t\gamma^{-1}|\Delta|\right] \le 136\exp\left(-\frac{\gamma^{-1}}{8}|\Delta|(\sqrt{1+t^2}-1)\right) \tag{5.24}
$$

Finally, a simple calculation shows that (5.24) implies that

$$
|I\!\!M f_\Delta^{(\mu^\pm,s^\pm)} - I\!\!E f_\Delta^{(\mu^\pm,s^\pm)}| \le 544\sqrt{\gamma^{-1}|\Delta|} \tag{5.25}
$$

and this implies the claim of Proposition 5.2. $\diamondsuit$

We will mainly use the Proposition 5.2 in the following form

Corollary 5.3: *If* $|\Delta| \leq \gamma^{-1} g(\gamma)$ *for some* $g(\gamma)$ *with* $g(\gamma) \downarrow 0$ *and* $\gamma^{-1} g(\gamma) > c$, *for all* γ *small enough then there exists a set* Ω_g *with* $I\!P[\Omega_g] \geq 1 - K e^{-c(g(\gamma))^{-1/2}}$ *for some positive constants* c *and* K, *such that for all* $\omega \in \Omega_g$

$$\left| \ln Z_{\Delta}^{(\mu^{\pm}, s^{\pm})} - I\!E\left[\ln Z_{\Delta}^{(\mu^{\pm}, s^{\pm})} \right] \right| \leq \beta \gamma^{-1} (g(\gamma))^{1/4} \tag{5.26}$$

Proof: The Corollary follows from Proposition 5.2 by choosing $t = \gamma^{1/2} |\Delta|^{-1/2} (g(\gamma))^{-1/4}$ $\diamondsuit$

6. Localization of the Gibbs measures II: The block-scale

6.1. Finite volume, free boundary conditions

Instead of dealing with the measures $\mathcal{G}^{(\mu_\pm,s_\pm)}_{\beta,\gamma,\Lambda}[\omega]$ immediately, we will first consider the simpler case of Gibbs measures in a finite volume $\Lambda \equiv [v_-, v_+]$ of order $|\Lambda| = o(\gamma^{-1})$ with *free* (Dirichlet) boundary conditions. This will be considerably simpler and the result will actually be needed as a basic input in order to deal with the full problem. On the other hand, the result may be seen as interesting in its own right and exhibits, to a large extent, the main relevant features of the model. This may indeed satisfy many readers who may not wish to follow the additional technicalities. With this in mind, we give a more detailed exposition of this case.

Our basic result here will be that the free boundary conditions measure in volumes small compared to γ^{-1} are concentrated on "constant profiles" with very large probability. More precisely, we have

Theorem 6.1: *Assume that $\gamma|\Lambda| \downarrow 0$, β large enough ($\beta > 1$) and $\gamma M(\gamma) \downarrow 0$. Then we can find $\gamma^{-1} \gg \hat L \gg 1$ and $\hat\zeta \downarrow 0$, such that on a subset $\Omega_\Lambda \subset \Omega$ with $I\!P(\Omega_\Lambda^c) \leq e^{-cg^{-1/2}(\gamma)}$ where $g(\gamma) \downarrow 0$ and $\gamma^{-1} g(\gamma) > c$, we have that for all $\omega \in \Omega_\Lambda$*

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\exists_{u\in\Lambda}\eta_{\hat\zeta,\hat L}(u,\sigma) = 0\right) \leq e^{-\hat L h(\hat\zeta)} \tag{6.1}$$

and

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\exists_{u\in\Lambda}\eta_{\hat\zeta,\hat L}(u,\sigma) \neq \eta_{\hat\zeta,\hat L}(u+1,\sigma)\right) \leq e^{-\hat L h(\hat\zeta)} \tag{6.2}$$

where $h(\zeta) = c\beta\zeta\epsilon(\zeta)$ and $\epsilon(\zeta)$ is defined in (3.15).

The proof of this theorem relies on a large deviation type estimate for events that take place on a scale much smaller than the size of Λ. We will consider events F that are in the cylinder algebra with base $I = [u_-, u_+] \subset \Lambda$, where $|I| \ll 1/(\gamma\ell)$ is very small compared to Λ and that in addition are measurable with respect to the sigma-algebra generated by the variables $\{m_\ell(\sigma,x)\}_{x\in I}$. Let us define the functions $U_\Delta^{s^\pm,\mu^\pm}$ and $\mathcal{F}^{s^\pm,\mu^\pm}_{\Delta,\beta,\rho}$ by

$$\begin{aligned} U_\Delta^{s^\pm,\mu^\pm}(m_\ell) \equiv & \gamma\ell \sum_{x,y\in\Delta} J_{\gamma\ell}(x-y)\frac{\|m_\ell(x) - m_\ell(y)\|_2^2}{4} \\ & + \gamma\ell \sum_{x\in\Delta, y\in\partial\Delta} J_{\gamma\ell}(x-y)\frac{\|m_\ell(x) - m^{(\mu^\pm,s^\pm)}\|_2^2}{2} \end{aligned} \tag{6.3}$$

and

$$\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) \equiv U_\Delta^{(\mu^\pm,s^\pm)}(m_\ell) + \gamma\ell\sum_{x\in\Delta} f_{x,\beta,\rho}(m_\ell(x)) \tag{6.4}$$

where

$$f_{x,\beta,\rho}(m_\ell(x)) \equiv -\frac{1}{\beta\ell}\ln I\!E_\sigma e^{\frac{\beta\ell}{2}\|m_\ell(\sigma,x)\|_2^2} 1\!\!I_{\{\|m_\ell(\sigma,x) - m_\ell(x)\|_2 \leq \rho\}} \tag{6.5}$$

For any $\delta > 0$ define the δ-covering F_δ of F as $F_\delta \equiv \{\sigma | \exists_{\sigma' \in F} : \forall_{x \in I} \|m_\ell(\sigma, x) - m_\ell(\sigma', x)\|_2 < \delta\}$.

With these notations we have the following large deviation estimates:

Theorem 6.2: *Let F and F_δ be as defined above. Assume that $|\Lambda| \leq g(\gamma)\gamma^{-1}$ where $g(\gamma)$ satisfies the hypothesis of Corollary 5.3. Then there exist ℓ, L, ζ, R all depending on γ and a set $\Omega_\Lambda \subset \Omega$ with $I\!P[\Omega_\Lambda^c] \leq Ke^{-c(g(\gamma))^{-1/2}} + e^{-cR/\gamma}$ such that for all $\omega \in \Omega_\Lambda$*

$$
\begin{aligned}
&\frac{\gamma}{\beta} \ln \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F) \\
&\leq - \inf_{\mu^\pm, s^\pm, \pm(w_\pm - u_\pm) \leq R} \left[\inf_{m_\ell \in F} \mathcal{F}^{(\mu^\pm, s^\pm)}_{[w_-, w_+], \beta, \gamma}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[w_-, w_+], \beta, \gamma}(m_\ell) \right] + er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{6.6}
$$

and for any $\delta > 0$, for γ small enough

$$
\begin{aligned}
&\frac{\gamma}{\beta} \ln \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F_\delta) \\
&\geq - \inf_{\mu^\pm, s^\pm, \pm(w_\pm - u_p m) \leq R} \left[\inf_{m_\ell \in F} \mathcal{F}^{(\mu^\pm, s^\pm)}_{[w_-, w_+], \beta, \gamma}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[w_-, w_+], \beta, \gamma}(m_\ell) \right] - er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{6.7}
$$

where $er(\ell, L, M, \hat{\zeta}, R)$ is a function of $\alpha \equiv \gamma M$ that tends to zero as $\alpha \downarrow 0$.

Proof: Relative to the interval I we introduce again the partition S from Section 4. While we will use again the estimate (4.14) we treat the terms corresponding to S_R somewhat differently. Let us introduce the constrained partition functions

$$
Z_{\beta,\gamma,\Lambda}[\omega](F) \equiv \mathcal{G}_{\beta,\gamma,\Lambda}[\omega](F) Z_{\beta,\gamma,\Lambda}[\omega]
\tag{6.8}
$$

Just as in Proposition 4.1 we have that

$$
\begin{aligned}
Z_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \leq\; & Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-, \sigma) = s^- e^{\mu^-}\}) Z^{(\mu^\pm, s^\pm)}_{\beta,\gamma,\Delta}(F) Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+, \sigma) = s^+ e^{\mu^+}\}) \\
& \times e^{8\gamma^{-1}(\zeta + 2\gamma L)}
\end{aligned}
\tag{6.9}
$$

and

$$
\begin{aligned}
Z_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \geq\; & Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-, \sigma) = s^- e^{\mu^-}\}) Z^{(\mu^\pm, s^\pm)}_{\beta,\gamma,\Delta}(F) Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+, \sigma) = s^+ e^{\mu^+}\}) \\
& \times e^{-8\gamma^{-1}(\zeta + 2\gamma L)}
\end{aligned}
\tag{6.10}
$$

where $\Delta = [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$ and $\Lambda^\pm$ are the two connected components of the complement of Δ in Λ. Using the trivial observation that

$$
Z_{\beta,\gamma,\Lambda} \geq Z_{\beta,\gamma,\Lambda}(\mathcal{A}(\mu^\pm = 1, s^\pm = 1, w_\pm))
\tag{6.11}
$$

this combines to

$$
\begin{aligned}
\mathcal{G}_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \leq & \frac{Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}} \\
& \times \frac{Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\})}{Z_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma) = e^1\})} \frac{Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma) = s^+ e^{\mu^+}\})}{Z_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,s) = e^1\})} \\
& \times e^{16\gamma^{-1}(\zeta+2\gamma L)}
\end{aligned}
\tag{6.12}
$$

The point is now that the ratios of partition functions on $\Lambda^\pm$ are in fact "close" to one. Indeed we have

Lemma 6.3: *Let $\Lambda = [w_- - \frac{1}{2}, w_+ + \frac{1}{2}]$ with $|\Lambda| \leq \gamma^{-1} g(\gamma)$, where $g(\gamma) \downarrow 0$ and $g(\gamma)/\gamma \geq c > 0$. Then*

$$
\left| \ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) - \ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = e^1\}) \right| \leq \beta\gamma^{-1} \left[(g(\gamma))^{1/4} + 10\zeta + 48\gamma L \right]
\tag{6.13}
$$

with probability greater than $1 - e^{-c\gamma^{-1}} - K e^{-c(g(\gamma))^{-1/2}}$.

Proof: Introducing a carefully chosen zero and using the triangle inequality, we see that

$$
\begin{aligned}
& \left| \ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) - \ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = e^1\}) \right| \\
& \leq \left| \ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) - \ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda\backslash w_-} + \ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda\backslash w_-} - \ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = e^1\}) \right| \\
& + \left| \ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda\backslash w_-} - I\!E \ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda\backslash w_-} \right| \\
& + \left| I\!E \ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda\backslash w_-} - I\!E \ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda\backslash w_-} \right| \\
& + \left| I\!E \ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda\backslash w_-} - \ln Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda\backslash w_-} \right|
\end{aligned}
\tag{6.14}
$$

The third term on the right hand side of (6.14) is zero by symmetry, while the second and fourth are bounded by Corollary 5.3 by $\gamma^{-1}(g(\gamma))^{-1/4}$ with probability at least $1 - e^{-c\gamma^{-1}} - K e^{-c(g(\gamma))^{-1/2}}$. To bound the first term we proceed as in the proof of Proposition 4.1, part i, that is we use the same decomposition as in (4.4) and (4.53). This gives that

$$
\begin{aligned}
\ln Z_{\beta,\gamma,\Lambda}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) - \ln Z^{(0,0,\mu^-,s^-)}_{\beta,\gamma,\Lambda\backslash w_-} &= \ln Z_{w_-,\beta,\gamma}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) \\
& + O\left(4\gamma^{-1}\left(\zeta + 2\gamma L\right)\right)
\end{aligned}
\tag{6.15}
$$

The constraint partition function on the block w_- is easily dealt with. First, we note that by (2.5) with probability greater than $1 - \exp(-c\gamma^{-1})$ we can replace the Hamiltonian by its blocked version on scale L at the expense of an error of order $\gamma^{-1}(16\gamma L)$. Then we can repeat the steps (4.58) to (4.60) to get that with the same probability,

$$
\ln Z_{w_-,\beta,\gamma}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) \geq -\beta\gamma^{-1}\left[\phi(a(\beta)) + \zeta^2 + \tfrac{\ln 2}{\ln L}\right] - \beta\gamma^{-1}(16\gamma L)
\tag{6.16}
$$

provided $\zeta \geq 2\sqrt{\frac{M}{L}}$. Using (4.61) and the large deviation bound (3.3), we also get

$$\ln Z_{w_-,\beta,\gamma}(\{\eta(w_-,\sigma) = s^- e^{\mu^-}\}) \leq -\beta\gamma^{-1}\left[\phi(a(\beta)) - \tfrac{1}{2}\zeta^2\right] + \beta\gamma^{-1}(16\gamma L) \tag{6.17}$$

The same bounds hold of course for the the term with (s^-, μ^-) replaced by $(1,1)$, so that we get an upper bound

$$\beta\gamma^{-1}\left[48\gamma L + 8\zeta + \frac{3}{2}\zeta^2\right] \tag{6.18}$$

for the first term on the right of (61.9). Putting all things together, we arrive at the assertion of the lemma.$\diamondsuit$

Lemma 6.3 asserts that to leading order, only the first ratio of partition functions is relevant in (6.12). On the other hand, since by Proposition 4.1, part (ii), we only need to consider the case $|\Delta| \leq R$, we can use the block approximation on scale ℓ for those, committing an error of order $\beta\gamma^{-1}(R\gamma\ell)$ only. We will make this precise in the next lemma.

Lemma 6.4: *For any $(s^\pm, \mu^\pm, w_\pm)$ and $I \subset \Delta \subset \Lambda$ and any F that is measurable with respect to the sigma algebra generated by $\{m_\ell(\sigma, x)\}_{x\in I}$*

$$\begin{aligned}\frac{\gamma}{\beta}\ln\frac{Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}} \leq &-\inf_{m_\ell\in F}\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) + \inf_{m_\ell}\mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell)\\ &+ c'\left(|\Delta|\gamma\ell + |\Delta|\gamma M|\ln\tfrac{2\ell}{M}| + |\Delta|\tfrac{M}{\ell}\right)\end{aligned} \tag{6.19}$$

and $\forall\delta > 0$ for sufficiently small γ

$$\begin{aligned}\frac{\gamma}{\beta}\ln\frac{Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F_\delta)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}} \geq &-\inf_{m_\ell\in F}\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) + \inf_{m_\ell}\mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell)\\ &+ c'\left(|\Delta|\gamma\ell + |\Delta|\gamma M|\ln\tfrac{2\ell}{M}| + |\Delta|\tfrac{M}{\ell}\right)\end{aligned} \tag{6.20}$$

with probability greater than $1 - e^{-c|\Delta|/\gamma}$.

Proof: Using Lemma 2.1, we see that

$$\begin{aligned}Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F) \leq\ & I\!E_\sigma 1\!\!1_{\{m_\ell(\sigma)\in F\}} e^{-\beta\gamma^{-1}\left[E^\ell_{\gamma,\Delta}(m_\ell(\sigma)) + E^{\ell,L}_{\gamma,\Delta}\left(m_\ell(\sigma_\Delta), m^{(\mu^\pm,s^\pm)}\right)\right]}\\ &\times e^{\beta\gamma^{-1}40|\Delta|\gamma\ell}\end{aligned} \tag{6.21}$$

and

$$\begin{aligned}Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta}(F) \geq\ & I\!E_\sigma 1\!\!1_{\{m_\ell(\sigma)\in F\}} e^{-\beta\gamma^{-1}\left[E^\ell_{\gamma,\Delta}(m_\ell(\sigma)) + E^{\ell,L}_{\gamma,\Delta}\left(m_\ell(\sigma_\Delta), m^{(\mu^\pm,s^\pm)}\right)\right]}\\ &\times e^{-\beta\gamma^{-1}40|\Delta|\gamma\ell}\end{aligned} \tag{6.22}$$

Now

$$
\begin{aligned}
&E^{\ell}_{\Delta}\left(m_\ell(\sigma_\Delta)\right)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)\\
&=E^{\ell}_{\Delta}\left(m_\ell(\sigma_\Delta)\right)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)+\gamma\ell\sum_{x\in\Delta}\frac{\|m_\ell(\sigma,x)\|_2^2}{2}+\gamma\ell\sum_{x\in\partial\Delta}\frac{[a(\beta)]^2}{2}\\
&-\gamma\ell\sum_{x\in\Delta}\frac{\|m_\ell(\sigma,x)\|_2^2}{2}-\gamma\ell\sum_{x\in\partial\Delta}\frac{[a(\beta)]^2}{2}\\
&=-\frac{1}{2}\gamma\ell\sum_{(x,y)\in\Delta\times\Delta}J_{\gamma\ell}(x-y)\left(m_\ell(\sigma,x),m_\ell(\sigma,y)\right)-\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\left(m_\ell(x,\sigma),m^{(\mu^\pm,s^\pm)}\right)\\
&+\gamma\ell\sum_{x\in\Delta}\frac{1}{2}\left(m_\ell(x,\sigma),m_\ell(x,\sigma)\right)+\gamma\ell\sum_{x\in\partial\Delta}\frac{1}{2}\left(m^{(\mu^\pm,s^\pm)},m^{(\mu^\pm,s^\pm)}\right)\\
&-\gamma\ell\sum_{x\in\Delta}\frac{\|m_\ell(\sigma,x)\|_2^2}{2}-\gamma\ell\sum_{x\in\partial\Delta}\frac{[a(\beta)]^2}{2}
\end{aligned}
\tag{6.23}
$$

On the other hand

$$
\begin{aligned}
&\gamma\ell\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)\frac{\|m_\ell(\sigma,x)-m_\ell(\sigma,y)\|_2^2}{4}+\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{\|m_\ell(\sigma,x)-m^{(\mu^\pm,s^\pm)}\|_2^2}{2}\\
&=-\gamma\ell\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}\left(m_\ell(\sigma,x),m_\ell(\sigma,y)\right)-\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}\left(m_\ell(\sigma,x),m^{(\mu^\pm,s^\pm)}\right)\\
&+\gamma\ell\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}\|m_\ell(\sigma,x)\|_2^2+\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\left(\frac{1}{2}\|m_\ell(\sigma,x)\|_2^2+\frac{1}{2}[a(\beta)]^2\right)\\
&=-\gamma\ell\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}\left(m_\ell(\sigma,x),m_\ell(\sigma,y)\right)-\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}\left(m_\ell(\sigma,x),m^{(\mu^\pm,s^\pm)}\right)\\
&+\gamma\ell\sum_{x\in\Delta}\frac{1}{2}\|m_\ell(\sigma,x)\|_2^2+\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}[a(\beta)]^2
\end{aligned}
\tag{6.24}
$$

Comparing (6.23) and (6.24) we find that

$$
\begin{aligned}
&E^{\ell}_{\Delta}\left(m_\ell(\sigma_\Delta)\right)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)+\gamma\ell\sum_{x\in\Delta}\frac{\|m_\ell(\sigma,x)\|_2^2}{2}+\gamma\ell\sum_{x\in\partial\Delta}\frac{[a(\beta)]^2}{2}\\
&=\gamma\ell\sum_{x,y\in\Delta}J_{\gamma\ell}(x-y)\frac{\|m_\ell(\sigma,x)-m_\ell(\sigma,y)\|_2^2}{4}+\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{\|m_\ell(\sigma,x)-m^{(\mu^\pm,s^\pm)}\|_2^2}{2}\\
&-\gamma\ell\sum_{x\in\Delta,y\in\partial\Delta}J_{\gamma\ell}(x-y)\frac{1}{2}[a(\beta)]^2\\
&\equiv U^{\mu^\pm,s^\pm}_{\Delta}\left(m_\ell(\sigma_\Delta)\right)-C(|\Delta|,\beta)
\end{aligned}
\tag{6.25}
$$

where $C(|\Delta|,\beta)$ is an irrelevant σ-independent constant that will drop out of all relevant formulas and may henceforth be ignored. For suitably chosen ρ we introduce a lattice $\mathcal{W}_{M,\rho}$ in $\mathbb{R}^M$ with spacing $\rho/\sqrt{M}$. Then for any domain $D\subset\mathbb{R}^M$, the balls of radius ρ centered at the points of $\mathcal{W}_{M,\rho}\cap D$ cover D. For reasons that should be clear from Section 3, we choose $\rho=2\sqrt{\frac{M}{\ell}}$. With probability greater than $1-\exp(-c\ell)$, $f_{x,\beta,\rho}(m_\ell(x))=\infty$ if $\|m\|_2^2>2$, while the number of lattice points within the ball of radius 2 are bounded by $\exp\left(M\ln\frac{2\ell}{M}\right)$. But this implies that

$$
\begin{aligned}
&\ln\left(\mathbb{E}_{\sigma_\Delta}\mathbb{1}_{\{m_\ell(\sigma)\in F\}}e^{-\beta\gamma^{-1}\left[E^\ell_\Delta(m_\ell(\sigma_\Delta)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)\right]}\right)\\
&\leq-\gamma^{-1}\beta\inf_{m_\ell\in F}\left[\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell)-C(|\Delta|,\beta)\right]+|\Delta|\left(M|\ln\tfrac{2\ell}{M}|+2\frac{M}{\ell}\right)
\end{aligned}
\tag{6.26}
$$

and also, if $\delta>2\sqrt{\frac{M}{\ell}}$,

$$
\begin{aligned}
&\ln\left(\mathbb{E}_{\sigma_\Delta}\mathbb{1}_{\{m_\ell(\sigma)\in F_\delta\}}e^{-\beta\gamma^{-1}\left[E^\ell_\Delta(m_\ell(\sigma_\Delta)+E^{\ell,L}_{\Delta,\partial\Delta}\left(m_\ell(\sigma_\Delta)|m^{(\mu^\pm,s^\pm)}\right)\right]}\right)\\
&\geq-\gamma^{-1}\beta\inf_{m_\ell\in F}\left[\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell)-C(|\Delta|,\beta)\right]-|\Delta|2\frac{M}{\ell}
\end{aligned}
\tag{6.27}
$$

Treating the denominator in the first line of (6.12) in the same way and putting everything together concludes the proof of the lemma.$\diamondsuit$

An immediate corollary of Lemma 6.4 is

Lemma 6.5: *For any $(s^\pm,\mu^\pm,w_\pm)$, $|\Lambda|\leq\gamma^{-1}g(\gamma)$ and any F that is measurable with respect to the sigma algebra generated by $\{m_\ell(\sigma,x)\}_{x\in I}$,*

$$
\begin{aligned}
\tfrac{\gamma}{\beta}\ln\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap\tilde{\mathcal{A}}(\mu^\pm,s^\pm,w_\pm))\leq&-\inf_{m_\ell\in F}\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell)+\inf_{m_\ell}\mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell)\\
&+c'\left(\gamma L+(g(\gamma))^{1/4}+\zeta+|\Delta|\gamma\ell+|\Delta|\gamma M|\ln\tfrac{2\ell}{M}|+|\Delta|\tfrac{M}{\ell}\right)
\end{aligned}
\tag{6.28}
$$

with probability greater than $1-Ke^{-c(g(\gamma))^{-1/2}}-2e^{-c/\gamma}$ for some finite positive numerical constants c,c',K.

Proof: This is an immediate consequence of (6.12) and Lemmata 6.3 and 6.4. $\diamondsuit$

We are now set to prove the upper bound in Theorem 6.2. Using the notation of Section 4 we have that

$$
\begin{aligned}
\ln\mathcal{G}_{\beta,\gamma,\Lambda}(F)&\leq\ln\left(\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R)+\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R^c)\right)\\
&=\ln\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R)+\ln\left(1+\frac{\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R^c)}{\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R)}\right)\\
&\leq4M^22R\sup_{\mu^\pm,s^\pm,\pm(w_\pm-u_\pm)\leq R}\ln\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm))+\ln\left(1+\frac{\exp\left(-c_2\beta LR\zeta\epsilon(\zeta)\right)}{\mathcal{G}_{\beta,\gamma,\Lambda}(F\cap S_R)}\right)
\end{aligned}
\tag{6.29}
$$

where we used (4.14). We see that the last term can be made irrelevantly small by choosing R sufficiently large. In fact, since we will consider events F those probability will be at least of order $\exp(-\gamma^{-1}\beta C)$, it will suffice to choose

$$R \gg \frac{1}{\gamma L\zeta\epsilon(\zeta)} \tag{6.30}$$

On the other hand, in order for the error terms in (6.19) to go to zero, we must assure that (note that $|\Delta| = |I| + 2R$ is of order R) $R(\gamma\ell + \frac{M}{\ell})$ tends to zero. With $\alpha \equiv \gamma M$, this means

$$R\left(\gamma\ell + \tfrac{\alpha}{\gamma\ell}\right) \downarrow 0 \tag{6.31}$$

From this we see that ℓ should be chosen as $\gamma\ell = \sqrt{\alpha}$ while R must satisfy $R\sqrt{\alpha} \downarrow 0$. (6.30) and (6.31) impose conditions on L and ζ, namely that

$$\tfrac{\sqrt{\alpha}}{\gamma L\zeta\epsilon(\zeta)} \downarrow 0 \tag{6.32}$$

Of course we also need that $\zeta \downarrow 0$ and $\gamma L \downarrow 0$, but clearly these constraints can be satisfied provided that $\alpha \downarrow 0$ as $\gamma \downarrow 0$. Thus the upper bound of Theorem 6.2 follows.

To prove the lower bound, we will actually need to make use of the upper bound. To do so, we need more explicit control of the functional $\mathcal{F}$, i.e. we have to use the explicit bounds on $f_{x,\beta,\rho}(m_\ell(x))$ in terms of the function Φ from Lemma 3.1.

Lemma 6.6: *The functional $\mathcal{F}$ defined in (6.4) satisfies*

$$\mathcal{F}^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\rho}(m_\ell) \geq U^{(\mu^\pm,s^\pm)}_{\Delta}(m_\ell) + \gamma\ell\sum_{x\in\Delta}\Phi_{x,\beta}(m_\ell(x)) - \frac{1}{2}|\Delta|\rho^2 \tag{6.33}$$

and

$$\inf_{m_\ell}\mathcal{F}^{(1,1,1,1)}_{\Delta,\beta,\rho}(m_\ell) \leq |\Delta|\phi_\beta(a(\beta)) + |\Delta|\frac{\ln 2}{\ell\beta} \tag{6.34}$$

where $\phi_\beta(a) \equiv \frac{a^2}{2} - \beta^{-1}\ln\cosh(\beta a)$.

Proof: Eq.(6.33) follows straightforward from (3.3). To get (6.34), just note that U is non-negative and is equal to zero for any constant m_ℓ, while from Lemma 3.1 it follows that

$$\begin{aligned}\inf_{m_\ell(x)} f_{x,\beta,\rho}(m_\ell(x)) &\leq \inf_{m_\ell(x)}\Phi_{x,\beta}(m_\ell(x)) + \frac{\ln 2}{\ell\beta}\\ &\leq \Phi_{x,\beta}(m^{(1,1)}) + \frac{\ln 2}{\ell\beta}\\ &= \phi_\beta(a(\beta)) + \frac{\ln 2}{\ell\beta}\end{aligned} \tag{6.35}$$

$\diamondsuit$

This concludes the derivation of the upper bound. We now turn to the corresponding lower bound. What is needed for this is an upper bound on the partition function that would be comparable to the lower bound (6.11). Now

$$\begin{aligned} Z_{\beta,\gamma,\Lambda} &= \sum_{(\mu^\pm,s^\pm)} I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} 1\!\!1_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}} \frac{Z_{\beta,\gamma,\Lambda}}{\sum_{(\mu^\pm,s^\pm)} I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} 1\!\!1_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}}} \\ &= \sum_{(\mu^\pm,s^\pm)} I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} 1\!\!1_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}} \frac{Z_{\beta,\gamma,\Lambda}}{I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} \left(1 - 1\!\!1_{\{\eta(w_\pm,\sigma)=0\}}\right)} \\ &= \sum_{(\mu^\pm,s^\pm)} I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} 1\!\!1_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}} \left[1 - \mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta(w_\pm,\sigma)=0\}\right)\right]^{-1} \end{aligned} \tag{6.36}$$

This is almost the same form as the one we want, except for the last factor. The point is now that we want to use our upper bound from Theorem 6.2 to show that $\mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta(w_\pm,\sigma)=0\}\right)$ is small, e.g. smaller than $1/2$. so that this entire factor is negligible on our scale. Remembering our estimate (4.14), one may expect an estimate of the order $\exp(-c_2\beta L\zeta\epsilon(\zeta))$, up to the usual errors. Unfortunately, these errors are of order $\exp(\pm\beta\gamma^{-1}(\zeta+\gamma L))$ and thus may offset completely the principle term. A way out of this apparent dilemma is given by our remaining freedom of choice in the parameters ζ and L; that is to say, to obtain the lower bound, we will use a $\hat\zeta$ and a $\hat L$ in such that first they still satisfy the requirement (6.32) while second $c_2\hat L\hat\zeta(\epsilon(\hat\zeta) \gg \gamma^{-1}\zeta + L$. This is clearly possible. With this in mind we get

Lemma 6.7: *With the same probability as in Lemma 6.5,*

$$\begin{aligned} &\frac{\gamma}{\beta}\ln\mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta_{\hat\zeta,\hat L}(w_\pm,\sigma)=0\}\right) \\ &\leq -\gamma\hat L\hat\zeta\frac{1-\delta}{2-\delta\hat\zeta}\epsilon(\delta\hat\zeta) + c'\left(\gamma L + (g(\gamma))^{1/4} + \zeta + R\gamma\ell + R\gamma M|\ln\tfrac{2\ell}{M}| + R\tfrac{M}{\ell}\right) \end{aligned} \tag{6.37}$$

Proof: The proof of this Lemma is very similar to the proof of (ii) of Proposition 4.1, except that in addition we use the upper bound of Lemma 6.5 to reduce the error terms. We will skip the details of the proof.$\diamondsuit$

Choosing $\hat L$ and $\hat\zeta$ appropriately, we can thus achieve that $\left[1 - \mathcal{G}_{\beta,\gamma,\Lambda}\left(\{\eta(w_\pm,\sigma)=0\}\right)\right]^{-1} \leq 2$ so that

$$\begin{aligned} Z_{\beta,\gamma,\Lambda} \leq& 2\sum_{(\mu^\pm,s^\pm)} I\!\!E_\sigma e^{-\beta H_\Lambda(\sigma_\Lambda)} 1\!\!1_{\{\eta(w_\pm,\sigma)=s^\pm e^{\mu^\pm}\}} \\ \leq& 2(2M)^2 \sup_{\mu^\pm,s^\pm} Z_{\beta,\gamma,\Lambda_-}\left(\{\eta(w_-\sigma)=s^-e^{\mu^-}\}\right) Z^{(\mu^\pm,s^\pm)}_{\beta,\gamma,\Delta} Z_{\beta,\gamma,\Lambda_+}\left(\{\eta(w_+\sigma)=s^+e^{\mu^+}\}\right) \\ & e^{+8\gamma^{-1}\beta(\hat\zeta+2\gamma\hat L)} \end{aligned} \tag{6.38}$$

(we will drop henceforth the distinction between $\hat L$ and L and $\hat\zeta$ and ζ). The first and third factor in the last line are, by Lemma 6.3, independent of $\mu^\pm, s^\pm$, up to the usual errors. The second partition function is maximal for $(\mu^+, s^+) = (\mu^-, s^-)$, (this will be shown later). Thus with probability greater than $1 - e^{-c\gamma^{-1}} - Ke^{-c(g(\gamma))^{-1/2}}$

$$\mathcal{G}_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^{\pm}, s^{\pm}, w_{\pm})) \geq \frac{Z^{(\mu^{\pm},s^{\pm})}_{\beta,\gamma,\Delta}(F)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Delta}} e^{-c'\beta\gamma^{-1}(\zeta+\gamma L+(g(\gamma))^{1/4})} \tag{6.39}$$

for some numerical constant c, c'. Using the second assertion of Lemma 6.4 allows us to conclude the poof of Theorem 6.2. $\diamondsuit\diamondsuit$

We are now ready to prove Theorem 6.1:

Proof of Theorem 6.1: Notice first that the first assertion (6.1) follows immediately from Lemma 6.7. Just note that

$$\mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\exists_{u\in\Lambda}\eta_{\hat{\zeta},\hat{L}}(u,\sigma)=0\right) \leq \sum_{u\in\Lambda} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(\{\eta_{\hat{\zeta},\hat{L}}(u,s)=0\}\right) \leq |\Lambda| e^{-c\beta\hat{L}\hat{\zeta}(\epsilon(\hat{\zeta}))} \tag{6.40}$$

for suitably chosen $\hat{L}, \hat{z}$. To prove (6.2), note that we need only consider the case where both $\eta(u,\sigma)$ and $\eta(u+1,\sigma)$ are non-zero. This follows then simply from the upper bound of Theorem 6.2 and the lower bound

$$\inf_{\mu^{\pm},s^{\pm}} \inf_{m_\ell:\eta(u,m_\ell)\neq\eta(u+1,m_\ell)\neq 0} U^{(\mu^{\pm},s^{\pm})}_{\Delta}(m_\ell) \geq \tfrac{1}{4}\gamma\ell \sum_{x\in u}\sum_{y\in u+1} J_{\gamma\ell}(x-y)\|m_\ell(x)-m_\ell(y)\|_2^2 \tag{6.41}$$

Using convexity, we see that

$$\begin{aligned}
&\gamma\ell \sum_{x\in u}\sum_{y\in u+1} J_{\gamma\ell}(x-y)\|m_\ell(x)-m_\ell(y)\|_2^2 \\
&\geq (\gamma\ell)^2 \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \sum_{x\in r}\sum_{y\in s} \|m_\ell(x)-m_\ell(y)\|_2^2 \\
&\geq (\gamma\hat{L})^2 \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \left\|\tfrac{\ell}{\hat{L}}\sum_{x\in r} m_\ell(x) - \tfrac{\ell}{\hat{L}}\sum_{y\in s} m_\ell(y)\right\|_2^2 \\
&= (\gamma\hat{L})^2 \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \left\|m_{\hat{L}}(r)-m_{\hat{L}}(s)\right\|_2^2
\end{aligned} \tag{6.42}$$

Inserting this inequality into (6.41) gives immediately that

$$\begin{aligned}
\inf_{\mu^{\pm},s^{\pm}} \inf_{m_\ell:\eta(u,m_\ell)\neq\eta(u+1,m_\ell)\neq 0} U^{(\mu^{\pm},s^{\pm})}_{\Delta}(m_\ell) &\geq \tfrac{1}{4} \sum_{\substack{r\in u, s\in u+1\\ |r-s|\leq(\gamma\hat{L})^{-1}-2}} \left((a(\beta))^2 - 2a(\beta)\hat{\zeta}\right) \\
&\geq \tfrac{1}{8}(1-2\gamma\hat{L})^2\left((a(\beta))^2 - 2a(\beta)\hat{\zeta}\right)
\end{aligned} \tag{6.43}$$

From here the proof of (6.2) is obvious. $\diamondsuit\diamondsuit$

This concludes our analysis of the free boundary condition measure in volumes of order $o(\gamma^{-1})$. We have seen that this measures are concentrated on constant profiles on some scale $\hat{L} \ll \gamma^{-1}$ (microscopic scale). In the next subsection we will analyse the measures with fixed equilibrium boundary conditions.

6.2 Finite volume, fixed symmetric boundary conditions

To proceed in order of increasing difficulty, we consider first the case where the boundary conditions are the same on both sides of the box Λ. Since these are compatible with one of the preferred constant profiles of the free boundary conditions measures and since the size of the box Λ we consider is so small that by our self-averaging results we know that the random fluctuations do not favour one of the constant values by a factor on the scale $\exp(\beta\gamma^{-1})$, we expect that the optimal profile will be the constant profile compatible with the boundary conditions. Indeed, we will prove

Theorem 6.8: *Assume that $|\Lambda| \leq g(\gamma)\gamma^{-1}$ where $g(\gamma)$ satisfies the hypothesis of Corollary 5.3. Then there exist ℓ, L, ζ, R all depending on γ and a set $\Omega_\Lambda \subset \Omega$ with $I\!P[\Omega_\Lambda^c] \leq Ke^{-c(g(\gamma))^{-1/2}} + e^{-cR/\gamma}$ such that for all $\omega \in \Omega_\Lambda$*

$$
\begin{aligned}
&\frac{\gamma}{\beta} \ln \mathcal{G}_{\beta,\gamma,\Lambda}^{(\mu,s,\mu,s)}[\omega](F) \\
&\leq - \inf_{\pm(w_\pm - u_\pm) \leq R} \left[\inf_{m_\ell \in F} \mathcal{F}_{[w_-,w_+],\beta,\gamma}^{(\mu,s,\mu,s)}(m_\ell) - \inf_{m_\ell} \mathcal{F}_{[w_-,w_+],\beta,\gamma}^{(1,1,1,1)}(m_\ell) \right] + er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{6.44}
$$

and for any $\delta > 0$, for γ small enough

$$
\begin{aligned}
&\frac{\gamma}{\beta} \ln \mathcal{G}_{\beta,\gamma,\Lambda}^{(\mu,s,\mu,s)}[\omega](F_\delta) \\
&\geq - \inf_{\pm(w_\pm - u_\pm) \leq R} \left[\inf_{m_\ell \in F} \mathcal{F}_{[w_-,w_+],\beta,\gamma}^{(\mu,s,\mu,s)}(m_\ell) - \inf_{m_\ell} \mathcal{F}_{[w_-,w_+],\beta,\gamma}^{(1,1,1,1)}(m_\ell) \right] - er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{6.45}
$$

where $er(\ell, L, M, \zeta, R)$ is a function of $\alpha \equiv \gamma M$ that tends to zero as $\alpha \downarrow 0$.

An immediate corollary of Theorem 6.8 is the analog of Theorem 6.1 for the measures $\mathcal{G}_{\beta,\gamma,\Lambda}^{(\mu,s,\mu,s)}[\omega]$:

Theorem 6.9: *Assume that $\gamma|\Lambda| \downarrow 0$, β large enough ($\beta > 1$) and $\gamma M(\gamma) \downarrow 0$. Then we can find $\gamma^{-1} \gg \hat{L} \gg 1$ and $\hat{\zeta} \downarrow 0$, such that on a subset $\Omega_\Lambda \subset \Omega$ with $I\!P(\Omega_\Lambda^c) \leq e^{-cg^{-1/2}(\gamma)}$ where $g(\gamma) \downarrow 0$ and $\gamma^{-1}g(\gamma) > c$, we have that for all $\omega \in \Omega_\Lambda$*

$$
\mathcal{G}_{\beta,\gamma,\Lambda}^{(\mu,s,\mu,s)}[\omega] \left(\exists_{u \in \Lambda} \eta_{\hat{\zeta},\hat{L}}(u,\sigma) \neq se^\mu \right) \leq e^{-\hat{L}g(\hat{\zeta})}
\tag{6.46}
$$

where $h(\zeta) = c\beta\zeta\epsilon(\zeta)$ and $\epsilon(\zeta)$ is defined in (3.15).

Remark: Eq. (6.46) implies that with $I\!P$-probability one

$$\lim_{\gamma\downarrow 0}\mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}[\omega]\left(\forall_{u\in\Lambda}\eta_{\hat\zeta,\hat L}(u,\sigma)=se^{\mu}\right)=1 \tag{6.47}$$

Proof of Theorem 6.8: Many of the technical steps in this proof are similar to those of the preceeding subsection, and we will stress only the new features here. Let us fix without restriction of generality $(\mu,s)=(1,1)$. We consider again the upper bound first. Proceeding as in (6.1), the first major difference is that (6.12) is replaced by

$$\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm))\leq\frac{Z^{(1,1,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\backslash w_-}}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda^-\backslash w_-}}\frac{Z^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\gamma}(F)}{Z^{(1,1,1,1)}_{\Delta,\beta,\gamma}}\frac{Z^{(\mu^+,s^+,1,1)}_{\beta,\gamma,\Lambda^+\backslash w_+}}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda^+\backslash w_+}}\times e^{c\gamma^{-1}(\zeta+\gamma L)} \tag{6.48}$$

where we have also used (6.15) through (6.17) to replace partition functions with boundary condition on one side and constraint on the other by partition functions with two-sided boundary conditions. While in the free boundary condition case, by symmetry, the ratios of partition functions on $\Lambda^\pm$ were seen to be negligible, we will show here that they favour $(\mu^\pm,s^\pm)=(1,1)$. To make this precise, define for any box $\Lambda\equiv[\lambda_-,\lambda_+]$ with $|\Lambda|=o(\gamma^{-1})$,

$$P^{(\tilde\mu,\tilde s,\mu,s)}_{\beta,\gamma,\Lambda}\equiv\frac{Z^{(\tilde\mu,\tilde s,\mu,s)}_{\beta,\gamma,\Lambda}}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} \tag{6.49}$$

In the case of symmetric boundary conditions, Corollary 5.3 provides the following estimates

$$e^{-c\beta\gamma^{-1}(g(\gamma))^{1/4}}\leq P^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}\leq e^{c\beta\gamma^{-1}(g(\gamma))^{1/4}} \tag{6.50}$$

All we need are thus estimates on the quantity $P^{(\tilde\mu,\tilde s,\mu,s)}_{\Lambda}$ for $(\tilde\mu,\tilde s)\neq(\mu,s)$. Without loss of generality we may consider the case $(\tilde\mu,\tilde s,\mu,s)=(1,1,2,1)$ only. As shown in the forthcoming lemma, the quantity

$$P_0\equiv\sup_{\substack{[w_-,w_+]\subset\Lambda\cup\partial\Lambda\\|w_--w_+|<2R}}\frac{Z^{(1,1,2,1)}_{\beta,\gamma,[w_-+1,w_+-1]}}{Z^{(1,1,1,1)}_{\beta,\gamma,[w_-+1,w_+-1]}} \tag{6.51}$$

with R chosen as in (6.31), will prove to be of special relevance in estimating $P^{(1,1,2,1)}_{\beta,\gamma,\Lambda}$. It has in a reasonable sense the interpretation of the probability of having a "jump". Note that the logarithm of P_0 is self-averaging so that, up to the usual error terms, by Corollary 5.3, the random quantity can be replaced by the following deterministic one

$$\bar P_0\equiv\sup_{\substack{[w_-,w_+]\subset\Lambda\cup\partial\Lambda\\|w_--w_+|<2R}}\exp\left(I\!E\ln Z^{(1,1,2,1)}_{\beta,\gamma,[w_-+1,w_+-1]}-I\!E\ln Z^{(1,1,1,1)}_{\beta,\gamma,[w_-+1,w_+-1]}\right) \tag{6.52}$$

With this notations we have the

Lemma 6.10: *Assume that R satisfies (6.30) and that $|\Lambda| < \gamma^{-1}g(\gamma)$ where $g(\gamma)$ is chosen as in Corollary 5.3. Then, there exists $\ell, L, \hat{L}, \zeta, \hat{\zeta}$ all depending on γ such that, with a probability greater than $1 - e^{-c'\gamma^{-1}R} - Ke^{-c''(g(\gamma))^{-1/2}}$, where K, c, c' and c'' are strictly positive numerical constants,*

$$P^{(1,1,2,1)}_{\beta,\gamma,\Lambda} \le \bar{P}_0 e^{\gamma^{-1} er'(\ell,L,M,\zeta,R)} \tag{6.53}$$

and, if in addition $|\Lambda| > R$, for γ small enough,

$$P^{(1,1,2,1)}_{\beta,\gamma,\Lambda} \ge \bar{P}_0 e^{-\gamma^{-1} er'(\ell,\hat{L},M,\hat{\zeta},R)} \tag{6.54}$$

where, $er'(\ell, \hat{L}, M, \zeta, R)$ is a function of α that tends to zero as $\alpha \downarrow 0$.

Remark: Lemma 6.10 states a very crucial result that can be paraphrased as follows: If the boundary conditions over a volume Λ with $|\Lambda| = o(\gamma^{-1})$ require a "jump", than this jump takes place somewhere in the volume over a region smaller than $2R$; in particular, and this will become evident in the proof, there will occur one single "jump". Note that we cannot determine the precise location of this jump. The optimal position will be determined by the randomness.

The proof of Lemma 6.10. relies on the important fact that, as stated in the next lemma, the quantity P_0 is exponentially small.

Lemma 6.11: *With the notations of Lemma 6.4 we have:*

i) With a probability greater than $1 - e^{-cM}$, for some constant $c > 0$,

$$P_0 \ge e^{-\frac{1}{2}\beta\gamma^{-1}a^2(\beta)} e^{-c\beta\gamma^{-1}\left(R\gamma\ell + R\gamma M|\ln\frac{2\ell}{M}| + R\frac{M}{\ell} + 2R\frac{\ln 2}{\ell}\right)} \tag{6.55}$$

ii) There exists $\tilde{\zeta}_0 > 0$ depending on β such that for all $\tilde{\zeta}_0 \ge \tilde{\zeta} \ge 2a(\beta)\sqrt{\frac{M}{\ell}}$, with a probability greater than $1 - e^{-c'M}$, for some constant $c' > 0$,

$$P_0 \le e^{-\beta\gamma^{-1}\sqrt{\epsilon(\tilde{\zeta})}\left(\sqrt{12((a(\beta))^2 - 4\tilde{\zeta}^2)} - 3\sqrt{\epsilon(\tilde{\zeta})}\right)} e^{c\beta\gamma^{-1}\left(R\gamma\ell + R\gamma M|\ln\frac{2\ell}{M}| + R\frac{M}{\ell}\right)} \tag{6.56}$$

We will assume in the sequel that the parameters ℓ, L, M and R satisfy the set of conditions (6.30) to (6.32) from Section 6.1. It is then clear that the parameter $\tilde{\zeta}$ in part ii) of Lemma 6.11 can always be chosen in such a way that the exponential decrease of the first term in the r.h.s. of (6.56) compensates the increase of the second one. We will postpone the proof of Lemma 6.11 to the end of this subsection.

Proof of lemma 6.10: Without loss of generality we will, for convenience, consider only sets Λ of the form $\Lambda \equiv [\lambda^- - \frac{1}{2}, \lambda^+ + \frac{1}{2}]$ where $\lambda^\pm$ are assumed to be integers. We start with the proof of the upper bound (6.53). Let us define the set

$$B \equiv \left\{\sigma : \forall_{u\in\Lambda} \eta(u,\sigma) \in \{0, e^1, e^2\}\right\} \tag{6.57}$$

We further define

$$u_1(\sigma) \equiv \begin{cases} \sup\left\{u \in [\lambda_- - \frac{1}{2}, \lambda_+ + \frac{1}{2}] \mid \eta(u,\sigma) = e^1\right\} & \text{, if such } u \text{ exists} \\ \lambda_- - 1 & \text{, otherwise} \end{cases} \tag{6.58}$$

$$u_2(\sigma) \equiv \begin{cases} \inf\left\{u \in (u_1(\sigma), \lambda_+ + \frac{1}{2}] \mid \eta(u,\sigma) = e^2\right\} & \text{, if such } u \text{ exists} \\ \lambda_+ + 1 & \text{, otherwise} \end{cases} \tag{6.59}$$

and we set

$$B(u_1,u_2) \equiv \{\sigma \in B \mid u_1(\sigma) = u_1, u_2(\sigma) = u_2\} \tag{6.60}$$

A piece of profile between locations $u_1(\sigma)$ and $u_2(\sigma)$ will be called a "jump" between equilibrium (1,1) and $(2,1)$. For R chosen as in (6.31), we will set moreover

$$C \equiv \bigcup_{\substack{\lambda_- - 1 \leq u_1 < u_2 \leq \lambda_+ + 1 \\ |u_2 - u_1| < 2R}} B(u_1,u_2) \tag{6.61}$$

and

$$D \equiv \bigcup_{\substack{\lambda_- - 1 \leq u_1 < u_2 \leq \lambda_+ + 1 \\ |u_2 - u_1| \geq 2R}} B(u_1,u_2) \tag{6.62}$$

Obviously,

$$Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda} = Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B) + Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B^c) \tag{6.63}$$

and

$$Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B) = Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(C) + Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(D) \tag{6.64}$$

Now, on the one hand, we have

$$\begin{aligned} \frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(D)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} &\leq \sum_{\substack{\lambda_- - 1 \leq u_1 < u_2 \leq \lambda_+ + 1 \\ |u_2 - u_1| \geq 2R}} \frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}\left(\forall_{u_1<u<u_2} \eta(u,\sigma) = 0\right)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} \\ &\leq \sum_{\substack{\lambda_- - 1 \leq u_1 < u_2 \leq \lambda_+ + 1 \\ |u_2 - u_1| \geq 2R}} \frac{Z_{\beta,\gamma,[u_1,u_2]}\left(\forall_{u_1<u<u_2} \eta(u,\sigma) = 0\right)}{Z^{(1,1,1,1)}_{\beta,\gamma,[u_1,u_2]}} e^{+c\beta\gamma^{-1}\left(\gamma L + \zeta + (g(\gamma))^{1/4}\right)} \end{aligned} \tag{6.65}$$

where we have proceeded by complete analogy with the proof of the upper bound of Theorem 6.2 (see (6.8)-(6.12) and Lemma 6.3) to chop out the partition functions in $[\lambda_- - \frac{1}{2}, u_1]$ and $[u_2, \lambda_+ + \frac{1}{2}]$, and where we have dropped the boundary conditions of $Z_{\beta,\gamma,[u_1,u_2]}$ in the numerator of the last line. This holds with a probability greater than $1 - Ke^{-c(g(\gamma))^{-1/2}}$. Up to some minor modifications, it then follows from the proof of Proposition 4.1, part ii), that, with a probability greater than $1 - e^{-c'M} - Ke^{-c(g(\gamma))^{-1/2}}$,

$$\frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(D)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} \leq |\Lambda|^2 e^{-\beta L R \zeta \epsilon(\zeta)} e^{c\beta\gamma^{-1}\left(\gamma L + \zeta + (g(\gamma))^{1/4}\right)} \tag{6.66}$$

On the other hand, we have also that, with a probability greater than $1 - Ke^{-c(g(\gamma))^{-1/2}}$,

$$\begin{aligned} \frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(C)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} &\le \sum_{\substack{\lambda_- -1 \le u_1 < u_2 \le \lambda_+ +1 \\ |u_2-u_1|<2R}} \frac{Z^{(1,1,2,1)}_{\beta,\gamma,[u_1+1,u_2-1]}\left(\forall_{u_1<u<u_2}\eta(u,\sigma)=0\right)}{Z^{(1,1,1,1)}_{\beta,\gamma,[u_1+1,u_2-1]}} \\ &\quad \times e^{c\beta\gamma^{-1}(\gamma L+\zeta+(g(\gamma))^{1/4})} \\ &\le 2R|\Lambda| P_0 e^{c\beta\gamma^{-1}\left(\gamma L+\zeta+(g(\gamma))^{1/4}\right)} \end{aligned} \tag{6.67}$$

From now on we will abstain from specifying the probability with which our various estimates hold; this will straightforwardly follow from the different results called into play. Now, using the lower bound (6.55) of Lemma 6.11 and recalling that R is chosen large enough to satisfy the constraint (6.30), we see that the r.h.s of (6.66) is negligibly small compared with that of (6.67). Combining this with (6.63) and using Corollary 5.3 we then arrive at

$$\frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} \le \bar{P}_0 e^{c'\beta\gamma^{-1}(\gamma L+\zeta+(g(\gamma))^{1/4})} \tag{6.68}$$

We are therefore left to consider the constrained partition function $Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B^c)$. By definition, for any $\sigma \in B^c$, there must exist $u \in \Lambda$ such that $\eta(u,\sigma) = se^{\mu}$, with $(s,\mu) \notin \{(1,1),(1,2)\}$. This means that we can define the four random locations

$$u_1^+(\sigma) = \sup\left\{u \in [\lambda_- - \tfrac{1}{2}, \lambda_+ + \tfrac{1}{2}] \mid \eta(u,\sigma) \notin \{0, e^2\}\right\} \tag{6.69}$$

$$u_2^+(\sigma) \equiv \begin{cases} \inf\left\{u \in [u_1^+(\sigma), \lambda_+ + \tfrac{1}{2}] \mid \eta(u,\sigma) = e^2\right\} & \text{, if such } u \text{ exists and} \\ \lambda_+ + 1 & \text{, otherwise} \end{cases} \tag{6.70}$$

and

$$u_1^-(\sigma) \equiv \inf\left\{u \in [\lambda_- - \tfrac{1}{2}, \lambda_+ + \tfrac{1}{2}] \mid \eta(u,\sigma) \notin \{0, e^1\}\right\} \tag{6.71}$$

$$u_2^-(\sigma) \equiv \begin{cases} \sup\left\{u \in [\lambda_- - \tfrac{1}{2}, u_2^-(\sigma)] \mid \eta(u,\sigma) = e^1\right\} & \text{, if such } u \text{ exists and} \\ \lambda_- - 1 & \text{, otherwise} \end{cases} \tag{6.72}$$

and can be sure that, for all $\sigma \in B^c$, $u_1^-(\sigma) \le u_1^+(\sigma)$. In other words, any configuration in B^c contains two "jumps". The following two events, $B^-(u_2^-)$ and $B^-(u_2^+)$, describe, respectively the leftmost and rightmost of these jumps. For $u_2^- > \lambda_- - 1$ we set

$$B^-(u_2^-) \equiv \left\{\sigma \middle| \forall_{\lambda_- -1<u<u_2^-}\eta(u,\sigma) \in \{0,e^1\},\, \eta(u_2^-,\sigma) = e^1,\, \forall_{u_2^-<u<u_1^-}\eta(u,\sigma)=0\right\} \tag{6.73}$$

while

$$B^-(\lambda_- - 1) \equiv \left\{\sigma \middle| \forall_{\lambda_- -1<u<u_1}\, \eta(u,\sigma) = 0\right\} \tag{6.74}$$

Similarly we set, for $u_2^+ < \lambda_+ + 1$,

$$B^+(u_2^+) \equiv \left\{\sigma \middle| \forall_{u_2^+<u<\lambda^+ -1}\eta(u,\sigma) \in \{0,e^2\},\, \eta(u_2^+,\sigma) = e^2\, \forall_{u_+^1<u<u_2^+}\eta(u,\sigma)=0\right\} \tag{6.75}$$

and

$$B^+(\lambda_+ + 1) \equiv \left\{\sigma \middle| \forall_{u^1_+ + 1 < u < \lambda^+ - 1} \eta(u,\sigma) = 0\right\} \tag{6.76}$$

Proceeding in the (by now) usual way, we see from here that

$$\begin{aligned} \frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B^c)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} \leq & \sup_{\lambda_- - 1 \leq u_2^- < u_1^- \leq u_1^+ < u_2^+ \leq \lambda_+ + 1} \sup_{\substack{(\mu^-,s^-)\neq(1,1)\\(\mu^+,s^+)\neq(1,2)}} \left\{ \frac{Z^{(\mu^-,s^-,\mu^+,s^+)}_{[u_1^- + \frac12, u_1^+ - \frac12],\beta,\gamma}}{Z^{(1,1,1,1)}_{\beta,\gamma,[u_1^- + \frac12, u_1^+ - \frac12]}} \right. \\ & \times \left. \frac{Z^{(1,1,\mu^-,s^-)}_{\beta,\gamma,[\lambda_- - \frac12, u_1^- - \frac12]}(B^-(u_2^-))}{Z^{(1,1,1,1)}_{\beta,\gamma,[\lambda_- - \frac12, u_1^- - \frac12]}} \frac{Z^{(\mu^+,s^+,2,1)}_{\beta,\gamma,[u_1^+ + \frac12, \lambda_+ + \frac12]}(B^+(u_2^+))}{Z^{(1,1,1,1)}_{\beta,\gamma,[u_2^+ + \frac12, \lambda_+]}} \right\} \\ & \times |\Lambda|^4 M^2 e^{c\beta\gamma^{-1}(\gamma L + \zeta + (g(\gamma))^{1/4})} \end{aligned} \tag{6.77}$$

Clearly, each of the two terms in the second line of (6.77) is bounded as in (6.68), so that, up to the error term, we get the relation[2]

$$\frac{Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(B^c)}{Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}} \leq (\bar P_0)^2 \sup_{\lambda_- - 1 \leq u_2^- < u_1^- \leq u_1^+ < u_2^+ \leq \lambda_+ + 1} \sup_{\substack{(\mu^-,s^-)\neq(1,1)\\(\mu^+,s^+)\neq(1,2)}} P^{(\mu^-,s^-,\mu^+,s^+)}_{\beta,\gamma,[u_1^- + \frac12, u_1^+ + \frac12]} \tag{6.78}$$

and combining this with (6.63) and (6.68) we arrive, still up to the error term, at

$$P^{(1,1,2,1)}_{\Lambda} \leq \bar P_0 + (\bar P_0)^2 \sup_{\lambda_- - 1 \leq u_2^- < u_1^- \leq u_1^+ < u_2^+ \leq \lambda_+ + 1} \sup_{\substack{(\mu^-,s^-)\neq(1,1)\\(\mu^+,s^+)\neq(1,2)}} P^{(\mu^-,s^-,\mu^+,s^+)}_{\beta,\gamma,[u_1^- + \frac12, u_1^+ + \frac12]} \tag{6.79}$$

We immediately see from this recursion that the supremum over the $\mu^\pm, s^\pm$ will be realized for $(\mu^-, s^-) = (\mu^+, s^+)$. But putting the estimates (6.50) together with the upper bound (6.56) of Lemma 6.11 and Corollary 5.3 we get that, for small enough γ, $\bar P_0 P^{(\mu,s,\mu,s)}_{\beta,\gamma,[u_1^- + \frac12, u_1^+ + \frac12]} << 1$. From this the upper bound (6.53) is readily obtained. $\diamondsuit$

We now turn to the proof of the lower bound (6.54). First note that for any $[w_-, w_+] \subset \Lambda$,

$$\begin{aligned} Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda} & \geq Z^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(\mathcal{A}(1,1,2,1,w_\pm)) \\ & \geq Z^{(1,1,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma) = e^1\}) Z^{(1,1,2,1)}_{\Delta,\beta,\gamma} Z^{(0,0,2,1)}_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma) = e^2\}) e^{-8\gamma^{-1}(\zeta + 2\gamma L)} \end{aligned} \tag{6.80}$$

where we obtained the second inequality by proceeding just as in (6.9). The difficulty thus lies in establishing a corresponding upper bound for the partition function $Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda}$. But this can be done by, basically, repeating the proof of the lower bound of Theorem 6.2. I.e., we first use the decomposition (6.36) to write

$$Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda} = \sum_{\mu^\pm, s^\pm} I\!\!E_\sigma e^{H_{\gamma,\Lambda}(\sigma_\Lambda) + W_{\gamma,\Lambda}(\sigma_\Lambda, m^{(\mu^\pm,s^\pm)})} 1\!\!\mathrm{I}_{\{\eta(w_\pm,\sigma^\pm) = s^\pm e^{\mu^\pm}\}} \left[1 - \mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}(\{\eta(w_\pm,\sigma^\pm) = 0\})\right]^{-1} \tag{6.81}$$

[2] Observe that this inequality shows in particular that the probability of having more than one jump is bounded by the square of the probability of having one jump.

anticipating that $\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}(\{\eta(w_\pm,\sigma^\pm)=0\})$ can be shown to be very small. We will prove that Lemma 6.7 still holds when the Gibbs measure with free boundary conditions in (6.37) is replaced by $\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}$. Assuming for the moment that this is true we get, as in (6.38) and with the same choices of the parameters $\hat\zeta$ and $\hat L$,

$$
\begin{aligned}
Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda} \leq & 2(2M)^2 \sup_{\mu^\pm,s^\pm} Z^{(1,1,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\}) Z^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\gamma} Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma)=s^+e^{\mu^+}\}) \\
& e^{-8\gamma^{-1}(\hat\zeta+2\gamma\hat L)}
\end{aligned}
\tag{6.82}
$$

Next, proceeding as in (6.48) to replace constrained partition functions with free boundary condition on one side, by partition functions with two-sided boundary conditions, we have

$$
\begin{aligned}
& \frac{Z^{(1,1,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=s^-e^{\mu^-}\})}{Z^{(1,1,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=e^1\})} \frac{Z^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\gamma}}{Z^{(1,1,1,1)}_{\Delta,\beta,\gamma}} \frac{Z^{(0,0,1,1)}_{,\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma)=s^+e^{\mu^+}\})}{Z^{(0,0,1,1)}_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma)=e^1\})} \\
\leq & P^{(1,1,\mu^-,s^-)}_{\beta,\gamma,\Lambda^-\backslash w_-} P^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\gamma} P^{(\mu^+,s^+,1,1)}_{\beta,\gamma,\Lambda^+\backslash w_+} e^{c\gamma^{-1}(\zeta+2\gamma L)} \\
\leq & e^{c'\gamma^{-1}(\zeta+2\gamma L+(g(\gamma))^{1/4}))}
\end{aligned}
\tag{6.83}
$$

where, to obtain the last line, we used (6.50) to treat terms with symmetric boundary conditions while we used, in the case of asymmetric boundary conditions, the upper bound (6.53) of Lemma 6.10 together with Lemma 6.11 and Corollary 5.3. From this and (6.82) it follows that, up to the error term,

$$
Z^{(1,1,1,1)}_{\beta,\gamma,\Lambda} \leq 2(2M)^2 \sup_{\mu^\pm,s^\pm} Z^{(1,1,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma)=e^1\}) Z^{(1,1,1,1)}_{\Delta,\beta,\gamma} Z^{(0,0,1,1)}_{\beta,\gamma,,\Lambda^+}(\{\eta(w_+,\sigma)=e^1\}) \tag{6.84}
$$

Following a procedure with which the reader is now well acquainted, (6.80) together with (6.84) easily yields (6.54) by choosing w_- and w_+ as those which satisfy the constrained supremum problem in (6.51). Of course we must ask that $|\Delta|>R$ to ensure that this choice is always possible.

To complete the proof of (6.54) it remains to show that Lemma 6.7 holds when replacing $\mathcal{G}_{\beta,\gamma,\Lambda}$ by $\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}$. To do so, all we need is to prove the analogous of Lemma 6.5 for the measure $\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}$. But this is an immediate consequence of (6.48). Indeed, we can for the present purpose be content with the bounds from Corollary 5.3. to estimate (although roughly) the first and third factor in the r.h.s. of (6.48), while using Lemma 6.4 to treat the middle term. This concludes the proof of Lemma 6.10.$\diamondsuit$

We are now ready to continue the proof of the upper bound of Theorem 6.8. Remember that we were left in (6.48) to estimate the ratios of partition functions in $\Lambda^\pm$. In the case of asymmetric boundary conditions i.e., $(\mu^\pm,s^\pm)\neq(1,1)$, Lemma 6.10 enables us to replace these quantities by the corresponding ratios in boxes of length at least R. More precisely, consider two boxes Λ' and

Λ such that $\Lambda' \subset \Lambda$ and $R < |\Lambda'| < |\Lambda| < \gamma^{-1} g(\gamma)$ where $g(\gamma)$ is chosen as in Corollary 5.3. Then, Lemma 6.10 implies, for any $(\tilde\mu, \tilde s) \neq (\mu, s)$, that

$$P^{(\tilde\mu,\tilde s,\mu,s)}_{\beta,\gamma,\Lambda} \leq P^{(\tilde\mu,\tilde s,\mu,s)}_{\beta,\gamma,\Lambda'} e^{+\gamma^{-1}(er'(\ell,L,M,\zeta,R)+er'(\ell,L,M,\hat\zeta,R))} \tag{6.85}$$

Therefore, defining the boxes $\widetilde\Lambda^- \equiv [w_- + \frac{1}{2} - R, w_- + \frac{1}{2}]$ and $\widetilde\Lambda^+ \equiv [w_+ - \frac{1}{2}, w_+ - \frac{1}{2} + R]$, adjacent to Δ on its left, respectively right hand side, we have, up the the error terms,

$$\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \leq P^{(1,1,\mu^-,s^-)}_{\widetilde\Lambda^-\backslash w_-,\beta,\gamma} \frac{Z^{(\mu^\pm,s^\pm)}_{\Delta,\beta,\gamma}(F)}{Z^{(1,1,1,1)}_{\Delta,\beta,\gamma}} P^{(\mu^+,s^+,1,1)}_{\widetilde\Lambda^+\backslash w_+,\beta,\gamma} \tag{6.86}$$

By (6.50) a relation of the form (6.84), and hence (6.86), trivially holds in the case of symmetric boundary conditions. From here we can easily reconstruct the ratio of partition functions in $\widetilde\Delta \equiv \widetilde\Lambda^- \cup \Delta \cup \widetilde\Lambda^+$ with $(1,1,1,1)$-boundary conditions. I.e., proceeding much along the line of the proof of the upper bound of Lemma 6.10 (using in particular (6.84)) we obtain, up to the usual error term,

$$\begin{aligned}
\mathcal{G}^{(1,1,1,1)}_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) &\leq \frac{Z^{(1,1,1,1)}_{\widetilde\Delta,\beta,\gamma}(F \cap \{\eta(w_\pm,\sigma) = s^\pm e^{\mu^\pm}\})}{Z^{(1,1,1,1)}_{\widetilde\Delta,\beta,\gamma}} \\
&\leq \frac{Z^{(1,1,1,1)}_{\widetilde\Delta,\beta,\gamma}(F)}{Z^{(1,1,1,1)}_{\widetilde\Delta,\beta,\gamma}}
\end{aligned} \tag{6.87}$$

The upper bound of Theorem 6.8 then follows from (6.87) and Lemma 6.4 just as the upper bound of Theorem 6.2 follows from Lemma 6.5.

At this point, the proof of the lower bound (6.54) is a simple matter. In full generality, for arbitrary $\Delta \equiv [w_- + \frac{1}{2}, w_+ - \frac{1}{2}]$ and any (μ, s),

$$\mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}(F) \geq \frac{Z^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm = \mu, s^\pm = s, w_\pm))}{Z^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}} \tag{6.88}$$

Proceeding as in (6.9) to bound the numerator in (6.87) and using (6.84) to treat the denominator we get

$$\begin{aligned}
&\frac{Z^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}(F \cap \mathcal{A}(\mu^\pm = \mu, s^\pm = s, w_\pm))}{Z^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}} \\
&\approx \frac{Z^{(\mu,s,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma) = se^\mu\})}{Z^{(\mu,s,0,0)}_{\beta,\gamma,\Lambda^-}(\{\eta(w_-,\sigma) = e^1\})} \frac{Z^{(\mu,s,\mu,s)}_{\Delta,\beta,\gamma}(F)}{Z^{(\mu,s,\mu,s)}_{\Delta,\beta,\gamma}} \frac{Z^{(0,0,\mu,s)}_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma) = se^\mu\})}{Z^{(0,0,\mu,s)}_{\beta,\gamma,\Lambda^+}(\{\eta(w_+,\sigma) = e^1\})}
\end{aligned} \tag{6.89}$$

Again, we recognise in the first and last factor above the quantities $P^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}$ for which we have the estimates (6.50). Thus, up to the usual error term,

$$\mathcal{G}^{(\mu,s,\mu,s)}_{\beta,\gamma,\Lambda}(F) \geq \frac{Z^{(\mu,s,\mu,s)}_{\Delta,\beta,\gamma}(F)}{Z^{(\mu,s,\mu,s)}_{\Delta,\beta,\gamma}} \tag{6.90}$$

Now, (6.90) and Lemma 6.4 yield (6.45) by choosing $w_\pm$ as the solutions of the variational problem in (6.44). This completes the proof of Theorem 6.8. $\diamondsuit$

We finally are left to give the proof of Lemma 6.11

Proof of lemma 6.11: To prove the lower bound (6.55), just note that for any event F satisfying the assumptions of Lemma 6.4, and, making use of the lower bound (6.20), we have.

$$\begin{aligned} P_0 \geq \exp &\left(-\beta\gamma^{-1} \left[\inf_{m_\ell \in F} \mathcal{F}^{(1,1,2,1)}_{[\lambda_-,\lambda_+]}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[\lambda_-,\lambda_+]}(m_\ell) \right] \right) \\ &\times e^{-c\beta\gamma^{-1}\left(R\gamma\ell + R\gamma M |\ln \frac{2\ell}{M}| + R\frac{M}{\ell}\right)} \end{aligned} \tag{6.91}$$

Now, choosing the event F as

$$F \equiv \left\{ \{m_\ell(x)\}_{x\in I} \middle| m_\ell(x) = a(\beta)e^1 \forall x \leq 0, m_\ell(x) = a(\beta)e^2 \forall x > 0 \right\} \tag{6.92}$$

it easily follows from the definition (6.4) of $\mathcal{F}$ together with the estimates of Lemma 3.1 and Proposition 3.2 that, under their respective assumptions,

$$\begin{aligned} \inf_{m_\ell \in F} \mathcal{F}^{(1,1,2,1)}_{[\lambda_-,\lambda_+]}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[\lambda_-,\lambda_+]}(m_\ell) &\leq \inf_{m_\ell \in F} U^{(1,1,2,1)}_{[\lambda_-,\lambda_+]}(m_\ell) + 2R\frac{\ln 2}{\beta\ell} + R\frac{M}{\ell} \\ &\leq \frac{a^2(\beta)}{2} + 2R\frac{\ln 2}{\beta\ell} + R\frac{M}{\ell} \end{aligned} \tag{6.93}$$

from which (6.55) follows. To prove the lower bound (6.56) we make use of the bound (6.19) of Lemma 4 to write

$$\begin{aligned} P_0 \leq \sup_{\substack{\lambda_- - 1 \leq u_1 < u_2 \leq \lambda_+ + 1 \\ |u_2 - u_1| < 2R}} &\exp\left(-\beta\gamma^{-1} \left[\inf_{m_\ell} \mathcal{F}^{(1,1,2,1)}_{[u_1+1,u_2-1]}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{[u_1+1,u_2-1]}(m_\ell) \right] \right) \\ &\times e^{c\beta\gamma^{-1}\left(R\gamma\ell + R\gamma M |\ln \frac{2\ell}{M}| + R\frac{M}{\ell}\right)} \end{aligned} \tag{6.94}$$

(6.56) is then an immediate consequence of (6.94) together with the following proposition

Proposition 6.12: *There exists $\tilde{\zeta}_0 > 0$ depending on β such that for all $\tilde{\zeta}_0 \geq \tilde{\zeta} \geq 2a(\beta)\sqrt{\frac{M}{\ell}}$ and for all boxes Δ*

$$\inf_{m_\ell} \mathcal{F}^{(1,1,2,1)}_{\Delta}(m_\ell) - \inf_{m_\ell} \mathcal{F}^{(1,1,1,1)}_{\Delta}(m_\ell) \geq \sqrt{\epsilon(\tilde{\zeta})} \left(\sqrt{12((a(\beta))^2 - 4\tilde{\zeta}^2)} - 3\sqrt{\epsilon(\tilde{\zeta})} \right) \tag{6.95}$$

with probability greater than $1 - e^{-cM}$.

The proof of Proposition 6.12, which is somewhat technical, will be the object of Section 7. With this, the proof of Lemma 6.11 is concluded. $\diamondsuit$

6.3 Finite volume, fixed asymmetric boundary conditions

In this last subsection we consider the case where the boundary conditions to the right and to the left of the box Λ are distinct. We would expect here that the optimal profile will be the "jump" profile compatible with these conditions. We will prove

Theorem 6.13: *Assume that $|\Lambda| \leq g(\gamma)\gamma^{-1}$ where $g(\gamma)$ satisfies the hypothesis of Corollary 5.3. Then there exist ℓ, L, ζ, R all depending on γ and a set $\Omega_\Lambda \subset \Omega$ with $I\!P[\Omega_\Lambda^c] \leq Ke^{-c(g(\gamma))^{-1/2}} + e^{-c/\gamma}$ such that for all $\omega \in \Omega_\Lambda$, for any $(\tilde{\mu}, \tilde{s}) \neq (\mu, s)$,*

$$
\begin{aligned}
&\frac{\gamma}{\beta} \ln \mathcal{G}_{\beta,\gamma,\Lambda}^{(\tilde{\mu},\tilde{s},\mu,s)}[\omega](F) \\
&\leq - \inf_{\pm(w_\pm - u_\pm) \leq R} \left[\inf_{m_\ell \in F} \mathcal{F}_{[w_-,w_+]}^{(\tilde{\mu},\tilde{s},\mu,s)}(m_\ell) - \inf_{m_\ell} \mathcal{F}_{[w_-,w_+]}^{(1,1,2,1)}(m_\ell) \right] + er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{6.96}
$$

and for any $\delta > 0$, for γ small enough,

$$
\begin{aligned}
&\frac{\gamma}{\beta} \ln \mathcal{G}_{\beta,\gamma,\Lambda}^{(\tilde{\mu},\tilde{s},\mu,s)}[\omega](F_\delta) \\
&\geq - \inf_{\pm(w_\pm - u_\pm) \leq R} \left[\inf_{m_\ell \in F} \mathcal{F}_{[w_-,w_+]}^{(\tilde{\mu},\tilde{s},\mu,s)}(m_\ell) - \inf_{m_\ell} \mathcal{F}_{[w_-,w_+]}^{(1,1,2,1)}(m_\ell) \right] - er(\ell, L, M, \zeta, R)
\end{aligned}
\tag{6.97}
$$

where $er(\ell, L, M, \hat{\zeta}, R)$ is a function of $\alpha \equiv \gamma M$ that tends to zero as $\alpha \downarrow 0$.

Proof: The proof of Theorem 6.13 presents no additional technical difficulties compared to that of Theorem 6.8. We shall thus be very brief and restrict ourselves to detail the only subtle step. This one enters in the proof of the upper bound for the quantity $\mathcal{G}_{\beta,\gamma,\Lambda}^{(\tilde{\mu},\tilde{s},\mu,s)}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm))$. From now on we will place ourselves on the subset of the probability space on which our various estimates from Section 6.1 and 6.2 hold. Without loss of generality we may only consider the case $(\tilde{\mu}, \tilde{s}, \mu, s) = (1, 1, 2, 1)$. It is a simple matter to establish that

$$
\mathcal{G}_{\beta,\gamma,\Lambda}^{(1,1,2,1)}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \leq \frac{Z_{\beta,\gamma,\Lambda^- \backslash w_-}^{(1,1,\mu^-,s^-)}}{Z_{\beta,\gamma,\Lambda^- \backslash w_-}^{(1,1,1,1)}} \frac{Z_{\Delta,\beta,\gamma}^{(\mu^\pm,s^\pm)}(F)}{Z_{\Delta,\beta,\gamma}^{(1,1,2,1)}} \frac{Z_{\Lambda^+ \backslash w_+,\beta,\gamma}^{(\mu^+,s^+,2,1)}}{Z_{\Lambda^+ \backslash w_+ \beta,\gamma}^{(2,1,2,1)}} \times e^{c\gamma^{-1}(\zeta + \gamma L)}
\tag{6.98}
$$

Just as in (6.86) we replace the ratios of partition functions in boxes $\Lambda^\pm$ above by the corresponding ratios in boxes $\widetilde{\Lambda}^\pm$ of length R. Thus, up to negligible errors,

$$
\mathcal{G}_{\beta,\gamma,\Lambda}^{(1,1,2,1)}(F \cap \mathcal{A}(\mu^\pm, s^\pm, w_\pm)) \leq P_{\widetilde{\Lambda}^- \backslash w_-,\beta,\gamma}^{(1,1,\mu^-,s^-)} \frac{Z_{\Delta,\beta,\gamma}^{(\mu^\pm,s^\pm)}(F)}{Z_{\Delta,\beta,\gamma}^{(1,1,2,1)}} P_{\widetilde{\Lambda}^+ \backslash w_+,\beta,\gamma}^{(\mu^+,s^+,2,1)}
\tag{6.99}
$$

From this we want to reconstruct the Gibbs measure in $\widetilde{\Delta} \equiv \widetilde{\Lambda}^- \cup \Delta \cup \widetilde{\Lambda}^+$ with $(1,1,2,1)$-boundary conditions. Treating the numerator just as in (6.87), all we need is to show that, still up to negligible errors,

$$
Z_{\widetilde{\Lambda}^-,\beta,\gamma}^{(1,1,0,0)}(\{\eta(w_-,\sigma) = e^1\}) Z_{\Delta,\beta,\gamma}^{(1,1,2,1)} Z_{\widetilde{\Lambda}^+,\beta,\gamma}^{(0,0,2,1)}(\{\eta(w_-,\sigma) = e^2\}) \geq Z_{\widetilde{\Delta},\beta,\gamma}^{(1,1,2,1)}
\tag{6.100}
$$

To prove this we start by piecing together the first and second term in the l.h.s. of (6.100). Using in turn the lower bound (6.54) and the upper bound (6.53),

$$
\begin{aligned}
Z^{(1,1,0,0)}_{\widetilde{\Lambda}^-,\beta,\gamma}(\{\eta(w_-,\sigma)=e^1\})Z^{(1,1,2,1)}_{\Delta,\beta,\gamma} &= Z^{(1,1,0,0)}_{\widetilde{\Lambda}^-,\beta,\gamma}(\{\eta(w_-,\sigma)=e^1\})P^{(1,1,2,1)}_{\Delta,\beta,\gamma}Z^{(1,1,1,1)}_{\Delta,\beta,\gamma} \\
&\geq Z^{(1,1,1,1)}_{\widetilde{\Lambda}^-\cup\Delta,\beta,\gamma}\bar{P}_0 e^{-\gamma^{-1}er'(\ell,L,M,\hat{\zeta},R)} \\
&\geq Z^{(1,1,2,1)}_{\widetilde{\Lambda}^-\cup\Delta,\beta,\gamma}e^{-\gamma^{-1}(er'(\ell,L,M,\hat{\zeta},R)+er'(\ell,L,M,\zeta,R))}
\end{aligned} \tag{6.101}
$$

where we have in addition used (6.84) in the first inequality. In the same manner

$$
Z^{(1,1,2,1)}_{\widetilde{\Lambda}^-\cup\Delta,\beta,\gamma}Z^{(0,0,2,1)}_{\widetilde{\Lambda}^+,\beta,\gamma}(\{\eta(w_-,\sigma)=e^2\}) \geq Z^{(1,1,2,1)}_{\widetilde{\Delta},\beta,\gamma}e^{-\gamma^{-1}(er'(\ell,L,M,\hat{\zeta},R)+er'(\ell,L,M,\zeta,R))} \tag{6.102}
$$

Combining (6.101) and (6.102) thus gives (6.100) and making use of the later with (6.99) yields, up to the usual error term,

$$
\mathcal{G}^{(1,1,2,1)}_{\beta,\gamma,\Lambda}(F\cap\mathcal{A}(\mu^\pm,s^\pm,w_\pm)) \leq \frac{Z^{(1,1,2,1)}_{\widetilde{\Delta},\beta,\gamma}(F)}{Z^{(1,1,2,1)}_{\widetilde{\Delta},\beta,\gamma}} \tag{6.103}
$$

The upper bound (6.96) of Theorem 6.13 now simply follows from (6.103) just as the upper bound of Theorem 6.8 follows from (6.87). The proof of the lower bound is a mere repetition of that of the lower of Theorem 6.8: simply substitute the boundary conditions $(1,1,1,1)$ by $(1,1,2,1)$ and use (6.100) instead of (6.83). This concludes the proof of Theorem 6.13.$\diamondsuit$.

Finally, we want to give a characterization of the typical profile in the case of asymmetric boundary conditions. The relevant estimates and notations for this have been introduced already in the proof of Lemma 6.10.

Let us define the events

$$
E^{(\mu,s,\mu',s')}_{1,\Lambda} \equiv \left\{\sigma \,\middle|\, \exists_{\substack{u_0\leq u_1\in\Lambda\\ u_1-u_0\leq 2R}} \forall_{\lambda_-\leq u<U_0}\eta_{\hat{\zeta},\hat{L}}(u,\sigma)=se^{\mu} \wedge \forall_{u_0<v\leq\lambda_+}\eta_{\hat{\zeta},\hat{L}}(u,\sigma)=s'e^{\mu'}\right\} \tag{6.104}
$$

Theorem 6.14: *Assume that $\gamma|\Lambda|\downarrow 0$, β large enough ($\beta>1$) and $\gamma M(\gamma)\downarrow 0$. Then we can find $\gamma^{-1}\gg\hat{L}\gg 1$ and $\hat{\zeta}\downarrow 0$, such that on a subset $\Omega_\Lambda\subset\Omega$ with $I\!P(\Omega^c_\Lambda)\leq e^{-\gamma^{-1}f(\zeta')}$ we have that for all $\omega\in\Omega_\Lambda$*

$$
\mathcal{G}^{(\mu,s,\mu',s')}_{\beta,\gamma,\Lambda}[\omega]\left(E^{(\mu,s,\mu',s')}_{1,\Lambda}\right) \geq 1-2Re^{-\hat{L}c(\hat{\zeta})} \tag{6.105}
$$

Remark: This theorem implies that for any volume Λ such that $\gamma|\Lambda|\downarrow 0$, we have $I\!P$-almost surely,

$$
\lim_{\gamma\downarrow 0}\mathcal{G}^{(\mu,s,\mu',s')}_{\beta,\gamma,\Lambda}[\omega]\left(E^{(\mu,s,\mu',s')}_{1,\Lambda}\right)=1 \tag{6.106}
$$

(Here one may, to avoid complications with the "almost sure" statement due to the uncountability of the number of possible sequences γ_n, assume for simplicity that $\lim_{\gamma\downarrow 0}$ is understood to be taken along some fixed discrete sequence, e.g. $\gamma_n = 1/n$. To show that the convergence holds also with probability one for *all* sequences tending to zero, one can use a continuity result as given inLemma 2.3 of [BGP2]).

Proof: The proof of this Theorem follows from Lemma 6.10 and its proof. We leave the details to the reader. $\diamondsuit$

We are now ready to state a precise version of the main result announced in the introduction. We define the events

$$E_{0,\Lambda}^{(\mu,s)} \equiv \left\{\sigma \,\middle|\, \forall_{u\in\Lambda}\eta_{\hat\zeta,\hat L}(u,\sigma) = se^{\mu}\right\} \tag{6.107}$$

and set

$$E_{0,\Lambda} \equiv \cup_{(\mu,s)} E_{0,\Lambda}^{(\mu,s)} \tag{6.108}$$

$$E_{1,\Lambda} \equiv \cup_{(\mu,s)\neq(\mu',s')} E_{1,\Lambda}^{(\mu,s,\mu',s')} \tag{6.109}$$

This this notation we have

Theorem 6.15: *For any macroscopic box V such that $\lim_{\gamma\downarrow 0}\gamma|V\| = 0$, $I\!P$-almost surely,*

$$\lim_{\gamma\downarrow 0}\lim_{\Lambda\uparrow \mathbb{Z}} \mathcal{G}_{\beta,\gamma,\Lambda}[\omega]\left(E_{0,V}\cup E_{1,V}\right) = 1 \tag{6.110}$$

Proof: This theorem follows immediately Corollary 4.2 and the Theorems 6.9 and 6.14. The remark after (6.106) also applies here.$\diamondsuit$

7. Conclusions and conjectures

In the preceeding sections we have labored hard to prove that typical profiles in the one dimensional Kac-Hopfield model are constant on a scale of the order $o(\gamma^{-1}$. The careful reader will have noticed the conspicuous absence of any argument that would proof that they are non-constant on any larger scale. The reason for the absence of such an argument lies in Section 5. There, we prove upper bounds on the fluctuations of the quantities $f_\Delta^{(\mu^\pm,s^\pm)}$ that imply that they are typically not larger than $\sqrt{\gamma^{-1}|\Delta|}$. What is not shown, and what would be needed, is that these fluctuations are actually of that size, and in particular that for $\mu\neq\mu'$, $f_\Delta^{(\mu,s,\mu,s)} - f_\Delta^{(\mu',s,\mu',s)}$ typically differ by a random amount of that order. We certainly believe that this is true, but rigorous proofs of such statements are notoriously difficult to obtain and many problems in the theory of disordered systems are unsolved for very similar reasons. To our knowledge the only known method in this direction is the work of Aizenman and Wehr [AW] that yields, however, no good quantitative results for finite volume objects. In fact, it appears that even the uniqueness of the Gibbs state in two dimensions (which one should expect to be provable with this method) cannot be shown using their approach (just as, and for similar reasons, is the case in the two dimensional spin glass). A general method that would allow to get lower bounds on fluctuations corresponding to Theorem 5.1 is thus still a great desideratum.

A natural question that poses itself is of course "What about dimensions greater than one?". Here, again, conjectures come easy, but at some of them may be provable. First, as mentioned, we would expect that in dimension $d=2$ we still have a unique Gibbs state. This is motivated by the fact that at least the block-approximation looks very much like a multi-state random field model, for which this result would follow from Aizenman-Wehr. But as for a proof, see above..... The same argument suggests, on the basis of the results of Bricmont-Kupiainen [BrKu] and Bovier-Külske [BK] that in dimension $d\geq 3$ we will have many Gibbs states, at least one for every pattern and its mirror image. We would expect that this can actually proven, although technically this would be quite hard. To our surprise, it turns out that such a result is not even known in the ferromagnetic Kac model (see Cassandro, Marra and Presutti [CMP] for a conjecture), and techniques to take into account the the weak but long range interaction in proofs of phase-stability have still to be worked out. However, this problem appears to be solvable. This entire line of research is very interesting and will be pursued in forthcoming publications.

Appendix A: Proof of Proposition 6.12

In this section we prove a lower bound on the infimum of the free energy functional over all the profiles that form an interface between a "phase" where the local overlaps are close to $a(\beta)s^-e^{\mu^-}$ and another one where they are close $a(\beta)s^+e^{\mu^+}$. In the case of the ferromagnetic Kac model, the shape of the interface was described in [COP] chap. 6. In the case of the Kac-Hopfield model due to our restricted knowledge of the Hopfield model free energy with fixed overlaps, we cannot perform such a detailed analysis.

Instead of working with the full free energy functional $\mathcal{F}$ defined in (6.4) we will replace it by a lower bound (that is also suitably normalized to have its minimal value equal to zero) defined as follows: Given a macroscopic volume $\widetilde{\Delta}$ that could be chosen without lost of generality to be $[1, u_3]$ we denote by

$$\widetilde{\mathcal{F}}_{\widetilde{\Delta}}^{\mu^\pm,s^\pm} \equiv V_{\widetilde{\Delta}}^{\mu^\pm,s^\pm} + \gamma\ell \sum_{x\in \text{int}\,\widetilde{\Delta}} \Phi^T(m_\ell(x)) \tag{8.1}$$

where

$$\begin{aligned} V_{\widetilde{\Delta}}^{\mu^\pm,s^\pm} \equiv & \gamma\ell \sum_{x,y\in \text{int}\,\widetilde{\Delta}} J_{\gamma\ell}(x-y)\frac{\|m_\ell(x)-m_\ell(y)\|_2^2}{4} \\ & + \gamma\ell \sum_{x\in \text{int}\,\widetilde{\Delta}, y\in\partial\widetilde{\Delta}} J_{\gamma\ell}(x-y)\frac{\|m_\ell(x)-m^{(\mu^\pm,s^\pm)}\|_2^2}{2} \end{aligned} \tag{8.2}$$

and for any $\zeta \geq 2a(\beta)(\frac{M}{\ell})^{1/2}$ (cf Proposition 3.1),

$$\Phi^T(m_\ell(x)) \equiv \begin{cases} 0, & \text{if } \exists_{\mu,s} \|m_\ell(x) - a(\beta)se^\mu\|_2 \leq \zeta; \\ \epsilon(\zeta), & \text{otherwise.} \end{cases} \tag{8.3}$$

The set of profiles that form an interface between the (s^-,μ) and the (s^+,μ^+) within the volume $\tilde{\Delta}$ is denoted by

$$\mathcal{T}^\pm(\widetilde{\Delta}) \equiv \{m_\ell(x), x\in\widetilde{\Delta} \,\big|\, m_\ell(x) = a(\beta)s^-e^{\mu^-}, \forall_x \leq 0, m_\ell(x) = a(\beta)s^+e^{\mu^+}, \forall_x \geq y_3\} \tag{8.4}$$

where $y_3 \equiv \sup\{y | y \in u_3\}$

Proposition 6.12 then follows immediately from

Proposition 7.1: *There exists a $\zeta_0 \equiv \xi(\beta, M, \ell)$ such that for all ζ, $\zeta_0 \geq \zeta \geq 2a(\beta)(\frac{M}{\ell})^{1/2}$, we have*

$$\inf_{m_\ell\in\mathcal{T}^\pm(\widetilde{\Delta})} \widetilde{\mathcal{F}}_{\widetilde{\Delta}}^{\mu^\pm,s^\pm} \geq \sqrt{\epsilon(\zeta)}\left(\sqrt{12((a(\beta))^2-4\zeta^2)} - 3\sqrt{\epsilon(\zeta)}\right) \tag{8.5}$$

Proof:

For any given profile in $\mathcal{T}^{\pm}(\widetilde{\Delta})$, we denote by

$$y_1 \equiv \sup\{x \geq 0 \big| \|m_\ell(x-1) - a(\beta)s^- e^{\mu^-}\|_2 \leq \zeta\} \tag{8.6}$$

the last exit of the ζ neighborhood of the (s^-, μ^-) phase and

$$y_2 = \inf\{x \geq y_1, \|m_\ell(x) - a(\beta)s^+ e^{\mu^+}\|_2 \leq \zeta\} \tag{8.7}$$

the first entrance in the ζ neighborhood of the (s^+, μ^+) phase *after* y_1. Notice that by definition of $\mathcal{T}^{\pm}(\widetilde{\Delta})$, y_1 and y_2 exist and satisfy $0 \leq y_1 \leq y_2 \leq y_3$. We defined also the overlap increments:

$$D(x) \equiv m_\ell(x) - m_\ell(x-1) \tag{8.8}$$

We write for $1 = y_0 \leq x_1 \leq y_1$

$$m_\ell(x_1) - a(\beta)s^- e^{\mu^-} = \sum_{x=1}^{x_1} D(x) \tag{8.9}$$

and for $i = 1, 2$ and all $y_i \leq x_{i+1} \leq y_{i+1}$

$$\begin{aligned} m_\ell(y_i) - m_\ell(x_i) &= \sum_{x=x_i+1}^{y_i} D(x) \\ m_\ell(x_{i+1}) - m_\ell(y_i) &= \sum_{x=y_i+1}^{x_{i+1}} D(x) \end{aligned} \tag{8.10}$$

at last

$$a(\beta)s^+ e^{\mu^+} - m_\ell(x_3) = \sum_{x=x_3+1}^{y_3} D(x) \tag{8.11}$$

We define now the quantity

$$\mathcal{L} \equiv \sum_{i=1}^{3} \sum_{x_i=y_{i-1}+1}^{y_i} \left\| \sum_{x=y_{i-1}+1}^{x_i} D(x) \right\|_2^2 + \left\| \sum_{x=x_i+1}^{y_i} D(x) \right\|_2^2 \tag{8.12}$$

we first show that $\mathcal{L}$ can be bounded from above in term of $V_{\underset{\sim}{\Delta}}^{\mu^{\pm}, s^{\pm}}$. Then we will bound from below $\mathcal{L}$ by solving elementary variational problems. Putting those two bounds together will give a lower bound for $V_{\underset{\sim}{\Delta}}^{\mu^{\pm}, s^{\pm}}$.

Let us start with the upper bound. We first perform for each value of $b \in 1, \ldots, [1/\gamma\ell]$ a block summation with blocks of length b, the location of the leftmost part of the first block being a point $z \in 1, \ldots, b$. Explicitly, calling

$$D(u, b, z) \equiv \sum_{x=(u-1)b+z+1}^{ub+z} D(x) \tag{8.13}$$

we write

$$\sum_{x=1}^{x_1} D(x) = \sum_{x=1}^{z} D(x) + \sum_{u=1}^{\left[\frac{x_1-z}{b}\right]} D(u,b,z) + D_+(x_1,b,z) \tag{8.14}$$

where

$$D_+(x_1,b,z) = \sum_{x=\left[\frac{x_1-z}{b}\right]b+z+1}^{x_1} D(x) \tag{8.15}$$

The second sum in the r.h.s of (8.14) will be called the bulk term, while the first sum and D_+ will be called boundary terms. We have also

$$\sum_{x=x_i+1}^{y_i} D(x) = D_-(x_i,b,z) + \sum_{u=\left[\frac{x_i+1-z}{b}\right]+2}^{\left[\frac{y_i-z}{b}\right]} D(u,b,z) + D_+(y_i,b,z) \tag{8.16}$$

with the 'boundary' terms

$$D_-(x_i,b,z) = \sum_{x=x_i+1}^{\left[\frac{x_i+1-z}{b}\right]b+b+z+1} D(x) \tag{8.17}$$

and

$$D_+(y_i,b,z) = \sum_{x=\left[\frac{y_i-z}{b}\right]b+z+1}^{y_i} D(x) \tag{8.18}$$

We have also

$$\sum_{x=y_{i-1}+1}^{x_i} D(x) = D_-(y_{i-1},b,z) + \sum_{u=\left[\frac{y_{i-1}+1-z}{b}\right]+2}^{\left[\frac{x_i-z}{b}\right]} D(u,b,z) + D_+(x_i,b,z) \tag{8.19}$$

here the 'boundary' terms are:

$$D_-(y_{i-1},b,z) \equiv \sum_{x=y_{i-1}+1}^{\left[\frac{y_{i-1}+1-z}{b}\right]b+b+z+1} D(x) \tag{8.20}$$

and

$$D_+(x_i,b,z) \equiv \sum_{x=\left[\frac{x_i-z}{b}\right]b+z+1}^{x_i} D(x) \tag{8.21}$$

For a given b and z, the Schwarz Inequality implies

$$\left\| \sum_{x=y_{i-1}+1}^{x_i} D(x) \right\|_2^2 \leq$$

$$\left(\frac{x_i - y_{i-1} + 1}{b} + 2\right) \left(\|D_-(y_{i-1},b,z)\|_2^2 + \sum_{u=\left[\frac{y_{i-1}+1-z}{b}\right]+2}^{\left[\frac{x_i-z}{b}\right]} \|D(u,b,z)\|_2^2 + \|D_+(x_i,b,z)\|_2^2 \right) \tag{8.22}$$

We want to take the mean of the two sides of (8.22) over all the possible choices of block lengths b in $1, \ldots, [(\gamma\ell)^{-1}$ and z in $1, \ldots, b$. To do this we use a weighted mean for the block lengths and an uniform mean for the $z \in 1, \ldots b$. We use

$$\begin{aligned}\sum_{b=1}^{[(\gamma\ell)^{-1}]} b^2 =& \frac{[(\gamma\ell)^{-1}]\left([(\gamma\ell)^{-1}]+1)(2\left[(\gamma\ell)^{-1}\right]+1\right)}{6} \\ &= \left([(\gamma\ell)^{-1}]\right)^3 \frac{1}{3}\left(1+O(\gamma\ell)\right)\end{aligned} \tag{8.23}$$

to define a weighted mean on $1, \ldots, [(\gamma\ell)^{-1}$. Performing explicitly these weighted means gives

$$\begin{aligned}&\left\| \sum_{x=y_{i-1}+1}^{x_i} D(x) \right\|_2^2 \leq \\ &\frac{3}{[(\gamma\ell)^{-1}]^3 (1+O(\gamma\ell))} \left(x_i - y_{i-1} + 1 + \frac{2}{\gamma\ell} \right) \\ &\sum_{b=1}^{[(\gamma\ell)^{-1}]} \sum_{z=1}^{b} \left(\|D_-(y_{i-1}, b, z)\|_2^2 + \sum_{u=\left[\frac{y_{i-1}+1-z}{b}\right]+2}^{\left[\frac{x_i-z}{b}\right]} \|D(u,b,z)\|_2^2 + \|D_+(x_i, b, z)\|_2^2 \right)\end{aligned} \tag{8.24}$$

and by the very same argument

$$\begin{aligned}&\left\| \sum_{x=x_i+1}^{y_i} D(x) \right\|_2^2 \leq \\ &\frac{3}{[(\gamma\ell)^{-1}]^3 (1+O(\gamma\ell))} \left(y_i - x_i + 1 + \frac{2}{\gamma\ell} \right) \\ &\sum_{b=1}^{[(\gamma\ell)^{-1}]} \sum_{z=1}^{b} \left(\|D_-(x_i, b, z)\|_2^2 + \sum_{u=\left[\frac{x_i-z}{b}\right]+2}^{\left[\frac{y_i-z}{b}\right]} \|D(u,b,z)\|_2^2 + \ D_+(y_i, b, z)\|_2^2 \right)\end{aligned} \tag{8.25}$$

Collecting the 'bulk' terms in (8.24) and (8.25) to bound $\mathcal{L}$, it is not difficult to check that

$$\begin{aligned}&\frac{3}{[(\gamma\ell)^{-1}]^3 (1+O(\gamma\ell))} \sum_{x_i=y_{i-1}}^{y_i} \sum_{b=1}^{[(\gamma\ell)^{-1}]} \sum_{z=1}^{b} \\ &\left(x_i - y_{i-1} + 1 + \frac{2}{\gamma\ell} \right) \sum_{u=\left[\frac{y_{i-1}+1-z}{b}\right]+2}^{\left[\frac{x_i-z}{b}\right]} \|D(u,b,z)\|_2^2 + \left(y_i - x_i + 1 + \frac{2}{\gamma\ell} \right) \sum_{u=\left[\frac{x_i-z}{b}\right]+2}^{\left[\frac{y_i-z}{b}\right]} \|D(u,b,z)\|_2^2 \\ &\leq 3\,[\gamma\ell]^3 (1+O(\gamma\ell))\,(y_i - y_{i-1}) \left(y_i - y_{i-1} + \frac{2}{\gamma\ell} \right) \sum_{b=1}^{[(\gamma\ell)^{-1}]} \sum_{z=1}^{b} \sum_{u=\left[\frac{y_{i-1}+1-z}{b}\right]}^{\left[\frac{y_i-z}{b}\right]} \|D(u,b,z)\|_2^2 \\ &\leq 3\,(1+O(\gamma\ell))\,(\gamma\ell(y_i - y_{i-1}))\,(\gamma\ell(y_i - y_{i-1}) + 2) \sum_{y_{i-1} \leq x, y \leq y_i} J_{\gamma\ell}(x-y)\,\|m_\ell(x) - m_\ell(y)\|_2^2\end{aligned} \tag{8.26}$$

It remains to consider the "boundary" terms, putting together the terms $D_+(x_i,b,z)$ and $D_-(x_i,b,z)$, it is not too difficult to check that

$$
\begin{aligned}
&\frac{3}{[(\gamma\ell)^{-1}]^3\,(1+O(\gamma\ell))}\sum_{x_i=y_{i-1}}^{y_i}\sum_{b=1}^{[(\gamma\ell)^{-1}]}\sum_{z=1}^{b}\\
&\left(x_i-y_{i-1}+1+\frac{2}{\gamma\ell}\right)\left\|\sum_{x=[\frac{x_i-z}{b}]b+z+1}^{x_i}D(x)\right\|_2^2+\left(y_i-x_i+1+\frac{2}{\gamma\ell}\right)\left\|\sum_{x=x_i+1}^{[\frac{x_i+1-z}{b}]b+b+z+1}D(x)\right\|_2^2\\
&\le 3\left((y_i-y_{i-1}+\frac{2}{\gamma\ell})\gamma\ell\right)\sum_{y_{i-1}\le x,y\le y_i}J_{\gamma\ell}(x-y)\,\|m_\ell(x)-m_\ell(y)\|_2^2
\end{aligned}
\tag{8.27}
$$

Therefore we get

$$
\mathcal{L}\le 4\sum_{i=1}^{3}\gamma\ell(y_i-y_{i-1})(\gamma\ell(y_i-y_{i-1})+3)\sum_{y_{i-1}\le x,y\le y_i}J_{\gamma\ell}(x-y)\,\|m_\ell(x)-m_\ell(y)\|_2^2 \tag{8.28}
$$

which is the upper bound we wanted.

Now we want to bound from below $\mathcal{L}$. Notice first that by solving explicitly the variational problem we have that for all $m_1,m_2\in I\!R^M$

$$
\begin{aligned}
&\inf_{m_\ell(x)}\{\|m_\ell(x)-m_1\|_2^2+\|m_\ell(x)-m_2\|_2^2\}\\
&\ge\frac{1}{2}\|m_1-m_2\|_2^2
\end{aligned}
\tag{8.29}
$$

using (8.6) and (8.7) and convexity, we get

$$
\begin{aligned}
\mathcal{L}&\ge(y_1-y_0)\frac{\zeta^2}{2}+\frac{1}{2}(y_2-y_1)\|m_\ell(y_1)-m_\ell(y_2)\|_2^2+(y_3-y_2)\frac{\zeta^2}{2}\\
&\ge\frac{1}{2}(y_2-y_1)\left((a(\beta))^2-4\zeta^2\right)
\end{aligned}
\tag{8.30}
$$

On the other hand, *c.f* (8.3), we have

$$
\gamma\ell\sum_{x\in\mathrm{int}\,\widetilde{\Delta}}\Phi^T(m_\ell(x))\ge\gamma\ell(y_2-y_1)\epsilon(\zeta) \tag{8.31}
$$

therefore, introducing the macroscopic variables $u_i=\gamma\ell y_i$ we get

$$
\begin{aligned}
\widetilde{\mathcal{F}}_{\widetilde{\Delta}}^{\mu^\pm,s^\pm}&\ge(u_2-u_1)\epsilon(\zeta)+\frac{12}{u_2-u_1+3}\left((a(\beta))^2-4\zeta^2\right)\\
&\ge\sqrt{\epsilon(\zeta)}\left(\sqrt{12\left((a(\beta))^2-4\zeta^2\right)}-3\sqrt{\epsilon(\zeta)}\right)
\end{aligned}
\tag{8.32}
$$

where the last step follows from the explicit computation of the infimum over all possible values of u_2-u_1.$\diamondsuit$

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