LANDAU HAMILTONIANS WITH UNBOUNDED RANDOM POTENTIALS

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Abstract

We prove the almost sure existence of pure point spectrum for the two-dimensional Landau Hamiltonian with an unbounded Anderson-like random potential, provided that the magnetic field is sufficiently large. For these models, the probability distribution of the coupling constant is assumed to be absolutely continuous. The corresponding density g has support equal to $I\!R$, and satisfies $\sup_{\lambda \in I\!R}\{\lambda^{3+\epsilon} g(\lambda)\} < \infty$, for some $\epsilon > 0$. This includes the case of Gaussian distributions. We show that the almost sure spectrum Σ is $I\!R$, provided the magnetic field $B \neq 0$. We prove that for each positive integer n, there exists a field strength B_n, such that for all $B \geq B_n$, the almost sure spectrum Σ is pure point at all energies $E \leq (2n+1)B - \mathcal{O}(B^{-1})$ and outside of intervals of width $\mathcal{O}(B^{-1})$ about each Landau level $E_m(B) \equiv (2m+1)B$, for $m < n$. We also prove that the integrated density of states is Lipschitz continuous away from the Landau energies $E_n(B)$. The proof relies on a new Wegner estimate for the finite-area magnetic Hamiltonians with random potentials and exponential decay estimates for the finite-area Green's functions. As in a previous work, the proof of the decay estimates for the Green's functions uses fundamental results from two-dimensional bond percolation theory.

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1 Introduction and the Main Result

Several recent discussions of the integer quantum Hall effect (IQHE) [1, 2, 3, 4, 6, 8] require certain spectral properties of the one-particle Schrödinger operator. As proved in [5], the one-particle model of a free electron moving in two dimensions in the presence of a constant, transverse magnetic field will not exhibit the IQHE. At very low temperatures, when dissipative effects are small, it is commonly believed that random impurities in the sample provide a necessary mechanism for the IQHE. Typically, the hypotheses on the spectral properties of one-particle Hamiltonian include 1) the existence of regions of localized states between the Landau levels, and 2) the divergence of the localization length at some energy between these regions. Here, we prove the first of these hypotheses on the existence of localized states. In a first paper [10], we proved the existence of localized states near the band edges, and to within $\mathcal{O}(B^{-1})$ of the Landau energies, for fixed disorder and large magnetic field. For these models, there are spectral gaps between the bands of length $\mathcal{O}(B)$. The corresponding eigenfunctions decay exponentially with respect to distance and the magnetic field strength B. Kunz's proof [3] of the IQHE requires the existence of spectral gaps and localized states near the band edges with finite localization length. A consequence of his analysis is a proof that the localization length diverges at some energy in each band. It is believed that for neutral samples, the localization length should diverge precisely at the Landau energy.

In a recent review article by Bellissard, van Elst, and Schulz-Baldes [5], these authors point-out that in experiments, the disorder is strong enough to fill the gap between the Landau levels. In their proof of the IQHE, they require regions of localized states between the Landau energies in order that the quantum Hall conductivity exhibits a plateau region. They do not require a spectral gap. They prove that if the quantum Hall conductivity jumps by an integer (as a function of the filling factor), then the localization length must diverge at some energy between the localized state regions. In this paper, we prove localization between Landau energies, and to within $\mathcal{O}(B^{-1})$ of the Landau energies, for Landau Hamiltonians with unbounded, Anderson-type random potentials and large magnetic fields. For these models, there is typically no spectral gap. As in [10], we prove that the integrated density of states is Lipschitz continuous at all energies but the Landau energies. This indicates that the localization length may diverge there, but we provide no such estimate here.

We now provide a description of the one-particle Hamiltonian and state the main results. We consider a one-particle Hamiltonian which describes an electron in two-dimensions (x_1, x_2) subject to a constant magnetic field of strength $B > 0$ in the perpendicular x_3-direction, and a random potential V_ω. The Hamiltonian H_ω has the form

$$H_\omega = (p - A)^2 + V_\omega, \tag{1.1}$$

on the Hilbert space $L^2(I\!R^2)$, where $p \equiv -i\nabla$, and the vector potential A is

$$A = \frac{B}{2}(x_2, -x_1), \tag{1.2}$$

so the magnetic field $B = \nabla \times A$ is in the x_3-direction. The random potential V_ω is Anderson-like having the form

$$V_\omega(x) = \sum_{i \in \boldsymbol{Z}^2} \lambda_i(\omega) u(x - i). \tag{1.3}$$

We make the following assumptions on the single-site potential u and the coupling constants $\{\lambda_i(\omega)\}$.

(V1) $u \geq 0$, $u \in C^2$, supp $u \subset B(0, \frac{1}{\sqrt{2}})$, and $\exists C_0 > 0$ and $r_0 > 0$ s.t. $u|B(0, r_0) > C_0$.

(V2) $\{\lambda_i(\omega)\}$ is an independent, identically distributed family of random variables with common distribution $g \in L^\infty([-M, M])$, for some $0 < M \leq \infty$, s.t. $g(-\lambda) = g(\lambda)$, $g(\lambda) > 0$ Lebesgue a.e. $\lambda \in [-M, M]$, and for some $\epsilon > 0$, $\sup_{\lambda \in \boldsymbol{R}} \{\lambda^{3+\epsilon} g(\lambda)\} < \infty$.

The condition on the decay of g implies that the first two moments are finite, i.e. $\int |\lambda|^k g(\lambda) d\lambda < \infty$ for $k = 0, 1, 2$. The above condition does not require any differentiability of the density g. If we require g to be twice differentiable, we can replace (V2) by another condition.

(V2)' $\{\lambda_i(\omega)\}$ is an independent, identically distributed family of random variables with common distribution $g \in C^2([-M, M])$, for some $0 < M \leq \infty$, s.t. $g(-\lambda) = g(\lambda)$, $g(\lambda) > 0$ Lebesgue a.e. $\lambda \in [-M, M]$, $\|g^{(p)}\|_1 < \infty$, for $p = 0, 1, 2$, and $\int_{-M}^{M} \lambda^2 g(\lambda) < \infty$.

We remark for future use that either condition (V2) or (V2)' implies that

$$\mathbb{P}\{\lambda \geq \xi > 0\} = \mathcal{O}(\xi^{-N}), \text{for all } \xi > 0 \text{ large and some } N \geq 2. \tag{1.4}$$

An important example is the case of Gaussian distributed coupling-constants for which $g(\lambda) = (\alpha\pi)^{-1/2}e^{-\alpha\lambda^2}$, for some $\alpha > 0$. The case of finite M was dealt with in [10], so here we consider only the case $M = \infty$.

We denote by $H_A \equiv (p - A)^2$, the Landau Hamiltonian. As is well-known, the spectrum of H_A consists of an increasing sequence $\{E_n(B)\}$ of eigenvalues, each of infinite multiplicity, given by

$$E_n(B) = (2n + 1)B, \ n = 0, 1, 2, \ldots \tag{1.5}$$

We will call $E_n(B)$ the n^{th} Landau level and denote by P_n the projection onto the corresponding subspace. The orthogonal projection is denoted by $Q_n \equiv 1 - P_n$.

The family of random potentials defined in (1.3) is unbounded. Consequently, we must show that the corresponding family of Schrödinger operators is self-adjoint with probability one. It is easy to show, by a Borel-Cantelli type argument, that the potential is almost surely bounded below by $-C_0(1+\|x\|^2)$, for some finite constant C_0. It then follows that the family H_ω is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ with probability one. We refer to [11] for the details. The family of random, self-adjoint Schrödinger operators defined above on $L^2(\mathbb{R}^2)$ is an ergodic, measurable family. For general densities g, it follows from Appendix 2 of [10] that the almost sure spectrum $\Sigma(B)$ of the family H_ω is a closed subset of $\mathbb{R}$ and may have spectral gaps of width at most $\mathcal{O}(B^{-1/2})$. When $\mathrm{supp} g = \mathbb{R}$, we prove below that $\Sigma(B) = \Sigma = \mathbb{R}$. Our main theorem concerning localization is the following.

Theorem 1.1 *Let H_ω be the family given in (1.1) with vector potential A satisfying (1.2), $B > 0$, and the random potential V_ω as in (1.3) and satisfying (V1) and either (V2) or (V2)' with $M = \infty$. Let $I_n(B)$ denote the unbounded set of energies*

$$I_n(B) \equiv \left(-\infty, B - \mathcal{O}(B^{-1})\right] \cup \bigcup_{j=0}^{n} \left[E_j(B) + \mathcal{O}(B^{-1}), E_{j+1}(B) - \mathcal{O}(B^{-1})\right],$$

where the term $\mathcal{O}(B^{-1})$ depends on n. For each integer $n > 0$, there exists $B_n \gg 0$ such that for $B > B_n$,

$$\Sigma \cap I_n(B)$$

is pure point and the corresponding eigenfunctions decay exponentially.

We remark that we can also prove localization at energies between the Landau levels for fixed, nonzero $B > 0$ for the family $H_\omega(g) \equiv H_A + gV_\omega$, with a coupling constant $g > 0$, provided that we work in the small coupling regime. This result does not require percolation theory, but we cannot control the spectrum to within $\mathcal{O}(B^{-1})$ of the Landau levels. We refer the reader to section 6 of [11] where results of this type are proved in general situations. As with previous work, the Wegner estimate for the finite-area Hamiltonians allows us to control the integrated density of states.

Theorem 1.2 *Let H_ω be the family given in (1.1) with vector potential A satisfying (1.2), $B > 0$, and the random potential V_ω as in (1.3) and satisfying (V1) and either (V2) or (V2)'. Then, the integrated density of states is Lipschitz continuous on $I\!R \backslash \sigma(H_A)$.*

This note is organized as follows. We first prove that the almost sure spectrum is the entire real line when supp $g = I\!R$ for any $B \neq 0$. The proof of Theorem 1.1 is similar to the proof in [10]. However, since the potential is unbounded, we need a refined Wegner estimate. In section 3, we give a new Wegner estimate for random perturbations of Landau Hamiltonians. This technique is generalized and applied to band-edge localization in [11]. We sketch the proof of Theorem 1.1 in section 4, providing the details of the modifications of the estimates in [9, 10] required for unbounded potentials. We mention that the proof of localization at energies below the first Landau level B and in the center of the gap does not require percolation estimates, since one is well within the spectral gap of the unperturbed operator H_A. For these energies, localization may be proved using the Wegner estimate and the Combes-Thomas method as in [11]. In order to prove localization for the energy intervals of Theorem 1.1, i.e. close to the Landau energies, the percolation estimates of [10] are needed.

We mention that Wang [12] has obtained results similar to those of [10] in the case that the density g has bounded support. A discussion of the single-band approximation, including the case of densities with unbounded support, is given in the papers of Dorlas, Macris, and Pulé [13, 14].

2 Location of the Spectrum

Let $\Sigma(B)$ be the almost sure spectrum of the Landau Hamiltonian as defined in (1.1)-(1.3). For general density g with compact support, the nature of $\Sigma(B)$ may be quite complicated (cf. [10]). If, however, $\operatorname{supp} g = I\!\!R$, we can characteristize $\Sigma(B)$ quite easily.

Proposition 2.1. *Assume conditions (V1) and either (V2) or (V2)', and that* $\operatorname{supp} g = I\!\!R$. *Then* $\Sigma(B) = I\!\!R, \quad \forall B \neq 0$.

Proof.

1. We will prove that $I\!\!R \backslash \sigma(H_A) \subset \Sigma$, which proves the result since $\sigma(H_A)$ is a discrete set. According to the theory of ergodic Schrödinger operators with Anderson-type potentials (see cf. [16] section 6, Theorem 1), for any configuration $\omega \in \operatorname{supp} I\!\!P$, the spectrum $\sigma(H_\omega) \subset \Sigma(B)$. Let Ω_1 be the set of configurations having measure one for which H_ω is essentially self-adjoint on $C_0^\infty(I\!\!R^2)$. We define Ω_0 to be $\operatorname{supp} I\!\!P$ intersected with this set Ω_1. Then, we have $I\!\!P(\Omega_0) = 1$. The set Ω_0 clearly contains the set $(Z\!\!\!Z^2)_0^{I\!\!R}$ of all configurations ω which assume a constant value outside some fixed square $\tilde{\Lambda} \subset Z\!\!\!Z^2$. Consequently, for any $E \in I\!\!R \backslash \sigma(H_A)$, we need to produce a configuration $\omega(E) \in (Z\!\!\!Z^2)_0^{I\!\!R}$ such that $E \in \sigma(H_\omega)$. It then follows from the above comments that $E \in \Sigma(B)$. We separately consider $E < B$ and $E > B$ but $E \neq E_n(B), \quad n = 0, 1, 2, \ldots.$

2. $E < B$. For a fixed $\tilde{\Lambda} \subset Z\!\!\!Z^2$ (nonempty and bounded) and any finite $M > 0$, define a configuration $\omega(E) \in (Z\!\!\!Z^2)_0^{I\!\!R}$ by setting $\lambda_i = -M, \quad i \in \tilde{\Lambda}$, and $\lambda_j = 0$, otherwise. The potential $V_{\omega(E)} \leq 0$ and has compact support. Let $\lambda \in I\!\!R^+$ and consider

$$H_{\omega(E)}(\lambda) = H_A + \lambda V_{\omega(E)} = H_A - \lambda |V_{\omega(E)}|. \tag{2.1}$$

   Note that because $\operatorname{supp} g = I\!\!R$, the family $H_{\omega(E)}(\lambda) = H_{\lambda\omega(E)}$ and $\lambda\omega(E) \in \Omega_0 \quad \forall \lambda \in I\!\!R^+$. Since $E < B$ and the sign of the potential is constant, we can apply Birman-Schwinger theory. The operator $H_{\omega(E)}(\lambda)$ has an eigenvalue E if and only if $K(E) \equiv W^{1/2}(H_A -$

$E)^{-1}W^{1/2}$, where $W \equiv -V_{\omega(E)} \geq 0$, has an eigenvalue $1/\lambda \geq 0$. The operator $K(E)$ is a compact, self-adjoint, nonnegative operator bounded above by $||W||_\infty (B-E)^{-1}$. It has an infinite dimensional kernel but we will show that it is a nonzero operator. For any ϕ such that $W^{1/2}\phi \neq 0$, we use the spectral representation of $H_A = \sum_{n\geq 0} E_n(B) P_n$ to write

$$\langle \phi, K(E)\phi \rangle = \sum_{n\geq 0} (E_n(B) - E)^{-1} \| P_n W^{1/2} \phi \|^2. \tag{2.2}$$

Now suppose that $K(E) = 0$. Since $(E_n(B) - E) > 0$, for all $n \geq 0$, it follows from (2.2) that $\|P_n W^{1/2}\phi\| = 0$, for all $n \geq 0$. By the completeness of the projectors P_n, we conclude that $W^{1/2}\phi = 0$, a contradiction. Note that any $\phi \neq 0$ with supp $\phi \subset$ supp u satisfies $W\phi \neq 0$. Hence, the operator $K(E)$ has at least one strictly positive eigenvalue $\mu_0(E) > 0$. We adjust $\lambda > 0$ so that $\lambda^{-1} = \mu_0(E)$. The configuration $\tilde{\omega}(E) \equiv \mu_0(E)^{-1}\omega(E) \in (\mathbb{Z}^2)_0^{\mathbb{R}}$. This shows that $E \in \sigma(H_{\tilde{\omega}(E)}) \in \Sigma(B)$.

3. $E > B$ and $E \notin \sigma(H_A)$. We proceed as above and arrive at the statement: $H_{\omega(E)}(\lambda)$ has an eigenvalue E if and only if $K(E) \equiv W^{1/2}(H_A - E)^{-1}W^{1/2}$ has an eigenvalue $1/\lambda > 0$. Now, however, the compact self-adjoint operator $K(E)$ is no longer nonnegative. It still has an infinite dimensional kernel. We must show that $K(E)$ is nonzero. We first note that by the argument in (2.2), the operator $W^{1/2}(H_A - E)^{-2}W^{1/2}$ is nonzero. The operator $K_E(\epsilon) \equiv W^{1/2}(H_A - E - \epsilon)^{-1}W^{1/2}$ is a real analytic function of ϵ in a small interval about E. We will show that for all ϵ small, $K_E(\epsilon)$ is nonzero. By the second resolvent formula,

$$\begin{aligned} K_E(\epsilon) &= K(E) + \epsilon W^{1/2}(H_A - E - \epsilon)^{-1}(H_A - E)^{-1}W^{1/2} \\ &= K(E) + \epsilon S_E(\epsilon) \end{aligned}$$

Now, we know that

$$n - \lim_{\epsilon \to 0} S_E(\epsilon) = W^{1/2}(H_A - E)^{-2}W^{1/2} \neq 0,$$

so for all ϵ small, $\epsilon S_E(\epsilon) \neq 0$. If $K(E) = 0$, then we can find an open interval $I(E)\backslash E$ of nearby energies $E+\epsilon$ for which $K_E(\epsilon) = K(E+\epsilon) \neq$

0. This operator $K(E+\epsilon)$ has a positive eigenvalue $\mu_0(E+\epsilon)$. By adjusting $\lambda > 0$, we find $K(E+\epsilon)$ has an eigenvalue $1/\lambda$ and hence $E+\epsilon \in \Sigma(B)$. This implies $\overline{I(E)} \subset \Sigma(B)$. Finally, since $\Sigma(B)$ is closed, $\Sigma(B) = I\!\!R$. ■

3 The Wegner Estimate

We give a new proof of the Wegner estimate which holds for any $E \notin \sigma(H_A)$, for any field strength $B > 0$ and for any density g satisfying either (V2) or (V2)'. This result improves Theorem 3.1 of [10] in two ways. First, it applies to the case when supp $g = I\!\!R$. Secondly, it gives a better dependence of the upper bound on the field strength B. Let H_Λ be a local Hamiltonian (see below). In [10], the bound depends on B through $[\text{dist}\,(\sigma(H_A), E)]^{-2}B$, so if $|E - B| = \mathcal{O}(B^{-1})$, the estimate is $\mathcal{O}(B^3)$. Here, the dependence is given as $[\text{dist}\,(\sigma(H_A), E)]^{-1}B^{-1}$, so when $|E - B| = \mathcal{O}(B^{-1})$, the bound is $\mathcal{O}(1)$.

We define local Hamiltonians as relatively compact perturbations of the Landau Hamiltonian $H_A = (p - A)^2$. Let $\Lambda \subset I\!\!R^2$ denote an open connected region in $I\!\!R^2$. We let $\Lambda_\ell(x)$ denote a square of side ℓ centered at $x \in I\!\!R^2$,

$$\Lambda_\ell(x) \equiv \left\{ y \in I\!\!R^2 |\ |x_i - y_i| < \ell,\ i = 1, 2 \right\}.$$

Given $\Lambda \subset I\!\!R^2$, the local potential V_Λ is defined as follows. Let us denote the lattice points in Λ by $\tilde{\Lambda} \equiv Z\!\!\!Z^2 \cap \Lambda$. We define the local potential by $V_\Lambda(x) \equiv \sum_{i \in \tilde{\Lambda}} \lambda_i(\omega) u(x - i)$, and the local Hamiltonians by $H_\Lambda \equiv H_A + V_\Lambda$ on $L^2(I\!\!R^2)$. We denote by $I\!\!P_\Lambda$ and $I\!\!E_\Lambda$ the probability and expectation with respect to the random variables in Λ. Note that $\sigma_{\text{ess}}(H_\Lambda) = \sigma_{\text{ess}}(H_A)$, since V_Λ is relatively compact. We denote the resolvent of H_A by $R_A(z) \equiv (H_A - z)^{-1}$, whenever it exists.

For two general regions $\Lambda_i, i = 1, 2$, the local Hamiltonians H_{Λ_i} are not independent since supp$u$ is not contained in a unit cube. This lack of independence is characteristic of models on $I\!\!R^d$. However, since supp$u$ is compact, this does not pose any problem as explained in [10]. Henceforth, we will assume that the potentials V_{Λ_1} and V_{Λ_2} are independent if $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Theorem 3.1 *Assume (V1) and either (V2) or (V2)'. For any* $E_0 \notin \sigma(H_A)$*, and for any* $\eta > 0$ *such that* $\eta < 1/2 \; dist(E_0, \sigma(H_A))$*, and for all* $B > 0$*,*

$$I\!\!P_\Lambda\{\text{dist }(\sigma(H_\Lambda),\ E_0) < \eta\} \leq \frac{\eta}{\pi B} C_g |\Lambda| C(B, E_0, \eta), \tag{3.1}$$

where, if (V2) holds, C_g *depends upon* $supp\ \{\lambda^{3+\epsilon} g(\lambda)\}$*, or, if (V2)' holds,* C_g *depends upon* $\|g^{(p)}\|_1$*, for* $p = 0, 1, 2$*, and the second moment*

$$I\!\!E(\lambda^2) \equiv \int_{I\!\!R} g(\lambda)\lambda^2 d\lambda. \tag{3.2}$$

Assuming, without loss of generality, that $E_N(B) < E_0 < E_{N+1}(B)$*, the constant in (3.1) is given by*

$$C(B, E_0, \eta) \equiv \left[\frac{2}{E_0 - \eta - (2N+1)B} - \frac{1}{E_0 - \eta - B} + \frac{1}{E_0 - \eta - (2N+3)B} \right]. \tag{3.3}$$

Proof

Let $I_\eta \equiv [E_0 - \eta, E_0 + \eta]$, so that by the hypotheses, $I_\eta \cap \sigma(H_A) = \emptyset$. We write R_0 for the resolvent $(H_A - E_0)^{-1}$. By the definition of η, it follows that $\eta \|R_0\| \leq 1/2$. Chebyshev's inequality shows that it is sufficient to estimate $I\!\!E_\Lambda(trE_\Lambda(I_\eta))$, where tr is the trace on $L^2(I\!\!R^2)$. Let $E \in I_\eta$ be an eigenvalue of H_Λ and let P_E be the corresponding finite-rank projector. We define an operator $K_0 \equiv V_\Lambda R_0$. From the eigenvalue equation, we easily derive

$$P_E = -K_0 P_E + (E - E_0) R_0 P_E, \tag{3.4}$$

and the related, adjoint equation,

$$P_E = -P_E K_0^* + (E - E_0) P_E R_0. \tag{3.5}$$

Since the spectrum of H_Λ in I_η is discrete, we find from (3.4) that

$$\begin{aligned} trE_\Lambda(I_\eta) &= \sum_{E \in I_\eta \cap \sigma(H_\Lambda)} trP_E \\ &\leq |E - E_0| \, |tr(R_0 E_\Lambda(I_\eta))| + |tr(K_0 E_\Lambda(I_\eta))| \\ &\leq \eta \|R_0\| \, \|E_\Lambda(I_\eta)\|_1 + \|K_0 E_\Lambda(I_\eta)\|_1. \end{aligned} \tag{3.6}$$

By the positivity of the projector $E_\Lambda(I_\eta)$, we have

$$trE_\Lambda(I_\eta) = \|E_\Lambda(I_\eta)\|_1.$$

By the fact that $\eta\|R_0\| \leq 1/2$, it follows from (3.6) that

$$trE_\Lambda(I_\eta) \leq 2\|K_0 E_\Lambda(I_\eta)\|_1. \tag{3.7}$$

We next use equation (3.5) to find an expression for $K_0 E_\Lambda(I_\eta)$,

$$K_0 P_E = -K_0 P_E K_0^* + (E - E_0) K_0 P_E R_0. \tag{3.8}$$

Upon taking the trace norm of (3.8) and using the same argument as above, we find that

$$\|K_0 E_\Lambda(I_\eta)\|_1 \leq 2\|K_0 E_\Lambda(I_\eta) K_0^*\|_1. \tag{3.9}$$

It follows from equations (3.7) and (3.9) that

$$trE_\Lambda(I_\eta) \leq 4tr(K_0 E_\Lambda(I_\eta) K_0^*). \tag{3.10}$$

Hence, it remains to evaluate the trace on the right side (3.10).

For any $E \notin \sigma(H_A)$ and any pair $i, j \in \tilde{\Lambda}$, define the operator $K_{ij}(E)$ by

$$K_{ij}(E) \equiv u_i^{1/2} R_A(E)^2 u_j^{1/2}. \tag{3.11}$$

We will prove below that this operator, which is analytic in E on $\mathbb{C}\backslash\sigma(H_A)$, is trace class. Let $\{\phi_k^{ij,E}\}$ and $\{\psi_k^{ij,E}\}$ be two complete orthonormal sets of functions occurring in the canonical representation of $K_{ij}(E)$, and let $\{\mu_k^{ij,E}\}$ be the singular values of $K_{ij}(E)$. Then the canonical representation of $K_{ij}(E)$ is

$$K_{ij}(E) = \sum_{k\in\mathbb{N}} \mu_k^{ij,E} |\phi_k^{ij,E}\rangle\langle\psi_k^{ij,E}|. \tag{3.12}$$

We write the potential as $V_\Lambda = \sum_{i\in\tilde{\Lambda}} \lambda_i u_i$. By the cyclic property of the trace, the right side of (3.10) can be written as

$$\mathbb{E}_\Lambda\left\{\sum_{i,j\in\tilde{\Lambda}} \lambda_i\lambda_j \; tr[K_{ij}(E_0)(u_j^{1/2} E_\Lambda(I_\eta) u_i^{1/2})]\right\}. \tag{3.13}$$

Inserting the canonical representation (3.12) for $K_{ij}(E_0)$, into (3.13), and evaluating the trace in the orthonormal basis $\{\phi_k^{ij,E_0}\}$ yields

$$I\!\!E_\Lambda \left\{ \sum_{i,j\in\tilde{\Lambda}} \lambda_i \lambda_j \sum_k \mu_k^{ij,E_0} \langle \phi_k^{ij,E_0},\ (u_j^{1/2} E_\Lambda(I_\eta) u_i^{1/2}) \psi_k^{ij,E_0} \rangle \right\}. \tag{3.14}$$

Since $E_\Lambda(I_\eta)$ is a projector, the k-sum can be bounded above by

$$\begin{aligned} &\frac{1}{2} \sum_k \mu_k^{ij,E_0} I\!\!E_\Lambda \{ \lambda_j^2 \langle \phi_k^{ij,E_0},\ (u_j^{1/2} E_\Lambda(I_\eta) u_j^{1/2}) \phi_k^{ij,E_0} \rangle \\ &\qquad + \lambda_i^2 \langle \psi_k^{ij,E_0},\ (u_i^{1/2} E_\Lambda(I_\eta) u_i^{1/2}) \psi_k^{ij,E_0} \rangle \} \\ \leq\ & \frac{1}{2} \parallel K_{ij}(E_0) \parallel_1 \sup_k I\!\!E_\Lambda \{ \lambda_j^2 \langle \phi_k^{ij,E_0},\ (u_j^{1/2} E_\Lambda(I_\eta) u_j^{1/2}) \phi_k^{ij,E_0} \rangle \\ &\qquad + \lambda_i^2 \langle \psi_k^{ij,E_0},\ (u_i^{1/2} E_\Lambda(I_\eta) u_i^{1/2}) \psi_k^{ij,E_0} \rangle \}. \end{aligned} \tag{3.15}$$

The spectral averaging theorem of [15] can be applied to the matrix elements in (3.15). Recalling the definition of the constant C_g given in the theorem, we obtain, for example,

$$I\!\!E_\Lambda \left(\lambda_j^2 \langle \phi_k^{ij,E_0},\ (u_j^{1/2} E_\Lambda(I_\eta) u_j^{1/2}) \phi_k^{ij,E_0} \rangle \right) \leq C_g. \tag{3.16}$$

Consequently, the right side of (3.10) is bounded above by

$$\left(\sum_{i,j\in\tilde{\Lambda}} \parallel K_{ij}(E_0) \parallel_1 \right) C_g. \tag{3.17}$$

The trace norm in (3.17) is evaluated in the same way as in [10] using the exponential decay of the localized resolvent $R_A(E)$, for any $E \notin \sigma(H_A)$. This exponential decay can be obtained by the Combes-Thomas method. Let χ_x be a bounded function with compact support localized near $x \in I\!\!R^2$. Let $\mathrm{dist}(\sigma(H_A), E) \equiv d(E)$. Then there exist finite, positive constants $C_1, C_2 > 0$, depending only on E, such that for $x, y \in I\!\!R^2$ with $\|x-y\|$ large enough, such that

$$\|\chi_x R_A(E) \chi_y\| \leq C_1 d(E)^{-1} e^{-C_2 d(E) \|x-y\|}. \tag{3.18}$$

Following [10], we first consider $i, j \in \tilde{\Lambda}$ such that $\parallel i-j \parallel < 2$. Let χ_{ij} be the characteristic function on the $\mathrm{supp}(u_i + u_j)$. We can then write

$$\sum_{\|i-j\|<2} \parallel K_{ij}(E) \parallel_1 \leq \sum_{\|i-j\|<2} \parallel \chi_{ij} R_A(E)^2 \chi_{ij} \parallel_1 . \tag{3.19}$$

Since the operator on the right in (3.19) is positive, we can evaluate the trace using the integral kernel

$$\begin{aligned} \| \chi_{ij} R_A(E)^2 \chi_{ij} \|_1 &= \sum_n \int \chi_{ij}^2(x) P_n(x,x)(E_n(B)-E)^{-2} dx \\ &\leq |\mathrm{supp}\chi_{ij}| C(E,B), \end{aligned} \tag{3.20}$$

where $C(E,B) \equiv \sum_n (E_n(B)-E)^{-2}$. Consequently, this gives

$$\sum_{\|i-j\|<2} \| K_{ij}(E) \|_1 \leq C(E,B). \tag{3.21}$$

As for the complementary set of indices, $i,j \in \tilde{\Lambda}$ such that $\| i-j \| > 2$, let $\chi_{ij}^+(x)$ be the characteristic function on the set $\{x \in \tilde{\Lambda} \mid \; \| x-i \| < \| x-j \| \}$, and define $\chi_{ij}^- \equiv 1 - \chi_{ij}^+$. Using the identity,

$$\| AB \|_1 \leq \| A \|_{HS} \| B \|_{HS}, \tag{3.22}$$

where HS denotes the Hilbert-Schmidt norm, we write

$$\begin{aligned} \| K_{ij}(E) \|_1 &= \| u_i^{1/2} R_A(E)((\chi_{ij}^+)^2 + (\chi_{ij}^-)^2) R_A(E) u_j^{1/2} \|_1 \qquad (3.23) \\ &\leq \| u_i^{1/2} R_A(E) \chi_{ij}^+ \|_{HS} \| \chi_{ij}^+ R_A(E) u_j^{1/2} \|_{HS} \\ &\quad + \| u_i^{1/2} R_A(E) \chi_{ij}^- \|_{HS} \| \chi_{ij}^- R_A(E) u_j^{1/2} \|_{HS} . \end{aligned}$$

As in [10], the condition on supp$u$ implies that there exists a constant $0 < a < 2$ such that

$$\mathrm{dist}(\mathrm{supp}\chi_{ij}^+, u_j) \geq a \| i-j \| . \tag{3.24}$$

It follows from this and the Combes-Thomas estimate (3.18) that there exists a positive constant C_E, depending only on E, such that

$$\| \chi_{ij}^+ R_A(E) u_j^{1/2} \|_{HS} \leq d(E)^{-1} \| u \|_1^{1/2} C_E e^{-ad(E)\|i-j\|/2}, \tag{3.25}$$

and

$$\| u_i^{1/2} R_A(E) \chi_{ij}^+ \|_{HS} \leq d(E)^{-1} \| u \|_1^{1/2} C_E, \tag{3.26}$$

and similarly for the other two terms in (2.18). So for the second sum we obtain

$$\sum_{\|i-j\|>2} \| K_{ij}(E) \|_1 \leq d(E)^{-2} C(a,E) |\mathrm{supp} u| \; |\Lambda|. \tag{3.27}$$

This estimate and (3.21) give the desired upper bound on (3.17). The constant $C(E,B)$, defined above, can be approximated by an integral giving (3.3). This proves the theorem. ■

4 Sketch of the Proof of Localization

The proof of localization for the model (1.1)-(1.3) follows the basic ideas of [9]. Once we control the decay of the localized resolvents of the infinite-volume Hamiltonian, the absence of continuous spectrum is proved using the Simon-Wolff [17] type argument in [9]. Hence, in this section, we will comment on the modifications of sections 4 and 5 of [10] and of the multiscale analysis of [9] necessary when the potential is unbounded (see also section 6 of [11]).

In addition to the Wegner estimate, Theorem 3.1, we need to verify an initial decay hypothesis on the localized resolvent of the finite-area Hamiltonian H_Λ. We fix the band index n. For each $E \in I_n(B) = (-\infty, B - \mathcal{O}(B^{-1})] \bigcup \{\cup_{j=0}^{n} [E_j(B) + \mathcal{O}(B^{-1}), E_{j+1}(B) - \mathcal{O}(B^{-1})]\}$, we show that the following condition holds. For $\delta > 0$ small, let $\chi_{l,\delta}$ be a smoothed characteristic function so that supp $\chi_{l,\delta} \subset \Lambda_l$, $\chi_{l,\delta} | \{x \in \Lambda_l | \text{dist}\,(x, \partial\Lambda_l) \geq \delta\}$, and supp $\nabla\chi_{l,\delta} \subset \{x \in \Lambda_l | \text{dist}\,(x, \partial\Lambda_l) \leq \delta\}$. Let $W(\chi_{l,\delta}) \equiv [-\Delta, \chi_{l,\delta}]$ be a first-order differential operator localized near the boundary of χ_l.

[H1](γ_0, l_0) For some $\gamma_0 > 0$, $l_0 > 1$ such that $l_0\gamma_0 > 1$, $\exists\, B_n > 0$ such that $\forall\, B > B_n$, $\exists\, \xi > 4$ such that

$$I\!\!P \left\{ \sup_{\epsilon > 0} ||W(\chi_{l_0,\delta}) R_{\Lambda_{l_0}}(E + i\epsilon)\chi_{l_0/3}|| < e^{-\gamma_0 l_0} \right\} \geq 1 - l_0^{-\xi} \tag{4.1}$$

Note that the size of the magnetic field B_n depends on the number n of bands treated (but not on the energy E).

This estimate (4.1) is established following the lines in [10]. We use percolation theory to prove the existence of effective magnetic barriers with large probability. Geometric perturbation theory is then used to prove the decay of the localized resolvent across these barriers. The main new feature in the case of densities with unbounded support is that the potential V_ω is no longer bounded. However, since we work only with finite-area Hamiltonians, we can compute the probability that V_{Λ_l} is bounded by a constant increasing with l. This probability approaches one as $l \to \infty$, and can be absorbed into the probability on the right side of [H1](γ_0, l_0). As a consequence of this l-dependent bound on V_{Λ_l} , the constants at each step in the multiscale analysis depend polynomially on l. We will indicate below how this is easily absorbed into the exponent. The second main difference is that we will prove localization at all energies below $B - \mathcal{O}(B^{-1})$.

We first turn to the percolation and decay estimates of section 4 in [10]. We recall that we associate the lattice $\Gamma \equiv e^{i\pi/4}\sqrt{2}\ Z\!\!\!Z^2$ with $Z\!\!\!Z^2$. To each site $j \in Z\!\!\!Z^2$ we associated a bond $b_j \in \Gamma$. For simplicity of exposition we will work with $E \in I_0(B)$. The case of any n is analogous.

We first consider $E < B$. We define a bond $b_j \in \Gamma$ *occupied* if $\lambda_j(\omega) > (E-B)/2$. Note that the probability p that b_j is occupied is $p = \int_{\frac{E-B}{2}}^{\infty} g(\lambda) d\lambda > \frac{1}{2} = p_c$, the critical probability. We define a ribbon $\mathcal{R}$ associated with a closed circuit $\mathcal{C}$ of occupied bonds in Γ as in [10]. Given that such a closed circuit $\mathcal{C}$ exists, we have for all $x \in \mathcal{R}$,

$$V(x) + B - E > (B - E)/2 \equiv a > 0, \tag{4.2}$$

with a good probability. For $E > B$, we consider the bond b_j to be *occupied* if $\lambda_j(\omega) < (E - B)/2$. The probability that b_j is occupied is $p = \int_{-\infty}^{(E-B)/2} g(\lambda)\, d\lambda > \frac{1}{2} = p_c$. On a closed ribbon of occupied bonds we have

$$V(x) + B - E < (B - E)/2 = -a < 0. \tag{4.3}$$

Percolation theory (cf. [18]) states that if $p > p_c$ then the probability that there exists a closed circuit $\mathcal{C}$ of occupied bonds lying in the annular region a_l between a square of side l and a square of side $3l$ is $\geq 1 - Cle^{-m(1-p)l}$, where $m(p)$ is a strictly positive, monotone function satisfying $m(p) \downarrow 0$ as $p \uparrow p_c$. Consequently, the following proposition (Proposition 4.1 of [10]) is valid with the probability p given above. For an open region $\Omega \subset I\!\!R^2$, we define $inrad \Omega \equiv \sup \{R > 0 \mid B_R(x) \subset \Omega, \forall x \in \Omega\}$.

Proposition 4.1 *Assume hypotheses (V1)–(V2) (or (V2)') and suppose that supp* $u \subset B_{r_u}(0)$ *for some* $r_u < 1/\sqrt{2}$. *Let* $l > \sqrt{2}$, $E \in (-\infty, 3B)\backslash\{B\}$, *and* $a > 0$. *Then for* $m(p)$ *and the constant* C *introduced above, there exists a ribbon* $\mathcal{R}$ *satisfying*

$$inrad\, \mathcal{R} \geq 2\left(\frac{1}{\sqrt{2}} - r_u\right); \tag{4.4}$$

$$dist\,(\mathcal{R}, \partial r_{3l}),\ dist\,(\mathcal{R}, \partial r_l) \geq \frac{1}{\sqrt{2}} + r_u; \tag{4.5}$$

$$\mathcal{R} \subset a_l, \tag{4.6}$$

and such that (4.2)–(4.3) hold with a probability larger than

$$1 - Cle^{-m(1-p)l}, \tag{4.7}$$

where

$$p \equiv \int_{-\infty}^{a} g(\lambda) d\lambda = \int_{-a}^{\infty} g(\lambda) d\lambda. \tag{4.8}$$

As for the decay estimates of section 4.2 of [10], we recall that the potential V of that section is supposed to be bounded, to have compact support, and satisfy either condition (4.2) or (4.3) $\forall x \in \mathbb{R}^2$. When we apply this to the finite-area Hamiltonians H_Λ, the potential V of section 4.2 is the local potential V_Λ. As we have seen above, this potential satisfies (4.2) or (4.3), depending on the energy, with a probability given in (4.8). As for the boundedness of V_Λ, the probability P_M that $|V_{\Lambda_l}| < M$ is given by

$$P_M \equiv \mathbb{P}\{|V_{\Lambda_l}(x)| < M \quad \forall x \in \Lambda_l\} \geq \left[1 - 2\int_M^\infty g(\lambda)\, d\lambda\right]^{|\Lambda_l|}. \tag{4.9}$$

We are interested in the asymptotics as $l \uparrow \infty$ and $M \uparrow \infty$. Using the decay on the probability given in (1.4), we find

$$P_M \geq 1 - C_0 l^2 M^{-N}, \tag{4.10}$$

for some $N \geq 2$. We want this to be larger than $1 - l^{-\xi}$, for some $\xi > 4$. Since $N \geq 2$, we take $M = \mathcal{O}(l^{3+\epsilon})$, for some $\epsilon > 0$.

As a consequence, Lemma 4.1 of [10] is modified as follows. Each local Hamiltonian $H_{\Lambda_l} = H_A + V_{\Lambda_l}$ admits an analytic continuation onto a strip $S = \{\alpha \in \mathbb{C} |\ |\text{Im } \alpha| < \eta_p B^{\frac{1}{2}}\}$, whose width is independent of l, with a probability $\geq 1 - l^{-\xi}$, for $\xi > 4$ as above. The decay estimates on the analytic family $H_{\Lambda_l}(\alpha)$, as given in Theorem 4.3, hold with probability $\geq 1 - l^{-\xi}$. The constant $C_2 \leq \eta_p$, and hence γ, are independent of l. With regard to the a priori estimates on the projections given in Appendix 1, we have, for example, $\forall\ \alpha$ with $|\text{Im } \alpha| < \gamma$,

$$||P(\alpha) V_{\Lambda_l} Q(\alpha)|| < C_l B^{-\frac{1}{2}}, \tag{4.11}$$

with probability $\geq 1 - l^{-\xi}$, some $\xi > 4$, and $C_l = \mathcal{O}(l^{3+\epsilon})$. The constant in Lemma A.2 is independent of l. We obtain the following version of Corollary 4.1.

Proposition 4.2. *Let $\chi_{l/3}$ be the characteristic function on $\Lambda_{l/3}$. For $\delta > 0$ small, let $\chi_{l,\delta}$ be the characteristic function for $\{x \in \Lambda_l | dist\,(x, \partial\Lambda_l) < \delta\}$. There exists constants $C_0 > 0$ and $C_1 > 0$ such that for any $l \gg 1$, $\exists\ B_0(l) = \mathcal{O}(l^{3+\epsilon})$, such that $\forall\ B > B_0(l)$ and $\forall\ E \in (-\infty, B - \mathcal{O}(B^{-1})] \cup [B + \mathcal{O}(B^{-1}),\ \ 3B - \mathcal{O}(B^{-1})]$, we have*

$$\sup_{\epsilon \neq 0} \|\chi_{l,\delta}(H_A + V_{\Lambda_l} - E - i\epsilon)^{-1}\chi_{l/3}\| \leq C_0 l^{3+\epsilon} \max\{a^{-1}, B^{-1}\} e^{-\gamma l},$$

with probability $\geq 1 - l^{-\xi}$, $\xi > 4$, where $\gamma \equiv C_1 \min\{B^{\frac{1}{2}}, aB\}$ and $a = (B-E)/2$ if $E < B$ and $a = (E-B)/2$ if $E > B$.

This proposition is the main component in verifying [H1](γ_0, l_0). Note that there is a gradient term in [H1](γ_0, l_0) coming from the operator $W(\chi_{l,\delta})$. This is now controlled as in Lemma 5.1 of [10] with a constant $\mathcal{O}(l^{3+\epsilon})$ and with a probability $\geq 1 - l^{-\xi}$, $\xi > 4$. We have the following analog of Proposition 5.1 of [10].

Proposition 4.3. *For any $l_0 \gg \sqrt{2}$ $\exists\ B_0 \equiv B_0(l_0)$ such that for any $E \in I_0(B)$, and for $a \equiv (B-E)/2$, if $E < B$, and $a \equiv (E-B)/2$, if $E > B$, and $a = \mathcal{O}(B^{-1+\sigma})$, for any $\sigma > 0$, hypothesis* [H1](γ_0, l_0) *holds for some $\xi > 4$ and for some $\gamma_0 > C_0\gamma$, where $\gamma = C_1\, min\{B^{\frac{1}{2}}, B^{\sigma}\}$.*

As for the multiscale analysis of [9], at each step of the iteration, we use (4.4)-(4.5) so we can work with a bounded local potential $V_{\Lambda_{l_n}}$. As a consequence, the coefficient of the estimate on the resolvent of the finite-volume Hamiltonian is $\mathcal{O}(l_n^{3+\epsilon})$. This, however, contributes a vanishing small amount $\mathcal{O}(l_{n+1}^{-1}\log l_n)$ to the new decay constant γ_{n+1}, and hence the iteration is unchanged. The probability estimate is also modified by an amount $\geq l_n^{-\xi}$, $\xi > 4$, which guarantees the boundedness of the local potential. This, however, can be easily absorbed into the probability at the $(n+1)^{st}$ step.

Let us remark that for energies near the center of the unperturbed gaps, in $\sigma(H_A)$, the hypothesis [H1](γ_0, l_0) can be obtained using a Combes-Thomas type argument without percolation theory. We refer the reader to [11] for discussion of this method.

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