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LOCALIZATION NEAR BAND EDGES FOR RANDOM SCHRÖDINGER OPERATORS

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Abstract

In this article, we prove exponential localization for wide classes of Schrödinger operators, including those with magnetic fields, at the edges of unperturbed spectral gaps. We assume that the unperturbed operator H_0 has an open gap $I_0 \equiv (B_-, B_+)$. The random potential is assumed to be Anderson-type with independent, identically distributed coupling constants. The common density may have either bounded or unbounded support. For either case, we prove that there exists an interval of energies in the unperturbed gap for which the almost sure spectrum of the family $H_\omega \equiv H_0 + V_\omega$ is dense pure point with exponentially decaying eigenfunctions. We also prove that the integrated density of states is Lipschitz continuous in the unperturbed spectral gap I_0.

Key-Words: localization, random operators, Schrödinger operators

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1 Introduction

The phenomenon of exponential localization for various families of random Schrödinger operators $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^d)$, $d \geq 1$, near the bottom of the almost sure spectrum is now reasonably well understood, see for example [5],[22], [20], [11], [12], [19], [24]. The results for lattice Schrödinger operators on $\ell^2(\mathbb{Z}^d)$ can be found in the book of Carmona and Lacroix [4] and Pastur and Figotin [27]. Recently, there have been several results [6], [23], [13], [14], [15], [31], [20], [1] concerning band edge localization, i.e. the existence of pure point spectra near the edges of the spectral bands of the deterministic, unperturbed operator H_0. In this paper, we prove that band edge localization is a rather general phenomenon. We study the perturbation of fixed, background Schrödinger operators $H_0 = (-i\nabla - A)^2 + V_0$ on $L^2(I\!\!R^d)$, $d \geq 1$, with an open spectral gap $I_0 = (B_-, B_+)$, by random potentials V_ω of Anderson-type

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u(x - i) \ . \tag{1.1}$$

The coupling constants $\{\lambda_i(\omega) | i \in \mathbb{Z}^d\}$ are assumed to be independent and identically distributed with common density h. We assume $u \geq 0$ and $\operatorname{supp} u$ is compact. There are three main results. First, we assume that $\operatorname{supp} h$ is compact and that the almost sure spectrum Σ of the family $H_0 + V_\omega$ has an open spectral gap $(\widetilde{B}_-, \widetilde{B}_+)$ with $B_- < \widetilde{B}_- < \widetilde{B}_+ < B_+$. We prove that near the band edges $\widetilde{B}_-$ and $\widetilde{B}_+$, the spectrum Σ consists of only pure point spectrum with exponentially decaying eigenfunctions. This result requires that h decays sufficiently rapidly near the edges of its support.

Secondly, we consider the case when $\operatorname{supp} h$ is unbounded so there is no spectral gap in Σ near I_0. We add a coupling constant $g \geq 0$ and let $\Sigma(g)$ denote the deterministic spectrum of $H_0 + gV_\omega$. For any energies $E_\pm$ with $B_- < E_- < E_+ < B_+$, we prove that there exists $g_0 > 0$ such that $\Sigma(g) \cap (E_-, E_+)$ is pure point for all $0 < g < g_0$, with exponentially decaying eigenfunctions. Thirdly, we prove that the integrated density of states for each of the families H_ω and $H_\omega(g)$ is Lipschitz continuous in the spectral gap I_0 of H_0 in both cases.

In the case of bounded perturbations V_ω, the localization result follows from the fact that the spectrum is "thin" near the band edges $\widetilde{B}_\pm$ (provided the density h decays sufficiently rapidly). In fact, for local Hamiltonians $H_{\Lambda,\omega} \equiv H_0 + (V_\omega | \Lambda)$, associated with bounded regions $\Lambda \subset \mathbb{R}^d$, we prove that the eigenvalues remain at a strictly positive distance from $\widetilde{B}_-$ and $\widetilde{B}_+$ with a good probability. This fact allows one to apply the Combes-Thomas argument [8] in order to prove decay estimates on the resolvent of $H_{\Lambda,\omega}$ with a good probability. This initial scale estimate, together with an improved Wegner estimate, are the starting points for the multiscale analysis of [5] which results in almost sure decay estimates for the localized resolvent of the infinite volume Hamiltonian. When the density h has unbounded support, we must add a coupling constant g and work in the weak coupling regime.

We also prove in this paper an improved version of the Combes-Thomas estimate [8] on the decay of localized resolvents. This result may be of independent interest. Suppose H is a self-adjoint operator with a spectral gap (B_-, B_+). The usual Combes-Thomas result gives an upper bound on the spatial decay of the resolvent $(H-E)^{-1}$, $E \in (B_-, B_+)$, with a decay constant proportional to $\operatorname{dist}(E, \sigma(H))$. We prove here that the decay constant is proportional to $\sqrt{\Delta_+(E)\Delta_-(E)}$, where $\Delta_+(E) \equiv \operatorname{dist}(E, \{\lambda \in \sigma(H) | \lambda \geq B_+\})$ and $\Delta_-(E) \equiv \operatorname{dist}(E, \{\lambda \in \sigma(H) | \lambda \leq B_-\})$. Note that when E is close to $\sigma(H)$, the decay is approximately $[\operatorname{dist}(E, \sigma(H))]^{1/2}$. This is similar to the case when $E \leq \inf \sigma(H)$ and the decay constant is proportional to $[\operatorname{dist}(E, \sigma(H))]^{1/2}$.

In [23], Klopp studied localization induced by random perturbations of a periodic Schrödinger operator $H_0(h) = -h^2\Delta + V_0$ in the semiclassical regime. He proved exponential localization near the band edges of the first band of $H_0(h)$ for h small. Figotin and Klein [13] studied band edge localization for perturbations of periodic lattice Schrödinger operators $H(g) = H_0 + gv$, on $\ell^2(\mathbb{Z}^d)$, in the weak coupling regime. In [14], these results were extended to lattice models of acoustic and electromagnetic waves propagating in random media. These results for wave propagation were extended to continuum models in [15]. In all these cases, the random perturbation is Anderson-type. Aizenman [1] gave an elementary proof of band edge localization on the lattice in the weak disorder regime. He studied Anderson-type perturbations $H_\omega = H_0 + \lambda V_\omega$, of a background operator $H_0 = T + U_0$, where T is a bounded self-adjoint operator with exponentially decaying matrix elements and U_0 is periodic. Aizenman proved that the a.s. spectrum of H_ω is pure point near $\sigma(H_0)$ for λ in a certain regime of small values. He utilized an extension of the ideas of Aizenman and Molchanov [2] which avoids multiscale analysis (Unfortunately, it is not clear how to extend [2] to continuous models). Our own interest in band edge localization originated with our study of localization for the randomly perturbed Landau Hamiltonian on $L^2(\mathbb{R}^2)$ ([6], [3], [31], [11], [12]). We discuss this model in Example 2.1 of the next section.

This paper is organized as follows. In section 2, we present the main hypotheses and results. We provide several examples of models satisfying these hypotheses. An improved version of the Combes-Thomas estimate is presented in section 3. Section 4 contains a new proof of the Wegner estimate which can be applied to models with unbounded random potentials (see [3]). In section 5, we give estimates on the location of the spectrum of the finite-volume Hamiltonians H_Λ, with good probability. These results, with those of section 3, allow us to verify the initial decay hypothesis $[\mathrm{H1}](\gamma_0, \ell_0)$ of [5]. By the multiscale analysis and perturbation theory of [5], we then establish band edge localization when $\operatorname{supp} h$ is bounded. The case of $\operatorname{supp} h$ unbounded is discussed in section 6. We present certain technical trace ideal estimates in the appendix, section 7.

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2 The Models and the Main Results

We study random families of Schrödinger operators $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^d)$, $d \geq 1$. The unperturbed Schrödinger operator H_0 has the form

$$H_0 = (-i\nabla - A)^2 + V_0 \ , \tag{2.1}$$

where A is a vector potential and V_0 is a background electrostatic potential. We first list the assumptions on H_0 and present our main results. We then discuss the assumptions and give several examples. Let $R_0(z) \equiv (H_0 - z)^{-1}$ denote the resolvent of H_0.

(H1) The operator H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

(H2) The spectrum of H_0, $\sigma(H_0)$, is semibounded and contains an open gap, that is, there exist finite constants $C_0 \geq 0$ and $-C_0 \leq B_- < B_+ \leq \infty$ such that

$$\sigma(H_0) \subset (-C_0, B_-] \cup [B_+, \infty) \ .$$

(H3) The operator H_0 is strongly locally compact in the sense that for any $f \in L^\infty(\mathbb{R}^d)$ with compact support, the operator $f(H_0+C_0+1)^{-1} \in \mathcal{J}_q$, for some even integer q, $1 \leq q < \infty$.

(H4) Let $\rho(x) \equiv (1 + \|x\|^2)^{1/2}$. The operator

$$H_0(\alpha) \equiv e^{i\alpha\rho} H_0 e^{-i\alpha\rho} \ ,$$

defined for $\alpha \in \mathbb{R}$, admits an analytic continuation as a type-A analytic family to a strip

$$S(\alpha_0) \equiv \{x + iy \in \mathbb{C} \mid |y| < \alpha_0\} \ ,$$

for some $\alpha_0 > 0$.

We now describe the random perturbations V_ω. We assume V_ω is Anderson-type of the form

$$V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u(x - i) \ . \tag{2.2}$$

The coupling constants $\{\lambda_i(\omega)\}$ and the single-site potential u are assumed to satisfy the following conditions.

(H5) The coupling constants $\{\lambda_i(\omega) \mid i \in \mathbb{Z}^d\}$ form a family of independent, identically distributed (iid) random variables. The common distribution has a density h satisfying $0 \leq h \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. There exist not necessarily finite, positive constants $0 < m, M$ such that $\operatorname{supp} h \subset [-m, M]$ and $h(0) > 0$.

(H6) The density h decays sufficiently rapidly near $-m$ and near M in the following sense. If $0 < m \leq M < \infty$, then

$$0 < \mathbb{P}\{|\lambda + m| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

$$0 < \mathbb{P}\{|\lambda - M| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

for some $\beta > 0$. In the case that either m or M is infinite, we require that for some $r > \max(q, d/2)$.

$$C_h \equiv \sup_\lambda h(\lambda)|\lambda|^{r+2} < \infty.$$

We take $\Omega \equiv [\operatorname{supp} h]^{\mathbb{Z}^d}$ to be the probability space equipped with the probability measure $\mathbb{P}$ induced by the finite product measure. The single site potential u in (2.2) is assumed to satisfy.

(H7) The single-site potential u has compact support and $0 \leq u \in L^\infty(\mathbb{R}^d)$.

At the level of generality maintained so far, we need some hypotheses on the spectral properties of the random family H_ω.

(H8) The family $\{H_\omega | \omega \in \Omega\}$ has deterministic spectrum Σ in the sense that $\exists \Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for $\omega \in \Omega_0$, $\sigma(H_\omega) = \Sigma$.

According to whether $\operatorname{supp} h$ is bounded or unbounded, we need to consider the nature of Σ near the unperturbed spectral gap (B_-, B_+).

(H9) Suppose supp h is bounded, i.e. $0 < m$, $M < \infty$. Then, $\exists$ constants $B'_\pm$ satisfying $B_- < B'_- < B'_+ < B_+$ such that

$$\Sigma \cap \{(B_-, B'_-) \cup (B'_+, B_+)\} \neq \emptyset\ .$$

We remark that in the presence of ergodicity (H8) is known (cf.[27]). In the unbounded case (H5) and (H7) imply that, the deterministic spectrum Σ fills the gap (B_-, B_+) entirely (see Proposition 6.4). Given (H9), we define the perturbed band edges $\widetilde{B}_\pm$, satisfying $B_- < \widetilde{B}_- \leq B'_-$ and $B'_+ \leq \widetilde{B}_+ < B_+$ by

$$\widetilde{B}_- \equiv \sup \left\{ E \in \Sigma \mid E \leq B'_- \right\} \tag{2.3}$$

and

$$\widetilde{B}_+ \equiv \inf \left\{ E \in \Sigma \mid E \geq B'_+ \right\} \tag{2.4}$$

We can now state our main results.

Theorem 2.1

Assume (H1) - (H9) and that $\operatorname{supp} h$ *is bounded, i.e.* $0 < m, M < \infty$. *There exist constants* $E_\pm$ *satisfying* $B_- \leq E_- < \widetilde{B}_-$ *and* $\widetilde{B}_+ < E_+ \leq B_+$ *such that* $\Sigma \cap (E_-, E_+)$ *is pure point with exponentially decaying eigenfunctions.*

In the case that $\operatorname{supp} h$ is unbounded, we must introduce a coupling constant g and work in the weak disorder regime of small g.

Theorem 2.2

Let $H_\omega(g) \equiv H_0 + gV_\omega$. *Suppose that* $\operatorname{supp} h$ *is unbounded and assume the hypotheses (H1) - (H8). For any energies* $E_\pm$ *satisfying* $B_- < E_- < E_+ < B_+$, $\exists\, g_0 = g_0(E_\pm) > 0$ *such that for all* $0 < g < g_0$, *we have* $\Sigma \cap (E_-, E_+)$ *is pure point with exponentially decaying eigenfunctions.*

Finally, the Wegner estimate of section 4 provides the following regularity result for the integrated density of states (IDS) in the unperturbed spectral gap.

Theorem 2.3

Assume (H1) - (H9) and $\operatorname{supp} h$ *bounded or (H1) - (H8) and* $\operatorname{supp} h$ *unbounded. In either case, the integrated density of states is Lipschitz continuous on* (B_-, B_+).

We remark that if $h \in C^k$, then we believe that the IDS $N(E) \in C^k((B_-, B_+))$. Such a result for $k \geq 3d/2$ would allow us to remove hypothesis (H6) in the case that supp h is compact.

Let us make a few remarks on the hypotheses. We refer to the review of Simon[29] and the book by Cycon, Froese, Kirsch and Simon[9] for further details. A theorem of Leinfelder and Simader[25] states that if $V \in L^2_{loc}$, $V_- \in K_d$, and $A \in L^4_{loc}$, then $C_0^\infty(\mathbb{R}^d)$ is a core for H_0, which is condition (H1). Let $H_A \equiv (-i\nabla - A)^2$ be the pure magnetic Hamiltonian. If $A \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, it is easy to see (cf [29]) that $D(H_A) \subset H^2(\mathbb{R}^d) = D(-\Delta)$. Let us suppose also that V_0 is relatively $-\Delta$-bounded with relative bound < 1. Then, $(-\Delta + V_0)$ is semibounded by some $-C_0 > -\infty$. The diamagnetic inequality (see [29]) implies that H_0 is also semibounded with the same constant. The strong local compactness condition (H3) is immediate under these conditions. Indeed, it suffices to prove that for all $f \in L^\infty$ with compact support, the operator $f(-\Delta + C_0 + 1)^{-1} \in \mathcal{J}_q$, for all q such that $\infty > q > [d/2]$ as in (H3). This follows from the standard estimate (see [30]):

$$f(x)g(-i\nabla) \in \mathcal{J}_q \quad if \quad f, g \in L^q(\mathbb{R}^d) \quad \text{for} \quad \infty > q > [d/2] \ .$$

The analyticity condition (H4) is also satisfied for general (A, V_0). For $\alpha \in \mathbb{R}$, we have

$$H_0(\alpha) \equiv e^{i\alpha\rho} H_0 \ e^{-i\alpha\rho} = H_0 - 2\alpha\nabla\rho \cdot (-i\nabla - A) + i\alpha\Delta\rho + \alpha^2|\nabla\rho|^2 \ ,$$

with $|\nabla\rho|$ and $\Delta\rho$ bounded. Assuming that V_0 is relatively H_A-bounded, it suffices for analyticity in α to show that for some $z \in \rho(H_A)$, the operator

$$\left\{-2\alpha\nabla\rho \cdot (-i\nabla - A) + i\alpha\Delta\rho + \alpha^2|\nabla\rho|^2\right\}(H_A - z)^{-1} \ , \tag{2.5}$$

is bounded with norm less than one for some $z \in \rho(H_A)$. Since the operator in (2.5) is bounded above by

$$\operatorname{dist}(\sigma(H_A), z)^{-1}\left\{2|\alpha|\,|\nabla\rho| \max\left\{1, |z|^{1/2}\right\} + |\alpha|\,|\Delta\rho| + \alpha^2|\nabla\rho|^2\right\} \ ,$$

it follows that for any fixed $\alpha_0 > 0$, this bound can be made $< 1/2$ by taking $z = -i\sigma$, $\sigma > 0$ sufficiently large. This shows that $H_0(\alpha)$ has a continuation to any strip $S(\alpha_0)$, $\alpha_0 > 0$.

We now present several examples satisfying these conditions and hypotheses (H2), (H8) and (H9).

Example 2.1

Landau Hamiltonians in $d = 2$ dimensions. We take $V_0 = 0$ and $A = \frac{B}{2}(-x_2, x_1)$. In this case, the unperturbed spectrum of H_A is pure point $\sigma(H_A) = \{E_n(B) = (2n+1)B,\ n = 0, 1, 2, ...\}$. When $\operatorname{supp} h$ is compact, the existence of localized states away from a region of size $\mathcal{O}(B^{-1})$ centered at the Landau energies $E_n(B)$, and for B, large was proved in [6]. Theorem 2.1 applied to this case avoids the restriction that B is large. The analog of Theorem 2.2, when $\operatorname{supp} h$ is unbounded, is proved in [3].

Example 2.2

Periodic Schrödinger Operators. We set $A = 0$ *so* $H_0 = -\Delta + V_0$ *and assume that* V_0 *is a real, bounded, periodic function with an open gap (see [28]).The random family* H_ω *has deterministic spectrum provided the lattice group of* V_0 *is commensurate with* $\mathbb{Z}^d$*. In the case of* $\operatorname{supp} h$ *bounded, condition (H9) can be guaranteed by writing* $H_\omega(\lambda) = H_0 + \lambda V_\omega$ *and taking* λ *small enough.*

Example 2.3

Pure Magnetic Field Hamiltonians. We take $H_0 = H_A(\lambda) \equiv (-i\nabla - \lambda A)^2$ *with* $\lambda \in \mathbb{R}$*, and* $A \in C^1(\mathbb{R}^d, \mathbb{R}^d)$*. Amongst other questions, Hempel and Herbst [17] studied the case when the magnetic field* $B = dA$ *is periodic with respect to* $\mathbb{Z}^d (d \geq 2)$*. Let* $M_B \equiv \{x | B(x) = 0\}$ *and* $M_A \equiv \{x | A(x) = 0\}$*. Under the condition that* $|M_B \backslash M_A| = 0$*, they show that* $H_A(\lambda)$ *converges in the norm resolvent sense as* $\lambda \to \infty$ *to the Dirichlet Laplacian on* M*. Based on this, they construct examples of pure magnetic hamiltonians* $H_A(\lambda)$ *in dimensions* $d \geq 2$ *with periodic magnetic fields and with open spectral gaps for all* λ *sufficiently large. As discussed above, our hypotheses hold for these models.*

Example 2.4

Magnetic Hamiltonians with periodic Potentials. Nakamura [26] (inspired by [17]) studied the existence of open spectral gaps for more general magnetic Schrödinger operators, for $d \geq 2$*, of the form*

$$H_0(\lambda) = (-i\nabla - \lambda A)^2 + V_0 \ ,$$

with $A \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ *and* V_0 *real and bounded. The result of [26] of interest to us is the following. Suppose* $B = dA$ *and* V *are periodic with respect to a lattice subgroup* Γ *of* $\mathbb{Z}^d$ *with bounded fundamental domain* Ω_Γ*. Let* $H_0^D(\lambda)$ *be the restriction of* $H_0(\lambda)$ *to* Ω_Γ *with Dirichlet boundary conditions on* $\partial\Omega_\Gamma$*. The spectrum* $\Sigma(\lambda)$ *of* $H_0^D(\lambda)$ *is discrete. Under the assumption that the largest eigenvalue of the matrix* $(B_{ij}(x))$ *restricted to* $\partial\Omega_\Gamma$ *is strictly positive, Nakamura proves that for all* λ *large* $\sigma(H_0(\lambda))$ *lies in neighborhoods of size* $\mathcal{O}(e^{-\alpha\sqrt{\lambda}})$*, for some* $\alpha > 0$*, about* $\Sigma(\lambda)$*. Hence, there are open spectral gaps in* $\sigma(H_0(\lambda))$ *and Theorems 2.1 and 2.2 apply to random perturbations of these* *operators. Note that if A is periodic, the operator $H_0(\lambda)$ has band spectrum and the width of the bands is $\mathcal{O}(e^{-\alpha\lambda})$.*

Example 2.5

Combes-Hislop model revisited. Theorem 2.1 can be applied to the Anderson-type models studied in [5] improving the result proven there. Let $H_\omega \equiv -\Delta + V_\omega$, where V_ω is given in (2.2). In [5], we assumed (H5) with $m = 0$, $M < \infty$, and that $u \geq C_0 \chi_{\Lambda_0}$, where χ_{Λ_0} is the characteristic function on the unit cube. The present work allows us to remove this last assumption on the single-site potential u. We choose a constant $C_1 > 0$ satisfying $C_1 < M$ and write

$$\begin{aligned} H_\omega &= \{-\Delta + \sum_{i \in \mathbb{Z}^d} C_1 u_i\} + \sum_{i \in \mathbb{Z}^d} (\lambda_i - C_1) u_i \\ &= H_0 + \tilde{V}_\omega \ , \end{aligned} \tag{2.6}$$

where $u_i(x) \equiv u(x - i)$. The operator H_0 is a periodic Schrödinger operator with a positive potential. and hence inf $\sigma(H_0) = \Sigma_0 > 0$. The potential $\tilde{V}_\omega$ is an Anderson-type potential with coupling constants $\tilde{\lambda}_i(\omega) \equiv \lambda_i(\omega) - C_1$. The density of these random variables has support in $[-C_1, M - C_1]$. Theorem 2.1 now shows that there is a small interval of energy of the form $[0, E_0]$, for some $E_0 > 0$, in which the spectrum is pure pont almost surely. Theorem 2.3 guarantees the Lipschitz continuity of the integrated density of states in the interval $[0, \Sigma_0]$.

3 Improved Resolvent Decay Estimates

In this section, we present an alternative form of the Combes-Thomas method [8] which allows an improvement on the rate of decay of the resolvent which is of independent interest. The basic technical result is the following.

Lemma 3.1

Let A and B be two self-adjoint operators such that $d_\pm \equiv \mathrm{dist}(\sigma(A) \cap \mathbb{R}^\pm, 0) > 0$, and $\|B\| < 1$. Then,

(i) For $\beta \in \mathbb{R}$ s.t. $|\beta| < \frac{1}{2}\sqrt{d_+ d_-}$, one has $0 \in \rho(A + i\beta B)$,

(ii) For $\beta \in \mathbb{R}$ as in (i),

$$\left\|(A+i\beta B)^{-1}\right\| \leq 2\sup(d_+^{-1}, d_-^{-1}).$$

Proof:

Let $P_\pm$ be the spectral projectors for A corresponding to the sets $\sigma(A)\cap\mathbb{R}^\pm$, respectively and define $u_\pm \equiv P_\pm u$. By the Schwarz inequality one has

$$\begin{aligned} \|u\|\;\|(A+i\beta B)u\| &\geq Re\,\langle (u_+ - u_-), (A+i\beta B)(u_+ + u_-)\rangle \\ &\geq d_+\|u_+\|^2 + d_-\|u_-\|^2 - 2\beta Im\,\langle u_+, Bu_-\rangle \\ &\geq \tfrac{1}{2}(d_+\|u_+\|^2 + d_-\|u_-\|^2), \end{aligned} \tag{3.1}$$

where we again used the Schwarz inequality to estimate the inner product. It follows that

$$\|(A+i\beta B)u\| \geq \frac{1}{2}\min(d_+, d_-)\|u\|,$$

and since this is independent of the sign of β, the lemma follows.

Proposition 3.2

Let $\widetilde{H}$ be a semibounded self-adjoint operator with a spectral gap $G \equiv (E_-, E_+) \subset \rho(\widetilde{H})$. Let W be a symmetric operator such that $D(W) \supset D((\widetilde{H}+C_0)^{\frac{1}{2}})$ and $\|(\widetilde{H}+C_0)^{-\frac{1}{2}}W(\widetilde{H}+C_0)^{-\frac{1}{2}}\| < 1$, for some C_0 such that $\widetilde{H}+C_0 > 1$. For any $E \in G$, let $\Delta_\pm \equiv \mathrm{dist}(E_\pm, E)$. Then, we have

(i) The energy $E \in \rho(\widetilde{H}+i\beta W)$ for all real β satisfying

$$|\beta| < \frac{1}{2}\left\{\frac{\Delta_+\Delta_-}{(E_+ + C_0)(E_- + C_0)}\right\}^{\frac{1}{2}};$$

(ii) for any real β and energy E as in (i),

$$\|(\widetilde{H}+i\beta W - E)^{-1}\| \leq 2\sup\left(\frac{E_+ + C_0}{\Delta_+}, \frac{E_- + C_0}{\Delta_-}\right).$$

Proof:

Let $E \in G$ and C_0 be as above. Define a self-adjoint operator $A \equiv (\widetilde{H}+C_0)^{-1}(\widetilde{H}-E)$ and $B \equiv (\widetilde{H}+C_0)^{-\frac{1}{2}}W(\widetilde{H}+C_0)^{-\frac{1}{2}}$. By hypothesis, the operator B is self-adjoint and satisfies $\|B\| < 1$. Note that $0 \in \rho(A)$ and

$$d_\pm \equiv \mathrm{dist}(\sigma(A) \cap \mathbb{R}^\pm, 0) = \Delta_\pm (E_\pm + C_0)^{-1} > 0 \tag{3.2}$$

Applying Lemma 3.1 to these operators A and B, we see that for β as in (i), $0 \in \rho(A + i\beta B)$ and that

$$\|(A+i\beta B)^{-1}\| \leq 2 \sup \left(\frac{E_+ + C_0}{\Delta_+}, \frac{E_- + C_0}{\Delta_-} \right) .$$

Let $P_\pm$ be as in the proof of Lemma 3.1. For any $w \in D(\widetilde{H})$,

$$\begin{aligned} \|(\widetilde{H} + i\beta W - E)w\| &= \|(\widetilde{H}+C_0)^{\frac{1}{2}}(A+i\beta B)(\widetilde{H}+C_0)^{\frac{1}{2}}w\| \\ &\geq \|(A+i\beta B)(\widetilde{H}+C_0)^{\frac{1}{2}}w\| , \end{aligned}$$

since $(\widetilde{H}+C_0) \geq 1$. We now repeat estimate (3.1) taking $u \equiv (\widetilde{H}+C_0)^{\frac{1}{2}}w$. This gives

$$\begin{aligned} \|(\widetilde{H} + i\beta W - E)w\| &\geq \tfrac{1}{2}\|(\widetilde{H}+C_0)^{\frac{1}{2}}u\|^{-1} \Big(d_+\|P_+(\widetilde{H}+C_0)^{\frac{1}{2}}w\|^2 \\ &\quad +(d_-\|P_-(\widetilde{H}+C_0)^{\frac{1}{2}}w\|^2\Big) \\ &\geq \tfrac{1}{2}\min(d_+, d_-)\|(\widetilde{H}+C_0)^{\frac{1}{2}}w\| . \end{aligned} \tag{3.3}$$

Since $\|(\widetilde{H}+C_0)^{\frac{1}{2}}w\| \geq \|w\|$ and $d_\pm$ are defined in (3.2), result (ii) follows from (3.3) and Lemma 3.1.

Theorem 3.3 *Let H_0 be given as in (2.1) satisfying (H1) and (H4). Let V be H_0-bounded with relative bound less than 1 and define the self-adjoint operator $H = H_0 + V$. Then the dilated operator $H(\alpha) \equiv e^{i\alpha\rho}He^{-i\alpha\rho}$, $\alpha \in \mathbb{R}$, admits an analytic continuation to a type-A family on the strip $S(\alpha_0)$ (ρ and $S(\alpha_0)$ are defined in (H4)). Suppose H_0 satisfies (H2) and that H has a spectral gap $G \equiv (E_-, E_+) \subset (B_-, B_+)$ $(E_- \neq E_+)$. For $E \in G$, define $\Delta_\pm \equiv \mathrm{dist}(E_\pm, E)$. Then there exist finite constants C_1, $C_2 > 0$, depending only on H_0 and V, such that*

(i) for any real β satisfying $|\beta| < \min(\alpha_0, C_1\sqrt{\Delta_+\Delta_-}, \sqrt{\Delta_+/2})$, the energy $E \in \rho(H(i\beta))$;

(ii) for any real β as in (i),

$$\|(H(i\beta) - E)^{-1}\| \leq C_2 \max(\Delta_+^{-1}, \Delta_-^{-1}) \ . \tag{3.4}$$

Proof:

As in the discussion of (H4) in section 2, we have,

$$H(\alpha) = H + \alpha^2 |\nabla \rho|^2 + \alpha W \ ,$$

where $\alpha \in \mathbb{R}$ and $W = -(\nabla\rho.(p - A) + (p - A).\nabla\rho)$ is symmetric. Note that $\|\nabla\rho\|_\infty = 1$ and $\|\Delta\rho\|_\infty = 1$ and that V and V_0 are relatively $(p - A)^2$-bounded. Consequently, (2.1) is less than $1/2$ for $|Imz|$ large enough. This proves analyticity of $H(\alpha)$ in $S(\alpha_0)$. Taking $\alpha = i\beta, \beta$ real and $|\beta| < \alpha_0$, we have

$$H(i\beta) = H - \beta^2 |\nabla\rho|^2 + i\beta W \ .$$

We apply Proposition 3.2 to this operator taking $\widetilde{H} \equiv H - \beta^2|\nabla\rho|^2$. This operator has a spectral gap which contains $(\widetilde{E}_-, \widetilde{E}_+)$, where $\widetilde{E}_- = E_-$ and $\widetilde{E}_+ = E_+ - \beta^2$. In order that $\widetilde{\Delta}_+ \equiv \text{dist}(\widetilde{E}_+, E) > (\Delta_+/2)$, we require $|\beta| < \sqrt{\Delta_+/2}$. (Note that $\widetilde{\Delta}_- = \Delta_-$). We can now apply Proposition 3.2 to conclude $E \in \rho(H(i\beta))$ for $|\beta| < \min\left\{\alpha_0, C_1\sqrt{\Delta_+\Delta_-}, \ \sqrt{\Delta_+/2}\right\}$ and that (3.4) holds.

4 The Wegner Estimate

In this section, we prove a Wegner estimate for local Hamiltonians valid at all energies in the spectral gap of H_0. This estimate holds in the case of unbounded potentials as will be discussed in section 6 (see also [3] for an application of this estimate to 2-dimensional Landau Hamiltonians with unbounded potentials). Let $\Lambda \subset \mathbb{R}^d$ be a bounded region and denote by $H_{\Lambda,\omega} = H_0 + (V_\omega|\Lambda)$. Since $(V_\omega|\Lambda)$ is a relatively compact perturbation of H_0, the spectrum $\sigma(H_{\Lambda,\omega}) \cap (B_-, B_+)$ is discrete. Let $\mathbb{P}_\Lambda$ and $\mathbb{E}_\Lambda$ denote the probability and expectation with respect to the random variables associated with $\Lambda \cap \mathbb{Z}^d \equiv \widetilde{\Lambda}$. We denote by Tr the trace on $L^2(\mathbb{R}^d)$. Let $R_\Lambda(z)$ and $E_\Lambda(.)$ denote the resolvent and the spectral projection for $H_{\Lambda,\omega}$ respectively;

we often surpress the ω and write H_Λ for $H_{\Lambda,\omega}$, V_Λ for $(V_\omega|\Lambda)$, and $R_0(z)$ for $(H_0 - z)^{-1}$.

Theorem 4.1

Assume (H1)-(H3), (H5) and (H7)-(H8). For any $E_0 \in (B_-, B_+)$ and for any $\eta < \frac{1}{2}\,\mathrm{dist}(E_0, \sigma(H_0))$, $\exists$ finite constant C_{E_0}, depending on $[\mathrm{dist}(\sigma(H_0), E_0)]^{-1}$ such that:

$$\mathbb{P}_\Lambda \left\{ \mathrm{dist}(\sigma(H_{\Lambda,\omega}), E_0) < \eta \right\} \leq C_{E_0}\eta\,|\Lambda|\ , \tag{4.1}$$

where q is the integer of (H3).

Proof:

Let $I_\eta = [E_0 - \eta, E_0 + \eta]$. We write H_Λ for $H_{\Lambda,\omega}$, V_Λ for $(V_\omega|\Lambda)$ and R_0 for $R_0(E_0)$. By Chebishev's inequality the left hand side of (4.1) is bounded above by

$$\mathbb{E}_\Lambda \left(Tr(E_\Lambda(I_\eta)) \right).$$

To control the trace, we recall that any eigenfunction ψ_E of $H_\Lambda \psi_E = E\,\psi_E$, $E \in I_\eta$, satisfies

$$K_0 \psi_E = -\psi_E + R_0 \left(H_\Lambda - E_0 \right) \psi_E\ , \tag{4.2}$$

where $K_0 \equiv R_0 V_\Lambda$. ¿From (4.2), it follows easily that:

$$E_\Lambda(I_\eta) = -K_0 E_\Lambda(I_\eta) + R_0(H_\Lambda - E_0) E_\Lambda(I_\eta)\ . \tag{4.3}$$

Hence, noting that $E_\Lambda(I_\eta)$ is a positive trace class operator,

$$\begin{aligned} Tr(E_\Lambda(I_\eta)) &= \|E_\Lambda(I_\eta)\|_1 \\ &\leq |Tr(K_0 E_\Lambda(I_\eta))| + \eta \|R_0\|\,\|E_\Lambda(I_\eta)\|_1\ , \end{aligned}$$

and by the assumption on η:

$$Tr(E_\Lambda(I_\eta)) \leq 2\,|Tr(K_0 E_\Lambda(I_\eta))|. \tag{4.4}$$

A first consequence of (4.4) is, by the Hölder inequality:

$$\begin{aligned} \mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1) &\leq 2\,\mathbb{E}_\Lambda(\|K_0 E_\Lambda(I_\eta)\|_1) \\ &\leq 2\,\mathbb{E}_\Lambda(\|K_0\|_q \|E_\Lambda(I_\eta)\|_p) \quad (\frac{1}{q} + \frac{1}{p} = 1) \\ &\leq 2\,\{\mathbb{E}_\Lambda(\|K_0\|_q{}^q)\}^{(1/q)}\,\{\mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_p{}^p)\}^{(1/p)}, \end{aligned}$$

where $\| \ \|_q$ denote the norm in the Schatten class $\mathcal{J}_q$. By Proposition 7.4 and the fact that $\mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_p{}^p) = \mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1$, since all the eigenvalues of the spectral projector are equal to one, we obtain,

$$\mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1) \leq 2C\,|\Lambda|, \tag{4.5}$$

from which the existence of the integrated density of states, for energies in the unperturbed spectral gap, follows. Now, we use the adjoint of formula (4.3) to derive

$$K_0 E_\Lambda(I_\eta) = -K_0 E_\Lambda(I_\eta) K_0^\star + K_0 E_\Lambda(I_\eta)(H_\Lambda - E_0) R_0,$$

which implies

$$\begin{aligned} |Tr(K_0 E_\Lambda(I_\eta))| &\leq \|K_0 E_\Lambda(I_\eta)\|_1 \\ &\leq Tr(K_0 E_\Lambda(I_\eta) K_0^\star) + \eta \|R_0\| \, \|K_0 E_\Lambda(I_\eta)\|_1. \end{aligned}$$

Hence, by (4.4),

$$\mathbb{E}_\Lambda(Tr(E_\Lambda(I_\eta)) \leq 4\,\mathbb{E}_\Lambda(Tr(K_0 E_\Lambda(I_\eta) K_0^\star). \tag{4.6}$$

If $q > 2$, one continues this procedure and writes:

$$K_0 E_\Lambda(I_\eta) K_0^\star = -K_0 E_\Lambda(I_\eta)(K_0^\star)^2 + K_0 E_\Lambda(I_\eta)(H_\Lambda - E) R_0 K_0^\star. \tag{4.7}$$

One has by Hölder's inequality,

$$\begin{aligned} |Tr(K_0 E_\Lambda(I_\eta)(H_\Lambda - E_0) R_0 K_0^\star)| &= \|K_0 E_\Lambda(I_\eta)(H_\Lambda - E_0) R_0 K_0^\star\|_1 \\ &\leq \eta \|R_0\| \|K_0 E_\Lambda(I_\eta)\|_{q/(q-1)} \|K_0^\star\|_q \\ &\leq \eta \|R_0\| \|K_0\|_q^2 \|E_\Lambda(I_\eta)\|_{q/(q-2)}. \end{aligned} \tag{4.8}$$

Taking the expectation and again using Hölder's inequality, Proposition 7.4, and (4.5), one can bound the expectation of the left hand side of (4.8) by:

$$(2C)^{\frac{q-2}{q}} \eta \, \|R_0\| \, |\Lambda| \ .$$

Consequently, equations (4.6)-(4.8) imply

$$\begin{aligned} \mathbb{E}_\Lambda(Tr(E_\Lambda(I_\eta)) \leq \ & 4\,\mathbb{E}_\Lambda(|Tr(K_0 E_\Lambda(I_\eta)(K_0^\star)^2|) \\ & + 4\,(2C)^{\frac{q-2}{q}} \eta \, \mathrm{dist}(E_0, \sigma(H_0))^{-1} |\Lambda|. \end{aligned}$$

If $q > 3$, one repeats this procedure again. Finally, one obtains:

$$\mathbb{E}_\Lambda(Tr(E_\Lambda(I_\eta)) \leq 4\,\mathbb{E}_\Lambda(|Tr(K_0 E_\Lambda(I_\eta)(K_0^\star)^{q-1}|) + \tilde{C}\,\eta\,\mathrm{dist}(E_0,\sigma(H_0))^{-1}|\Lambda|, \tag{4.9}$$

where $\tilde{C}$ depends on q and the constant C of Proposition 7.4.

To estimate the first term on the right hand side of (4.9), we expand the potential

$V_\Lambda = \sum_{i\in\widetilde{\Lambda}} \lambda_i u_i$, where $u_i(x) \equiv u(x-i)$. For each q-tuple of indices $\{i\} \equiv (i_1, ..., i_q) \in \widetilde{\Lambda}^q$, we define:

$$K_{i_1\ldots i_q} \equiv u_{i_2}^{\frac{1}{2}} R_0 u_{i_3} R_0 u_{i_4} .... u_{i_q} R_0^2 u_{i_1}^{\frac{1}{2}} \tag{4.10}$$

We prove in the appendix, section 7, that (H3) implies that $K_{\{i\}} \equiv K_{i_1\ldots i_q} \in \mathcal{J}_1$. In terms of this operator, the first term on the right side of (4.9) becomes

$$\mathbb{E}_\Lambda\left\{\sum_{i_1,\ldots i_q\in\widetilde{\Lambda}} \lambda_{i_1}(\omega)...\lambda_{i_q}(\omega) Tr\left\{K_{\{i\}}(u_{i_1}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_2}^{\frac{1}{2}})\right\}\right\}. \tag{4.11}$$

Since $K_{\{i\}}$ is compact, we write it in terms of its singular value decomposition. For each multi-index $\{i\}$, there exists a pair of orthonormal bases, $\left\{\phi_k^{\{i\}}\right\}$ and $\left\{\psi_k^{\{i\}}\right\}$, and nonnegative numbers $\left\{\mu_k^{\{i\}}\right\}$, all independent of ω, such that

$$K_{\{i\}} = \sum_{k=1}^{\infty} \mu_k^{\{i\}} \left|\phi_k^{\{i\}}\rangle\langle\psi_k^{\{i\}}\right|. \tag{4.12}$$

Inserting the representation (4.12) into (4.11) and expanding the trace in $\{\phi_k^{\{i\}}\}$, we obtain

$$\mathbb{E}_\Lambda\left\{\sum_{\{i\}\in\widetilde{\Lambda}^q}\sum_{k\geq 1} \lambda_{\{i\}}(\omega)\mu_k^{\{i\}}\langle\psi_k^{\{i\}}, (u_{i_1}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_2}^{\frac{1}{2}})\phi_k^{\{i\}}\rangle\right\}, \tag{4.13}$$

where $\lambda_{\{i\}}(\omega) \equiv \lambda_{i_1}(\omega)...\lambda_{i_q}(\omega)$. Recalling that $E_\Lambda(I_\eta) \geq 0$, we bound the k-sum above by:

$$\begin{aligned} \tfrac{1}{2}\sum_{k\geq 1} \mu_k^{\{i\}} \mathbb{E}_\Lambda & \left\{|\lambda_{\{i\}}(\omega)|\langle\psi_k^{\{i\}}, (u_{i_1}^{\frac{1}{2}} E_\Lambda(I\eta) u_{i_1}^{\frac{1}{2}})\psi_k^{\{i\}}\rangle\right. \\ & \left. + |\lambda_{\{i\}}(\omega)|\langle\phi_k^{\{i\}}, (u_{i_2}^{\frac{1}{2}} E_\Lambda(I\eta) u_{i_2}^{\frac{1}{2}})\phi_k^{\{i\}}\rangle\right\}. \end{aligned} \tag{4.14}$$

¿From independance of the λ_i's, the spectral averaging result (see [5] or [7]) applied to each term in (4.14) gives for the first term:

$$\mathbb{E}_\Lambda \left\{ |\lambda_{\{i\}}(\omega)| \langle \psi_k^{\{i\}}, (u_{i_1}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_1}^{\frac{1}{2}}) \psi_k^{\{i\}} \rangle \right\} \leq C_1 \, \eta. \tag{4.15}$$

where C_1 is finite according to (H6). From (4.13)-(4.15), we obtain an upper bound for the first term on the right hand side of (4.9),

$$C_1' \, \eta \sum_{i_1,\ldots,i_q \in \widetilde{\Lambda}} \left(\|K_{\{i\}}\|_1 \right) \ . \tag{4.16}$$

In the appendix, we prove in Proposition 7.2 that $\exists$ finite constant $C_{E_0} > 0$, depending only on $\mathrm{dist}(\sigma(H_0), E_0)^{-1}$ and the dimension $d \geq 1$, such that (4.16) is bounded above by

$$C_1' \, C_{E_0} \, \eta |\Lambda| \ . \tag{4.17}$$

Results (4.9) and (4.17) prove the theorem.

5 Verification of [H1](γ_0, ℓ_0)

The goal of this section is to prove the hypothesis [H1](γ_0, ℓ_0) for finite volume Hamiltonians corresponding to the models introduced in section 2. We let $\Lambda \subset \mathbb{R}^d$ denote a bounded open region and $\Lambda_\ell(x_0) \equiv \{x \in \mathbb{R}^d | |x_i - x_{0,i}| < \ell/2, i = 1, ..., d\}$. When $x_0 = 0$, we will write Λ_ℓ for simplicity. The potential depending only on the λ_i in a region Λ is denoted $V_{\Lambda,\omega} = (V_\omega | \Lambda)$.The finite volume Hamiltonians $H_{\Lambda,\omega}$ are defined as $H_{\Lambda,\omega} \equiv H_0 + V_{\Lambda,\omega}$. Since $V_{\Lambda,\omega}$ has compact support, it is a relatively compact pertubation of H_0 and hence $\sigma_{ess}(H_0) = \sigma_{ess}(H_{\Lambda,\omega})$. One of our first tasks is to locate precisely the eigenvalues of $H_{\Lambda,\omega}$ in the gap (B_-, B_+) with good probability.

The condition [H1](γ_0, ℓ_0) on the resolvent of $H_{\Lambda,\omega}$, written $R_\Lambda(z) = (H_{\Lambda,\omega} - z)^{-1}$, when it exists, is the following. For any $\chi \in C^2$, define the first order differential operator $W(\chi)$ by

$$W(\chi) \equiv [-\Delta, \chi] = -\nabla.\nabla\chi - \nabla\chi.\nabla \tag{5.1}$$

This operator is localized on the support of $\nabla\chi$. Fix any $\delta > 0$ small and let $\Lambda_{\ell,\delta} \equiv \{x \in \Lambda_\ell | \mathrm{dist}(\partial\Lambda_\ell, \chi) > \delta\}$. We will use χ_ℓ to denote a function satisfying $\chi_\ell | \Lambda_{\ell,\delta} = 1$, $\mathrm{supp}\,\chi_\ell \subset \Lambda_\ell$ and $\chi_\ell \geq 0$. It follows that $\mathrm{supp}\,\nabla\chi_\ell \subset \Lambda_\ell \backslash \Lambda_{\ell,\delta}$ and $W(\chi_\ell)$ is also localized in this region. The condition we must verify is

$[\mathrm{H1}](\gamma_0, \ell_0)$:
$\exists \gamma_0 > 0$ and $\ell_0 >> 1$ such that $\gamma_0 \ell_0 >> 1$ and $\mathbb{P}\{\|W(\chi_\ell) R_{\Lambda_\ell}(E + i\varepsilon)\chi_{\ell/3}\| < e^{-\gamma_0 \ell_0}\|\} \geq 1 - \ell_0^{-\xi}$, for E near the band edges $\widetilde{B}_\pm$ and for some $\xi > 2d$.

We do this in two steps. We first prove that for $\delta > 0$ small, dist $(\sigma(H_{\Lambda,\omega}), \widetilde{B}_\pm) > \delta$ with good probability. We can then apply the Combes-Thomas result of section 3 to conclude exponential decay at energies $E \in (\widetilde{B}_- - \delta/2, \widetilde{B}) \cup (\widetilde{B}_+, \widetilde{B}_+ + \delta/2)$ with a good probability. We then verify $[\mathrm{H1}](\gamma_0, \ell_0)$ for an appropriate choice of γ_0 and ℓ_0.

We now discuss the location of the spectrum of the finite volume Hamiltonians $H_{\Lambda,\omega}$ in the unperturbed spectral gap. Recall that by (H8) the family $\{H_\omega\}$ has an almost sure spectrum Σ. The probabitity space is $\Omega = (\mathrm{supp}\,h)^{\mathbb{Z}^d}$.

Lemma 5.1 *Suppose $\mu \equiv \mu_{\Lambda,\omega_0} \in \sigma_d(H_{\Lambda,\omega_0}) \cap (B_-, B_+)$ for some $\omega_0 \in \Omega$, then $\mu \in \Sigma$.*

Proof: let ψ_{ω_0} be an eigenfunction of H_{Λ,ω_0} with eigenvalue $\mu_{\Lambda,\omega_0} \equiv \mu$: $H_{\Lambda,\omega_0}\psi_{\omega_0} = \mu\psi_{\omega_0}$, $\|\psi_{\omega_0}\| = 1$. For any R such that $\Lambda \subset\subset \Lambda_R$, consider the following events (for any $\nu > 0$)

$$I_{R,\nu} \equiv \left\{\omega \in \Omega |\ |\lambda_i(\omega_0) - \lambda_i(\omega)| \leq \nu(6|\Lambda|\ \|u\|_\infty)^{-1},\ \forall i \in \widetilde{\Lambda}\right\},$$

and

$$E_{R,\nu} \equiv \left\{\omega \in \Omega |\ |\lambda_i(\omega) f| < \nu(6|\widetilde{\Lambda}_R \backslash \widetilde{\Lambda}|\ \|u\|_\infty)^{-1},\ \forall i \in \widetilde{\Lambda}_R \backslash \widetilde{\Lambda}\right\}\ .$$

Set $B_{R,\nu} \equiv I_{R,\nu} \cap E_{R,\nu}$. Let $\chi \in C^2$ be a smoothed characteristic function with supp $\chi \subset \Lambda_2$, $\chi \leq 1$, and $\chi|\Lambda_1 = 1$. For $R > 1$, set $\chi_R(x) \equiv \chi(R^{-1}x)$ so that $\|\partial^\alpha \chi_R\| = \mathcal{O}(R^{-|\alpha|})$, for $|\alpha| = 0, 1, 2$. Choose R_1 sufficiently large so $\|\chi_{R_1}\psi_{\omega_0}\| > \frac{1}{2}$, and for $R > R_1$ define $\psi_R \equiv \|\chi_R\psi_{\omega_0}\|^{-1}\,\chi_R\psi_{\omega_0}$ so $\|\psi_R\| = 1$. Then, by the definition of ψ_R and the local Hamiltonians,

$$(H_\omega - \mu)\psi_R = (H_{\Lambda,\omega_0} - \mu)\psi_R + \sum_{i\in\widetilde{\Lambda}}(\lambda_i(\omega) - \lambda_i(\omega_0))u_i\psi_R + \sum_{i\in\widetilde{\Lambda}_R\setminus\Lambda} \lambda_i(\omega)u_i\psi_R \ ,$$

and it follows that for all $\omega \in B_{R,\nu,}$

$$\|(H_\omega - \mu)\psi_R\| \le 2\| \, [H_0, \chi_R]\psi_{\omega_0}\| + \frac{1}{3}\nu \tag{5.2}$$

The commutator is estimated as follows: as $H_0 = (p-A)^2 + V_0$, we have

$$[(p-A)^2, \chi_R]\psi_{\omega_0} = -2i\Delta\chi_R(p-A)\psi_{\omega_0} - (\Delta\chi_R)\psi_{\omega_0} \ . \tag{5.3}$$

Now ψ_{ω_0} is an eigenfunction of H_{Λ,ω_0} and, in particular, $\psi_{\omega_0} \in D(H_0)$, so

$$(p-A)_j\psi_{\omega_0} = (p-A)_j(H_0 - z)^{-1}(\mu - z - V_{\Lambda,\omega_0})\psi_{\omega_0}.$$

Setting $z = i\delta$, $\delta > 0$, we obtain

$$\|(p-A)_j\psi_{\omega_0}\| \le \delta^{-1}\|\mu - i\delta - V_{\Lambda,\omega_0}\|_\infty \ . \tag{5.4}$$

which is independent of R. Hence, by taking R sufficiently large, it follows from (5.4) that

$$\|(H_\omega - \mu)\psi_R\| \le \frac{2}{3}\nu \ .$$

This shows that for any $\nu > 0$, $\sigma(H_\omega) \cap [\mu - \nu, \mu + \nu] \neq \emptyset$ with probability $\mathbb{P}(B_{R,\nu}) = \mathbb{P}(E_{R,\nu})\mathbb{P}(I_{R,\nu}) > 0$. Since the spectrum of $\{H_\omega\}$ is deterministic, this implies $\mu \in \Sigma$.

Lemma 5.2 *Let $\mu_{\Lambda,\omega} \equiv \mu \in \sigma_d(H_{\Lambda,\omega}) \cap (B_-, B_+)$, with eigenfunction $\phi_\omega, \|\phi_\omega\| = 1$. Assume that $V_{\Lambda,\omega} \ge 0$. Then we have*

$$\langle \phi_\omega, V_{\Lambda,\omega}\phi_\omega\rangle \ge [\mathrm{dist}(\mu, \sigma(H_0))]^2 \, M_\infty^{-1} \ .$$

Proof: Since $M_\infty V_{\Lambda,\omega} \ge (V_{\Lambda,\omega})^2$ under the hypothesis that $V_{\Lambda,\omega} \ge 0$, we have

$$\begin{aligned} \langle \phi_\omega, V_{\Lambda,\omega}\phi_\omega\rangle &= M_\infty^{-1}\langle \phi_\omega, M_\infty V_{\Lambda,\omega}\phi_\omega\rangle \\ &\ge M_\infty^{-1}\|V_{\Lambda,\omega}\phi_\omega\|^2 \ . \end{aligned}$$

The eigenvalue equation gives $V_{\Lambda,\omega}\phi_\omega = -(H_0 - \mu)\phi_\omega$, so that

$$\begin{aligned}\langle \phi_\omega, V_{\Lambda,\omega}\phi_\omega\rangle &\geq M_\infty^{-1}\|(H_0-\mu)\phi_\omega\|^2 \\ &\geq M_\infty^{-1}\left[\mathrm{dist}(\sigma(H_0),\mu)\right]^2 \ .\end{aligned}$$

Proposition 5.3 *Let* $\delta_\pm \equiv \frac{1}{2}|\widetilde{B}_\pm - B_\pm|$, *and for any* $0 < \delta < \frac{1}{2}M_\infty^{-1}\min(\delta_+,\delta_-)$, *assume that* $\lambda_i(\omega) < (1-\delta M_\infty[\min(\delta_+,\delta_-)]^{-2})M, \forall i \in \widetilde{\Lambda}$. *Then we have*

$$\sup\left\{\sigma(H_{\Lambda,\omega}) \cap (-\infty, \widetilde{B}_-)\right\} < \widetilde{B}_- - \delta$$

and

$$\inf\left\{\sigma(H_{\Lambda,\omega}) \cap (\widetilde{B}_+, \infty)\right\} > \widetilde{B}_+ + \delta \ .$$

Proof: Without loss of generality, we assume $H_{\Lambda,\omega}$ has an eigenvalue $\mu_{\Lambda,\omega} \equiv \mu \in [\widetilde{B}_- - \delta, \widetilde{B}_-]$. Furthermore, we can assume that $V_{\Lambda,\omega} \geq 0$, since by Lemma 5.1, we always have $\mu \leq \widetilde{B}_-$ and the eigenvalues of $H_{\Lambda,\omega}$ are increasing functions of the coupling constants $\left\{\lambda_i(\omega) | i \in \widetilde{\Lambda}\right\}$. This fact follows, for example, from the Feynman-Hellman formula and the positivity of u. Indeed, if ϕ_ω is an eigenfunction of $H_{\Lambda,\omega}$, so that $H_{\Lambda,\omega}\phi_\omega = \mu\phi_\omega$, then

$$\begin{aligned}\frac{\partial \mu_{\Lambda,\omega}}{\partial \lambda_i} &= \langle \phi_\omega, \frac{\partial H_{\Lambda,\omega}}{\partial \lambda_i}\phi_\omega\rangle \\ &= \langle \phi_\omega, u_i\phi_\omega\rangle > 0 \ .\end{aligned}$$

The family $T(\theta) \equiv H_0 + \theta V_{\Lambda,\omega}$, for θ in a small neighborhood of $\theta_0 = 1$, is an analytic type A family which is self-adjoint for θ real. If μ has multiplicity m, there are at most m functions $\mu^{(k)}(\theta)$, analytic in θ for θ near $\theta_0 = 1$, and which satisfy $\lim_{\theta\to\theta_0=1} \mu^{(k)}(\theta) = \mu$. Let $\phi^{(k)}(\theta)$ be an eigenfunction for $\mu^{(k)}(\theta)$, with $\|\phi^{(k)}(\theta)\| = 1$ for θ real and $|\theta - 1|$ small. Applying the Feynman-Hellman formula again, we find

$$\begin{aligned}\frac{d\mu^{(k)}(\theta)}{d\theta} &= \langle \phi(\theta), V_{\Lambda,\omega}\phi(\theta)\rangle \\ &= \theta^{-1}\langle \phi(\theta), (\theta V_{\Lambda,\omega})\phi(\theta)\rangle \ .\end{aligned} \tag{5.5}$$

We now assume $\lambda_i(\omega) < (1-\delta M_\infty[\min(\delta_+,\delta_-)]^{-2})M, \forall i \in \widetilde{\Lambda}$, and fix

$$\theta_1 = \min_{i\in\widetilde{\Lambda}} \left(\frac{M}{\lambda_i(\omega)}\right) \;\geq\; \left(1-\delta M_\infty \left[\min(\delta_+,\delta_-)\right]^{-2}\right)^{-1}$$
$$> \; 1\;.$$

Applying Lemma 5.2 to $V_{\Lambda,\omega}$ under these conditions yields

$$\frac{d\mu^{(k)}(\theta)}{d\theta} \geq \theta^{-1} M_\infty^{-1} \left[\mathrm{dist}(\mu^{(k)}(\theta),\sigma(H_0))\right]^2,$$

Upon integrating over $[1,\theta_1]$, we get, by monotonicity of $\mu^{(k)}(\theta)$:

$$\begin{aligned}\mu^{(k)}(\theta_1) \;&\geq\; \mu + (log\theta_1) M_\infty^{-1} \min\left\{\left[\mathrm{dist}(\mu^{(k)}(\theta_1),\sigma(H_0))\right]^2 \;,\; \left[\mathrm{dist}(\mu,\sigma(H_0))\right]^2\right\} \\ &\geq\; \mu+\delta > \widetilde{B}_- \;.\end{aligned}$$

This shows that $\left(H_0 + \sum_{i\in\widetilde{\Lambda}} M u_i\right)$ has an eigenvalue outside of Σ which contradicts Lemma 5.1.

This proposition is the main technical result. We can now easily compute the probability that $\mathrm{dist}(\sigma(H_{\Lambda,\omega}),\, \widetilde{B}_\pm) > \delta$.

Corollary 5.4 *For* $0<\delta<\frac{1}{2}M_\infty^{-1}\min(\delta_+,\delta_-)$, *we have*

$$\sup\left\{\sigma(H_{\Lambda,\omega})\cap(-\infty,\widetilde{B}_-)\right\} < \widetilde{B}_- - \delta,$$

and

$$\inf\left\{\sigma(H_{\Lambda,\omega})\cap(\widetilde{B}_+,\infty)\right\} > \widetilde{B}_+ + \delta,$$

with a probability larger than

$$1-|\Lambda| \max_{X=m,M}\left|\int_{1-\delta M_\infty[\min(\delta_+,\delta_-)]^{-2}X}^{X} h(s)ds\right|$$

Proof: The probability that $\lambda_i(\omega) < (1-\delta M_\infty[\min(\delta_+,\delta_-)]^{-2})M\;,\;\forall i\in\widetilde{\Lambda}$, is given by $\left[1-\int_{(1-\delta M_\infty \Delta^{-2})M}^{M} h(s)ds\right]^{|\Lambda|}$. The corollary now follows by expanding this probability and from Proposition 5.3.

We verify [H1](γ_0,ℓ_0) by combining Corollary 5.4 on the location of the spectrum of $H_{\Lambda_\ell,\omega}$ and the exponential decay estimate of Theorem 3.3. We note that hypothesis (H6) on the decay of the tail of the density h near the endpoints of its support m and M is essential in order to control the probability in corollary 5.4. We first give the decay estimate for the localized resolvent and then comment on the gradient term.

Proposition 5.5 *Let $\chi_i, i = 1, 2$, be two functions with $\|\chi_i\|_\infty \leq 1$, supp $\chi_1 \subset \Lambda_{\ell/3}$ and supp χ_2 localized near $\partial\Lambda_\ell$ and $\delta_\pm \equiv \frac{1}{2}|\widetilde{B}_+ - \widetilde{B}_-|$. For $\beta > 0$ as in (H6), consider any $\nu > 0$ such that $0 < \nu < 4\beta(2\beta + 3d)^{-1}$. Then $\exists \ell_0^\star \equiv \ell_0^\star(M_\infty, \delta_+, \delta_-, M)$ such that $\forall \ell_0 > \ell_0^\star$ and $\forall E \in (\widetilde{B}_- - \ell_0^{\nu-2}, \widetilde{B}_-] \cup [\widetilde{B}_+, \widetilde{B}_+ + \ell_0^{\nu-2})$,*

$$\sup_{\varepsilon>0} \|\chi_2 R_{\Lambda_{\ell_0}}(E + i\varepsilon)\chi_1\| \leq e^{-\ell_0^{\nu/3}},$$

with probability $\geq 1 - \ell_0^{-\xi}$, for some $\xi > 2d$.

Proof: ¿From Corollary 5.4 and (H6), we compute the probability that $\sigma\left(H_{\Lambda_{\ell_0},\omega}\right)$ is at a distance $\delta = 2\ell_0^{\nu-2}$ from $\widetilde{B}_\pm$,

$$\mathbb{P}\left\{\text{dist}\left(\sigma\left(H_{\Lambda_{\ell_0},\omega}\right), \widetilde{B}_\pm\right) > 2\delta\right\} \geq 1 - \ell_0^d \left(2\ell_0^{\nu-2} M_\infty \left[\min(\delta_+, \delta_-)\right]^{-2} X\right)^{3d/2+\beta}, \tag{5.6}$$

where $X = m$ for $\widetilde{B}_-$ and $X = M$ for $\widetilde{B}_+$. A simple computation shows that the right side of (5.6) is bounded below by $1 - \ell_0^{-\xi}$ for some $\xi > 2d$ provided ν satisfies $0 < \nu < 4\beta(2\beta + 3d)^{-1}$. We now apply Theorem 3.3 to $H_{\Lambda_{\ell_0},\omega}$. Let $E \in [\widetilde{B}_- - \ell_0^{\nu-2}, \widetilde{B}_-)$ and, following the notation of Theorem 3.3, let $\Delta_- \equiv \text{dist}(\widetilde{B}_- - \delta, E) > \delta/2 = \ell_0^{\nu-2}$ and $\Delta_+ \geq |\widetilde{B}_+ - \widetilde{B}_-|$. Since $\text{dist}(\text{supp}\,\chi_2, \text{supp}\,\chi_1) \geq \ell_0/3$ (in dimension $d > 9$, this is no longer true; one has to replace $\ell_0/3$ by $\ell_0/(3\sqrt{d})$, for the diameter of the inner cube), we obtain

$$\begin{aligned} \|\chi_2 R_{\Lambda_{\ell_0}}(E + i\varepsilon)\chi_1\| &\leq C_2 \sup\left(|\widetilde{B}_+ - \widetilde{B}_-|^{-1}, \ell_0^{2-\nu}\right) \\ &\times \; e^{-\inf(\alpha_0, C_1 \ell_0^{\nu/2-1}|\widetilde{B}_+ - \widetilde{B}_-|^{1/2})\ell_0/6} \end{aligned}$$

The result follows by taking ℓ_0 large.

Corollary 5.6 $\exists \ell_0^\star$ *such that* $\forall \ell_0 > \ell_0^\star$, *hypothesis* [H1] (γ_0, ℓ_0) *holds* $\forall E \in (\widetilde{B}_- - \ell_0^{\nu-2}, \widetilde{B}_-] \cup [\widetilde{B}_+, \widetilde{B}_+ + \ell_0^{\nu-2})$ *and any ν satisfying $0 < \nu < 4\beta(2\beta + 3d)^{-1}$, β as in (H6).*

Proof: As in Lemma 5.1, of [6], we write

$$\begin{aligned}\|W(\chi_{\ell,\upsilon})R_{\Lambda_\ell}\chi_{\ell/3}\| \leq\ & \|(\Delta\chi_{\ell,\upsilon})R_{\Lambda_\ell}\chi_{\ell/3}\| \\ & +2\textstyle\sum_{j=1}^d \|(\partial_j\chi_{\ell,\upsilon})(p-A)_j R_{\Lambda_\ell}\chi_{\ell/3}\|\end{aligned} \tag{5.7}$$

for a function $\chi_{\ell,\upsilon}$ localized within distance υ of $\partial\Lambda_\ell$. Let χ_i, $i=1,2$, be smooth functions such that $\chi_i\chi_{\ell,\upsilon}=\chi_{\ell,\upsilon}$, $\chi_2\chi_1=\chi_1$, and $\operatorname{supp}\chi_i$ is localized within a distance 2υ for $i=1$ and 3υ for $i=2$, of $\partial\Lambda_\ell$. Then, we write for each j and any $u\in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\|(\partial_j\chi_{\ell,\upsilon})(p-A)_j R_{\Lambda_\ell}u\|^2 &\leq C_0\langle(p-A)_j R_{\Lambda_\ell}u, \chi_1(p-A)_j\chi_2 R_{\Lambda_\ell}u\rangle \\ &\leq C_0\|\chi_2 R_{\Lambda_\ell}u\|\ \|(p-A)_j\chi_1(p-A)_j R_{\Lambda_\ell}u\|\end{aligned}$$

Taking $u=\chi_{\ell/3}f$, we see that (5.7) is bounded above as in Proposition 5.5 (taking $\ell_0^\star$ larger) provided we have $\|(p-A)^2R_{\Lambda_{\ell_0}}u\|$ bounded. This follows with a probability $\geq 1-\ell_0^{-\xi}$, since V_0 is relatively bounded and $V_\omega^{\Lambda_{\ell_0}}$ is bounded.

6 The Case of Unbounded Random Potentials

We indicate here the modifications necessary when $\operatorname{supp} h$ is unbounded and satisfies the second part of (H6). To control the location of $\sigma(H_\Lambda)$ with a good probability, we must work in the weak coupling regime. Consequently, we study the family $H(g)=H_0+gV_\omega$, for $|g|$ sufficiently small. We assume conditions (H1)-(H8) in this section.

Proposition 6.1 *The random family of Schrödinger operators $H_\omega(g)=H_0+gV_\omega$ is essentialy self-adjoint on $C_0^\infty(\mathbb{R}^d)$ with probability 1.*

Proof: We refer to Hinz and Stolz [18] for a discussion of essential self-adjointness. It suffices to prove that $|V_\omega(x)|=\mathcal{O}(|x|^2)$ as $|x|\to\infty$ with probability one. We define events $A_k, k\in\mathbb{Z}^d$, by

$$A_k \equiv \left\{\omega|\ |\lambda_k(\omega)|\geq 1+|k|^2\right\}.$$

¿From (H6), we have for any bounded set $B \subset \mathbb{R}^d$ containing the origin,

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d \setminus B} \mathbb{P}(A_k) &= \sum_{k \in \mathbb{Z}^d \setminus B} \Big(\int_{|\lambda| > 1 + |k^2|} h(\lambda) d\lambda \Big) \\ &\leq \frac{C_h}{r+1} \sum_{k \in \mathbb{Z}^d \setminus B} |k|^{-2(r+1)} \\ &< \infty, \end{aligned}$$

since $r > \frac{d}{2} - 1$ according to (H6). The Borel-Cantelli lemma then states that $\mathbb{P}(\overline{\lim} A_k) = 0$. So $\forall k \in \mathbb{Z}^d$, $|k|$ sufficiently large, and for a.e.$\omega \in \Omega$, $\exists$ finite $C_\omega > 0$ s.t. $|\lambda_k(\omega)| \leq C_\omega(1 + |k|^2)$.

We next turn to the Wegner estimate. The only change in Theorem 4.1 is a factor g^q on the right side of (4.1). The multiscale analysis of [5] requires a simple modification. For a constant C_ℓ, depending on the length ℓ , assumption (H6) on the decay of $h(\lambda)$ implies

$$\mathbb{P}\left\{ |V_{\Lambda_\ell}| < C_\ell \right\} \geq 1 - \frac{2}{r+1} \|u\|_\infty^r C_h \, \ell^d C_\ell^{-(r+1)} \tag{6.1}$$

for C_h and r as in (H6). It is easy to verify that for $C_\ell = \mathcal{O}(\ell^6)$, we have (6.1) bounded below by $1 - \ell^{-\xi}$, for some $\xi > 2d$. This can be absorbed into the probability of exponential decay. At each stage of the multiscale analysis the constant is $\mathcal{O}(\ell_n^6)$. A careful check of the calculations in the appendix of [5] shows that this changes the decay constant γ by a vanishing amount of $\mathcal{O}((log \ell_n) \ell_n^{-1})$ at each step. Hence, the results remain unchanged. Next, we indicate how the small coupling constant g allows us to obtain estimates on $\sigma(H_{\Lambda_\ell, \omega}(g))$, which replace those of section 5. In fact, the results are simpler in this case.

Proposition 6.2 *Let $E_m \equiv (\frac{1}{2})(B_+ - B_-)$, $\Delta \equiv (\frac{1}{2})|B_+ - B_-|$, and fix $K > 2$. Then $\exists g_0(K) > 0$ such that $\forall g < g_0(K)$ and for $l_0 = g^{-6}$,*

$$\mathbb{P}\left\{ \text{dist}\left(\sigma(H_{\Lambda_{\ell_0}, \omega}(g)), E_m \right) > K^{-1}(K-1)\Delta \right\} \geq 1 - l_0^{-\xi}$$

for some $\xi > 2d$.

Proof: It is clear that $\|gV_{\Lambda_\ell}\|_\infty < K^{-1}\Delta$, if $\forall i \in \widetilde{\Lambda}_l$, the coupling constants satisfy

$$|\lambda_i(\omega)| \leq (gKu_\infty)^{-1}\Delta \equiv \nu(\Delta) \ ,$$

where $u_\infty \equiv \| \sum_{i\in\mathbb{Z}^d} u_i\|_\infty$. If this condition holds, then $\sigma(H_{\Lambda_{\ell_0},\omega})$ is a distance $K^{-1}(K-1)\Delta$ from E_m. The probability of this occurring is

$$p_0 \equiv \mathbb{P}\left\{\mathrm{dist}(\sigma(H_{\Lambda_{\ell_0},\omega}(g)), E_m) \geq K^{-1}(K-1)\Delta\right\} \geq 1 - \ell_0^d \int_{|\lambda|>\nu(\Delta)} h(\lambda)d\lambda.$$

Using (H6), we obtain the estimate for the probability

$$p_0 \geq 1 - C_h \ell_0^d g^{q+1}(\Delta(Ku_\infty)^{-1})^{-(q+1)},$$

where q and C_h are as in (H6). By choosing l_0 large enough and $g = \mathcal{O}(\ell_0^{-6})$, we can bound p_0 from below by $1 - \ell_0^{-\xi}$, for some $\xi > 2d$.

Finally, we formulate the analog of Proposition 5.5 in this case. We note that Corollary 5.6 is immediate since we know $V_{\Lambda_{\ell_0},\omega}$ can be bounded with a good probability.

Proposition 6.3 *Let $\chi_i, i = 1, 2$, be the functions defined in Proposition 5.5 and fix $K > 2$ as in Proposition 6.2. There exist finite positive constants $g_0(K)$, ℓ_0^* and C_3, C_4 (depending only on $\chi_i, K,$ and Δ), such that for all $\ell_0 >> \ell_0^*$ and any $E \in [B_- + 2K^{-1}\Delta, B_+ - 2K^{-1}\Delta]$, we have*

$$\sup_{\varepsilon>0} \|\chi_2 R_{\Lambda_{\ell_0}}(E + i\varepsilon)\chi_1\| \leq C_3 e^{-C_4\ell_0}$$

with probability $\geq 1 - l_0^{-\xi}$, for some $\xi > 2d$.

Proof: For E as in the proposition, define

$$\Delta_- \equiv \inf_{B\in\{B_-,B_+\}} (\mathrm{dist}(E, B)),$$

$$\Delta_+ \equiv \sup_{B\in\{B_-,B_+\}} (\mathrm{dist}(E, B)).$$

¿From Proposition 6.2, it follows that for $g < g_0(K)$ and $\ell_0 > g^{-6}$, we have $\Delta_\pm \geq (\Delta/4)$ with a probability $\geq 1 - \ell_0^\xi$, for some $\xi > 2d$. We can now apply Theorem 3.1 directly. The analog of Corollary 5.6 now follows since we can control $V_\omega^{\Lambda_l}$ with a good probability ($\geq 1 - \ell^\xi$), as indicated above.

It remains to provide some examples which show that theorem 2.2 is not empty. We prove that if $\mathrm{supp}\, h = \mathbb{R}$ and hypotheses (H1)-(H8) are satisfied, then the almost sure spectrum $\Sigma(g)$ fills in the spectral gaps of H_0; an example of this is the Gaussian distribution. We prove the following.

Proposition 6.4 *Let H_0 satisfy (H1)-(H3) and assume (H5), (H7)-(H8). Let $\Sigma(g)$ be the a.s. spectrum of $H_\omega(g) = H_0 + gV_\omega$ and assume $\operatorname{supp} h = \mathbb{R}$. Then we have*

$$\mathbb{R}\backslash\sigma(H_0) \subset \Sigma(g)\,,\ g \neq 0\ .$$

Proof: We fix $g = 1$ without loosing generality and consider $\mu_0 \in (B_-, B_+)$; by Lemma 5.1 one has $\mu_0 \in \Sigma(g)$ if $\mu_0 \in \sigma_d(H_{\Lambda,\omega_0})$ for some finite volume hamiltonians H_{Λ,ω_0} and some $\omega_0 \in \Omega$. Given any fixed ball $\Lambda \subset \mathbb{R}^d$ there exists by (H5) and (H7) an $\omega_0 \in \Omega$ such that V_{Λ,ω_0} is positive; consider then $H(\lambda) \equiv H_0 + \lambda V_{\Lambda,\omega_0}$, i.e., $H(\lambda) = H_{\Lambda,\lambda\omega_0}$; by (H3) and (H7) one has $\sigma_{ess}(H(\lambda)) = \sigma_{ess}(H_0)$ for all λ and the operator $K(\mu) = |V_{\Lambda,\omega_0}|^{\frac{1}{2}}(H_0 - \mu)^{-1}|V_{\Lambda,\omega_0}|^{\frac{1}{2}}$ is compact for all $\mu \in \rho(H_0)$. It is well known that $\mu_0 \in \sigma_d(H(\lambda))$ iff $-1/\lambda \in \sigma(K(\mu_0))$; so unless $K(\mu_0) = 0$ one has $\mu_0 \in \overline{\cup_{\lambda\in\mathbb{R}}\sigma_d(H(\lambda))}$. On the other hand, if $K(\mu_0) = 0$, then either there exists a sequence $(\mu_n)_n$ converging to μ_0 such that $K(\mu_n) \neq 0$ in which case μ_0 obviously belongs to $\overline{\cup_{\lambda\in\mathbb{R}}\sigma_d(H(\lambda))}$; or $K(\mu) = 0$ for all μ in an open neighbourhood of μ_0 ; but then $\frac{d}{d\mu}K(\mu)_{|\mu=\mu_0} = |V_{\Lambda,\omega_0}|^{\frac{1}{2}}(H_0 - \mu_0)^{-2}|V_{\Lambda,\omega_0}|^{\frac{1}{2}}$ is zero i.e. $(H_0 - \mu_0)^{-1}|V_{\Lambda,\omega_0}|^{\frac{1}{2}} = 0$; but $\mu_0 \in \rho(H_0)$ so this is possible only if $|V_{\Lambda,\omega_0}|^{\frac{1}{2}} = 0$. Since Lemma 5.1 implies that $\overline{\cup_{\lambda\in\mathbb{R}}\sigma_d(H(\lambda))} \subseteq \Sigma(g)$, the proof is complete.

We remark that this technique also applies to show (H9) in case $\operatorname{supp} h$ is compact but large. Essential in the proof is positivity of V_{Λ,ω_0} which follows from positivity of u (assumption (H7)); if there is no magnetic field, $A = 0$, we could also use results of Deift and Hempel to get the result without this positivity assumption ([10], [16]).

7 Appendix

We prove estimates on the operator $K_{\{i\}}$ defined in (4.10) which are needed in the proof of Theorem 4.1. Let $\widetilde{\Lambda} \equiv \Lambda \cap \mathbb{Z}^d$ and recall that $\{i\}$ is a q-tuple of elements of $\widetilde{\Lambda}$. The following lemma is easily proved using Hölder's inequality for trace ideals (see, for example, Theorem 2.8 of [30]).

Lemma 7.1 *Assume (H1) - (H3) and (H7). Then $K_{\{i\}}$ is trace class provided $\{i\}$ is a q-tuple, with q as in (H3). There exists a finite constant $\widetilde{C}_{E_0} > 0$, depending only on $\|u\|_\infty$, dist$(\sigma(H_0), E_0)^{-1}$, and $d \geq 1$, such that $\|K_{\{i\}}\|_1 \leq \widetilde{C}_{E_0}$.*

The main result of this appendix is the following proposition which establishes (4.17).

Proposition 7.2 *Under the assumptions of Lemma 7.1, for any $E_0 \in (B_-, B_+)$, $\exists$ finite constant $C_{E_0} > 0$ such that*

$$\sum_{i_1,\ldots,i_q \in \widetilde{\Lambda}} \|K_{\{i\}}\|_1 \leq C_{E_0} |\Lambda| \tag{7.1}$$

provided $\{i\}$ is a q-tuple, with q as in (H3). The constant C_{E_0} depends on the dimension $d \geq 1$, $\|u\|_\infty$, and $\mathrm{dist}(\sigma(H_0), E_0)^{-1}$.

The work in this appendix concerns only the unperturbed Hamiltonian $H_0 = (p - A)^2 + V_0$ and $E_0 \in (B_-, B_+) \subset \rho(H_0)$. To simplify the notation, we write $R_0 \equiv (H_0 - E_0)^{-1}$.

Lemma 7.3 *Assume (H1) - (H3) and (H7). Suppose $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^d)$, with supp χ_1 compact and that supp χ_2 lies in a half-space disjoint from supp χ_1, $\|\chi_i\|_\infty = 1$, dist(supp χ_1, supp χ_2) $\geq a > 0$, for some $a > 0$.*

Then, the operator $\chi_1 R_0 \chi_2 \in \mathcal{J}_1$. Furthermore, there exist finite constants $D > 0$, $\alpha > 0$ such that

$$\|\chi_1 R_0 \chi_2\|_1 \leq D \ e^{-\alpha a} \ , \tag{7.2}$$

where D and α depend only on dist$(\sigma(H_0), E_0)^{-1}$.

Proof: Let H_1 be the half-space containing $\mathrm{supp}\,\chi_1$ and such that

$$H_1 \, \{x | \mathrm{dist}(x, \mathrm{supp}\,\chi_1) < \mathrm{dist}(x, \mathrm{supp}\,\chi_2)\}$$

Let L be the straight line minimizing $\mathrm{dist}(\mathrm{supp}\,\chi_1, \mathrm{supp}\,\chi_2)$ so $|L| = a$ and $\partial H_1 \perp L$. Let T_λ, $\lambda > 0$, denote the parallel translate of H_1 along L, that is, $T_\lambda H_1 = \{x \mid \lambda^{-1}\mathrm{dist}(\mathrm{supp}\,\chi_1, \, x) < \mathrm{dist}(\mathrm{supp}\,\chi_2, \, x)\} \equiv H_\lambda$. For any $\lambda > 1$, we can choose $\chi_\lambda \in C_0^\infty$ such that $\chi_1 \chi_\lambda = \chi_1$ and $\mathrm{dist}(\mathrm{supp}\,\chi_\lambda, \mathrm{supp}\,\chi_2) =$

$a(1-\frac{1}{\lambda})$. Note that $\chi_\lambda \chi_2 = 0$ if $\lambda > 1$. Iterating the geometric resolvent equation $2q$-times, we find

$$\chi_1 R_0 \chi_2 = \chi_1 R_0 W(\chi_{\lambda_1}) R_0 W(\chi_{\lambda_2}) ... W(\chi_{\lambda_{2q}}) R_0 \chi_2 \tag{7.3}$$

for any sequence $\lambda_1 > \lambda_2 > ... \lambda_{2q} > 1$, where

$$W(\chi_\lambda) = [H_0, \chi_\lambda] = [(p-A)^2, \chi_\lambda] \ . \tag{7.4}$$

For each λ_i, we can find $\tilde{\chi}_{\lambda_i} \in C_0^\infty(\mathbb{R}^d)$ such that $W(\chi_{\lambda_i})\tilde{\chi}_{\lambda_i} = W(\chi_{\lambda_i})$. It then follows for q as in (H3) and the boundedness of $W(\chi_\lambda) R_0 W(\chi_{\lambda'})$ that

$$\tilde{\chi}_{\lambda_i} R_0 W(\chi_{\lambda_{i+1}}) R_0 W(\chi_{\lambda_{i+2}}) \in \mathcal{J}_q \tag{7.5}$$

In exactly the same way as in the proof of Lemma 7.1, we use the Hölder inequality to conclude that $\chi_1 R_0 \chi_2 \in \mathcal{J}_1$. To prove the exponential decay estimate (7.2), we use the Combes-Thomas estimate. If $\delta(E_0) \equiv \mathrm{dist}(E_0, \sigma(H_0))^{-1}$, then there exist finite constants $C > 0$, $\tilde{\alpha} > 0$ depending on δ, s.t.

$$\|\chi_1 R_0 \chi_2\| \leq C \ e^{-\tilde{\alpha} a} \ . \tag{7.6}$$

By the same argument as above, we can choose $\tilde{\chi}_1 \in C_0^\infty(\mathbb{R}^d)$ such that $\tilde{\chi}_1 \chi_1 = \chi_1$, $\tilde{\chi}_1 \chi_2 = 0$ and

$$\mathrm{dist}(\mathrm{supp}\, \chi_1, \ \mathrm{supp}\, \nabla \tilde{\chi}_1) > a/3 \ ,$$

$$\mathrm{dist}(\mathrm{supp}\, \tilde{\chi}_1, \ \mathrm{supp}\, \chi_2) > a/3 \ .$$

Again, by the support properties,

$$\chi_1 R_0 \chi_2 = \chi_1 R_0 W(\tilde{\chi}_1) R_0 \chi_2 \ ,$$

and we estimate the trace norm by

$$\|\chi_1 R_0 \chi_2\|_1 \leq \|\chi_1 R_0 \ \approx\!\!\!\!\chi_1 \ \| \ \|W(\tilde{\chi}_1) R_0 \chi_2\|_1 \ , \tag{7.7}$$

where $\overset{\approx}{\chi}_1$ is the characteristic function for $\mathrm{supp}\,(\nabla \tilde{\chi}_1)$.

To prove the finiteness of the second factor, we note

$$W(\tilde{\chi}_1) R_0 \chi_2 = i \nabla \tilde{\chi}_1 \cdot (p-A) R_0 \chi_2 + \Delta \tilde{\chi}_1 R_0 \chi_2 \ . \tag{7.8}$$

The second term on the right in (7.8) is trace class by the first part of the theorem. The first term is trace class by the argument of (7.3) - (7.5). Applying the estimate (7.6) to the first factor in (7.7) gives the result (7.2). Note that C depends on $\|W(\chi_{\lambda_1})R_0 W(\chi_{\lambda_2})\|$, which is proportional to $\mathrm{dist}(E_0, \sigma(H_0))^{-1}$.

Remark : We will only use this lemma for $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^d)$. In the proof of Proposition 7.2 below, we simplify notation and write

$$K_{\{i\}} = u_{i_1} R_0 u_{i_2} R_0 u_{i_3} ... u_{i_{l-1}} R_0^2 u_{i_q} \ ,$$

and assume $u_{i_k}^2 = u_{i_k}$, when convenient.

Proof of Proposition 7.2: Fix q so that $K_{i_1 ... i_q} \in \mathcal{J}_1$ according to Lemma 7.1. We separate the multiple sum over $\widetilde{\Lambda}^q = \{(i_1, ..., i_q) | i_k \in \widetilde{\Lambda}\}$ into two parts. By (H5), we can choose a finite $a > 0$ so that if $\eta \equiv 2diam(\mathrm{supp}\, u)$, we have $\eta < a < 2\eta$ and $\|i_n - i_m\| > a$ implies $\mathrm{dist}(\mathrm{supp}\, u_{i_n}, \mathrm{supp}\, u_{i_m}) > a/2 > 0$. We define a subset I_1 of $\widetilde{\Lambda}^q$ as follows. An q-tuple $\{i\} \in \widetilde{\Lambda}^q$ is in I_1 if $\|i_{k-1} - i_k\| < a \ \forall k = 2, ..., q$. Let $I_2 \equiv \widetilde{\Lambda}^q \backslash I_1$ be the complementary set of indices. If $\{i\} \in I_2$, then there exist at least one pair of consecutive indices (i_{k-1}, i_k) s.t. $\|i_k - i_{k-1}\| > a$. We use this pair of indices for the exponential decay. From Lemma 7.1, we have an estimate for the sum over I_1,

$$\sum_{\{i\} \in I_1} \|K_{\{i\}}\|_1 \leq \widetilde{C}_{E_0} \, |I_1| \ , \tag{7.9}$$

where $\widetilde{C}_{E_0}$ is the constant appearing in Lemma 7.1. To estimate $|I_1|$, if we fix i_1, there are a finite number $C(a, d)$ of possible i_2 terms so that $\|i_1 - i_2\| < a$. This number depends only on the constant a and d, for all large Λ. Hence, for fixed i_1, there are $C(a, d)^{q-1}$ terms satisfying the closeness condition since there are $|\widetilde{\Lambda}|$ choices for i_1, we obtain the bound

$$|I_1| \leq \widetilde{C}(a, d) |\Lambda|. \tag{7.10}$$

We now turn to the sum over I_2. We write $i_j \cap i_k$ if $\|i_j - i_k\| < a$ and $i_j \not\cap i_k$ if $\|i_j - i_k\| > a$. We sum successively from i_1 to i_q, for $\{i\} \in I_2$. We sum first over all i_1 such that $\exists (i_2, i_3, ..., i_q) \in \widetilde{\Lambda}^{q-1} s.t. (i_1, i_2, ..., i_q) \in I_2$. We separate the i_1 sum into two parts :

$$\sum_{i_1} \|K_{\{i\}}\|_1 = \Big(\sum_{i_1 \cap i_2} + \sum_{i_1 \not\cap i_2} \Big) \|K_{\{i\}}\|_1 \tag{7.11}$$

The sum over i_1 s.t. $i_1 \cap i_2$ is bounded above by

$$\sum_{i_1 \cap i_2} \|u_{i_1} R_0 u_{i_2}\| \; \|u_{i_2} R_0 \cdots {R_0}^2 u_{i_q}\|_1 \leq C(a,d)\delta(E_0) \; \|u_{i_2} R_0 \cdots R_0^2 u_{i_q}\|_1 \quad (7.12)$$

where $\delta(E_0) \equiv (\mathrm{dist}(\sigma(H_0), E_0))^{-1}$ and $C(a,d)$ are the numbers introduced above. To evaluate the sum for which $i_1 \not\cap i_2$, we use (7.2) of lemma 7.3, and obtain an upper bound,

$$\sum_{i_1 \not\cap i_2} \|u_{i_1} R_0 u_{i_2}\|_1 \; \|u_{i_2} R_0 .... {R_0}^2 u_{i_q}\| \leq \left(\sum_{i_1 \not\cap i_2} D e^{-\alpha \|i_1 - i_2\|} \right) \|u_{i_2} R_0 .... {R_0}^2 u_{i_q}\| \|u\|_\infty^2 . \quad (7.13)$$

The sum in (7.13) is finite and independent of $|\Lambda|$. Note that in (7.13), we only need to continue the estimate in the operator norm. We now pass to the sum over i_2 . There are two terms, one coming from (7.12) and one from (7.13). From (7.12), we sum over all i_2 such that $(i_2, i_3, ...i_q) \in \widetilde{\Lambda}^{q-1} \cap I_2$. Separating the sum into 2 terms as in (7.11), we obtain,

$$\begin{aligned} \sum_{i_2} \|u_{i_2} R_0 .... R_0^2 u_{i_q}\|_1 \;\; & \leq \;\; \sum_{i_2 \cap i_3} \|u_{i_2} R_0 u_{i_3}\| \; \|u_{i_3} R_0 .... {R_0}^2 u_{i_q}\|_1 \\ & \quad + \sum_{i_2 \not\cap i_3} \|u_{i_2} R_0 u_{i_3}\|_1 \|u_{i_3} R_0 .... {R_0}^2 u_{i_q}\| . \quad (7.14) \end{aligned}$$

Each term is estimated as in (7.12)-(7.13). As for the $i_2 - sum$ in (7.13), we have

$$\begin{aligned} & \sum_{i_2} \|u_{i_2} R_0 u_{i_3} \cdots {R_0}^2 u_{i_q}\| \\ & \quad \leq \;\; \sum_{i_2 \cap i_3} \|u_{i_2} R_0 u_{i_3}\| \; \|u_{i_3} R_0 \cdots {R_0}^2 u_{i_q}\| \\ & \quad + \;\; \sum_{i_2 \not\cap i_3} \|u_{i_2} R_0 u_{i_3}\| \; \|u_{i_3} R_0 \cdots {R_0}^2 u_{i_q}\| \qquad (7.15) \end{aligned}$$

Since the trace norm has been evaluated in (7.13); the usual Combes-Thomas result (7.6) (see section 3) can be used for the second term of (7.15). As above, the bound on both terms is $|\Lambda|$-independent. We continue to sum over $i_3, \cdots i_{q-2}$. In (7.14) the trace norm is pushed through each pair when (i_{j-1}, i_j) satisfy $\|i_{j-1} - i_j\| < a$, and it is evaluated using lemma 7.3 otherwise. Similarly, in (7.15), we use the Combes-Thomas result (7.6) of section 3 to control the operator norm of pairs $(i_{j-1}, i_j) s.t. \|i_{j-1} - i_j\| > a$. We obtain in this way 2^{q-2} terms and a coefficient depending only on $d, \|u\|_\infty, C(a,d)$, and $\delta(E_0)$. Finally, there are 2 remaining terms to evaluate : one from (7.14),

$$\sum_{i_{q-1}, i_q} \|u_{i_{q-1}} {R_0}^2 u_{i_q}\|_1, \tag{7.16}$$

and the other from (7.15),

$$\sum_{i_{q-1}, i_q} \|u_{i_{q-1}} {R_0}^2 u_{i_q}\| \tag{7.17}$$

for the trace norm in (7.16), we recall that the remaining indices, coming from I_2, satisfy $i_{q-1} \not\cap i_q$ (that is, the only remaining trace norm is from those $q - tuples$ $\{i\}$ for which only the pair (i_{q-1}, i_q) satisfy $\|i_{q-1} - i_q\| > a$). Because $\mathrm{dist}(\mathrm{supp}\, u_{i_{q-1}}, \mathrm{supp}\, u_{i_q}) \geq \frac{a}{2}$, we can find $\chi \in C_0^\infty(\mathbb{R}^d) s.t. \chi u_{i_{q-1}} = u_{i_{q-1}}, \mathrm{dist}(\mathrm{supp}\, \chi, \mathrm{supp}\, u_{i_q}) \geq \frac{a}{4}$, and $\mathrm{dist}(\mathrm{supp}\, \nabla\chi, \mathrm{supp}\, u_{i_{q-1}}) \geq \frac{a}{4}$. We then have:

$$\begin{aligned} \sum_{i_{q-1} \not\cap i_q} \|u_{i_{q-1}} {R_0}^2 u_{i_q}\|_1 \;&\leq\; \sum_{i_{q-1} \not\cap i_q} \|u_{i_{q-1}} R_0\| \|\chi R_0 u_{i_q}\|_1 \;. \\ &+ \sum_{i_{q-1} \not\cap i_q} \|u_{i_{q-1}} R_0 W(\chi)\|_1 \|{R_0}^2 u_{i_q}\| \end{aligned} \tag{7.18}$$

The trace norms in both terms on the right in (7.18) are exponentially bounded by Lemma 7.3 and the operator norms are bounded by a power of $\delta(E_0)$. Consequently, the $i_{q-1} - sum$ is controlled and the i_q-sum results in a factor of $|\Lambda|$. Finally, we estimate (7.17). We separate the sum into $i_{q-1} \cap i_q$ and $i_{q-1} \not\cap i_q$. The nearest neighbour sum is bounded by $C(a,d)\delta(E_0)^2|\Lambda|$. The sum over disjoint pairs is estimated as in (7.18) using the usual Combes-Thomas estimate (7.6) for the operator norm. This completes the proof.

Proposition 7.4 *Let $K_0 \equiv R_0 V_\Lambda$, then there exists a finite constant $C > 0$, as in Proposition 7.2, such that*

$$\mathbb{E}(\|K_0\|_q{}^q) \leq C\,|\Lambda|. \tag{7.19}$$

Proof: The proof uses almost exactly the same arguments as in Proposition 7.2 and we will indicate how to reduce the expression to those calculations.

By hypothesis (H3), we write $q = 2p$, for some integer p. ¿From the definition of the norms, we have

$$\|K_0\|_q{}^q = \||K_0|^{q/2}\|_1 = \|(K_0^* K_0)^p\|_1. \tag{7.20}$$

Since $K_0^* K_0 = R_0 V_\Lambda^2 R_0$, we must estimate

$$\mathbb{E}_\Lambda \{Tr(V_\Lambda^2 R_0^2 \cdots V_\Lambda^2 R_0^2)\}. \tag{7.21}$$

Expanding the potential as before (4.9), we obtain the analog of (4.10),

$$K_{\{i\}} = u_{i_1} R_0^2 u_{i_2} u_{i_3} R_0^2 u_{i_4} u_{i_5} ... u_{i_{q-1}} R_0^2 u_{i_q} \ . \tag{7.22}$$

The trace norm of this operator is estimated as in the proof of Proposition 7.2. We carry out the summation over the indices in the same manner using Lemma 7.3 to control $\|u_i R_0^2 u_j\|_1$ when $u_i u_j = 0$. Note that the intermediate terms like $u_{i_4} u_{i_5}$ actually vanish when the supports are disjoint. The proof then proceeds as in the proof of Proposition 7.2.

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