A SHORT COURSE ON THE PIROGOV – SINAI THEORY

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Abstract. This is the first draft of the Lecture Notes of the "Short course of the Pirogov – Sinai theory" given by M. Zahradník at the University Roma II, February - March 1996.

In this series of lectures I explain in some detail the method (of investigation of the phase picture of low temperature spin models of statistical mechanics) called the Pirogov Sinai Theory. The Pirogov Sinai theory is a general method which culminates one of the lines of study of the low temperature phases, whose origins go back to the fundamental Peierls paper [P].

Namely, it was the very notion of a *contour* which first appeared in that paper and which is still the main keyword of the theory. The Peierls type approach is still one of the leading ideas in the development of rigorous Statistical Mechanics.

One has to note that the descriptions like "Peierls type approach", "Peierls argument" have a broader meaning now and are used also in situations different from those belonging to the range of possible application of Pirogov – Sinai theory at the present time. Very roughly speaking, the possibility of estimating the probability of some "contours" in the given model does not mean that we are already within the scope of the Pirogov – Sinai theory. In addition to the yet mentioned keyword – the notion of a "contour" – the possibility of an "expandability" of the partition functions of the model (at least of some of them) is the second very important feature of the models suggesting a useful application of the Pirogov – Sinai type reasonings to them.

Some historical notes: After the publication of the Peierls paper in late thirties, there was almost no additional progress in this line of study until mid sixties – when Dobrushin and Griffiths started the new, rigorous investigation of the subject. An important milestone in its development was then the work [MS] of Minlos and Sinai. This was the first and systematic investigation of the notion of a "contour model"; the construction which played crucial role in the development of the Pirogov – Sinai

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theory later. The papers [MS] are restricted to the case of an Ising model; more importantly they are devoted to the study (in a great depth!) of the case of the phase *coexistence*.

The emphasis on the study of phase coexistence was understandable in those early days of matematically rigorous statistical mechanics: not only the coexistence of phases was considered to be the most interesting (and new in its mathematical description) phenomenon but also, even more importantly, the coexistence regime offers some *symmetries* which are not present in the general case. (This does not mean that there are no portions of the papers [MS] giving nontrivial information also about the case of Ising model with nonzero magnetic field.) Once the notion of a contour model appeared in [MS], it was a question of a short time to generalize it also to other models, offering sufficient symmetries of their Hamiltonian. Gertzik obtained such results [G] (formulating, for the first time, a sufficiently general version of the condition now commonly called the Peierls condition) but the method of Pirogov and Sinai [PS] which appeared in the same time has proved to be far more reaching. The important novelty of the new method of [PS] was their ability to handle also the *nonsymmetric case*. They achieved this by introducing another important notion of a *contour model with a parameter*. Their method turned out to be a powerful and general tool for obtaining rigorous results about the low temperature phase diagram of many lattice spin models.
The original version of the theory is described in [PS] and also in the book [S].

However, from the beginning there was quite a widespread attitude towards this theory by outsiders, describing it as a "complicated tool which should be used only if all the more common methods (reflection positivity, correlation inequalities, ...) fail to apply". This attitude has changed only slowly in the last 20 years. The feeling that Pirogov – Sinai theory should be considered (at least after necessary simplifications – which inevitably will appear if the theory is sufficiently widely used) *as a standard tool* (like, say, the high temperature expansions) is still not widespread among the people working in the field of rigorous statistical mechanics.[1]

Of course I can not describe this theory in all its aspects and I also cannot mention various applications which already appeared in the literature. I will concentrate, in my lectures, on a version of this theory which was developed in Prague, starting from the paper [Z] and which was then applied, for example, to the study of the interface problem. (The latter will not be mentioned here in detail but, nevertheless, namely *this* problem was important for us to understand the need to have a more flexible and more powerful version of the Pirogov – Sinai theory.) After some years of our attempts to apply the Pirogov – Sinai theory (in its version described in [Z]) to various situations, we recently developed another, simpler and at the same time more powerful, version of this theory in the paper [HZ] (which exists as a preprint in mp_arc, Texas). The method of [HZ] originated as the result of our attempt to find an appropriate method to deal with the case when some "stratified phases" (with one or more interfaces) appear. It turns out, however, that the method of [HZ] has wider applications. Even when applied to the ordinary proplems solved by previous versions of the Pirogov – Sinai theory it gives a

[1]There is a hope that the Pirogov – Sinai theory can be extremely useful also in *nonrigorous studies*. Unfortunately, there was little activity in this direction up to now.

new, simpler and more detailed description of the situation.

So, I apologize that I shall not describe in detail the original Pirogov – Sinai theory here, but instead, I will describe the working version of that theory which we are using at the present time.

Of course, the important ideas of the original Pirogov – Sinai approach (and some additional simplifying ideas, like that of the "metastable ensemble", which originally appeared in [Z]) are not much changed in the new approach I will present here. I simply believe that these ideas can be now expressed in a simpler and stronger form than before. In particular, instead of stressing the importance of the *estimates* of various (so called) diluted partition functions appearing in the theory we now supplement these estimates by more accurate (and more informative) *expansions* of these partition functions. In a sense, bounds for partition functions (and *any* estimates in general, except of statements of the type " the terms of these series converge quickly to zero") now disappeared from our version of the theory – being replaced by expansion formulas.

The plan of my lectures is the following. In lectures 1 and 2, I explain (after introducing some typical examples for which the presented theory is a useful tool) the core of the original Pirogov – Sinai theory in the special case when the *maximal number of coexisting phases exists.* I mention the simplifying role of the *symmetry* in these considerations, but the emphasis will be always on the nonsymmetric case.

Namely, the strength of the Pirogov – Sinai theory lies in its ability to describe *nonsymmetric* applications, also in the situations when "not all of the candidates for the translation invariant phases survive the thermodynamic limit resp. survive a small change of the parameters of the model".

Note. It is erroneous, I think, to consider the Pirogov – Sinai theory just as a mere "means to construct the phase diagram". What is far more important is that this theory gives really a detailed *control* over the behaviour of all "phases" in all possible situations (when these "phases" become either "stable" or "instable") in infinite but also (not less importantly) in any *finite* volume.[2] There will be few statements like "if the volume goes to infinity" resp. "if the temperature goes to zero"; the assumptions on the size of a volume resp. on the smallnes of the temperature (for which the formulated statement is valid) will be usually given explicitly.

To summarize the contents of Lecture 1 and 2: they cover the simplest application of the theory, namely the case of a hamiltonian with a finite number of "local"ground states, fulfilling the (so called) Peierls condition. It answers the question: what are the conditions on the hamiltonian (resp. how to adjust its parameters) such that a maximal number of possible phases (each phase corresponding to a particular local ground state) would coexist? We introduce here the fundamental notion of the Pirogov – Sinai *contour functional*, and also the notion of a *contour model.* The latter is mentioned, however, mainly from the "historical" reasons. In our later study, we will deviate (in Lectures 3 and 4) from the concept

[2]The question what happens in a given finite volume (under given boundary conditions) is nontrivial even in the (typical!) situation where only *one* infinite volume phase exists. The case when several phases coexist is just marginal from such a general point of view, and of course it is desirable to be able to treat such a case together with the previous one. This is something which the Pirogov – Sinai theory does!

of a contour model, but our deviation would not be felt seriously in the special situation investigated (after giving some general overview of the considered models in Lecture 1) in Lecture 2.

Lecture 3 gives then a general course on the Pirogov – Sinai theory (in its present "Prague" form). The version of the theory given in Lecture 3 follows essentially the approach of [Z] but there are some important recent simplifications and strengthenings taken from [HZ].

We left here the notion of a contour model completely, and also the behaviour of the "unstable phases" is studied in a more concise way. Instead of the notion of a contour model (and the estimates of its partition functions) the concept of an *expansion* of considered partition functions is stressed now.

Note. Cluster expansions were always an important tool in the Pirogov – Sinai theory. However, in the previous versions of this theory, the expansions of the considered partition functions were viewed merely as some "auxilliary" technique which was applied to the study of the contour models. One could think that the cluster expansion method could be possibly replaced by something else, giving comparably nice expressions of the partition functions. Namely, the possibility to work with precise *decompositions of the considered partition functions* into the "bulk" and the "boundary" (surface tension) terms was always quite characteristic for the Pirogov – Sinai theory but still, one could hope that such a precise information about the partition functions could be obtained by other methods (than cluster expansion).

This is not so in the approach I present here, where the concept of an expansion enters even the basic notions and the basic ideology of the theory. It can be really said that the Pirogov – Sinai theory, in the version I present, is *just the method of organizing the expansions of low temperature partition functions* and the Pirogov – Sinai contour functional F (more specifically, $\exp(-F)$) is the most important quantity entering these expansions. As a byproduct, our attitude is selfcontained in the sense that no references to the literature on cluster expansions are necessary. Applied to so called "polymer models", our method is just another approach to the question of finding the expansions of partition functions of polymer models. Nevertheless, some previous knowledge of the cluster expansion theory is useful here and our treatment of the problems purely related to the cluster expansion theory of the – so called – polymer models will be quite sketchy. However, there is an extensive literature on the subject, see [M],[ES],[KP],[DC], ...

This is developed in detail in Lecture 4 – where also some additional, "topological" investigations of the structure of systems of "large" contours (appearing under "unstable boundary conditions) are made. This concludes the development of the Pirogov – Sinai theory in its basic form.

First Lecture. General Setting.

Hamiltonians, Gibbs States

We study the Gibbs states on a lattice $\mathbb{Z}^\nu$ in dimensions $\nu \geq 2$. We consider some norm on $\mathbb{Z}^\nu$ e.g. the norm

$$|t| = \max_i |t_i|$$

and say that a subset $\Lambda \subset \mathbb{Z}^\nu$ is connected if for any $t, s \in Y$ there is a connecting path $\{t_i\}, i = 1, 2, \ldots, n$ such that $t_0 = t,\ t_n = s$ and $|t_i - t_{i+1}| = 1$ for each $i = 1, 2, \ldots, n$.

Our basic set of *configurations* will be the set

$$\mathbb{X} = S^{\mathbb{Z}^\nu}$$

where S is some (finite) set of "spins". One usually takes the σ – algebra $\mathcal{B}$ generated by the set of all "cylindrical" events, i.e. events measurable in terms of some finite projection

$$x \in \mathbb{X} \to x_\Lambda \in S^\Lambda$$

where Λ is some finite subset of $\mathbb{Z}^\nu$.

By a *state* we mean a probability measure on $(\mathbb{X}, \mathcal{B})$.

A *hamiltonian* on $\mathbb{X}$ will be usually given by some family of *interactions* i.e. functions Φ_A defined on S^A, $A \subset \mathbb{Z}^\nu$. These interactions will usually satisfy some further requirements like the *translation invariancy* – if Φ_A commute with all the shifts of the sets A (sometimes only the invariancy with respect to some subgroup of $\mathbb{Z}^\nu$, possibly of lower dimension, will be assumed) and the *finite range* – namely if $\Phi_A = 0$ whenever the diameter of A is bigger than some integer r. A hamiltonian of a configuration x_Λ in a finite volume $\Lambda \subset \mathbb{Z}^\nu$, under a boundary condition x_{Λ^c} will be defined as

$$H(x_\Lambda | x_{\Lambda^c}) = \sum_{A \not\subset \Lambda^c} \Phi_A(x_A).$$

Given a state P on $(\mathbb{X}, \mathcal{B})$ and boundary condition x_{Λ^c} given on a set Λ^c, or at least on the set

$$\partial\Lambda^c = \{t \in \mathbb{Z}^\nu : \text{dist}(t, \Lambda) \le r\}$$

for a finite Λ, consider the conditional probability

$$P((\cdot)|x_{\Lambda^c})$$

which is defined uniquely for P almost all x_{Λ^c}. We say that P is a *Gibbs state* with respect to the hamiltonian H if this conditioned probability satisfies, for almost all x_{Λ^c}, the condition

$$P(x_\Lambda | x_{\Lambda^c}) = Z^{-1}(\Lambda, x_{\Lambda^c}) \exp(-H(x_\Lambda | x_{\Lambda^c})$$

where

$$Z(\Lambda, x_{\Lambda^c}) = \sum \exp(-H(x_\Lambda | x_{\Lambda^c})$$

the latter sum being over all $x_\Lambda \in S^\Lambda$.

Note. We always incorporate the inverse temperature $\frac{1}{T}$ into our hamiltonian. In other words, the temperature will be just one of the parameters in the hamiltonian H (usually not to be explicitly mentioned below). The fact that we are actually studying the *low temperature case* will be formulated later, by the Peierls condition.

A Gibbs state P is said to be an *extremal* one if it is moreover indecomposable in the sense that there are no other Gibbs states $\tilde{P}, \tilde{P}'$ such that P would be a nontrivial convex combination of $\tilde{P}, \tilde{P}'$. In the following we will have to work almost exclusively with the extremal Gibbs states. Namely, these will appear as the limits of finite volume Gibbs states $P((\cdot), x_{\Lambda^c})$ under a special choice of (constant) boundary conditions (yielding an extremal Gibbs state, as we will see).

We will usually omit the adjective "extremal" in the following, and call these Gibbs states also "pure phases" or simply "phases" (especially if one has in mind the *translation invariant* Gibbs states). Later we will call them occasionally also as the "stable phases" – as opposed to the "unstable phases" which apppear only in volumes of a limited size. The change of the parameters in the hamiltonian affects usually the "stability" of the considered "phases". This is the main question to be clarified by the Pirogov – Sinai theory.

One of the basic keywords of the Pirogov – Sinai theory is the notion of a (local) ground state. This is used for the configurations x satisfying the property that whenever we change it "locally" – in a set not exceeding a prescribed size – then its energy increases i.e.

$$H(\tilde{x}) - H(x) = \sum_A (\Phi_A(\tilde{x}_A - \Phi_A(x_A)) > 0. \tag{0.0}$$

Notes. 1. Roughly speaking, such a "local" ground state x usually turns out to be the true ground state of a suitable "original", "unperturbed" hamiltonian $\tilde{H}$ (whose slight perturbation the given "perturbed" hamiltonian H is).
2. This is noted here just for an intuition; we will *not* use the formal notion of a local ground state below! Even the notion of a ground state – appearing when we drop any requirements on the size of the perturbation (except of its finiteness) of $\tilde{x}$ with respect to x – will not be employed below (on the formal level). We notice that all the formal requirements to be later used in connection with the idea of a (local) ground state will be *contained in the formulation of the Peierls condition* below.

To avoid misunderstanding: Of course, the idea to construct Gibbs states around some "local ground states of the given model" (and these "local" ground states are often the true ground states of some "unperturbed" hamiltonian) lies in the very heart of the Pirogov – Sinai theory. However, the very notion of a ground state is *not* used by the theory.[3]
3. By a *degeneracy* of a (local) ground state one usually has in mind the fact that *several* (local) ground states exist for a given hamiltonian. This is the interesting case; Namely, in the regions of phase unicity, well developed methods based

[3]In practice, one usually starts with a given hamiltonian (the "unperturbed" one, often having some additional symmetries) whose ground states are to be found – because one expects that no other (even local) ground states will appear even for the slightly *perturbed* hamiltonian. One moreover expects to be able to construct Gibbs states around *some*, at least, of these ground states, also for the perturbed hamiltonians. Of course, the determination of all the ground states of a given "unperturbed" hamiltonian is a very important and often notrivial task. We are *not* discussing this question here (see e.g. [BS]) – because this is an investigation which should *precede* the application of the Pirogov – Sinai theory (giving proper candidates for the elements of the "reference set" Q introduced below & showing that the Peierls condition will be valid). However, such an investigation is not, strictly speaking, the part of this theory; it just opens a way to its application!

essentially on the Dobrushin' s unicity theorem (and later complete analyticity investigations by [DS]) are available.

On the other hand, in the regions where phase coexistence is expected no serious alternative to the Pirogov – Sinai theory exists reaching a comparable level of generality and universality of its applications.

The idea behind the Pirogov – Sinai theory is the following one: under the condition that there are "sufficiently strong energetical barriers" between different local ground states (formally, such a requirement will be formulated as the Peierls condition – see below) one should construct the expected (generally "unstable") "phases" as some perturbations of the corresponding local ground states. Moreover, one should acquire a full control on how these "unstable" phases turn out to be "stable"(i.e. giving rise to the true, infinite volume Gibbs states) under the change of the parameters in the hamiltonian.

The specification of the "stable phases" thus constructed (for any values of the parameters in the hamiltonian) is what is usually called the *phase diagram* of the given model.

Contours

The central notion of the Pirogov – Sinai theory is the notion of a *contour* (see below). The idea is to consider contours as some "barriers" (more precisely connected components of these barriers) separating the regions occupied by various local ground states. Though very important, such a notion cannot obviously have some "canonical" definition. On the contrary, the notion of a contour in various situation can differ, and one could tailor it with respect to the peculiarities of the considered model. For example, in the Ising model, contours are traditionally defined as some connected paths separating the regions occupied by + or − spins. Having this in mind (the fact that the definition of a contour can be adapted to particular features of the model or to the precise formulation of the problem, which has to be solved) we present here a general definition of a ("thick") contour suitable for all the situations outlined above. This general definition appeared first in [PS]. To begin, we have to specify the collection of the local ground states of the general model above. Let us assume that we have already found this collection. Denote their elements as $\{x_q, q \in Q\}$ where $Q \subset S$ and x_q denotes the constant configuration $\{x_t = q, t \in \mathbb{Z}^\nu\}$.

Notes. 1. This setting is sufficiently general to cover also the case of *periodical* local ground states. Namely, if there are also some periodical local ground states of the given model then it can be assumed that they have the same period G; G is understood here as a subgroup of $\mathbb{Z}^\nu$ such that the factor group $D = \mathbb{Z}^\nu/G$ is finite. By defining the blocks of original spins $x_{D'}$ (D being identified with a suitable subset of $\mathbb{Z}^\nu$, $D' = D + t$ where $t \in G$) as the spins of the new (blocked) model (with the spin space S^D), the original model is converted to a new one having only constant local ground states.

2. We recall that the concept of a local ground state is not defined precisely. The decision whether a given constant (or periodical) configuration *is* a local ground state is therefore sometimes a little bit arbitrary. This should not cause substantial problems: namely the "phases" constructed around such "suspect" local ground

states will be so "highly unstable" that even the very notion of an unstable "phase" will lose a reasonable sense here.
3. The nontrivial task of finding all the local ground states (of the given model) is completely left out here. The fact is that the Pirogov – Sinai theory actually only *starts* at the moment when the collection of all local ground states is already given. The specification of the family x_q is of course a very important (in spite of some arbitrariness contained in the choice of x_q) but *preliminary* step of the investigation of the given model. We just call by the Pirogov – Sinai theory everything which *follows* the finding of the local ground states of the model.

Now let us define the central notion of the Pirogov – Sinai theory:

Definition of a contour. Say that a point $t \in \mathbb{Z}^\nu$ is a q – correct point of a configuration $x \in \mathbb{X}$ if for all the points $s \in \mathbb{Z}^\nu$ from the r – neighborhood of t (recall that r denotes the range of interactions of the given model), $x_s = q$. A point which is correct for *no* $q \in Q$ will be called incorrect. The collection, denoted by $B(x)$, of all incorrect points of $x \in \mathbb{X}$ will be splitted into connected components. Having such a component $B \subset B(x)$, the restriction of x to B will be called a *contour* of x. Contours will be denoted by symbols $\mathbf{\Gamma}$. The above mentioned set B will be called the support $B = \operatorname{supp} \mathbf{\Gamma}$ of the corresponding contour $\mathbf{\Gamma}$.

Diluted configurations. A configuration x will be called diluted, more precisely q – diluted, if all the contours of x have finite supports and moreover, if we denote by ext the infinite component of the complement of $\cup \operatorname{supp} \mathbf{\Gamma}$ then such a component is unique and all the components of the set ext^c are also finite and we have $x = q$ on ext.

Any collection $\{\mathbf{\Gamma}_i\}$ of contours which is the collection of all contours of some diluted configuration will be called an *admissible* collection of contours. On the other hand, any admissible collections determines uniquely some diluted configuration i.e. we can use the notions of an admissible system of contours and of a configuration from $\mathbb{X}$ as synonyma.

In the sequel, we will work almost exclusively with finite volumes. In such a case we will consider diluted configurations, which are equal to some $q \in Q$ outside of the given finite volume. More specifically, if $\Lambda \subset \mathbb{Z}^\nu$ is given and x is a q diluted configuration such that all its contours have supports in Λ having a distance at least 2 from Λ^c we will say that x is q – diluted *in* Λ.

Diluted partition functions. The partition function over all configurations x_Λ which are q – diluted in Λ,

$$Z^q(\Lambda) = \sum_{x_\Lambda} \exp(-H(x_\Lambda | x^q_{\Lambda^c})) \tag{1.0}$$

will be called the diluted partition function.

This will be our *main object of study* in the sequel.

Note. The reader maybe expects some "telescopic", recurrent relations, connecting the diluted partition functions in various volumes – in a way analogous to the classical DLR equations. We will minimize the use of such relations in the sequel. In fact, they will never be used as relations between various *diluted* partition functions in various volumes.[4]

In what follows, the volume Λ will be *fixed* for most of the exposition (better speaking, all possible shifts of Λ will be considered at once) and only at the very end the limit $\Lambda \to \mathbb{Z}^\nu$ will be considered. Thus, most of our effort will be devoted to the study of situations appearing in a given finite volume (simultaneously with respect to all its shifts) and it will be quite sensible even not to think about the infinite volume limit which should be (possibly) taken at the very end, after finishing the development of all the important constructions. Just imagine that the set Λ in (1.0) has the cardinality of 10^{27} or so. The philosophy of the Pirogov – Sinai theory is that it should give full answers to all reasonable questions about the behaviour of the system in any given finite volume, not only in the infinite volume limit!

Reformulation of the hamiltonian

As we already noted the idea is to consider the contours as some "barriers" separating regions occupied by various local ground states. We will now define the *energy* of such a barrier, relating it also to the "ground energy" of the coresponding "underlying configuration" x_q. Define first the latter notion:

Density of energy of x_q. Put

(1.1) $$e_q = \sum_{A:\ A \ni 0} |A|^{-1} \Phi_A(x_A) \quad ; \quad e = \min\{e_q\}.$$

Note. There is some arbitrariness in the precise form of this formula. For example the following quantity has the same value as before:

(1.1') $$e_q = \sum_A \Phi_A(x_A)$$

where the sum in (1.1') is over A having 0 as its *first* point, in the lexicographical order on $\mathbb{Z}^\nu$.

In the following we will say that $\mathbf{\Gamma}$ has an "external colour q" if $\mathbf{\Gamma}$ is the only contour of some q – diluted configuration; the latter will be denoted by $x_{\mathbf{\Gamma}}$ in the sequel, and we willl write $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ in such a case.

Contour energy. Put

(1.2) $$\Phi(\mathbf{\Gamma}) = \sum_{A: A \cap \operatorname{supp} \mathbf{\Gamma} \neq \emptyset} \Phi_A(x_A) |\operatorname{supp} \mathbf{\Gamma} \cap A| \, |A|^{-1}.$$

The following quantities obtained by subtracting, from $\Phi(\mathbf{\Gamma})$, the "ground energy" $e|\operatorname{supp} \mathbf{\Gamma}|$ will be more relevant in sequel: Put

(1.3) $$E(\mathbf{\Gamma}) = \Phi(\mathbf{\Gamma}) - e|\operatorname{supp} \mathbf{\Gamma}| \quad , \quad E_q(\mathbf{\Gamma}) = \Phi(\mathbf{\Gamma}) - e_q|\operatorname{supp} \mathbf{\Gamma}|$$

Notice that for a q contour $\mathbf{\Gamma} = \mathbf{\Gamma}^q$; $E_q(\mathbf{\Gamma}) = E(\mathbf{\Gamma}) + (e_q - e)|\operatorname{supp} \mathbf{\Gamma}|$.

With these notations one has the following expression of the hamiltonian, which will play a fundamental role in the following.

[4]DLR equations will be used only at the very end of our exposition. Having achieved the control over the "external behaviour" of considered configurations (and *this* is the main theme of the Pirogov – Sinai theory) we can supplement the additional information (on what happens inside of the external contours) from the usual DLR equations.

Theorem 1. *Let x be a configuration which is q diluted in Λ (or, more generally, such that all its contours have supports in Λ). Then*

$$(1.4)\qquad H(x_\Lambda|x^q_{\Lambda^c}) = \sum_{q'\in Q}\sum_{t\in\Lambda_{q'}} e_{q'} + \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|) + C(q,\Lambda)$$

$$= \sum_{q'\in Q}\sum_{t\in\Lambda_{q'}} e_{q'} + \sum_{\mathbf{\Gamma}}\sum_{t\in\operatorname{supp}\mathbf{\Gamma}} e + \sum_{\mathbf{\Gamma}} E(\mathbf{\Gamma}) + C(q,\Lambda) \quad i.e.$$

$$(1.4q)\qquad H(x_\Lambda|x^q_{\Lambda^c}) = \sum_{q'} \Big(\sum_{t\in\Lambda_{q'}\cup(\cup\operatorname{supp}\mathbf{\Gamma}^{q'})} e_{q'} + \sum_{\mathbf{\Gamma}^{q'}} E_{q'}(\mathbf{\Gamma}^{q'})\Big) + C(q,\Lambda)$$

where $\Lambda_{q'}$ denotes the collection of all points of Λ which are q' – correct. The first sum on the right hand side of (1.4) resp. of (1.4q) is over all $q'\in Q$ (including q), and the second sum is over all contours $\mathbf{\Gamma}$ of x resp. over all q' contours $\mathbf{\Gamma}^{q'}$ of x. The constant is equal to $C(q,\Lambda) = H(x^q_\Lambda|x^q_{\Lambda^c}) - e_q|\Lambda|$.

The Abstract Pirogov – Sinai Model.

Let us rewrite once again (1.4), from now on without the constant term $C(q,\Lambda)$ (which has no effect on the corresponding Gibbs measure):

$$(1.5)\qquad H(x_\Lambda|x^q_{\Lambda^c}) = \sum_{q'}\sum_{t\in\Lambda_{q'}} e_{q'} + \sum_{\mathbf{\Gamma}}(E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|).$$

Recall that $\Lambda_{q'}$ denotes the collection of all points of Λ which are q' – correct and the sum is over all $q'\in Q$ including the "external colour" q of x_Λ.

Forget now the way how contours were constructed and apply the following, more general approach to the problem of studying the "diluted Gibbs measures" P^q_Λ corresponding to the hamiltonian (1.5), with the partition function over all q – diluted configurations in Λ:

$$(1.6)\qquad Z^q(\Lambda) = \sum_{x_\Lambda} \exp(-H(x_\Lambda|x^q_{\Lambda^c})).$$

Namely, imagine that contours are some abstract, "connected" objects (the exact meaning of the word "connected" can be specified for any particular model; at the moment we may assume that it means the usual connectedness of the supports of contours; as above) which are "colored" on its boundary by colors from Q, such that any component of the boundary of the set $\operatorname{supp}\mathbf{\Gamma}$ has a *constant* colour $q'\in Q$.

Admissible family of contours. By an admissible family of contours we will mean a family $\{\mathbf{\Gamma}_i\}$ of contours which has the following two properties: 1) Contours of the system do not mutually "touch" (e.g. in the sense that $\operatorname{dist}(\operatorname{supp}\mathbf{\Gamma}_i, \operatorname{supp}\mathbf{\Gamma}_j) \geq 2$ if $i\neq j$) and 2) The prescriptions of the colours outside $\cup\operatorname{supp}\mathbf{\Gamma}_i$ are *not in conflict* i.e. there is a mapping from $(\cup\operatorname{supp}\mathbf{\Gamma}_i)^c$ to Q which is constant on each component of $(\cup\operatorname{supp}\mathbf{\Gamma}_i)^c$ and which is also in accordance with the "colour" of the given component, induced by the neighboring contours of the system.

Abstract Pirogov – Sinai model. Having defined a family of "colours" Q, a family $\mathcal{G} = \{\mathbf{\Gamma}\}$ of allowable contours of the model, some quantities $\{e_q, q \in Q\}$ and $\{E(\mathbf{\Gamma})\}$ we consider, in any volume Λ, a model whose configurations space is the collection of all admissible systems of contours in Λ and whose hamiltonian is given by (1.5). This will be called the abstract Pirogov – Sinai model (corresponding to the given choice of $\mathcal{G}$ and the given quantities $\{e_q, q \in Q\}$ and $\{E(\mathbf{\Gamma})\}$).

If $\mathcal{G}$ and also all the quantities e_q and $E(\mathbf{\Gamma})$ are translation invariant then we will speak about the translation invariant abstract Pirogov – Sinai model. This will be mainly the case considered in these lectures.

Note. We are proposing here the following ideology: do *not* think about the problem in the usual language of hamiltonians (1.0), DLR equations etc. *Forget* how contours were defined (they could be possibly defined also in another way than before, tailored better to the particular situation) and work with them as with some abstract objects.[5]

Of course, one needs some assumptions about the behaviour of the quantities e_q and $E(\mathbf{\Gamma})$. These assumptions can be formulated in an extremely simple way: Namely, the following is really the *only assumption* which is needed to apply the Pirogov – Sinai machinery (developed below) on a given abstract Pirogov – Sinai model.

The Peierls Condition

Assumption. Assume that there is some $\tau >> 1$ such that for any contour $\mathbf{\Gamma} \in \mathcal{G}$ we have the inequality[6]

$$E(\mathbf{\Gamma}) > \tau |\operatorname{supp} \mathbf{\Gamma}|. \tag{1.8}$$

Notes. 1. Recall that, whenever we are in the case of a reformulated spin model, we are including the inverse temperature into the hamiltonian. Typically, τ is then of the order $\frac{J}{T}$ where J denotes the "strength of the interactions" and T is the temperature of the original spin model. See the examples in the forthcoming section.

2. If the "connectedness" of $\mathbf{\Gamma}$ has some "less standard" meaning (than above) then the right hand side of (1.8) should be replaced by another quantity, having the meaning of a "minimal number of points needed to make the set $\operatorname{supp} \mathbf{\Gamma}$ connected".

3. Actually, what will be really needed instead of (1.8) will be the inequality (with another $\tilde{\tau}$)

$$\sum_{\mathbf{\Gamma} : \operatorname{supp} \mathbf{\Gamma} = G} \exp(-E(\mathbf{\Gamma})) \leq \exp(-\tilde{\tau}|G|). \tag{1.9}$$

[5] Once again, the ideology we propose is: Forget the notion of a spin and of a configuration from $\mathbb{X}$ and replace these primitive notions by other primitive notions of a contour & of an admissible system of contours!

[6] Often, in the literature, such a condition is formulated only with respect to all true ground states of the original "unperturbed hamiltonian", requiring a lower bound $C|\operatorname{supp} \mathbf{\Gamma}|$ for the right hand side of (0.0). However, the formulation (1.8) (based on the notions of e_q and $E(\mathbf{\Gamma})$) is more adequate.

It is apparent that for contours constructed from the spin model as above, this means (for $|S|$ not too big and $\tilde{\tau}$ slightly smaller than τ) practically the same as (1.8). However, for large $|S|$ (or for contours defined in some more exotic way) it is sometimes advisable to work directly with (1.9).
4. The exact meaning of the relation $\tau >> 1$ depends on the dimension of $\mathbb{Z}^\nu$. For $\nu = 3$ this means something like $\tau > 5$. See below.
5. The important observation is that the Peierls condition *remains to be valid* for all sufficiently *small perturbations* of the given model (assuming that for that model this condition was already established).
6. Peierls condition sounds quite natural and there was even a hypothesis for some time that such a condition *always holds* in sufficiently low temperatures (to make τ sufficiently big). The counterexample is due to Pecherski [PE]. However, in practise, the nonvalidity of the Peierls condition usually means that we just *failed to find some local ground states* of the given model! In other words, our choice of Q was inappropriate. Either too small or, possibly, too big. To illustrate the former case (the latter one can also appear – see the discussion below – but is less important in practice) imagine the Ising model at a small temperature, with an external field whose intensity is even smaller (than the temperature). It is a completely erroneous idea to construct the abstract Pirogov – Sinai model only around the true ground state in such a situation! The other, "slightly unstable" local ground state *must* be also included into Q; otherwise the constant in the Peierls condition (1.8) would be too poor (for contours marking large droplets of the slightly instable state) to allow, except of extremely small temperatures, the applicability of the Pirogov – Sinai method.
7. The collection Q must not be also too big. Namely, it can happen that with a too big reference set Q one also has difficulties with establishing (1.8). Here, we do not mean the possibility that we would not be able to check (1.8) even for those contours $\mathbf{\Gamma}^q$ with e_q roughly equal to e (this would be a real catastrophe; see the point 6 above) but we have in mind the case $e_q - e >> 1$. Then the remedy is easy (if the contour energy $E(\mathbf{\Gamma}^q)$ is not *very low*): We just remove these q from Q and include also the collection of all q – correct points (for such q) of any configuration x to the set $B(x)$ of its incorrect points. An example of such a situation is the Ising model with a very strong magnetic field.

Some Examples

Ising model.

This is the model where the notion of a contour was invented (for the ferromagnetic case; by Peierls, 1936). Notice that the usual definition of an Ising contour in the ferromagnetic case (as a path in the dual lattice) is *not* identical to the definition of a contour given above, when applied to the Ising model ($r = 2$). However, the interpretation of the usual "contour model" (constructed for the Ising model) as an abstract Pirogov – Sinai model is of course possible, if the notions like "support of a contour" resp. the property "contours do not touch each other" are defined properly.[7] It is probably well known to any reader of this text which was already

[7] There are several ways how to do that; for example the connectivity "over the corners" may be or may be not considered etc.

exposed to some presentation of the classical Peierls argument that for the Ising model, the Peierls condition is valid for sufficiently small temperatures. In fact, the energy $E(\mathbf{\Gamma})$ is proportional to the *length* of the contour $\mathbf{\Gamma}$ in this case.[8]

Notice that any finite range *perturbation* of the ordinary Ising model can be transcripted to a suitable abstract Pirogov – Sinai model. If the perturbation is sufficiently weak then the validity of the Peierls condition will not be destroyed.

Blume Capel Models.

This example consists of a variety of models whose properties depend on the choice of the numerical constants in the sequel. We conside the three spin case, the spin space being denoted by $S = \{-, 0, +\}$. The interactions are again the nearest neighbor ones, and they are given as follows :

$$\Phi_{\{t,s\}}(\pm, \pm) = \Phi_{\{t,s\}}(0,0) = 0,\ \Phi_{\{t,s\}}(\pm, \mp) = \alpha > 0,\ \Phi_{\{t,s\}}(\pm, 0) = \alpha_{\pm} > 0.$$

The one spin interaction ("external field") is

$$\Phi_t(q) = a_q \quad \text{where } q \in S.$$

i) For $a_{\pm} = 0$ and $a_0 > 0$, this is a simplest discrete version of the "two wells model" (which appeared first in the euclidean field theory): the reasonable choice of the set Q is then (for a_0 not too small and for sufficiently small temperatures) $Q = \{+, -\}$.[9] We have

$$e_+ = e_- = 0; \quad (\&\ \text{possibly} \quad e_0 = a_0)$$

and (the reader is kindly asked to check this inequality)

$$E(\mathbf{\Gamma}) > C|\operatorname{supp} \mathbf{\Gamma}|$$

where $C = \min\{a_0, \frac{\alpha}{2}, \frac{\alpha_{\pm}}{2}\}$ (resp. $C = \min\{\frac{\alpha}{2}, \frac{\alpha_{\pm}}{2}\}$ if 0 is not included to Q).

ii) Rather trivial case is obtained for $a_0 = 0$ if $a_{\pm} > 0$ are large. Taking $Q = \{0\}$ we obtain an object called a "polymer model", where compatibility of contours means just the compatibility of supports and no "outside colours" of the contours have to be discussed.[10]

iii) The case when all the quantities a_0, a_+, a_- are small (compared to $\alpha_{\pm}$, α) is the most interesting one. It was first treated by Bricmont and Slawny [BS]. Then it is natural to take $Q = S$ and for any sufficiently large $\alpha, \alpha_{\pm}$ (sufficiently large is meant after being divided by the temperature; this remark should be applied everywhere in the following examples!) one can adjust $a_{\pm}, a$ near zero such that the coexistence of *all three phases* takes place.

Consider, to be more specific, the case $1 << \alpha_+ = \alpha_- << \alpha$. Then the 0 –th phase has obviously a "more freedom" to make smallest (one point) perturbations.

[8] Of course, the *antiferromagnetic* case can be also treated by contour methods.

[9] If a_0 is small then it may be advisable to include 0 into Q. See the point iii).

[10] Similar situation is obtained for the Ising model with a strong magnetic field. Then it is advisable to take $Q = \{\text{sign } h\}$.

More specifically these perturbations (from 0 to $\pm$) "cost the same" as 0 perturbations of the $+$ resp. $-$ regime[11] but (because of the double possibility of the choice $\pm$ of the sign) the entropy of these perturbations in the 0 regime is almost twice as big as the entropy of the 0 perturbations of the $+$ resp. $-$ regime.

This, of course, should be compensated (to keep the coexistence of all three phases) by requiring that a_0 is suitably (slightly!) greater than $a_+ = a_-$.

Finally, in the very special case when all the quantities $\alpha, \alpha_\pm$ are the same then also the quantities $a, a_\pm$ should be the same to obtain the phase coexistence. The model is symmetric in that case.

Polymer models.

The class of so called polymer models is a very special case of the abstract Pirogov – Sinai model, for Q containing *one* element only. Of course there are no phase transitions in such a model – but the question of constructing the *cluster expansions* for these models is very important for us – as we will see below.[12]

Below, in Lecture 2, we will assume some familiarity with the cluster expansion theory. However, later in Lecture 3 we will present another, more general, self-contained approach to the Pirogov – Sinai theory which *incorporates* a variant of the usual cluster construction methodology (presented here in a spirit of the recent reference [DC]). When applied to polymer models, our method of Lecture 3 will be just a variant of such a cluster expansion method.

Other examples.

Among many other examples (see the literature on the subject) of models where it is advisable to use the Pirogov – Sinai theory we mention here the *Potts model* and the *"double well" models* of the quantum field theory.[13]

Our next strategy.

The exposition given so far in Lecture 1 will be in the sequel developed in *two separate* (and more or less independent) *ways*. The first one (presented in the forthcoming Lecture 2) will give a brief explanation of the essence of the *original Pirogov – Sinai approach* in the situation where "all the possible phases coexist". By the last statement one means the situation where for each $q \in Q$ there exists a "q – like phase" characterized by the following property: Almost all its configurations have the following structure: their spins have "mostly the value q" and the "islands" formed by contours of the given configuration are relatively "rare" (though they are distributed with a uniform density throughout the whole lattice).

Namely, this is the case where the original Pirogov – Sinai idea of constructing the so called contour functional (and the contour model) is seen in a *most simple and most characteristic* situation.

[11] We omit, in this approximate discussion, the energetically more costly perturbations going from $+$ to $-$ and vice versa.

[12] We ignore here completely the fact that polymer models are important also in the study of *high temperature* situations.

[13] On a discrete spin space; the continuous spin case is more technical and will be mentioned briefly later, in Lecture 5.

We will not follow here the other original Pirogov – Sinai constructions like the parametric contour models. Instead, we will rather apply, *starting from Lecture 3*, quite a *different* (and arguably, a stronger one) approach based on [Z] and [HZ].

Second Lecture. The Coexistence of All Phases. The Contour Model

Notations. Given a contour $\mathbf{\Gamma}$ with a support $\operatorname{supp}\mathbf{\Gamma}$ denote by $\operatorname{ext}(\mathbf{\Gamma})$ resp. $\operatorname{inn}(\mathbf{\Gamma})$ the only infinite component of the set $(\operatorname{supp}\mathbf{\Gamma})^c$ resp. the collection of all finite components of $(\operatorname{supp}\mathbf{\Gamma})^c$. Denote by

$$V(\mathbf{\Gamma}) = (\operatorname{ext}\mathbf{\Gamma})^c = \operatorname{supp}\mathbf{\Gamma} \cup \operatorname{inn}\mathbf{\Gamma}. \tag{2.1}$$

Contour partition functions. Denote by $\mathbb{X}_{\mathbf{\Gamma}}$ the collections of all configurations $y \in \mathbb{X}$ which satisfy the property that $\mathbf{\Gamma}$ belongs to the collection of all their contours and moreover all the other contours of y are *inside* $V(\mathbf{\Gamma})$. Let q be the external colour of $\mathbf{\Gamma} = \mathbf{\Gamma}^q$. For any $x \in \mathbb{X}_{\mathbf{\Gamma}}$ and any volume $V \supset V(\mathbf{\Gamma})$ take the hamiltonian $H(x_V | x^q_{V^c})$ where q denotes the external colour of $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ and the hamiltonian is given, as always in the following, by (1.5). Notice that we have the value $e|\operatorname{supp}\mathbf{\Gamma}|$ "below" $\operatorname{supp}\mathbf{\Gamma}$ (in addition to $E(\mathbf{\Gamma})$). Take the corresponding partition function

$$Z(\mathbf{\Gamma}) = Z^q(\mathbf{\Gamma}) = \sum_{x \in \mathbb{X}_{\mathbf{\Gamma}}} \exp(-H(x_V | x^q_{V^c}). \tag{2.2}$$

This partition function is called "crystallic" in the original Pirogov – Sinai notation.

The following is really the *central notion of the Pirogov – Sinai theory*. However, the definition given below causes no difficulties only in the "coexistence regime" we are studying below; other aspects of this notion will be discussed later, in Lecture 3.

The contour functional. Define the quantity, called the contour functional of $\mathbf{\Gamma}$

$$F(\mathbf{\Gamma}) = \log Z^q(\operatorname{inn}\mathbf{\Gamma}) - \log Z^q(\mathbf{\Gamma}) - e_q|\operatorname{supp}\mathbf{\Gamma}|. \tag{2.3}$$

The following statement follows immediately from the definition of $F(\mathbf{\Gamma})$ and from the *additivity* of the hamiltonian (1.5) as a function of a volume Λ:

Proposition. *The quantity $F(\mathbf{\Gamma})$ can be expressed as follows (recall (1.4)):*

$$F(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) + \log Z^q(\operatorname{inn}\mathbf{\Gamma}) - \sum_{q' \in Q} \log Z^{q'}(\operatorname{inn}_{q'}\mathbf{\Gamma}) \tag{2.4}$$

where $\operatorname{inn}_{q'}\mathbf{\Gamma}$ denotes the union of the components of $\operatorname{inn}\mathbf{\Gamma}$ *which "have the colour q'" and the last sum is over all $q' \in Q$ (including q).*

Notes. 0. The value(s) of q' for which $\operatorname{inn}_{q'}\mathbf{\Gamma}$ is nonvoid will be called the interior colour(s) of $\mathbf{\Gamma}$. Typically, $\operatorname{inn}_{q'}\mathbf{\Gamma}$ is nonvoid for only *one* $q' \in Q$. If, in such a case,

$q' = q$ then we will of course have the simple relation $F(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma})$.
1. In general, a first approximation to $F(\mathbf{\Gamma})$, $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ is the quantity

$$\tilde{F}(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) + \sum_{q' \in Q : q' \neq q} (e_{q'} - e_q) |\operatorname{inn}_{q'} \mathbf{\Gamma}|. \tag{2.5}$$

In fact, this is quite an accurate approximation of $F(\mathbf{\Gamma})$ for typical[14], not too big contours; it is even *exact* for contours not exceeding some limited size (such that there is *no* place for other contours in the volume $\operatorname{inn} \mathbf{\Gamma}$).
2. One could call the quantity $F(\mathbf{\Gamma})$ as the "work needed to install the contour $\mathbf{\Gamma}$".

The symmetric case.

Assume that we have some group of transformations G acting on Q in a transitive way i.e. such that any element of Q can be mapped by a suitable transformation from G to a selected element of Q. Assume that G acts also, in some way (which is in accordance to the already determined images of the outside and inside colours of $\mathbf{\Gamma}$), on the family of all contours of the model. (Notice that we formulate here the symmetry of our hamiltonian in a more general setting, in terms of the abstract Pirogov – Sinai model.) Assume that both e_q and $E(\mathbf{\Gamma})$ are *invariant* with respect to the action of G. Then also the partition functions $Z^q(\Lambda)$ do not change after applying the group actions from G to them and we have again (compare Note 0 above) the simple relation

$$F(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) = E(\mathbf{\Gamma}). \tag{2.4 S}$$

This includes the Ising ferromagnetic case with zero magnetic field and also some of the other popular examples – like the symmetric Blume Capel model (where $+, -, 0$ play the same role), Potts model etc. Of course, in such special cases the whole Pirogov – Sinai theory is reduced just to an ordinary "Peierls argument" which can be applied whenever the *Peierls condition* holds, with a sufficiently large τ. Our emphasis will *not* be on such (trivial from the point of view of development of the general Pirogov – Sinai theory) cases.

The contour model.

Given $q \in Q$ consider the polymer model formed by the contours $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ (having the external colour q) and the activities

$$k_{\mathbf{\Gamma}} = \exp(-F(\mathbf{\Gamma})) \tag{2.6}$$

where F is some "abstract functional of contours" (not necessarily always defined *exactly* by (2.3), see below).

We recall that configurations of such a polymer model are defined as arbitrary collections of contours $\{\mathbf{\Gamma}_i\}$ such that the sets $\operatorname{supp} \mathbf{\Gamma}_i$ do not touch each other i.e.

[14] By a "typical" contour we mean here a contour which is "not dangerously large" – in the sense that the volume term in the right hand side of (2.4) could not lower $E^q(\mathbf{\Gamma})$ substantially. We will see that for the contours which *are* large in such a sense (these will "mark the jumps to the droplets of other, more stable phases"), the quantities $F(\mathbf{\Gamma})$ will *not* be introduced as straightforwardly as above. These "residual contours" must be treated in a more cautious way.

the relation $\operatorname{dist}(\operatorname{supp}\mathbf{\Gamma}_i, \operatorname{supp}\mathbf{\Gamma}_j) > 1$ is fulfilled for any pair $i \neq j$, and the weight of any polymer configuration $\{\mathbf{\Gamma}_i\}$ is defined as the product

$$\prod_i k_{\mathbf{\Gamma}_i}. \tag{2.7}$$

We notice that we are studying here and everywhere (most notable exception will be the section dealing with general polymer models) mostly the translation invariant "hamiltonians" $k_{\mathbf{\Gamma}}$ i.e. the situation when the weights (2.6) are invariant with respect to shifts in $\mathbb{Z}^\nu$.

The polymer model with the weights (2.6), defined for all possible configurations of q –contours will be called below the q –th *contour model* corresponding to the hamiltonian (1.5). (This is in accordance with the terminology of [S]; at least in the coexistence regime.)

Note. This is, more or less, a formal algebraic object up to now. However, to obtain some genuine control over the behaviour of such models (in particular, to obtain some really useful information from the algebraic relations (2.10) below) one needs some information about the behaviour of the activities (2.6) – which will be available only later.

Thus, one should have in mind that the following relations hold *universally* but a really useful information can be extracted from them only in special cases studied below:

Proposition (Equivalence of ensembles). *Denote by Z^q_Λ the partition function*

$$Z^q_\Lambda = \sum_{\{\mathbf{\Gamma}_i\}} \prod_i k_{\mathbf{\Gamma}^q_i} \tag{2.8}$$

where the sum is over all admissible collections of polymers (i.e. q–contours) $\{\mathbf{\Gamma}^q_i\}$ and the right hand side is from (2.6); $F(\mathbf{\Gamma})$ being given by (2.3). Then

$$Z^q(\Lambda) = \exp(-e_q|\Lambda|)Z^q_\Lambda. \tag{2.9}$$

The *proof* is done by the induction over the *size* of Λ: assuming already the validity[15] of (2.9) for the interiors of contours appearing in Λ one can write the partition function $Z^q(\Lambda)$ as the sum (over all collections of external contours $\{\mathbf{\Gamma}_i\}$) of the products :

$$Z^q(\Lambda) = \sum_{\{\mathbf{\Gamma}_i\}} \exp(-e_q|\operatorname{ext}|) \prod_i Z(\mathbf{\Gamma}_i)$$

which is equal to (the products below are over all $\mathbf{\Gamma}_i$ and all $q' \in Q$)

$$\sum_{\{\mathbf{\Gamma}_i\}} \exp(-e_q|\operatorname{ext}|) \prod_i \exp(-E_q(\mathbf{\Gamma}_i) - e_q|\operatorname{supp}\mathbf{\Gamma}_i|) \prod_{q'} \prod_i Z^{q'}(\operatorname{inn}_{q'}\mathbf{\Gamma}_i)$$

and this last expresion is equal (after inserting (2.4)) to

$$\sum_{\{\mathbf{\Gamma}_i\}} \exp(-e_q|\operatorname{ext}|) \prod_i \exp(-F(\mathbf{\Gamma}_i) - e_q|\operatorname{supp}\mathbf{\Gamma}_i|) Z^q(\operatorname{inn}\mathbf{\Gamma}_i) = \exp(-e_q|\Lambda|)Z^q_\Lambda \tag{2.10}$$

by the induction assumption for $Z^q(\operatorname{inn}\mathbf{\Gamma}_i)$.

[15] The validity of (2.9) for very small volumes (not containing any contour) is trivial.

Corollary (Probabilities of external events). *The probability of any event of the type "* $\mathbf{\Gamma}$ *is an external contour of a configuration in* Λ*" is the same both in the diluted ensemble with the partition function* $Z^q(\Lambda)$ *as well as in the contour ensemble with the partition function* Z^q_Λ.

Note. This concept, namely the description of an "external behaviour of a configuration" is quite characteristic for the Pirogov – Sinai theory. The information about the behaviour of the given model (1.5) *inside* of the contours is, of course, *not* directly available from the corresponding contour model. However, knowing the probabilities of external contours one can compute the probabilities of the events inside just by computing the conditional Gibbs distributions.

Now, the question is what contour models offer a reasonable (for $\Lambda \to \mathbb{Z}^\nu$) "external behaviour". This, of course, depends on the values $F(\mathbf{\Gamma})$.[16]

A standard condition assuring the "stability of q" is the following Peierls type condition for the contour functional. In fact, the stability of q *requires* such a condition; this will be clarified later.

Peierls type assumption for F. Below we are studying only the situations where there is a $\tilde{\tau} >> 1$ such that for *any* contour $\mathbf{\Gamma}$ which can appear in the considered ensembles,

$$F(\mathbf{\Gamma}) > \tilde{\tau}|\operatorname{supp}\mathbf{\Gamma}|. \tag{2.11}$$

Namely, we will see that $F(\mathbf{\Gamma}) = F(\mathbf{\Gamma}^q)$ may *drop* almost to zero if q is "unstable" i.e. if the corresponding "q –th phase" does not exist. However, this is not the behaviour to be studied here, in Lecture 2:

The strategy of solving (2.4) in the coexistence case

The original Pirogov – Sinai strategy of solving this particular situation (all q "stable") can be now formulated as follows: If the hamiltonian $H = H(\lambda)$ in (1.5) depends on some n parameters $\lambda \in \mathbb{R}^n$ (in general one needs at least $n \geq |Q| - 1$ parameters to formulate the theorem below), adjust the parameters λ such that all q will become "stable" (in the above mentioned sense).

This will lead to some nonlinear integral equations for $F(\mathbf{\Gamma})$ – and we will look for the parameters λ for which these equations have a solution satisfying (2.11).

Note. Below, we are using some rudiments of the cluster expansion theory. The reader not acquainted with this theory can find more details in Lecture 4, where an independent exposition of the elements of this theory will be given.

In what follows (including Polymer Lemma below) we mean by a contour $\mathbf{\Gamma}$ a contour $\mathbf{\Gamma}^q$ with a *fixed* "colour" q.

[16] In Lecture 2, we are interested only in the behaviour which will be later called the "stability of all $q \in Q$"; in other words we assume that for any $q \in Q$ the corresponding "q –th phase" will exist; and, in any volume, a typical configuration of this phase will look like a "sea" of the values q with relatively rare "islands of perturbations", marked by the external contours of the given configuration.

Definition of a cluster (of contours). A cluster $\mathcal{T}$ of contours is an integer valued function $\phi : \mathcal{G} \to \mathbb{N}$ (with a connected support; see below) defined on the collection $\mathcal{G}$ of all contours satisfying the following condition: Denote by $\operatorname{Supp}\mathcal{T}$ resp. $\operatorname{supp}\mathcal{T}$ the collection resp. union of the supports of all contours of $\mathcal{G}$ for which $\phi(\mathbf{\Gamma}) > 0$. The *collection of sets* $\operatorname{Supp}\mathcal{T}$ will be called the "Support" of $\mathcal{T}$. By saying that $\mathcal{T}$ is a cluster we will mean that the support $\operatorname{supp}\mathcal{T}$ is *connected*.

The notion of a cluster and of its support can be defined also *recursively* as follows: i) any contour is a cluster ii) if $\mathcal{T}_i$, $i > 0$ are clusters and $\mathbf{\Gamma}$ is a contour such that $\operatorname{dist}(\operatorname{supp}\mathbf{\Gamma}, \operatorname{supp}\mathcal{T}_i) \le 1$ then the collection $\mathcal{T} = \mathbf{\Gamma} + \{\mathcal{T}_i\}$ is again a cluster; the $+$ operation should be understood in the sense of the indicators ϕ above. (If we replace contours by their *supports* we obtain above a recursive definition of the collection $\operatorname{Supp}\mathcal{T}$.)

The *cardinality* $|\mathcal{T}|$ of a cluster $\mathcal{T}$ represented by a function ϕ will now be defined as

$$|\mathcal{T}| = \sum_{\mathbf{\Gamma}\in\mathcal{T}} \phi(\mathbf{\Gamma})|\operatorname{supp}\mathbf{\Gamma}| \tag{2.12}$$

(or, recursively, as $|\operatorname{supp}\mathbf{\Gamma}| + \sum_i |\mathcal{T}_i|$).

Thus, one can interpret clusters as some "connected conglomerates of contours" i.e. some collections of contours[17] which are *indecomposable* into two subcollections – such that any contour of one subcollection would be distant from any contour of the other subcollection.

Polymer Lemma. *If the activities of the polymer model ("polymers" are just contours here and we denote them by symbols $\mathbf{\Gamma}$) satisfy assumptions of the type (with small ε)*

$$|\sum_{\mathbf{\Gamma}:\operatorname{supp}\mathbf{\Gamma}=T} k_{\mathbf{\Gamma}}| \le \varepsilon^{|T|} \tag{2.13}$$

then the polymer partition function $Z_\Lambda = \sum_{\{\mathbf{\Gamma}_i\}}\prod_i k_{\mathbf{\Gamma}_i}$ *can be expanded as the sum over clusters*

$$\log Z_\Lambda = \sum_{\mathcal{T}\subset\Lambda} \alpha_{\mathcal{T}} k_{\mathcal{T}} \tag{2.14}$$

where $\alpha_{\mathcal{T}}$ are some "combinatorial coefficients" satisfying a bound

$$|\alpha_{\mathcal{T}}| \le C^{|\mathcal{T}|} \tag{2.15}$$

for a suitable constant C depending only on the dimension ν. The quantities $k_{\mathcal{T}}$ are given by formulas

$$k_{\mathcal{T}} = k_{\mathbf{\Gamma}}\prod_i k_{\mathcal{T}_i} \quad \textit{resp.} \quad = \prod_{\mathbf{\Gamma}:\phi(\mathbf{\Gamma})>0} k_{\mathbf{\Gamma}}^{\phi(\mathbf{\Gamma})}$$

[17] Recall once again that contours in the cluster are not required to be mutually different; ϕ is in general an integer valued function having possibly also values > 1 i.e. multiple copies of the same $\mathbf{\Gamma}$ are allowed.

and therefore satisfy the estimates

$$|\sum k_{\mathcal{T}}| \leq (\varepsilon)^{|\mathcal{T}|} \tag{2.16}$$

where the summation is over all clusters with the same collection $\operatorname{Supp}\mathcal{T}$ *(of the supports of elements of* $\mathcal{T}$*). The sums in (2.14) are quickly convergent:*

$$\sum_{\mathcal{T}:\operatorname{supp}\mathcal{T}\supset S} |\alpha_{\mathcal{T}} k_{\mathcal{T}}| \leq (\varepsilon')^{|S|} \tag{2.16S}$$

with another small ε'. *Define the quantity* s_t *(the "density of free energy of the polymer model at the point* t*") as*

$$s_t = \sum_{\mathcal{T}:t\in\operatorname{supp}\mathcal{T}} |\operatorname{supp}\mathcal{T}|^{-1} k_{\mathcal{T}}. \tag{2.17}$$

Then the following important approximation for $\log Z_\Lambda$ *is obtained. Denote by* $\Delta(\Lambda)$ *the quantity defined by the equation*

$$\Delta(\Lambda) = \log Z_\Lambda - \sum_{t\in\Lambda} s_t. \tag{2.18}$$

Then $\Delta(\Lambda)$ *satisfies the bound, with another small* ε''

$$|\Delta(\Lambda)| \leq \varepsilon''|\partial\Lambda|. \tag{2.19}$$

If we moreover write the activities $k_{\mathbf{\Gamma}}$ *as* $k_{\mathbf{\Gamma}} = \exp(-F(\mathbf{\Gamma}))$ *and define the norm*

$$||F|| = \sup|F(\mathbf{\Gamma})(V(\mathbf{\Gamma}))^{-1}|$$

then all the quantities $s_t = s_t(F)$ *and* $\Delta(\Lambda) = \Delta(\Lambda, F)$ *are Lipschitz functions of* F*:*

$$|s_t(F+F') - s_t(F)| \leq \varepsilon||F'|| \ ; \ |\Delta(\Lambda, F) - \Delta(\Lambda, F+F')| \leq \varepsilon'||F'|| \tag{2.20}$$

assuming that both F *and* $F + F'$ *remain in the regime (2.11).*

Results of the type (2.14) – (2.16) are proven in almost any text on cluster expansions. (We will give an independent proof later, in Lecture 4.) The relation (2.19) is an easy consequence of the smallness of the terms $k_{\mathcal{T}}$ & of the quick convergence of the series (2.16). An analogous argument can be used for the sums of the *derivatives* of $k_{\mathcal{T}}$ with respect to F; this yields (2.20).

Notice that for a translation invariant model, all the quantities s_t and Δ are also translation invariant. For a q – contour model, we will write $s_t \equiv s$ more precisely as s^q in the following.

The investigation of (2.4) continued.

Let us conclude our investigation of the equation (2.4): ¿From now on we are again considering only the translation invariant case (the setting of Polymer Lemma was more general, also for translation noninvariant models!), and we are here also *assuming (2.11) to hold* all the time. Put

$$h^q = e^q - s^q. \tag{2.21}$$

Write (2.4) as follows: Using (2.10) for the expression of the partition functions on the right hand side of (2.4), using the expression (2.18) from the above lemma, and *forgetting the constant terms* $(h^q - h^{q'})\ |\operatorname{inn}_{q'} \mathbf{\Gamma}|$ we can rewrite the equation (2.4) as follows:

$$F_{\text{formal}}(\mathbf{\Gamma}) = E_q(\mathbf{\Gamma}) + \tilde{\Delta}(\mathbf{\Gamma}) \tag{2.22}$$

where, for $\mathbf{\Gamma} = \mathbf{\Gamma}^q$, $\tilde{\Delta}(\mathbf{\Gamma}) = \tilde{\Delta}(\mathbf{\Gamma}, F_{\text{formal}})$ is given as

$$\tilde{\Delta}(\mathbf{\Gamma}) = \Delta_q(\operatorname{inn} \mathbf{\Gamma}) - \sum_{q'} \Delta_{q'}(\operatorname{inn}_{q'} \mathbf{\Gamma}) \tag{2.23}$$

and where $\Delta_q(\operatorname{inn}\mathbf{\Gamma})$ is from (2.18). We consider F, written here more precisely as F_{formal}, as an "independent variable" not defined by (2.3),(2.4) but instead of it satisfying the integral equation (2.22). The quantity $\Delta(\mathbf{\Gamma})$ is then computed by (2.23)) from F_{formal}.

We stress that our forgetting of the (generally, of course, the most important!) terms $(h^q - h^{q'})\ |\operatorname{inn}_{q'} \mathbf{\Gamma}|$ above means that we are willing to give the interpretation to the results obtained below *only* in the case when *all* $h^q, q \in Q$ *are the same.*

Then, (2.22) can be really tracted not only as an integral equation for the "unknown functional" F_{formal} – which can be solved (iteratively) by the Banach fixed point theorem – but also as an equation from which the "physical value" of F, determined by (2.3) and written more precisely as $F = F_{\text{phys}}$ can be computed: Namely, we solve for any hamiltonian $H = H(\lambda)$ (from (1.5)) the equation (2.22); then we *compute the quantities* $\{h^q\}$ from the values F_{formal} and finally we *solve* the equation

$$h^q(\lambda) = h^{q'}(\lambda) \text{ whenever } \ q \neq q' \tag{2.24}$$

where the quantities h^q depend on λ through the functional F_{formal}.

Only those solutions of (2.22) which satisfy (2.24)[18] have a physical sense; these solutions correspond to the situations where all the q – like phases, $q \in Q$ coexist.

Let us conclude this discussion by some theorem. It can be proven by using the implicit function theorem and by establishing suitable differentiability properties of the mapping $\{H \ \to \ \{h^q\}\}$. (In particular, we stress the fact that s^q is a *slowly changing* function of the variable F in the norm $||F||$.)

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[18] It is rather straightforward to see (by induction over the volume) that the solution F_{formal} of (2.22) satisfying (2.24) must be equal also to the "physical" value of the functional F_{formal} given by (2.3) or (2.4). This would be, of course, no more valid if (2.24) were violated.

Theorem. *Let the hamiltonian $H = H^{(\lambda,\mu)}$ in (1.5) depend on some vector parameters (λ, μ) (one may or may not include the temperature into these parameters) where $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^n$. Assume that H is continuously differentiable (resp. smooth, analytic) in these parameters around the value $\lambda = 0, \mu = 0$ and assume that the hamiltonian $H^{(0,0)}$ has exactly $n+1$ ground states. Let the matrix of the partial derivatives of $\{e_q^{(\lambda,\mu)}\}$ around $\lambda = 0, \mu = 0$ satisfy the property that when completed by the column "1" (all entries in the column are 1), it has the rank $n+1$. Then, for a sufficiently small temperature, there is a continuously differentiable (smooth, analytic) mapping*

$$\{ \ \mu \to \lambda(\mu) \ \} \tag{2.25}$$

such that for each μ from some neighborhood of zero and for each $q \in Q$, all the values $h^q = h^q(F_{\text{formal}})$ are the same and therefore, $F_{\text{formal}} = F_{\text{phys}}$ and for each $q \in Q$ there is a "q – like" Gibbs state $P_q^{(\lambda(\mu),\mu)}$ of the hamiltonian $H^{(\lambda(\mu),\mu)}$, having the support in the set X^q of all q – diluted configurations.

Third lecture. The General Phase Picture

This lecture explains the core of the *general* Pirogov – Sinai theory. We will use essentially the approach of [Z] (however, with some important new modifications, made in the spirit of the paper [HZ]).

Note. We should warn the reader that no phase diagrams will be explicitly constructed and *no* analogy of the theorem above will be formulated here. This *can* be done, of course, similarly as the final theorem of the preceding lecture was more or less direct consequence of the construction of the quantities h^q . So, our emphasis will be again on the construction of (suitable variants of) the quantities h^q, however we are willing to do it now in the *general* case.[19]

We will see that from these quantities, more specifically from the mapping

$$\{ \ H(\lambda) \quad \to \quad \{h^q(\lambda)\} \ \}$$

(λ denotes the parameters on which the given hamiltonian H depends) everything important about the thermodynamics of the model can be computed, after all. The phase diagram of the model will be then just the specification (for any choice of the parameters λ in the hamiltonian) of all the $q \in Q$ for which h_q is *minimal* possible.

Concerning the "methodological" aspects of our approach, it will not be of much importance for us whether we are in a regime where at least *some* of the phases coexist or in the unicity regime. Maybe it sounds unusual to some readers (who may identify the Pirogov – Sinai theory as some strange method of construction of the phase diagram), but the essence of the Pirogov – Sinai theory is conserved even in the case when we are studying a *fixed hamiltonian*, even in the unicity regime.

While in Lecture 2 we have used essentially the original technique of [PS] (adapted to a nontrivial special case of the coexistence of the maximal number of phases; this case also "historically" arose as the first one in the course of the development

[19] These quantities will have the meaning of the free energy of some "metastable model".

of the paper [PS]) here we *depart* from it (and depart also from the more general notion of a parametric contour model – see [S]) and adapt the point of [Z] with some improvements adapted from [HZ]. We abandon here the notion of a contour model completely and replace it by another, more "physical" notion of a "metastable ensemble". What will *not* be abandoned is the crucial notion of a *contour functional* F. The latter notion will have the same meaning as before ($F_{\text{phys}} = F_{\text{formal}}$) in the regime of the coexistence of all phases; otherwise its meaning will be changed compared to [PS] (remaining, however, closer to its previous value F_{phys} from (2.3) than to the more formal contour functionals used in connection with the original Pirogov – Sinai parametric contour models).

Some intuitive background.

We are considering the abstract Pirogov – Sinai model everywhere in the following. That is, we have a general hamiltonian of the type (1.5), with the Peierls condition (1.8) resp. (1.9) being satisfied.

Recall for a moment the definition of a contour model from Lecture 2. The basic problem which appears in the study of such a model – and which we simply *avoided* in lecture 2 by studying only the regime of the coexistence of the maximal possible number of phases – is that the contour functional $F = F_{\text{phys}}$ from (2.4) does *not* in general satisfy the Peierls condition $F(\mathbf{\Gamma}) \geq \tilde{\tau}|\operatorname{supp}\mathbf{\Gamma}|$ from (2.11).

In fact, we will see that $F_{\text{phys}}(\mathbf{\Gamma})$ may drop almost to zero in the cases when the contour $\mathbf{\Gamma}$ marks some "jump to the more stable phase inside". (This can happen only if the volume $V(\mathbf{\Gamma})$ is sufficiently large). This observation will lead us to the *decomposition* of the class of all contours into two subclasses[20] with very *different behaviour*: The contours from the first class (of "small contours") will contribute to the *entropy* of the corresponding (stable or metastable) "phase" while the other contours (called "large" in the following) will mark possible large droplets of the "more favorable" phases. These droplets should be *so large*, that the jump into the (more favorable) phase inside (which requires an excessive energy spending around the boundary of the droplet) is either "recommended" or at least "disputable" – if the balance of the total free energy loss/gain around $\mathbf{\Gamma}$ is measured.[21]

There is no *entropy* gain from the latter contours as we will see; more precisely the possible entropy gain of these contours would be much smaller than the energy excess due to the dwelling in the "less favorable regime" outside of these large contours. Thus, these large contours tend to be (if they appear at all) *as large as possible* and there will be typically *one* such large contour (at most) in a "normally looking" volume Λ (e.g. in a cube Λ) if the outside boundary condition is unstable.

On the other hand, the *small* contours will be relatively "rare" but nevertheless they *will* appear with a regular (nozero) density throughout the whole $\mathbb{Z}^\nu$, in any phase. So the contribution, to the free density, of these small contours will be nonzero. This contribution thus may play a decisive role in the result of the "energy entropy fighting of various possible phases of the given model" which determines

[20]Notice however, that the exact borderline between these two subfamilies will be defined in somehow arbitrary manner.

[21]Namely, the loss of the energy $E(\mathbf{\Gamma})$ around the boundary of the droplet may be compensated by the free energy gain resulting from the more favorable regime *inside* of $\mathbf{\Gamma}$.

the behaviour of the given system . Its detailed computation really forms the core of the Pirogov – Sinai theory.

Expansion ("recoloring") of contours. The basic ideas.

To outline the technical constructions used below and to understand better their meaning we will explain first the very idea of a *partial expansion* (and the meaning of the notion of "recolorability" used in it) on a *simplified caricature* of the models considered by us. Imagine, for example, a model where for any $q \in Q$ only *one possible shape* of a contour $\mathbf{\Gamma}^q$ is permitted i.e. all allowed contours are of the type

$$\mathbf{\Gamma}^q + t \ , \ t \in \mathbb{Z}^\nu$$

where $\mathbf{\Gamma}^q$ is a fixed contour. Assume for brevity that $\operatorname{inn} \mathbf{\Gamma}^q$ has only one component, having a colour $q' \in Q$. Expand the partition function Z^q in such a case: Write

$$Z^q(\Lambda) = \sum_{\{t_i\}} \exp(-e^q | \operatorname{exxt} | \prod_i \exp(-E^q(\mathbf{\Gamma}_i) \exp(-e^{q'} | \operatorname{inn} \mathbf{\Gamma}_i|$$

where the sum is over all collections of points $\{t_i\}$ such that the contours $\mathbf{\Gamma}_i = \mathbf{\Gamma}^q + t_i$ mutually do not touch and where we denote by $\operatorname{exxt} = \Lambda \setminus (\cup_i \operatorname{inn} \mathbf{\Gamma}_i)$. This can be expressed as

$$Z^q(\Lambda) = \sum_{\{t_i\}} \exp(-e^q|\Lambda|)| \prod_i \exp(-F(\mathbf{\Gamma}_i)$$

where, of course, the contour functional $F(\mathbf{\Gamma}^q)$ is defined here simply as

$$F(\mathbf{\Gamma}_i) = E^q(\mathbf{\Gamma}_i) + (e^q - e^{q'})| \operatorname{inn} \mathbf{\Gamma}_i|.$$

Assuming that F satisfies a Peierls type bound (compare Polymer Lemma, Lecture 2!)

$$\exp(-F(\mathbf{\Gamma})) \leq \exp(-\tilde{\tau}| \operatorname{supp} \mathbf{\Gamma}|)$$

we know already from this lemma that the above partition function can be written as

$$Z^q(\Lambda) = \exp(-e^q|\Lambda| + \sum_T k_T^q) \tag{3.0}$$

with quickly decaying cluster terms k_T^q. Thus, already in this simplified case we may foresee the main problems which we will have to tackle in the following:
1) A kind of Peierls condition for functionals $F(\mathbf{\Gamma})$ is strongly desirable here. Contours satisfying such a condition will be called "recolorable" in the following text (roughly speaking) and the shift from the original model to the formula (3.0) (which we presented above in an extremely simplified situation; of course) will be called the *recoloring* of (all the shifts) of $\mathbf{\Gamma}$.[22]
2) Our desire will be then to *repeat* the expansion leading to formulas (3.0) *as many*

[22]Notice that the transition from $E^q(\mathbf{\Gamma})$ to $F(\mathbf{\Gamma})$ can be really visualized as the "change of the colour (putting q instead of q') inside of the contour $\mathbf{\Gamma}$".

times as possible. It is useful to introduce here a kind of a "generalized Pirogov – Sinai model" *whose structure will not be changed* after applying such a "recoloring" procedure. So we introduce, in the following, the important general notion of a "mixed model" which will be defined as an abstract Pirogov – Sinai model having an "additional cluster external field" $\{k_T\}$ i.e. its hamiltonian will be defined by the following analogy of (1.4):

$$H(x_\Lambda | x^q_{\Lambda^c}) = \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) + e|\operatorname{supp}\mathbf{\Gamma}|) + \sum_q \sum_{t\in\Lambda_q} e_q - \sum_q \sum_{T\subset\Lambda} k^q_T$$

where the last sum is over all "clusters" T whose support belongs to the set Λ^q (denoting the collection of q –correct points of the given configuration $x_\Lambda = \{\mathbf{\Gamma}_i\}$).
3) Having such an "invariant" (with respect to the procedure outlined in 1)) notion of a "mixed model", it is natural to apply the recoloring procedure above so many times such that it becomes finally *inactive* in the sense that there is "nothing to recolor" in the mixed model obtained so far. The question is whether "nothing to recolor" means already that "there are no contours in the final mixed model". Namely, the latter would mean that a *total expansion* of the partition functions of the model was obtained.

The answer to the last question will be "yes, at least for some q" (these will be called stable q). Then no contours $\mathbf{\Gamma}^q$ will remain in the model and a total control over its behaviour under boundary condition q will be obtained.

The meaning of our main theorem will be, more precisely, the following. We will find a simple constructive (at least in principle) criterion how to determine whether given q is "stable". Namely, looking at the quantities

$$h^q = e^q - \sum_{T:\ 0\in T} k^q_T |\operatorname{supp} T|^{-1}$$

constructed for the final ("most expanded") mixed model we will have the statement that q is stable (in the sense suggested above) if and only if h^q has its *smallest possible* value! To prove this, we will have to investigate in more detail the relation between the notion of a "recolorability" of $\mathbf{\Gamma}$ and another (more transparent!) new notion of a "smallness" of $\mathbf{\Gamma}$: A contour $\mathbf{\Gamma}$ will be called small if we are "absolutely sure" that the difference

$$\sum_{q'\in Q} (\log Z^{q'}(\operatorname{inn}_{q'}\mathbf{\Gamma}) - \log Z^q(\operatorname{inn}_{q'}\mathbf{\Gamma}))$$

of partition functions on the right hand side of (2.3) "cannot substantially erode the energy $E(\mathbf{\Gamma})$". It is now quite a delicate technical task how to define this on a *technical level.*

The way chosen below by us turns out to be quite passable: Denote by

$$a^q = h^q - h\ ,\ \ h = \min_q h^q.$$

We will see that whenever $\square$ is a cube such that, say

$$a^q|\square| \le \tau \operatorname{diam}\square$$

(and such a condition will hold for squares $\Box$ of *any* size if q is stable !!!) then no substantial erosion of the type above can happen for contours $\mathbf{\Gamma}^q$ *inside* of $\Box$.
4) Notice that the notion of a smallness (and recolorability) would not work well if restricted to *single contours only*. Namely, the very idea of recoloring requires its application only to *interior* contours of the model:

Imagine the following concentric system of two contours: let the external contour of the system mark a jump of the configuration from a stable phase outside to a "very instable" one residing in the "middle belt", and let the interior contour "jumps again to same stable phase (e.g. the same as outside) in the center". Clearly, it will be impossible to recolor the interior contour itself because its contour functional may violate the Peierls condition. However, it is obviously possible to recolor both contours *at once* (assuming that no other contours are present in the belt we recolor).

This example becomes even more instructive if we generalize it in such a way that *several* interior (& mutually external) contours are inside of a given external contour. It is clear that to speak about a "connectedness" of such a system requires some care.
5) Having established the need for a "simultaneous recoloring" of all contours in some admissible interior subsystem (like above) we want to claim now that *all* small objects (not only single contours but also their admissible collections) $\mathbf{\Gamma}$ are recolorable.

To see the importance of such a statement notice that for q stable it says that *all* systems $\mathbf{\Gamma}^q$ would be recolorable i.e. *nonexistent* in the final mixed model! This is quite obvious for *single* contours but requires some supplementary argumentation if general small (admissible, interior) *systems* of contours are considered.

This problem is essentially "topological" in its nature, as we will see later (in the subsection "Tight sets").[23]
6) Of course, the quantities h^q used in the argumentation below cannot be defined so simply in a general "recoloring step" (which was outlined above only in its simplest version). A useful idea here is to assume that h^q are defined, in general, as free energies of some "metastable models" where "dangerous" ("nonsmall", "nonrecolorable"; the difference between the choices of these adjectives is not crucial for the intuitive understanding of this notion) contours are simply *excluded*.

Now we develop the ideas outlined above in a rigorous way: We start with a precise definition what the metastable model should be :

The notion of a submodel.

Throughout the rest of Lecture 3, we will consider always the hamiltonian (1.5). The corresponding diluted partition functions $Z^q(\Lambda)$ will be defined by (1.6).

[23] Our very formulation of the "recoloring procedure" also for the *systems of contours* will free us from the necessity to combine both the lower and *upper* bounds for for considered partition functions

$$\exp(-h^q|\Lambda| - \varepsilon|\partial\Lambda|) \leq Z^q(\Lambda) \leq \exp(-h|\Lambda| + \varepsilon|\partial\Lambda|).$$

These bounds were very important in some previous versions of the Pirogov – Sinai theory. Compare "Main Lemma" in [Z]. We are not using these bounds in the development of the Pirogov – Sinai theory now. However, we will formulate them later (in a stronger form than before!) as a *corollary* of our Main Theorem.

In the following we will use a concept of a *submodel* of the model (1.5) (on the configuration space $\mathbb{X}$). This is quite a general concept: the determination of the submodel will be done below simply by specifying the corresponding "subset of allowed configurations"

$$\mathcal{M} \subset \mathbb{X} \tag{3.1}$$

of the given submodel.

The submodel residing on such a configuration space $\mathcal{M}$ will be denoted often also by the same symbol $\mathcal{M}$.

Note. We continue to use the symbol $\mathbb{X}$ for the collection of *all* admissible (finite or infinite) collections of contours in $\mathbb{Z}^\nu$. By a "submodel" we will, however, mean in the following usually something *more specific* than what was mentioned above: We will consider below some special submodels called "metastable" ones. They will be indexed by elements $q \in Q$. Each of these submodels will be defined in terms of exclusion, from the family of all q diluted configurations, of configurations where some "dangerous systems of large contours" (namely systems marking large droplets of stable phases appearing inside of the given unstable regime) appear.

The diluted partition functions $Z^q_{\mathcal{M}}(\Lambda)$ *of the submodel* $\mathcal{M}$ will be defined analogously as in (1.6) but with the additional requirement that $x_\Lambda \in \mathcal{M}$.

Extend now also the notion of a contour functional F to a submodel $\mathcal{M}$ just by putting (compare (2.3) and (2.4))

$$F_{\mathcal{M}}(\mathcal{D}) = \log Z^q_{\mathcal{M}}(\operatorname{inn}\mathcal{D}) - \log Z^q_{\mathcal{M}}(\mathcal{D}) - e_q|\operatorname{supp}\mathcal{D}|$$

i.e.

$$F_{\mathcal{M}}(\mathcal{D}) = E_q(\mathcal{D}) + \log Z^q_{\mathcal{M}}(\operatorname{inn}\mathcal{D}) - \sum_{q' \in Q} \log Z^{q'}_{\mathcal{M}}(\operatorname{inn}_{q'}\mathcal{D}). \tag{3.2}$$

It will be technically very important to generalize this notion below, by the same prescription as in (3.2), to the case when we have, instead of a single contour $\mathbf{\Gamma}$, a *general admissible system* $\mathcal{D}$ *of contours*. (This is the reason why we already used the different symbol $\mathcal{D}$ – instead of $\mathbf{\Gamma}$ – here.) Working with such a general system of contours we will need a notion describing "how much connected the system is ":

Definition. For any set $T \subset \mathbb{Z}^\nu$ denote by $\operatorname{con} T$ the minimal possible cardinality of a connected set $\tilde{T}$ such that $\tilde{T} \supset T$. Write $\operatorname{con}\mathcal{D}$ instead of $\operatorname{con}\operatorname{supp}\mathcal{D}$ (where, for $\mathcal{D} = \{\mathbf{\Gamma}_i\}$, we take $\operatorname{supp}\mathcal{D} = \cup_i \operatorname{supp}\mathbf{\Gamma}_i$).

The following three mutually related notions are *crucial* in our approach. The notion of metastability appeared first in [Z] and here it keeps its intuitive meaning. However, technically we define it now in a different way, based after all (this we will see below) also on the idea of a partial expansion (of a metastable model). Such a partial expansion will be based here always on an algorithm called the "recoloring of a contour".

All these notions (and also the partial expansion of the model) will be constructed now rigorously, in an *inductive* way:

Recolorability, Residuality, Metastability.

Let us start with another two auxiliary topological notions: Say that an admissible subsystem $\mathbf{\Gamma}$ of an admissible system $\mathcal{D}$ is an *interior* one if

$$V(\mathbf{\Gamma}) \cap (\mathcal{D} \setminus \mathbf{\Gamma}) = \emptyset.$$

Say that an admissible subsystem $\mathbf{\Gamma}$ of an admissible system $\mathcal{D}$ is an *exterior* one if there is another admissible subsystem $\tilde{\mathbf{\Gamma}} \subset \mathcal{D}$ such that

$$V(\mathbf{\Gamma}) \cap V(\tilde{\mathbf{\Gamma}}) = \emptyset$$

and moreover $\mathcal{D} \setminus (\mathbf{\Gamma} \cup \tilde{\mathbf{\Gamma}})$ consists only of (one or more) interior subsystems (denote them by $\mathbf{\Gamma}_i$) of $\mathcal{D}$ which are inside of $\mathbf{\Gamma}$ i.e. which satisfy the condition $V(\mathbf{\Gamma}_i) \subset V(\mathbf{\Gamma})$.

Note. One should not take here the word "external" in a too narrow sense: Even a "concentric" system $\mathbf{\Gamma}$ of several contours can be an external one in $\mathcal{D}$, if $V(\mathbf{\Gamma}_i)$ do not intersect $\operatorname{supp} \mathbf{\Gamma}$.

Definition.
i) Say that an admissible system $\mathcal{D}$ of contours is *residual* if it has no recolorable (see the point iii) of this definition) interior subsystems.
ii) Say that an admissible system $\mathcal{D}$ is *metastable* if no residual (see the point i) of this definition) exterior subsystems of $\mathcal{D}$ exist.
iii) Say that $\mathcal{D}$ is *recolorable* if

$$F_{\mathcal{M}}(\mathcal{D}) \geq \frac{\tau}{12\nu} \operatorname{con} \mathcal{D} \tag{3.4}$$

and moreover this holds also for any other $\tilde{\mathcal{D}}$ with the same external colour and the same support : $\operatorname{supp} \tilde{\mathcal{D}} = \operatorname{supp} \mathcal{D}$.[24] Here, $\mathcal{M}$ denotes the *metastable* model defined as the subset of the original model consisting of *all metastable* (see the point ii) of this definition) configurations. (Do not care now about the particular choice of the constant namely the value $\frac{\tau}{12\nu}$ on the right hand side of (3.4). We will see later why namely this choice is convenient here.)

Agreement. In what follows, the subscript $\mathcal{M}$ at the quantity $F_{\mathcal{M}}$ will always denote the metastable model mentioned above. We will usually denote the value of $F_{\mathcal{M}}(\mathbf{\Gamma})$ as $F_{\text{meta}}(\mathbf{\Gamma})$.

The following theorem contains the core of Lecture 3 (and the core of our present approach to the Pirogov – Sinai theory in its simplest application to the models with a finite number $|Q|$ of constant local ground states, satisfying the Peierls condition). We precede it by introducing another important notions, closely related to the notion of a recolorability:

[24] Thus, recolorability od $\mathcal{D}$ is rather the property of the *set* $\operatorname{supp} \mathcal{D}$ (and $\operatorname{inn} \mathcal{D}$) and of the *external colour* of $\mathcal{D}$, not the property of a particular contour resp. admissible system $\mathcal{D}$!

Densities of free energy. The Concept of a "Small" Volume.

Denote by h_q the free energy, $h_q = \lim_{\Lambda\to\mathbb{Z}^\nu}(|\Lambda|)^{-1} Z^q_{\text{meta}}(\Lambda)$ of the metastable model $\mathcal{M}_q$ (consisting of all q diluted metastable configurations). The existence of the limit will be obvious below. Put

$$h = \min_{q\in Q} h_q \quad ; \; a_q = h_q - h. \tag{3.5}$$

Say that $q \in Q$ is *stable* if $a_q = 0$. Say that a cube $\square \subset \mathbb{Z}^\nu$ is q –small if

$$a_q|\square| < C\tau \operatorname{diam} \square. \tag{3.6}$$

The constant τ is from (2.11) and we can take here $C = 1$ (or, better, slightly bigger $C > 1$ – see (4.28)) for example.[25] Say that an admissible system $\mathcal{D} = \mathcal{D}^q$ with the external colour q is small if it can be "packed by some q – small cube $\square$" namely if $\square \supset \operatorname{supp} \mathcal{D}$.

The forthcoming definition recalls the definition of a cluster, given already in Lecture 2, in a slightly more general context needed here: Namely, the primitive objects from which clusters will be formed now will not be single contours but some special ("recolorable", see below) admissible *systems* of contours.

Clusters of sets resp. of (systems of) contours.

Say that the two collections $\mathcal{T} = \{\mathbf{\Gamma}_i\}$, $\mathcal{T}' = \{\mathbf{\Gamma}_j\}$ of contours resp. of admissible systems of contours are compatible if any two pairs $\mathbf{\Gamma}_i, \mathbf{\Gamma}_j$ are compatible in the sense that their supports *do not touch*. Say that a collection $\{\mathbf{\Gamma}_i\}$ of contours resp.of admissible systems is (in)decomposable if it is (im)possible to split it into two compatible parts. Indecomposable collections of contours or of admissible systems of contours will be called *clusters* and denoted below by symbols $\mathcal{T}$.[26]

Main Theorem.

(1) *The quantity h is the free energy (in the Van Hove sense) of the model (1.5): For any $q \in Q$ we have*

$$\lim_{\Lambda\to\mathbb{Z}^\nu} |\Lambda|^{-1} Z^q(\Lambda) = h. \tag{3.7}$$

However, for the metastable partition functions we have

$$\lim_{\Lambda\to\mathbb{Z}^\nu} |\Lambda|^{-1} Z^q_{\text{meta}}(\Lambda) = h_q. \tag{3.7 q}$$

(2) *Small subsystems $\mathcal{D} = \mathcal{D}^q$ can not be residual. In particular there are no residual systems $\mathcal{D}^q$ for q stable. In other words, for stable q we have $\mathcal{M}^q = \mathbb{X}^q$ where $\mathbb{X}^q$ denotes the collection of all q–diluted configurations from $\mathbb{X}$.*

[25] Do not inquire about the particular choice of the constant $C \approx 1$ here. Any constant C such that $C\tau >> 1$ would do the job.

[26] This is closely related to the already used, slightly less general notion of a cluster introduced in Lecture 2.

(3) *The metastable partition functions* $Z^q(\Lambda)$ *can be more precisely expressed as*

$$\log Z^q_{\mathcal{M}}(\Lambda) = -e_q|\Lambda| + \sum_{\mathcal{T}\subset\Lambda} \alpha_{\mathcal{T}} k^q_{\mathcal{T}} \tag{3.8}$$

where $k^q_{\mathcal{T}}$ *are products, over the contours (resp. admissible systems)* $\mathbf{\Gamma}^q_i$ *which are elements of* $\mathcal{T}$*, of the values* $\exp(-F(\mathbf{\Gamma}_i))$ *and* $\alpha_{\mathcal{T}}$ *are some combinatorial coefficients depending only on the "topology" of the cluster* $\mathcal{T}$*, such that*

$$\alpha_{\mathcal{T}} \le C^{\ \sum_{\mathbf{\Gamma}\in\mathcal{T}} \operatorname{con}\mathbf{\Gamma}} \tag{3.9}$$

for a suitable constant C *depending only on the dimension* ν *and also on the meaning of the statement "contours* $\mathbf{\Gamma}$ *and* $\mathbf{\Gamma}'$ *do not touch"*[27].
The cluster series (3.8) quickly converge, like in (2.16S). In analogy to (2.17), the quantities h_q *can be computed also from the formulas*

$$h_q = e_q - \sum_{\mathcal{T}:\ 0\in\mathcal{T}} |\operatorname{supp}\mathcal{T}|^{-1} k_{\mathcal{T}}. \tag{3.10}$$

Corollary. *The hamiltonian (1.5) restricted to* $\mathcal{M}^q = \mathbb{X}^q$ *gives a probability measure* $P^q_{\mathcal{M}}$ *which can be interpreted, by taking the inclusion* $\mathcal{M}^q \subset \mathbb{X}$*, as the "q – th" Gibbs state on* $\mathbb{X}$*.*
The above is true for each stable q*.*

Notes. 0. See also Corollary at the end of Lecture 4 for some additional information and interpretation.
1. Of course, the support of the injection of $P_{\mathcal{M}}$ into $\mathbb{X}$ has the support in $\mathbb{X}^q$ and thus the measures $P^q_{\mathcal{M}}$ are really disjoint.[28] Therefore it has a reasonable sense to say that almost any configuration of $P_{\mathcal{M}}$ is "externally equal to q".
2. The properties of the *phase diagram* of the model can be extracted from the mapping

$$\{\ H \quad\to\quad \{h_q\}\ \}. \tag{3.11}$$

One should have in mind that while for stable q the quantity $h_q = h$ has the *unique possible meaning*, for unstable q there *is* some arbitrariness of the definition of h_q (steming from some arbitrariness in the definition of a recolorable subsystem $\mathcal{D}$, and from the related arbitrariness in the notion of a metastability). Thus (for example), if we study the differentiability properties of the phase diagram i.e. if the smoothness (as good as possible) of the mapping (3.11) is required than it may be advantageous to modify (to "smoothen" as possible) the notion of a recolorability. See Lecture 5.

[27] If the latter is meant in the most common sense $\operatorname{dist}(\mathbf{\Gamma},\mathbf{\Gamma}') \ge 2$ then C can be taken something like $C = 2\nu$.

[28] A support of a probability is a Borel set whose complement has measure zero.

The proof of Main Theorem

An earlier variant of the theorem was proven in [Z], using still some variant of the original Pirogov – Sinai concept of a contour model (applied there, however, to the study of the corresponding metastable ensemble). Here, we will apply a more recent approach of [HZ] – which seems to be more powerful and at the same time (at least conceptionally) more simple. Namely, the *bounds* for partition functions will be now replaced by the exact *expansions*, whenever possible. (In fact, *no* bounds for partition functions will now be employed below.)

Let us make some "philosophical" remark about the role of contour models in the Pirogov – Sinai theory: Analyzing the simplest possible answer to the question "why it is necessary to introduce the contour models (i.e. some auxiliary gases of contours)?" the possible answer could be the following[29] one: Contour models are *good for expansions*!

Asking further "why expansions are so needed here, in Pirogov – Sinai theory"? the reasonable answer could be that there are, at present, *no* other means enabling to prove the bounds (absolutely crucial for the Pirogov – Sinai theory; this we have seen already in Lecture 1) of the type

$$|\log Z^q_{\mathcal{M}}(\Lambda) - h_q|\Lambda|| \leq \varepsilon|\partial\Lambda|.$$

The essence of our present approach is the following. In the sequence of reasonings

(metastable) partition functions → their equivalent expression by contour models → expansions of these partition functions → extraction of useful *estimates* from them[30] we will now omit the middle term, namely the very *construction of the contour model*. Instead of the notion of a contour model, the keywords of our approach will be the following:

1) The notion of a *partial expansion* of the model. This will be called as a *mixed model* below. (This idea is, of course, not new; however in our approach it is really the cornerstone of the theory.)

2) A procedure describing the transition from a given partially expanded model to another, "more expanded" model.

"More expanded" means here that some contours will be "removed" from the new, more expanded model (i.e. they will be "recolored" – in the language we are using below). Of course, such a simplification of the configuration space (configuration space of the given partially expanded model) must be accompanied by a suitable *redefinition* of some cluster expansion terms appearing in the new, more expanded model.

It should be not surprising that these new cluster expansion terms are employing the quantity already well known to us – namely the contour functional F (of the contours which are just recolored).

[29] There are, of course, also relevant "historical" arguments explaining how the notion of a contour model emerged: The actual line of development of these ideas was as follows: Peierls argument → Minlos – Sinai contour models → Pirogov – Sinai contour models (in the maximal coexistence regime) → general Pirogov – Sinai theory.

[30] This byproduct of the expansion theory is sometimes – misleadingly! – viewed as the only thing which is absolutely indispensable in the Pirogov – Sinai theory.

The concept of as mixed model is defined as a natural generalization of the abstract Pirogov – Sinai model (1.5). We just generalize the notion of an external field suitably: namely, in addition to the quantities $E(\mathbf{\Gamma})$ and e_t we just assume that another "cluster field" $\{k_T\}$ (sitting on connected clusters T and depending also on the underlying "colour" induced on T by the configuration $\{\mathbf{\Gamma}_i\}$) is given.

In the following, it will be useful to simplify the notion of a cluster variable by putting, whenever we have an expansion of the type (3.8),

$$k_T^q = \sum_{\mathcal{T}} \alpha_{\mathcal{T}} k_{\mathcal{T}}^q \tag{3.13}$$

where the summation is over all clusters $\mathcal{T} = \{\mathbf{\Gamma}_i^q\}$ with the same collection of supports $T = \operatorname{Supp} \mathcal{T} = \{\operatorname{supp} \mathbf{\Gamma}_i\}$. In such situations the notion of a cluster will have the meaning of a collection of *sets* whose "connectedness" will be meant in the sense that

$$|k_T^q| \le \varepsilon^{\operatorname{con} T}$$

where we denote by

$$\operatorname{con} T = \sum_{\operatorname{supp} \mathbf{\Gamma}_i \in \operatorname{Supp} \mathcal{T} = T} \operatorname{con} \mathbf{\Gamma}_i. \tag{3.14}$$

We define the *new hamiltonian* $H_{\mathcal{M}}$ of the mixed model as follows.

$$H_{\mathcal{M}}(x_\Lambda | x_{\Lambda^c}^q) = \sum_q \sum_{t \in \Lambda_q} e_q + \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) + e|\operatorname{supp} \mathbf{\Gamma}|) - \sum_{T: T \cap (\cup \operatorname{supp} \mathbf{\Gamma}) = \emptyset} k_T^{q(x)} \tag{3.15}$$

where $q(x)$ denotes the "colour"(from Q) induced by x on T and the quantities $k_T^q = k_T^q(\mathcal{M})$ are the "external fields" of the given mixed model $\mathcal{M}$. In the following, any mixed model considered by us will be just an *expansion of the metastable model constructed so far*. More precisely we will have from (3.8) (whose validity we are assuming here) and (3.14) the relation

$$\log Z_{\mathcal{M}}^q(\Lambda) = \sum_{T \subset \Lambda} k_T^q \ ; \ k_T^q = k_T^q(\mathcal{M}) \tag{3.16}$$

where $\mathcal{M}$ denotes the metastable model.

Then, if (3.15) is such an expansion of the metastable model (notice that different metastable models live in the regions with different $q(x)$ and the quantities k_T^q are functions of q !) in the volume $\Lambda = \operatorname{inn} \mathcal{D}$ where $\mathcal{D}$ is some (admissible system of) contour(s) we can rewrite our quantity (3.2) as follows :

$$F_{\mathcal{M}}(\mathcal{D}) = E_q(\mathcal{D}) - \sum_{q' \in Q} \sum_{T \subset \operatorname{inn}_{q'} \mathcal{D}} (k_T^q - k_T^{q'}) + \sum_{q' \in Q} (e_q - e_{q'}) |\operatorname{inn}_{q'} \mathcal{D}|. \tag{3.17}$$

Extend now the notion of a recolorability, residuality, metastability also to *any mixed model*[31]: Just write the mixed hamiltonian $H_{\mathcal{M}}$ instead of H everywhere in the definitions above. In particular, (3.4) now says with the help of (3.17) that $F_{\mathcal{M}}(CalD)$ is

$$E_q(\mathcal{D}) - \sum_{q'\in Q} \sum_{T\subset \operatorname{inn}_{q'} \mathcal{D}} (k_T^q - k_T^{q'}) + \sum_{q'\in Q} (e_q - e_{q'})|\operatorname{inn}_{q'} \mathcal{D}| \geq \frac{\tau}{12\nu} \operatorname{con} \mathcal{D} \tag{3.18}$$

if the given mixed model is already the expansion of the metastable model in the volume $\operatorname{inn} \mathcal{D}$.

In fact, below we will be using the notion of a recolorable subsystem of a mixed model only for such a special class of admissible systems, which are the "smallest possible" in the following sense:

Removable systems of the model.

Say that an admissible system $\mathcal{D}$ of a given abstract Pirogov – Sinai mixed model is *removable* if it is recolorable and moreover it satisfies the condition that *no* admissible recolorable system $\mathcal{D}'$ with a smaller volume $V(\mathcal{D}')$ (smaller is meant in the sense that $V(\mathcal{D}') \subsetneq V(\mathcal{D})$) already exists in this mixed model.

Note. Thus, like recolorability, removability is a property of the *set* $\operatorname{supp} \mathcal{D}$ and of *external colour* of $\mathcal{D}$, not the property of a particular admissible system $\mathcal{D}$. By a removal of $\mathcal{D}$ we will in fact mean the removal of all $\tilde{\mathcal{D}}$ with the same support and the same external colour *at once*; see below.

The following technical lemma is absolutely crucial for our approach.

Equivalency of mixed models. Say that two mixed models are equivalent if all diluted partition functions are the *same* for both models.

Recoloring Lemma. *Consider a mixed model $\mathcal{M}$ from (3.15). If $\mathcal{D}$ is removable then it is possible to define a new, equivalent mixed model $\mathcal{M}_{\text{new}}$ on a smaller configuration space, with all configurations containing a shift of some $\tilde{\mathcal{D}}$ such that* $\operatorname{supp} \tilde{\mathcal{D}} = \operatorname{supp} \mathcal{D}$ *being excluded, and (together with the "old" quantities $k_T(\mathcal{M})$) with some new cluster quantities $k_{\mathcal{T}}$ satisfying a bound*

$$\sum_{\mathcal{T}:\ \operatorname{Supp} \mathcal{T} = T} k_{\mathcal{T}} \leq \varepsilon^{\operatorname{con} T} \tag{3.19}$$

where (see also (3.14)) $\operatorname{con} \mathcal{T} = \sum_{\mathbf{\Gamma}\in\mathcal{T}} \operatorname{con} \mathbf{\Gamma}$ *. (Recall that contours resp. admissible systems $\mathbf{\Gamma}$ are counted here with their multiplicity!). These new cluster quantities are defined only for clusters containing some shift of* $\operatorname{supp} \mathcal{D}$*, and they depend only on the values $F(\mathcal{D})$ and also on the "old" values $k_{\mathcal{T}'}$,* $\operatorname{supp} \mathcal{T}' \subset V(\mathcal{T})$.

[31] Recall again that we are using the same symbol $\mathcal{M}$ for the mixed as well as for the metastable model. Fortunately there will be no confusion in the notation of $F_{\mathcal{M}}(\mathcal{D})$ (which will be later denoted also as $F_{\text{meta}}(\mathcal{D})$) because our mixed model will be always defined as an expansion (in a given volume Λ) of the metastable model.

Notation. The passage from the original mixed model to the new one (as described in the theorem above) will be called the *recoloring* of $\mathcal{D}$.

Proof of Recoloring Lemma.

Given a configuration x of a given mixed model write it as

$$x = (\mathcal{D}^1 \cup \mathcal{D}^2 \cup \dots) \cup \tilde{\mathcal{D}} \tag{3.20}$$

where $\mathcal{D}^1, \mathcal{D}^2, \dots, \mathcal{D}^k$ are some shifts of $\mathcal{D}$ (which mutually do not touch and which are *interior subsystems* of x) and $\tilde{\mathcal{D}}$ is the admissible system of the "remaining contours of x" (which appears if all the above shifts of $\mathcal{D}$ are removed).

The fundamental observation – steming from (3.2) (which is rewritten by (3.17)) is the following one. We write it here only for $k = 1$; $\mathcal{M}$ is the mixed model given in Recoloring Lemma; $F_{\mathcal{M}}$ is the value of the contour functional in the model $\mathcal{M}$; we are, of course, working in some volume Λ but we omit the symbols Λ, x_Λ for the simplicity of notations:

$$\exp(-H_{\mathcal{M}}(\mathcal{D} \cup \tilde{\mathcal{D}}) = \exp(-F_{\mathcal{M}}(\mathcal{D}) \exp(-H_{\mathcal{M}}(\tilde{\mathcal{D}})) \exp(-\sum_T k_T^q) \tag{3.21}$$

where the sum in the exponent is over all T in Λ such that

$$T \not\subseteq \operatorname{inn} \mathcal{D} \quad \& \quad T \cap \operatorname{supp} \mathcal{D} \neq \emptyset$$

and where q denotes the external colour of $\mathcal{D}$.

Then the new Gibbs factor $\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}}))$ of the remaining configuration $\tilde{\mathcal{D}}$ should be equal to the sum of all corresponding "old" factors $\exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}}))$ and $\exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}} \cup \mathcal{D}^1 \cup \dots \cup \mathcal{D}^k))$ expressed by (3.21) above (for a general $k \geq 1$). In other words, we require that (we are writing below the result again for a general $k \in \mathbb{N}$)

$$\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}})) +$$

$$+ \sum_{\{\mathcal{D}^1, \dots, \mathcal{D}^k\}} \exp(- \sum_{\substack{T : T \not\subseteq \cup_i \operatorname{inn} \mathcal{D}^i \\ \& \ T \cap \cup_i \operatorname{supp} \mathcal{D}^i \neq \emptyset}} k_T^q) \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}} \cup \mathcal{D}^1 \cup \dots \cup \mathcal{D}^k)). \tag{3.22}$$

Write

$$k_D^q = \sum_{\mathcal{D}' : \operatorname{supp} \mathcal{D}' = D} \exp(-F_{\mathcal{M}}(\mathcal{D})) \tag{3.23}$$

where q denotes the external colour of $\mathcal{D}$ and the sum is over all removable $\mathcal{D}'$ which have the same support $\operatorname{supp} \mathcal{D}' = D$ and the same external colour as $\mathcal{D}$.[32]

[32] The terms $\exp(-F_{\mathcal{M}}(\mathcal{D}))$ will be thus "glued together", into one term k_D^q.

This expression can be written as follows. Let us first *ignore*, just for the clarity of the exposition below, the appearance of the cluster terms k^q_T above i.e. write (3.22) in a simplified form

(3.24)

$$\exp(-H_{\mathcal{M}_{\mathrm{new}}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\mathrm{old}}}(\tilde{\mathcal{D}})) + \sum_{\{\mathcal{D}^1,\ldots,\mathcal{D}^k\}} \exp(-H_{\mathcal{M}_{\mathrm{old}}}(\tilde{\mathcal{D}} \cup \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^k)).$$

See (4.22) (and below it) for the errata concerning this simplification of the formula (3.22). Using (3.23), the relation (3.24) can be written as

$$\exp(-H_{\mathcal{M}_{\mathrm{new}}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\mathrm{old}}}(\tilde{\mathcal{D}}))Z_{\mathrm{out}\,\tilde{\mathcal{D}}} \tag{3.25}$$

where

$$Z_M = \sum_{\{T_i\}} \prod_i k^q_{D_i} \tag{3.26}$$

is the polymer partition function and the activities k^q_D are given by (3.23). The set $\mathrm{out}\,\tilde{\mathcal{D}}$ is just the collection of all points of the given volume Λ which are outside $\mathrm{supp}\,\tilde{\mathcal{D}}$ and whose colour in $(\mathrm{supp}\,\tilde{\mathcal{D}})^c$ is the same as the external colour of the subsystems $\mathcal{D}_i$ which were just recolored.

Fourth Lecture. Elements of the Cluster Expansion Method. Conclusion of the proof of Main Theorem.

Now, to finish the proof of Recoloring Lemma we have to expand the *polymer* partition function $Z_{\mathrm{out}\,\tilde{\mathcal{D}}}$: This is essentially the polymer partition function for polymers which are just shifts of *points* of $\mathbb{Z}^\nu$ with some translation invariant compatibility relation $\approx$ between them. Notice, however, that *different* polymer models live in regions of $\tilde{\mathcal{D}}$ marked by different colours $q \in Q$.

Maybe it is useful to expose in some detail, in this section, the standard way how polymer partition functions of such a type are expanded: Write the activities k^q_{D+t} (where $D+t$ are various possible, mutually compatible shifts of $D = \mathrm{supp}\,\mathcal{D}$) simply as k_t and consider them as complex variables indexed by points of $\mathbb{Z}^\nu$. The exposition below was motivated by the paper [DE], however we use a slightly different approach here, not using Cauchy formulas and giving instead a more emphasis on the Kirkwood Salsburg equations (which *were* traditionally used in these studies) . We develop first the simplified (according to the simplifications made in (3.22) $\to$ (3.26)) case when the polymers "are" just points of the lattice. This is also for the "pedagogical reasons" (to obtain a maximal possible clarity of the formulation of the assumptions imposed on the contour "activities"). The general case is briefly commented at the end of the section, together with the necessary comments needed to clarify the omissions made in (3.22).

Main Lemma for (point) polymer models. *Let $\approx$ be some compatibility pair relation between points of $\mathbb{Z}^\nu$, such that $t \approx s$ whenever t and s are sufficiently distant: $|t-s| > r$. (The actual choice of r depends on the size of the contours above;*

recall that the points $t \in \mathbb{Z}^\nu$ actually represent various shifts of some recolorable system $\mathcal{D}$). Let

$$|k_t| \le \varepsilon \tag{4.0}$$

for each $t \in \mathbb{Z}^\nu$ where $\varepsilon \le \varepsilon(\approx)$ for some $\varepsilon(\approx)$ which is sufficiently small (see Note below). Then the partition function

$$Z_\Lambda = \sum_{\{t_1,\dots,t_k\}} \prod_i k_{t_i} \tag{4.1}$$

is nonzero and it can be expanded into a convergent sum

$$\log Z_\Lambda = \sum_{n_T} \frac{(-1)^n}{n_T!} \alpha_{n_T} k_{n_T} \tag{4.2}$$

where the summation is over all multiindices $n_T = \{n_t, t \in T\}$ indexed by "indecomposable" collections ("clusters") T of points. Here we denote by

$$k_{n_T} = \prod_{t\in T} (k_t)^{n_t}, \ n = \sum_t n_t, \ n_T! = \prod_t n_t! \tag{4.3}$$

for any multiindex n_T whose support $\{t : n_t \ne 0\}$ is a cluster $T = \{t_1, \dots, t_k\}$. The indecomposability of T is meant in the following sense: T cannot be splitted into two mutually compatible parts T_1, T_2 such that $t \approx s$ whenever $t \in T_1$ and $s \in T_2$.

The coefficients α_{n_T} satisfy the bound, with a constant $C = C(\approx)$ denoting the number of incompatible neighbors to a given point of $\mathbb{Z}^\nu$,

$$0 \le \alpha_{n_T} \le (1 + C\varepsilon)(Ce)^n \tag{4.4}$$

for any n_T. The series (4.2) quickly converge, e.g. we have

$$\sum_{n_T : n_t > 0 \text{ for all } t \in S} |\alpha_{n_T} k_{n_T}| \le (C'\varepsilon)^{|S|} \tag{4.4S}$$

with another constant $C' = C'(\approx)$.

Note. The statement "$\varepsilon(\approx)$ is sufficiently small" depends therefore on the dimension ν and also on the value r (more precisely it depends on the properties of the relation $\approx$). For example, for $\nu = 2$ and $r = 1$ one can take $\varepsilon(\approx)$ something like 1/8.

The proof of the lemma is given by taking the Taylor expansion of $\log Z_\Lambda$ with respect to the variables k_t . One has to show that the terms α_{n_T} with decomposable $T = T_1 \cup T_2$ disappear i.e. that the derivative of $\log Z_\Lambda$ at zero, with respect to the quantities $\{k_t, t \in T_1\}$ and $\{k_s, s \in T_2\}$ such that $s \approx t$ for each $t \in \mathcal{T}_1$ and $s \in \mathcal{T}_2$ is zero. It is immediate to see that the part of the infinite Taylor sum for $\log Z_\Lambda$ which does not disappear by taking the derivative above can be written as the logarithm of the *product* of the corresponding two quantities Z_{T_1} and Z_{T_2} ! Noticing that one of these two variables depends on $\{s \in T_1\}$ only and, analogously, the other only on $\{t \in T_2\}$ we conclude that the derivative of the logarithm of the product $Z_{T_1} Z_{T_2}$ (being the sum of two terms each depending on the corresponding group of variables) disappears.

It remains to prove the bound (4.4) for multiindices $n_{\mathcal{T}}$ with indecomposable $\mathcal{T}$.

Let us start with the *first order* derivatives (the general case will be then worked out with the help of the Cauchy formulas; see below):

Given $t \in \Lambda$ the first derivative of $\log Z_\Lambda$ with respect to k_t is equal to

$$\frac{\partial \log Z_\Lambda}{\partial k_t} = \frac{Z_{\Lambda \setminus \hat{t}}}{Z_\Lambda} \quad := \rho_{\hat{t}}^{\Lambda} \tag{4.5}$$

where $\hat{t}$ denotes, here and everywhere in this section, the collection of points which are incompatible with t. This is the "correlation function" of t.

For the estimate of the right hand side of this equation we will use the Kirkwood Salsburg equations: Assume that we have some some selection rule $\{\ A \mapsto t_A\ \}$ for any finite A. Such a selection rule induces also some partial ordering $\prec$ on the family of all finite sets A, extending the relation

$$A \setminus t_A \prec A \quad . \tag{4.6}$$

Write the partition functions $Z_{\Lambda \setminus A}$ as

$$Z_{\Lambda \setminus A} = Z_{(\Lambda \setminus A) \cup t_A} - k_{t_A} Z_{\Lambda \setminus (A \cup \hat{t}_A)} \tag{4.7}$$

and iterate this equation many times for all the terms $Z_{\Lambda \setminus A}$ appearing on the right hand side such that $A \neq \emptyset$. We get the following relation. First write down the result of the iterative substitution ($|A|$ times) of (4.7) into the *first* term on its right hand side:

$$Z_{\Lambda \setminus A} = Z_\Lambda - \sum_{B \prec A} k_{t_B} Z_{\Lambda \setminus (B \cup \hat{t}_B)} \tag{4.7B))}$$

Now substitute (4.7B) iteratively $n-1$ times into the each term on its right hand side which does *not* already contain the term Z_Λ:

$$Z_{\Lambda \setminus A} = Z_\Lambda \Big(1 + \sum_{k=1}^{n} (-1)^k \sum_{(B_1, \ldots, B_k): B_i \prec B_{i-1} \cup \hat{t}_{B_{i-1}}} \prod_{i=1}^{k} k_{t_{B_i}}\Big) + R_{n+1} \tag{4.8}$$

where the remainder R_{n+1} is

$$R_{n+1} = \ (-1)^{n+1} \sum_{(B_1, \ldots, B_{n+2}): B_i \prec B_{i-1} \cup \hat{t}_{B_{i-1}}} \prod_{i=1}^{n+1} k_{t_{B_i}} Z_{\Lambda \setminus B_{n+2}}. \tag{4.8R)}$$

Below we will see that $R_n \to 0$ for $n \to \infty$. This will yield the expression of $\rho_A^\Lambda = \frac{Z_{\Lambda \setminus A}}{Z_\Lambda}$ by the infinite sum

$$\rho_A^\Lambda = 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{(B_1, \ldots, B_k): B_i \prec B_{i-1} \cup \hat{t}_{B_{i-1}}} \prod_{i} k_{t_{B_i}}. \tag{4.9}$$

If we count the number $M(k)$ of the "chains" $(B_1,\dots,B_k)$, $k\ge 1$ in this sum we obtain a bound $N(k)\le C(k-1)+|A|$ for the number $N(k)$ of the consequent substitutions of the equation (4.7) yielding such a chain. Here, $C=C(\approx)$ denotes the maximal possible number of the points which are incompatible with a given point $t\in\Lambda$. Let us concentrate on the case $A=\hat{t}$ i.e. the case $N(k)\le Ck$. Surely, then, $\binom{N(k)}{k}$, which is $\le (Ce)^k$ is an upper bound for the quantity $M(k)$ and $(Ce)^k\varepsilon^k$ is therefore an upper bound for the contribution of all these chains to the sum (4.9). This proves the convergence of (4.9) and therefore also the relation $R_k\to 0$ for ε sufficiently small. The relation $Z_\Lambda\neq 0$ follows from (4.9) by induction over the volume, if we use the obvious relation

$$Z_\Lambda^{-1}=\prod_{M\prec\Lambda}\rho_{t_M}^M \tag{4.10}$$

and the fact (established by induction over volume from (4.9)) that ρ_t^Λ is nonzero (in fact close to 1). The conclusion of all these estimates is the following bound for the sets A of a small cardinality (the term $\varepsilon|A|$ corresponds to all chains of the length $k=1$, the remainder is the estimate of the contribution of all the longer chains):

$$|\rho_A^\Lambda-1|\le\varepsilon|A|(1+C'\varepsilon) \tag{4.11}$$

where C' is a suitable new constant. This proves, considering the special the case $A=\{t\}$) the desired relation (4.4) for all the multiindices (with lowest possible cardinality) $\{n_t=1 \text{ if } t=t_0\ ;\ n_s=0 \text{ otherwise }\}$.

Note. For $|A|>>1$ then it is advisable to substitute (4.10) with $|A|=1$ into an equation (generalization of (4.10))

$$\rho_A^\Lambda=\prod_{B\prec A}\rho_{t_B}^{\Lambda\setminus B}$$

to obtain a bound

$$|\log\rho_A^\Lambda|\le\varepsilon|A|(1+C'\varepsilon). \tag{4.12}$$

For higher multiindices, a most convenient way to establish an analogous bound for α_{n_T} is apparently to derive (4.9) with respect to the remaining variables k_t: Given a multiindex n_T, select a point t_0 with $n_{t_0}>0$ and denote by m_T the new multiindex $\{m_t=n_t \text{ if } t\neq t_0;\ m_{t_0}=n_{t_0}-1\}$. Put $m=\sum_t m_t$, $n=\sum_t n_t$.

Write

$$\frac{\partial^n\log Z_\Lambda}{\prod_t\partial(k_t)^{n_t}}(0)=\frac{\partial^m\rho_{t_0}}{\prod_t\partial(k_t)^{m_t}}(0).$$

We get, for $m\ge 1$, the following consequence of (4.9):

$$\alpha_{n_T}=\frac{\partial^m\rho_{t_0}}{\prod_t\partial(k_t)^{m_t}}(0)=\sum_{k=1}^{\infty}(-1)^k\sum_{(B_1,\dots,B_k):B_i\prec B_{i-1}\cup\hat{t}_{B_{i-1}}}1 \tag{4.13}$$

where the summation runs over those chains $(B_1, \ldots, B_k)$ only which contain any point $t \in T$ *exactly* m_t times. Therefore we have (analogously as in our study (4.11) of (4.9) above) from (4.13) the inequality

$$|\alpha_{n_T}| \leq M(m)|\rho^\Lambda_{t_0}| \tag{4.14}$$

where $M(m)$ denotes the number of such chains $(B_1, \ldots, B_k)$) and this is (4.4), if we take into account the already established bounds (4.11) and $M(m) \leq (Ce)^m$, $C = C(\nu, \approx)$.

Note. The coefficients α_{n_T} can be computed also from the Cauchy formulas (like in [DC]) but the estimates of these contour integrals do not seem to give better bounds for α_{n_T} than what we obtained above.

Cluster expansion of general polymer models.

Now we shortly mention the modifications needed to extend the results above to the case of general polymer models. Consider now a general polymer partition function

$$Z_\Lambda = \sum_{\{S_i\}} \prod_i k_{S_i} \tag{2.35}$$

where S_i are some connected "polymers" with some compatibility relation $\approx$ between them (typically $S \approx S'$ if and only if S and S' do not "touch'; the latter can have e.g. the meaning $\operatorname{dist}(S, S') > 1$). There is almost nothing requiring a nontrivial change in the formulation and the proof of a corresponding analogy of Main polymer lemma: Asssume that the polymers S are connected sets and their weights k_S satisfy a bound

$$|k_S| \leq \varepsilon^{|S|} \tag{4.16}$$

Notice that for any set $M \subset \Lambda$ we have the bound (where the constant $C = C(\nu)$ denotes the number of neighbors of a point)

$$|\hat{M}| \leq C|M| \tag{4.17}$$

where $\hat{M}$ is the 1 neighborhood of M. In particular, we need such a bound for sets of the type $M = \cup_i \operatorname{supp} T_i$ where $\{T_i\}$ is a "cluster" of polymers, appearing in the following equation (generalizing (4.9)) :

$$\rho^\Lambda_A = 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{\substack{(B_1, S_1, \ldots, B_k, S_k): \\ B_i \prec B_{i-1} \cup \hat{t}_{B_{i-1}} \& t_{B_i} \in S_i}} \prod_i k_{S_i}. \tag{4.18}$$

The number of such chains of the length k is estimated here (for $|A| = 1$) as $\binom{M(k)}{k}$ where $M(k) \leq C(\sum_i |S_i|)$ i.e. we have the upper bound $(Ce)^{\sum_i |S_i|}$ for the number of terms in the k th sum above.

Corollary. *Under the condition (4.16), the polymer partition sum (4.15) can be expanded as*

$$\log Z_\Lambda = \sum \alpha_{\mathcal{N}} k_{\mathcal{N}} \tag{4.19}$$

where for any "cluster" of polymers $\mathcal{N} = \{S_i; n_i\}$ (where n_i is the multiplicity of the polymer S_i and the statement "$\mathcal{N}$ is a cluster" has the meaning "$\cup S_i$ is connected") we write $k_{\mathcal{N}} = \prod_i (k_{S_i})^{n_i}$. The coefficients $\alpha_{\mathcal{N}}$ satisfy the bound

$$|\alpha_{\mathcal{N}}| \leq (1 + C\varepsilon)(Ce)^{\sum_i n_i |S_i|} \tag{4.20}$$

and the series (2.39) quickly converges. In particular we have, with another small ε', the bounds

$$\sum_{\mathcal{N} \in \mathcal{S}'} |\alpha_{\mathcal{N}} k_{\mathcal{N}}| \leq (\varepsilon')^{|S|} \tag{4.20'}$$

resp.

$$\sum_{\mathcal{N} \in \mathcal{S}''} |\alpha_{\mathcal{N}} k_{\mathcal{N}}| \leq (\varepsilon')|S| \tag{4.20''}$$

where $\mathcal{S}'$ resp. $\mathcal{S}''$ denotes the family of clusters whose support contains resp. intersects a given set S.

Recoloring. Conclusion

Return now to (3.24). Using Corollary of the preceding section we can finally write (3.24) as

$$\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}}))Z_{\text{out}\,\tilde{\mathcal{D}}} = \exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}}))\exp(\sum_{\mathcal{T}} k_{\mathcal{T}}) \tag{4.21}$$

where $\mathcal{T}$ are the new clusters formed by indecomposable collections of shifts of D. This essentially completes our recoloring procedure; however the simplification made at (3.24) must be supplemented now by a more complete discussion: Actually, instead of (3.24) one has a more precise relation (3.22):

$$\exp(-H_{\mathcal{M}_{\text{new}}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}}))+$$

$$+ \sum_{\{\mathcal{D}^1,\ldots,\mathcal{D}^k\}} \exp(-\sum_{\substack{T:T\not\subseteq\cup_i \operatorname{inn}\mathcal{D}^i \\ \& T\cap\cup_i \operatorname{supp}\mathcal{D}^i\neq\emptyset}} k_T^q)\exp(-H_{\mathcal{M}_{\text{old}}}(\tilde{\mathcal{D}}\cup\mathcal{D}^1\cup\cdots\cup\mathcal{D}^k)) \tag{4.22}$$

where the last sum is over all "old" clusters T whose supports are *not* subsets of a suitable shift $V(D_i)$ of $V(D) = V(\mathcal{D})$ but *intersect* the union of these shifts (which are all recolored at once): $\cup_i V(D_i) \cap T \neq \emptyset$. The modification which must be made here is the following one: writing

$$\exp(k_T) = 1 + \tilde{k}_T \tag{4.23}$$

one has in fact the following relation :

$$\exp(-H_{\mathcal{M}_{\rm new}}(\tilde{\mathcal{D}})) = \exp(-H_{\mathcal{M}_{\rm old}}(\tilde{\mathcal{D}}))\tilde{Z}_{\mathrm{out}\,\tilde{\mathcal{D}}} \tag{4.24}$$

where the new polymers acting in the partition functions $\tilde{Z}_{\mathrm{out}\,\tilde{\mathcal{D}}}$ are some connected *conglomerates* of the shifts of $\operatorname{supp}\mathcal{D}$ *and* sets T from (4.22) and the weight $k_{\mathcal{C}}$ is

$$k_{\mathcal{C}} = \prod_i k_{D_i} \prod_j \tilde{k}_{T_j} \tag{4.25}$$

for any such conglomerate $\mathcal{C} = \{D_i \& T_j\}$ of shifts D_i (of $D = \operatorname{supp}\mathcal{D}$) and of clusters T_j. Let us finally note that the convergence of the cluster series with terms decaying like (3.14) is studied in more detail in [HZ] (but this is essentially the type of convergence obtained already in classical Mayer expansions).

Conclusion of the proof of Main Theorem. The skeleton.

Let $\mathcal{D}^q$ be a residual admissible system in a cube $\Box$ which is q – small in some (temporary) mixed model (which we constructed up to now). We will show that such a system *is* recolorable (and, therefore, such a situation cannot happen in the context of Main Theorem where *total expansion* – with no recolorable systems left in the expanded model – can be considered.

Note. The sense of the recoloring procedure formulated above was that for any mixed model with some remaining recolorable systems, we can define an equivalent model on a *smaller configuration space*. Thus, applying such a procedure sufficiently many times, we may assume that an expansion of the model considered in Main Theorem was already found yielding *no removable* (no recolorable) contours in the corresponding mixed model. The core of the proof of Main Theorem lies then in establishing of the fact that there are also *no small* contours resp. admissible systems in such an expanded model. Namely this is proven in this last section.

So, consider a residual system $\mathcal{D}$ with an external colour q and assume that it is small i.e. there is some q –small cube $\tilde{\Box}$ packing $\mathcal{D}$: $V(\mathcal{D}) \subset \tilde{\Box}$. Let us show that $\mathcal{D}$ is recolorable.

Take successively some *nonsmall* (in their colour induced by $\mathcal{D}$; of course this interior colour is already different from q !) cubes $\Box_i$ which are disjoint from $\operatorname{supp}\mathcal{D} \cup (\tilde{\Box})^c$ and which are also *mutually* disjoint. The *skeleton* of $\mathcal{D}$ will be defined as a maximal possible collection $\{\Box_i\}$ of such cubes, giving no room for *additional* nonsmall cubes *not* intersecting $\operatorname{supp}\mathcal{D} \cup (\tilde{\Box})^c$ & the cubes $\Box_i$ already constructed. Writing

$$E(\Box_i) = \tau \operatorname{diam} \Box_i \tag{4.26}$$

one can treat the cubes of the skeleton as some "additional contours". This is just a suitable convention to be used below; we define the new, artificial "contours" $\Box_i$ and assign the energy $E(\Box_i)$ to them.

If a new, best possible colour q'' (minimizing $h_{q''}$) is assigned to a cube $\square_i = \square$ living, say, in a q' regime i.e. having the colour q' at the boundary of the complement of $\square$ then we have, using the nonsmallness of $\square$, the inequality[33]

$$\log Z^{q''}(\square) - \log Z^{q'}(\square) \geq a_{q'}|\square| - \varepsilon|\partial\square| \geq \tau \operatorname{diam}\square. \tag{4.27}$$

Imagine the new "admissible system" $\mathcal{D}^*$ which is defined as the original system $\mathcal{D}$ *enriched* by all the cubes $\square_i$ of the skeleton. Imagine that the configuration determined by $\mathcal{D}^*$ "jumps into the best possible q'' inside any cube of the skeleton" and the energy of any $\square_i$ from the skeleton is given by (4.26).[34] Then we have from (4.27) the inequality

$$F(\mathcal{D}^*) \leq F(\mathcal{D}). \tag{4.28}$$

Really, from (2.4) we have, inserting into it the relation (4.27), the inequality

$$F(\mathcal{D}^*) \geq F(\mathcal{D}) + \sum_i (\tau \operatorname{diam}\square_i - a_{q_i}|\square_i| - \varepsilon|\partial\square_i|)$$

where q_i denotes the colour of the square $\square_i$. This is, by (3.6), the desired bound (4.28).

On the other hand, the smallness of $\mathcal{D}$ (which we assume) implies even the smallness of the enriched system $\mathcal{D}^*$! Namely, $\mathcal{D}^*$ still belongs to $\tilde{\square}$. (This "minor" observation shows the advantage of our very definition of smallness, through the smallness of the covering cubes $\square$![35]) Thus we have the relation (notice that we have, the bound, say $|\operatorname{supp}\mathbf{\Gamma}| - \operatorname{diam}\mathbf{\Gamma} > 1/2|\operatorname{supp}\mathbf{\Gamma}|$)

$$F(\mathcal{D}^*) \geq \frac{\tau}{2}|\operatorname{supp}\mathcal{D}^*|. \tag{4.29}$$

Concerning the relation between $|\mathcal{D}^*|$ and $\operatorname{con}\mathcal{D}^*$ we will now have the following bound (see the topological section below, concluding the Lecture 4):

$$\operatorname{con}\mathcal{D}^* \leq 6\nu|\operatorname{supp}\mathcal{D}|. \tag{4.30}$$

It is clear that (4.29) and (4.30) would conclude the proof of the fact that a residual small $\mathcal{D}$ is recolorable. More precisely, noticing that the enrichened system of contours $\mathcal{D}^*$ is "tight" (see the next section), we obtain the bound

$$F(\mathcal{D}) \geq F(\mathcal{D}^*) \geq \frac{\tau}{2}|\operatorname{supp}\mathcal{D}^*| \geq \frac{\tau}{12\nu}\operatorname{con}\mathcal{D}^* \geq \frac{\tau}{12\nu}\operatorname{con}\mathcal{D} \tag{4.31}$$

i.e. $\mathcal{D}$ really is recolorable. This would, however, contradict the notion of residuality. Thus, there are *no* small residual systems and our Main Theorem is proven.

[33]This is just the consequence of the fact that the cubes of the skeleton are not small i.e. the condition (3.6) does *not* hold for them. The constant C from (3.6) should be taken such that (4.27) is fulfilled: take C (slightly) bigger than 1 such that $C(\tau - \varepsilon) > \tau$.

[34]This is just a play with values $\tau|\operatorname{diam}\square|$ versus $a^{q'}|\square|$; it is not at all necessary to interpret the cubes $\square_i$ as some "real" contours. One could not replace $\operatorname{diam}\square$ by the more "physical" quantity $\partial\square$ in this play without substantial changes in the notion of smallness.

[35]In connection with rather subtle considerations used here, when discussing the relation between the very notions of smallness and recolorability, one should make one general "philosophical" remark: Sometimes, one is fighting severe technical problems in the Pirogov – Sinai theory which however start to be relevant only in volumes which are really *astronomically large*. For example the problem discussed above i.e. establishing of the fact that extremally small implies recolorable could hardly be called nontrivial for "moderate" volumes of a size, say, 10^{27} !

A TOPOLOGICAL APPENDIX: TIGHT SETS

For any set $T \subset \mathbb{Z}^\nu$ denote by $\Box(T)$ the smallest cube containing T. Say that a set $S \subset T$ is isolated in T if

$$\operatorname{dist}(\Box(S), T \setminus S) \geq \operatorname{diam} \Box(S). \tag{4.32}$$

Say that a set $T \subset \Lambda$ is *tight* if it has *no* isolated subsets.

Lemma. *If T is tight then*

$$\operatorname{con} T \leq 6\nu |T|. \tag{4.33}$$

The proof of this topological statement can be found in [HZ].[36]

When applying this result to the enrichened collections $\mathcal{D}^*$ one has to identify the "supports" of cubes of $\mathcal{D}^*$ as suitable subsets of $\mathbb{Z}^\nu$ of cardinality $\operatorname{diam} \Box$. (Contours of $\mathcal{D}$ have supports which already are defined as connected subsets of $\mathbb{Z}^\nu$.) Let us assume e.g. that any square is identified with some of its "diagonal". Then clearly $\operatorname{supp} \mathcal{D}^*$ is tight in the sense of the definition above.

BOUNDS FOR DILUTED PARTITION FUNCTIONS

The bounds stated below summarize some of the main consequences of our Main Theorem. Such bounds were always important in the Pirogov – Sinai theory (see [S], [I], [Z]). They can be, however, formulated now in a sharper form than before: The volumes Λ considered below are assumed to be[37] simply connected.

Corollary. *The partition functions $Z^q(\Lambda)$ and the metastable partition functions $Z^q_{\mathrm{meta}}(\Lambda)$ can be estimated as follows (compare (3.8) – (3.10)):*
1) The metastable partition functions are expressed as

$$Z^q_{\mathrm{meta}}(\Lambda) = \exp(-h^q|\Lambda| + \Delta^q(\Lambda)) \tag{4.45}$$

where the boundary ("surface tension") terms defined as

$$\Delta^q(\Lambda) = - \sum_{T:\ T \cap \Lambda^c \neq \emptyset} \ \sum_{\mathcal{T}:\ \operatorname{supp} \mathcal{T} = T} \alpha_{\mathcal{T}} k_{\mathcal{T}} \frac{|T \cap \Lambda|}{|T|} \tag{4.46}$$

satisfy the obvious bound, with some small ε' :

$$|\Delta^q(\Lambda)| \leq \varepsilon' |\partial \Lambda^c|. \tag{4.47}$$

2) The general diluted partition functions differ from the metastable ones by factors

$$Z^q(\Lambda) - Z^q_{\mathrm{meta}}(\Lambda) = Z^q_{\mathrm{residual}}(\Lambda) \tag{4.48}$$

[36] The constant 6ν can be probably considerably improved.
[37] as always in this text!

where the difference $Z^q_{\text{residual}}(\Lambda)$ is expressed as the sum over all possible residual $\mathcal{D}$

$$Z^q_{\text{residual}}(\Lambda) = \sum_{\mathcal{D}} Z^q_{\text{meta}}(\operatorname{ext}\mathcal{D}) \exp(-E(\mathcal{D}) - e|\operatorname{supp}\mathcal{D}|) \prod_{q'\in Q} Z^{q'}_{\text{meta}}(\operatorname{inn}_{q'}\mathcal{D}).$$

This is equal (when using (4.48) for the expression of both the exterior and the interior[38] *partition functions $Z^q_{\text{meta}}(\text{ext})$ and $Z^{q'}_{\text{meta}}(\operatorname{inn}_{q'}\mathcal{D})$) to the following expression :*

$$Z^q_{\text{residual}}(\Lambda) = \exp(-h|\Lambda|) \sum_{\mathcal{D}} \exp(-a_q|\operatorname{ext}| - E^q(\mathcal{D})) \exp(\Delta'(\Lambda, \mathcal{D})) \tag{4.49}$$

with $\Delta'(\Lambda,\mathcal{D}) = \Delta^q(\operatorname{ext}\mathcal{D}) + \sum_{q'}(\Delta^{q'}(\operatorname{inn}_{q'}\mathcal{D}) - a_{q'}|\operatorname{inn}_{q'}\mathcal{D}|$. The quantity $Z^q_{\text{residual}}(\Lambda)$ (it is nonzero only if the above sum over $\mathcal{D}$ is nonvoid) satisfies the bound

$$Z^q_{\text{residual}}(\Lambda) \le \exp(-h|\Lambda| + \Delta^q(\Lambda)) \exp(-A^q_\tau(\Lambda)) \tag{4.50}$$

where $A^q_\tau(\Lambda)$ is defined as

$$A^q_\tau(\Lambda) = \min_T\{A^q_T(\Lambda)\} \tag{4.51}$$

with $A^q_T(\Lambda)$ being defined as

$$A^q_T(\Lambda) = a_q|\operatorname{ext} T \cap \Lambda^c| + (\tau - \varepsilon)|T| \tag{4.51T}$$

and with the minimum in (4.51) being taken over all connected $T \subset \Lambda$ whose diameter is greater or equal to $\frac{\tau}{a_q}$. (By $\operatorname{ext} T$ we denote the external component of T^c.) If there is no such T then we put $A^q_\tau = \infty$. In particular, for volumes of a size $\approx \frac{\tau}{a_q}$ we get

$$Z^q_{\text{residual}}(\Lambda) \le \exp(-h|\Lambda|) \exp(-C\frac{\tau^2}{a_q}) \tag{4.52}$$

with a suitable constant $C = C(\nu)$, whereas for volumes of the size $>> \frac{\tau}{a_q}$ the bound (4.50) can be written as

$$Z^q_{\text{residual}}(\Lambda) \le \exp(-h|\Lambda|) \exp(-(1-\varepsilon')\tau|\operatorname{supp}\mathcal{D}_\Lambda|)$$

where $\mathcal{D}_\Lambda$ is the smallest contour (or admissible system) "swallowing most of Λ".

The *proofs* of this statements follow from (3.8) and (3.10) (from which (4.48) follows just by resummation) and also from the fact – proven in the last section –

[38] *metastable* partition functions, by the very definition of residuality

namely that any residual (therefore nonsmall!) system $\mathcal{D}$ has a size[39] *at least* $\frac{\tau}{a_q}$. Therefore we have the inequality

$$E(\mathcal{D}) \geq \tau |\operatorname{supp} \mathcal{D}| \geq \frac{\tau^2}{a_q}. \tag{4.53}$$

It is easy to see that (for τ sufficiently large) with the help of this bound, the summation over all possible residual $\mathcal{D}$ gives the bound (4.52).

These bounds say, roughly speaking, that in the volumes of the size $\approx \frac{\tau}{a_q}$ (or smaller) there is no noticeable difference between the behaviour of the stable phases and the q – th "phase". Therefore, the presence of the large "bubbles" of the stable phases inside of the residual contours starts to be decisive (concerning the relative contribution, through these configurations, of $Z^q_{\text{residual}}(\Lambda)$ to the partition function $Z^q(\Lambda)$) only when the quantity $A^q_\tau(\Lambda)$ starts to be "considerably smaller than $a_q|\Lambda|$".

Note. This opens a way to the estimates of "finite size effects" for realistic boundary conditions (not only the "weak" ones like in [BK]). To do this, one of course needs a more precise evaluation of the quantities $Z^q_{\text{residual}}(\Lambda)$ – through the quantities of the type $A^q_\tau(\Lambda)$.

Some bounds of this type were already obtained in [ZA] . They can be apparently made more sharp now, using the expressions (4.48),(4.49) above.

Surface tension.

It is the characteristic feature of the Pirogov – Sinai theory that quantities like the free energy but also the *surface tension* around a *rigid* interface (and other, less dimensional "tensions", see below) can be computed by quickly converging cluster expansion formulas (and not only by limit procedures using logarithms of partition functions – which is far less flexible and often suspect if more delicate quantities than the "bulk" ones are considered). It can be even said that the *very control* over all these delicate quantities (like various surface tensions) is one of the most characteristic features of the Pirogov – Sinai theory, thus distinguishing it from the other, less detailed theories.

Simply speaking, all the above mentioned quantities are defined by suitable *resummation* of the cluster expansion formulas. This resummation uses formulas of the type

$$\sum_T k_T = \sum_t k_t \quad \text{where} \quad k_t = \sum_{T:\ t \in T} \frac{k_T}{|T|} \tag{4.54}$$

which were already used in many parts of these lectures, starting from the definition (1.1). We will not go into details of the computation of quantities like the surface

[39] More precisely one could argue that the ratio between $V(\mathcal{D})$ and $|\operatorname{supp} \mathcal{D}|$ for really "dangerous"(nonrecolorable) systems should be at least so big. However, such a stronger statement does not follow immediately from our definition of smallness. Apparently, to improve the bound (4.52) (and, therefore, to achieve the quantity of the order $\frac{\tau^\nu}{(a_q)^{\nu-1}}$ or so in the exponent) a suitable modification of the notion of smallness is desirable.

tension around a rigid interface (see [HZ]) but mention here only the most basic question: what are possible more precise variants of the formula (4.45), taking in account the parts of $\partial\Lambda$ which are *flat*? In a more detailed expression we could consider also the "$\nu - 2$ dimensional parts of $\partial\Lambda$" (the collection of all "edges of quadrants sticked to $\partial\Lambda$") etc.

Obviously, the formulas (4.46), (4.47) can be apparently made more precise for volumes whose boundary is sufficiently regular, containing flat pieces having a significant area: Consider the dimension $\nu = 3$ for the brevity, write (3.8) as

$$\log Z^q_{\rm meta}(\Lambda) = -e^q|\Lambda| + \sum_{T\subset\Lambda} k^q_T \tag{4.55}$$

and introduce the following six resp. fifteen quantities (which can be called the "plane" surface tensions, the "angle" surface tensions): Put, for example

$$\begin{aligned}\sigma_{3+} &= \sum_{T:0\in T} \frac{t^3_+}{t^3_0}\,\frac{k^q_T}{|T|}\\ \sigma_{\{2+,3+\}} &= -\sum_{T:0\in T}\frac{1}{t^{2,3}_{0,0}}\Big(t^{2,3}_{+,+} - \frac{t^3_+t^{2,3}_{+,0}}{t^3_0} - \frac{t^2_+t^{2,3}_{0,+}}{t^2_0}\Big)\,\frac{k^q_T}{|T|}\end{aligned} \tag{4.56}$$

where the quantities $t^3_+, t^3_0, t^{2,3}_{+,+}, t^{2,3}_{+,0}, t^{2,3}_{0,0}$ etc. are defined as follows:

$$\begin{aligned} t^3_+ = |\mathbb{Z}^3_+\cap T| \;\;&;\;\; t^3_0 = |\mathbb{Z}^3_0\cap T| \;\;;\\ t^{2,3}_{+,+} = |\mathbb{Z}^{2,3}_{+,+}\cap T| \;\;&;\;\; t^{2,3}_{0,0} = |\mathbb{Z}^{2,3}_{0,0}\cap T|\\ t^{2,3}_{+,0} = |\mathbb{Z}^{2,3}_{+,0}&\cap T| \;\;\text{etc.}\end{aligned} \tag{4.57}$$

where

$$\begin{aligned}\mathbb{Z}^3_+ &= \{t\in\mathbb{Z}^3 : t_3\geq 0\} \;\;;\mathbb{Z}^3_0 = \{t\in\mathbb{Z}^3 : t_3 = 0\}\\ \mathbb{Z}^{2,3}_{+,+} &= \{t\in\mathbb{Z}^3 : t_3\geq 0 \;\;\&\;\; t_2\geq 0\} \;\;;\;\; \mathbb{Z}^{2,3}_{0,0} = \{t\in\mathbb{Z}^3 : t_3 = 0 \;\;\&\;\; t_2 = 0\}\\ \mathbb{Z}^{2,3}_{+,0} &= \{t\in\mathbb{Z}^3 : t_3\geq 0 \;\;\&\;\; t_2 = 0\} \quad \text{etc.}\end{aligned}$$

Of course, if the hamiltonian of the given model is symmetric with respect to permutations and reflections of coordinates then we have the relations $\sigma_{3+} = \sigma_{1-}$, $\sigma_{\{2+,3-\}} = \sigma_{\{1-,2-\}}$ etc. Then (4.55) can be written as

$$\log Z^q_{\rm meta}(\Lambda) = -h^q|\Lambda| + \sigma_{3+}\,|\partial^3_+\Lambda| + \cdots + \sigma_{\{2+,3+\}}|\partial^{2,3}_{+,+}\Lambda| + \cdots + \Delta(|\partial_{\rm extr}\Lambda|) \tag{4.58}$$

where e.g. $|\partial^3_+\Lambda|$ denotes the overall cardinality of the flat part of $\partial\Lambda$ which is "of the type $\mathbb{Z}^3_+$" (including its boundary) i.e. the cardinality of that area of $\partial\Lambda$ where Λ "sticks to a halfspace parallel to Z^3_+ " and, analogously e.g. $|\partial^{2,3}_{+,+}\Lambda|$ denotes the overall length of all the segments of $\partial\Lambda$ which correspond to the "edges of quadrants parallel to $Z^3_{\{2+,3+\}}$ sticked (in the given segment) to Λ". The second resp. the third term on the right hand side of (4.56) thus represents, together with the other terms of this type, the "flat" resp. the "angular" part of the surface $\partial\Lambda$. (Of course, in higher dimensions there are more quantities of this type.) The last term in (4.58) corresponds to the sum, over T touching $\partial_{\text{extr}}\Lambda$, of the "rests of k_T" and it can be estimated as

$$|\Delta(\partial_{\text{extr}}\Lambda)| \le \varepsilon' |\partial_{\text{extr}}\Lambda| \tag{4.59}$$

where $\partial_{\text{extr}}\Lambda$ denotes the collection of all the extremal points of Λ i.e.the collection of points which do not belong to any segment of $\partial\Lambda$ of notrivial length.

Any corrections and remarks concerning this text (notices on missing information, references etc.) are welcome! I plan, in future, to add some additional chapters to this text dealing briefly with other aspects of the theory and other models not covered by the basic setting of Lectures 3 – 4: 1) Description of some details of [HZ] – giving more or less definite description of a wide class of "stratified" (depending on one coordinate only) phases of the Dobrushin type (with one or several *rigid* interfaces) appearing in translation invariant Ising type models or in their "stratified" versions. 2) Short description of some other aspects of the theory like the smoothness and local analyticity properties of the phase diagram. 3) Finally, I plan to mention some other applications to more complicated models like the models with a continuous spin, models with random impurities, long range models of Kac type.

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