THE ENTROPY OF UNIQUELY ERGODIC TILING SYSTEMS *

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ABSTRACT. We generalize the study of the entropy of uniquely ergodic subshifts of finite type with $\mathbf{Z^n}$ action to uniquely ergodic tiling systems of finite type with $\mathbf{R^n}$ action. We prove that the entropy of any such tiling system is zero.

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1. INTRODUCTION

The objective of this paper is to extend work done of Radin [12] on the entropy of uniquely ergodic subshifts of finite type with $\mathbf{Z^n}$ action. Tiling systems of finite type are a natural generalization of subshifts of finite type (when the action is not one dimensional). The elements of a tiling system are tilings of some Euclidean space, rather than configurations in a product space. One of the main reasons for the interest in tiling systems is their use in studying the properties of ground states in physical models [12, 13, 14]. Our goal is to extend Theorem 2 of [12], which shows that uniquely ergodic subshifts of finite type have zero entropy, to include, for example, the pinwheel [15] and Penrose-like tilings [13]. The method of proof in this paper holds in any dimension $d > 1$; for simplicity we only consider dimension $d = 2$.

2. NOTATION

We begin with a definition for a finite type tiling system ([2],[16]) in the plane $\mathbf{R^2}$. By a "prototile", p, we mean a homeomorphic image of the closed unit disk in $\mathbf{R^2}$. Assume given a finite set S of prototiles. Let E denote the Euclidean group of $\mathbf{R^2}$, the connected subgroup of the topological group of isometries of $\mathbf{R^2}$. The image of a prototile by an isometry of E will be called a "tile". A final requirement: defining a "swatch" of a given bounded set $\mathcal{B} \subset \mathbf{R^2}$ as a minimal collection of tiles with pairwise disjoint interiors which covers $\mathcal{B}$, we assume that any bounded set in the plane can only be covered by finitely many different swatches, up to isometry. We associate with given prototile set S the set $X(S)$ of all infinite collections of tiles, called "tilings", that have union $\mathbf{R^2}$, and are such that each pair of tiles in a tiling has disjoint interiors. We assume $X(S)$ is nonempty. We next define a topology on $X(S)$.

A typical element of a base for the topology on $X(S)$ is $C(P,\epsilon)$, where P is a swatch $\{T^{t_j}[R^{\theta_j}(p_j)] \,:\, 1 \leq j \leq n\}$, ($p_j$ being a prototile, R^{θ_j} a rotation about the origin by an angle θ_j, and T^{t_j} a translation by $t_j \in \mathbf{R^2}$), consisting of all tilings which contain a swatch $P^{'} = \{T_j^{'}[R_j^{'}(p_j) \,:\, 1 \leq j \leq n\}$ where $||R_j - R_j^{'}|| < \frac{\epsilon}{2}$ and $||T_j - T_j^{'}|| < \frac{\epsilon}{2}$. It is not hard to check that with this topology $X(S)$ is metrizable and compact, and that $\mathbf{R^2}$ acts continuously on it [16]. Consider the obvious action of a subgroup $G \subset E$ on the space $X(S)$ by automorphisms, denoted by $T : G \times X \to X$. Then for each $g \in G$, $T(g,\cdot) : X \to X$ is continuous and $T(g_1g_2,\omega) = T(g_1, T(g_2,\omega))$, $g_1, g_2 \in G$, $\omega \in X(S)$. Note that we write T^g for the map $T(g,\cdot)$, where $g \in G$. The "tiling system" associated with given S and action by the translations $\mathbf{R}^2 \subset E$ will be denoted simply by X.

Let μ be a translation invariant Borel probability measure on X. We assume μ is unique, that is, (X, T, μ) is a uniquely ergodic tiling system. We also assume that X has a metric d on it compatible with the topology. We will state two equivalent definitions [17, p110] of the topological entropy of X under a $\mathbf{Z^2}$ action and give a definition of the topological entropy of X under an $\mathbf{R^2}$ action by analogy to the case of an $\mathbf{R}$ action [4, p185].

(a) A subset $E \subset X$ is called a (n, ϵ)-separated set for T if $y, y' \in E$, $y \neq y'$ implies there exists $m = (m_1, m_2) \in \mathbf{Z^2}$, $0 \leq m_1, m_2 \leq n$, such that $d(T^m y, T^m y') > \epsilon$. Then

$$h_{top}(T, \mathbf{Z^2}) = \sup_{\epsilon > 0} \limsup_{n \to \infty} \frac{\log(\text{ maximum cardinality of } (n, \epsilon) - \text{separated set})}{n^2}.$$

(b) Given a finite open cover O of X, let $H(O)$ be the natural logarithm of the minimal number of open sets in any subcover of O. For every finite subset A of $\mathbf{Z}^2$, denote by O^A the open cover of X consisting of all sets $\bigvee_{g \in A} (T^g)^{-1} O$, where for any two covers $\mathcal{U}$, $\mathcal{B}$, we define $\mathcal{U} \bigvee \mathcal{B} := \{A \bigcap B : \quad A \in \mathcal{A},\ B \in \mathcal{U}\}$. Then

$$h_{top}(T, \mathbf{Z^2}) = \sup_{O} \lim_{l \to \infty} \frac{H(\bigvee_{||(j,k)||_\infty \leq l} T^{-(j,k)} O)}{(2l+1)^2}, \text{ where } (j,k) \in \mathbf{Z^2}.$$

¿From now on we will use (b) as our definition of the topological entropy of X under a $\mathbf{Z^2}$ action, but we use (a) and the topological pressure for an $\mathbf{R}$ action as used in [4, p185] as motivation for the following.

(c) A subset $E \subset X$ is called a (τ, δ)-separated set for T if $y, y' \in E$, $y \neq y'$ implies there exists $t = (t_1, t_2) \in \mathbf{R^2}$, $0 \leq t_1, t_2 \leq \tau$, such that $d(T^t y, T^t y') > \delta$. Then

$$h_{top}(T, \mathbf{R^2}) = \sup_{\delta > 0} \limsup_{\tau \to \infty} \frac{\log(\text{ maximum cardinality of } (\tau, \delta) - \text{separated set})}{\tau^2}.$$

Note that by a slight change in the proof of the variational principle for $\mathbf{R}$ action in [4, p185] we can get the variational principle for $\mathbf{R^2}$ action.

Let $N_l = 2l+1$. Denote the square of side length N_l centered at the origin by B_l ("B" for box) and let $D_l = B_l \setminus int(B_{l-1})$ ("D" for doughnut), where $int(B_{l-1})$ denotes the interior of B_{l-1}. Given A, a closed subset of $\mathbf{R^2}$, we denote by $C_A(\omega, \epsilon)$ ("C" for cylinder) the open set

$$C_A(\omega, \epsilon) = \{\tilde{\omega} \,:\, \Pi_A(T^\alpha R^\beta \tilde{\omega}) = \Pi_A(\omega), \ \text{ for some } ||\alpha||_\infty, |\beta| < \epsilon\}$$

where Π_A is the projection operator onto the set A. If $A = B_l$ we will simply write $C_l(\omega, \epsilon)$ instead of $C_{B_l}(\omega, \epsilon)$; if $A = D_l$ we will write $D_l(\omega, \epsilon)$ instead of $C_{D_l}(\omega, \epsilon)$.

3. STATEMENT OF RESULT

Theorem. *Any uniquely ergodic tiling systems of finite type with* $\mathbf{R^n}$ *action has zero topological entropy.*

4. PROOF OF THE THEOREM

In brief, seven lemmas are needed for a complete proof. The first three lemmas are used to find an estimate of the number of different $C_{B_l}(\omega, \epsilon)$ there are for given $l \in \mathbf{N}$. This is analogous to the number of different cylinder sets based in B_l for the square lattice systems in [12]. Lemmas 3, 4, 5 are used in the proof of Lemma 6. Roughly, Lemma 6 shows that if the topological entropy is greater than 0, then there exists some annular patch $P \subset \omega \in support(\mu)$ such that P covers D_l (for some

$l \in \mathbf{N}$) and there exist at least two ways to fill the inner disk of P. Finally we use this result and the Local Isomorphism Theorem [16] to complete the proof.

Shrinking Lemma. *Given $l \in \mathbf{N}$, $0 < \epsilon \leq 1/(2N_l - 1)$, if $\Pi_{B_l}(T^\alpha R^\beta \omega) = \Pi_{B_l}(\omega')$ for some $||\alpha||_\infty$, $|\beta| < \epsilon$, then $\Pi_{B_{l-1}}(\omega) = \Pi_{B_{l-1}}(R^{-\beta}T^{-\alpha}\omega')$.*

Proof. The proof follows directly from elementry geometry. Note that we have a similar result in any finite dimension.

The following lemma is analogous to the Lebesgue number theorem: roughly, given any open cover $\mathcal{A}$ of a compact metric space X, then any open ball of sufficiently small diameter will be contained in some element of $\mathcal{A}$.

Lemma 1. *Assume given $l' \in \mathbf{N}$, and an open cover $\{C_{l'}(\omega_i, \epsilon) : i = 1, \cdots, m\}$ of X. Then exists $l_0 \in \mathbf{N}$ such that given $\omega \in X$ we can find q such that*

$$C_{T^{(j,k)}B_{l+l'+3}}(\omega, \epsilon/N_{l+l'+3}) \subset C_{l'}(\omega_q, \epsilon)$$

for all $||(j,k)||_\infty \leq l$ and $l \geq l_0$.

Proof. The proof will be by contradiction, so suppose there is no $l_0 \in \mathbf{N}$ such that the above statement is true. Then for each $l \in \mathbf{N}$ there exist $\omega_l \in X$ and $||(j_l, k_l)||_\infty \leq l$ such that $C_{T^{(j_l,k_l)}B_{l+l'+3}}(\omega_l, \epsilon/N_{l+l'+3}) \not\subset C_{l'}(\omega_i, \epsilon)$ for all $i = 1, \cdots, m$. Thus there exists an infinite sequence of open sets $\{C_{T^{(j_l,k_l)}B_{l+l'+3}}(\omega_l, \epsilon/N_{l+l'+3})\}$ having the the property that none is contained in any element of $\{C_{l'}(\omega_i, \epsilon)\}$. By the compactness of X we may assume without loss of generality that $\omega_l \to \omega_0$ as $l \to \infty$. Since $\{C_{l'}(\omega_i, \epsilon) \,:\, i = 1, \cdots, m\}$ is an open cover of X there exists $1 \le i_0 \le m$ such that $\omega_0 \in C_{l'}(\omega_{i_0}, \epsilon)$. It is easy to check that there exists $0 < \epsilon_0 < \epsilon$ such that

$$C_{l'+1}(\omega_0, \epsilon_0) \subset C_{l'}(\omega_{i_0}, \epsilon).$$

Let $r = \epsilon/\epsilon_0$. Since $\omega_l \to \omega_0$ as $l \to \infty$ there exists a subsequence $\omega_{l_i} \in C_{l'+3}(\omega_0, \epsilon_0/i)$ for $i \ge 1$. But this implies:

$$\begin{aligned} C_{l'+3}(\omega_{l_i}, \epsilon/N_{l_i}) &= C_{l'+3}(\omega_{l_i}, r\epsilon_0/N_{l_i}) \\ &\subset C_{l'+1}(\omega_0, 2\epsilon_0(1/i + r/N_{l_i})) \\ &\subset C_{l'+1}(\omega_0, \epsilon_0) \qquad \text{for } i \text{ sufficiently large} \\ &\subset C_{l'}(\omega_{i_0}, \epsilon). \end{aligned}$$

The first inclusion above follows by the following calculation. By definition we have

$$\omega_{l_i} \in C_{l'+3}(\omega_0, \epsilon_0/i) \Longrightarrow \Pi_{B_{l'+3}}(T^{\alpha_{l_i}} R^{\beta_{l_i}} \omega_{l_i}) = \Pi_{B_{l'+3}}(\omega_0)$$

for some $||\alpha_{l_i}||_\infty$, $|\beta_{l_i}| < \epsilon_0/i$ and

$$x \in C_{l'+3}(\omega_{l_i}, r\epsilon_0/N_{l_i}) \Longrightarrow \Pi_{B_{l'+3}}(T^{\alpha} R^{\beta} x) = \Pi_{B_{l'+3}}(\omega_{l_i})$$

for some $||\alpha||_\infty$, $|\beta| < r\epsilon_0/N_{l_i}$. Therefore by two applications of the Shrinking Lemma we have

$$\Pi_{B_{l'+1}}(T^{\alpha_{l_i}} R^{\beta_{l_i}} T^{\alpha} R^{\beta} x) = \Pi_{B_{l'+1}}(\omega_0)$$

and so

$$\Pi_{B_{l'+1}}(T^{\alpha_{l_i}+R^{-\beta_{l_i}}(\alpha)}R^{\beta+\beta_{l_i}}x) = \Pi_{B_{l'+1}}(\omega_0).$$

Note that $||\alpha_{l_i} + R^{-\beta_{l_i}}(\alpha)||_\infty \leq ||\alpha_{l_i}||_\infty + ||R^{-\beta_{l_i}}(\alpha)||_\infty < \epsilon_0/i + r\epsilon_0/N_{l_i}$ and $||\beta + \beta_{l_i}||_\infty < \epsilon_0/i + r\epsilon_0/N_{l_i}$. Thus we have that $x \in C_{l'+1}(\omega_0, 2\epsilon_0(1/i + r/N_{l_i}))$. But x was arbitrary, so we have the first inclusion above.

Since $T^{(j,k)}B_{l+l'+3} \supset B_{l'+3}$ for all $||(j,k)||_\infty \leq l$ we have that

$$C_{T^{(j,k)}B_{l+l'+3}}(\omega_l, \epsilon/N_{l+l'+3}) \subset C_{B_{l'+3}}(\omega_l, \epsilon/N_{l+l'+3}) \subset C_{B_{l'+3}}(\omega_l, \epsilon/N_l).$$

Thus for i sufficiently large

$$C_{T^{(j,k)}B_{l_i+l'+3}}(\omega_{l_i}, \epsilon/N_{l_i+l'+3}) \subset C_{l'}(\omega_{i_0}, \epsilon)$$

which contradicts our original supposition. $\square$

We need to introduce the following notation. For all $l \in \mathbf{N}$ and $\epsilon > 0$ let

$$\epsilon_l = \frac{\epsilon}{N_l^2} \tag{1}$$

and let

$$C(l, \epsilon) = \{C_l(\omega_i, \epsilon_l) \,:\, i = 1, \cdots, m'\}$$

be an open cover of X satisfying

$$\omega_j \notin \bigcup_{i<j} C_l(\omega_i, \epsilon_l) \qquad \text{for all } j = 2, \cdots, m'. \tag{2}$$

Let

$$E(l, \epsilon) = \{\omega_i \,:\, i = 1, \cdots, m'\}$$

denote the collection of *centers* of $C(l,\epsilon)$.

Lemma 2. *For any tiling system*

$$h_{top}(T,\mathbf{R}^2) \leq \lim_{\epsilon\to 0^+} \limsup_{l\to\infty} \frac{\log|C(l,\epsilon)|}{N_l^2}.$$

Proof. We know that $h_\mu(T,\mathbf{R^2}) \leq h_\mu(T,\mathbf{Z^2})$ for μ $\mathbf{R^2}$-invariant [8]. By the variational principle we have that

$$h_{top}(T,\mathbf{Z^2}) = \sup_\mu h_\mu(T,\mathbf{Z^2}) \text{ and } h_{top}(T,\mathbf{R}^2) = \sup_\mu h_\mu(T,\mathbf{R}^2)$$

(where the supremum is taken over $\mathbf{Z^2}$ and $\mathbf{R^2}$ invariant measures, respectively). As the set of $\mathbf{Z^2}$-invariant measures contains the set of $\mathbf{R^2}$-invariant measures, it follows that $h_{top}(T,\mathbf{R^2}) \leq h_{top}(T,\mathbf{Z^2})$. Thus it is enough to consider $G=$ $\mathbf{Z^2}$ in what follows. So we take the sequence $\{A_l\ :\ \ l\in\mathbf{N}\}$ in $\mathbf{Z^2}$ as in the following:

$$A_l = \ \{(j,k)\in\mathbf{Z^2}\ :\ \ ||(j,k)||_\infty \leq l\}.$$

Let $C(l,\epsilon)$ be as above. Given l (l as in Lemma 1 is sufficient large), $l'\in\mathbf{N}$ and $\epsilon>0$ we will first show that

$$H(\bigvee_{||(j,k)||_\infty\leq l} T^{-(j,k)}C(l',\epsilon)) \leq \log|C(l+l'+3,\epsilon)|. \tag{3}$$

Given $\omega\in C_{l+l'+3}(\omega_i,\epsilon_{l+l'+3})$ there exist (by definition) $||\alpha_\omega||_\infty < \epsilon_{l+l'+3}$ and $|\beta_\omega| < \epsilon_{l+l'+3}$ such that

$$\Pi_{B_{l+l'+3}}(T^{\alpha_\omega}R^{\beta_\omega}\omega) = \Pi_{B_{l+l'+3}}(\omega_i)$$

$$\Longrightarrow \Pi_{T^{(j,k)}B_{l+l'+3}}(T^{(j,k)}T^{\alpha_\omega}R^{\beta_\omega}\omega) = \Pi_{T^{(j,k)}B_{l+l'+3}}(T^{(j,k)}\omega_i)$$

$$(4) \qquad \Longrightarrow \Pi_{T^{(j,k)}B_{l+l'+3}}(T^{\alpha_\omega+\beta_\omega(j,k)}R^{\beta_\omega}T^{(j,k)}\omega) = \Pi_{T^{(j,k)}B_{l+l'+3}}(T^{(j,k)}\omega_i)$$

where $R^{\beta_\omega}T^{(j,k)} = T^{R^{-\beta_\omega}(j,k)}R^{\beta_\omega} = T^{(j,k)-\beta_\omega(j,k)}R^{\beta_\omega}$. Note that

$$(5)\ ||\alpha_\omega + \beta_\omega(j,k)||_\infty \le ||\alpha_\omega||_\infty + ||\beta_\omega(j,k)||_\infty \le \epsilon_{l+l'+3} + 2l\epsilon_{l+l'+3} < N_{l+l'+3}\epsilon_{l+l'+3}$$

Thus for each $||(j,k)||_\infty \le l$ (l as in Lemma 1 is sufficiently large) we have that

$$\begin{aligned} T^{(j,k)}\omega &\in C_{T^{(j,k)}B_{l+l'+3}}(T^{(j,k)}\omega_i, N_{l+l'+3}\epsilon_{l+l'+3}) \qquad \text{(by 4 and 5)} \\ &= C_{T^{(j,k)}B_{l+l'+3}}(T^{(j,k)}\omega_i, \epsilon/N_{l+l'+3}) \\ &\subset C_{l'}(\omega_q, \epsilon) \qquad \text{(by Lemma 1)}, \end{aligned}$$

where $q = q(i,j,k)$, $C_{l'}(\omega_{q(i,j,k)}, \epsilon) \in C(l', \epsilon)$. Thus for each $\omega \in C_{l+l'+3}(\omega_i, \epsilon_{l+l'+3})$ we have that $\omega \in T^{-(j,k)}C_{l'}(\omega_{q(i,j,k)}, \epsilon)$ for all $||(j,k)||_\infty \le l$. Therefore

$$C_{l+l'+3}(\omega_i, \epsilon_{l+l'+3}) \subset \bigcap_{||(j,k)||_\infty \le l} T^{-(j,k)}C_{l'}(\omega_{q(i,j,k)}, \epsilon).$$

Whence the cover $\{\bigvee_{||(j,k)||_\infty \le l} T^{-(j,k)}C(l', \epsilon)\}$ has a subcover with cardinality at most $|C(l + l' + 3, \epsilon)|$. We use this fact in $(*)$ below.

Finally, given an open cover O of X there exist $l' \in \mathbf{N}$ and $\epsilon > 0$ such that $C(l', \epsilon)$ refines O. It follows that

$$\begin{aligned} &h_{top}(T,\ \mathbf{R^2}) \le h_{top}(T,\ \mathbf{Z^2}) \\ &= \sup_O \lim_{l\to\infty} \frac{1}{N_l^2} H(O^{A_l}) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{O} \lim_{l\to\infty} \frac{1}{N_l^2} H(\bigvee_{||(j,k)||_\infty \le l} T^{-(j,k)} O) \\
&= \sup_{l',\epsilon} \lim_{l\to\infty} \frac{1}{N_l^2} H(\bigvee_{||(j,k)||_\infty \le l} T^{-(j,k)} C(l',\epsilon)) \\
&\le \sup_{l',\epsilon} \limsup_{l\to\infty} \frac{\log |C(l+l'+3,\epsilon)|}{N_l^2} \quad (\text{by } *) \\
&= \sup_{\epsilon} \limsup_{l\to\infty} \frac{\log |C(l,\epsilon)|}{N_l^2} \\
&= \lim_{\epsilon\to 0^+} \limsup_{l\to\infty} \frac{\log |C(l,\epsilon)|}{N_l^2}. \ \square
\end{aligned}
$$

We need to introduce some further notation. Let

$$
\begin{aligned}
\eta &= \text{the number of prototiles} \\
\Delta &= \max_{1\le i\le \eta} \{diam(p_i)\} \\
A &= \min_{1\le i\le \eta} \{area(p_i)\} \\
a &= \frac{16(1+\Delta)}{A}
\end{aligned}
$$

Lemma 3. *Given* $0<\epsilon<1/2$, $l \in \mathbf{N}$, $\omega \in X$ *and* $E(l,\epsilon)$,

(6) $$|\{\omega_i \in C_l(\omega,\epsilon_l) \,:\, \omega_i \in E(l,\epsilon)\}| < 7^3 \eta^{aN_l}.$$

Proof.

Step 1. Suppose $C_{l+1}(\omega,\epsilon_l)$ contains exactly q different $\omega_i \in E(l,\epsilon)$, say $\omega_1,\cdots,\omega_q$, such that

(7) $$\omega_j \notin \bigcup_{i<j} C_l(\omega_i,\epsilon_l)$$

for all $j = 2, \cdots, q$. We will prove that $q \leq 7^3$.

By definition there exists for each $i = 1, \cdots, q$ some $||\alpha_i||_\infty < \epsilon_l$ and $|\beta_i| < \epsilon_l$ such that

$$\Pi_{B_{l+1}}(T^{\alpha_i} R^{\beta_i} \omega_i) = \Pi_{B_{l+1}}(\omega)$$

which implies

$$\Pi_{B_{l+1}}(T^{\alpha_i} R^{\beta_i} \omega_i) = \Pi_{B_{l+1}}(T^{\alpha_j} R^{\beta_j} \omega_j)$$

for all i, j. By the Shrinking Lemma we have then

$$\Pi_{B_l}(T^{R^{\beta_j}(\alpha_i - \alpha_j)} R^{\beta_i - \beta_j} \omega_i) = \Pi_{B_l}(\omega_j). \tag{8}$$

For each α_i and β_i choose j_i, k_i, $n_i \in \mathbf{Z}$ such that $||\alpha_i - (j_i, k_i)\epsilon_l/2||_\infty \leq \epsilon_l/4$ and $|\beta_i - n_i\epsilon_l/2| \leq \epsilon_l/4$. Note that $||(j_i, k_i)||_\infty < 3$ and $|n_i| < 3$. For each i define $\Delta(\alpha_i)$ and $\Delta(\beta_i)$ via

$$\alpha_i = (j_i, k_i)\epsilon_l/2 + \Delta(\alpha_i) \tag{9}$$

$$\beta_i = n_i\epsilon_l/2 + \Delta(\beta_i). \tag{10}$$

We claim that if $(j_r, k_r) = (j_s, k_s)$ and $n_r = n_s$ then $\omega_r = \omega_s$. Assume without loss of generality that $r > s$, then by (8), (9) and (10) we have

$$\Pi_{B_l}(T^{R^{\beta_s}(\Delta(\alpha_r) - \Delta(\alpha_s))} R^{\Delta(\beta_r) - \Delta(\beta_s)} \omega_r) = \Pi_{B_l}(\omega_s).$$

Note that $||R^{\beta_s}(\Delta(\alpha_r) - \Delta(\alpha_s))||_\infty < \epsilon_l$ and $|\Delta(\beta_r) - \Delta(\beta_s)| < \epsilon_l$. Thus we get $\omega_r \in C_l(\omega_s, \epsilon_l)$, which contradicts (7). So it is impossible for two different ω_i's to have the same j_i, k_i, n_i. The possible number of such triples (j,k,n) is $7^2 \cdot 7$. Thus $C_{N_l+1}(\omega, \epsilon_l)$ contains at most 7^3 different ω_i's satisfying (7); i.e., $q \leq 7^3$.

Step 2. Recall that $D_{l+1} = B_{l+1} \setminus int(B_l)$. Consider the collection of projections given by

$$\mathcal{F}_{l,\omega} = \{\Pi_{B_{l+1}}(\omega') \,:\, \omega' \in X,\, \Pi_{B_l}(\omega') = \Pi_{B_l}(\omega),\ \Pi_{D_{l+1}}(\omega') \neq \Pi_{D_{l+1}}(\omega)\}.$$

$\mathcal{F}_{l,\omega}$ represents the different ways of extending $\Pi_{B_l}(\omega)$ to a patch covering the square B_{l+1}. Associate to each different way one (and only one) $\omega' \in X$ and let $F_{l,\omega}$ denote this collection of tilings. Then $F_{l,\omega}$ has the property that $\Pi_{B_{l+1}}(F_{l,\omega}) = \mathcal{F}_{l,\omega}$ but $\Pi_{B_{l+1}}(\tilde{F}) \neq \mathcal{F}_{l,\omega}$ for all $\tilde{F} \subset F_{l,\omega}$. We claim that $|F_{l,\omega}| \leq \eta^{aN_l}$.

We note that the annulus $B_{l+1+\Delta} \setminus int(B_{l-\Delta})$ has area

$$\begin{aligned}(N_l + 2 + 2\Delta)^2 - (N_l - 2\Delta)^2 &= (2N_l + 2) \cdot 2(2\Delta + 1) \\ &< 8(1+\Delta)(N_l + 1) \\ &< 16(1+\Delta)N_l\end{aligned}$$

Thus the number of tiles which have nonempty intersection with $B_{l+1} \setminus int(B_l)$ is at most

$$\frac{16(1+\Delta)N_l}{A} = aN_l. \tag{11}$$

from which it follows immediately from the definition of $F_{l,\omega}$ that $|F_{l,\omega}| \leq \eta^{aN_l}$.

Step 3. We show that

$$C_l(\omega, \epsilon_l) = \bigcup_{\omega' \in F_{l,\omega}} C_{l+1}(\omega', \epsilon_l). \tag{12}$$

Let $x \in C_l(\omega, \epsilon_l)$. Then by definition there exist $||\alpha||_\infty < \epsilon_l,\ |\beta| < \epsilon_l$ such that

$$\begin{aligned}
&\Pi_{B_l}(T^\alpha R^\beta x) = \Pi_{B_l}(\omega) = \Pi_{B_l}(\omega') \qquad \text{for all } \omega' \in F_{l,\omega} \\
&\quad \Longrightarrow \Pi_{D_{l+1}}(T^\alpha R^\beta x) = \Pi_{D_{l+1}}(\omega'_x) \qquad \text{for some } \omega'_x \in F_{l,\omega} \\
&\quad \Longrightarrow \Pi_{B_{l+1}}(T^\alpha R^\beta x) = \Pi_{B_{l+1}}(\omega'_x) \\
&\quad \Longrightarrow x \in C_{l+1}(\omega'_x, \epsilon_l).
\end{aligned}$$

Therefore $C_l(\omega, \epsilon_l) \subset \bigcup_{\omega' \in F_{l,\omega}} C_{l+1}(\omega', \epsilon_l)$. The opposite inclusion is immediate. This establishes (12).

Step 4. Finally, given $\omega_i \in C_l(\omega, \epsilon_l)$, $\omega_i \in E(l, \epsilon)$, by (12) we have that

$$\omega_i \in C_{l+1}(\omega', \epsilon_l)$$

for some $\omega' \in F_{l,\omega}$. But for each fixed $\omega' \in F_{l,\omega}$ there exist at most 7^3 $\omega_i \in E(l, \epsilon)$ such that $\omega_i \in C_{l+1}(\omega', \epsilon_l)$. Since $|F_{l,\omega}| \le \eta^{aN_l}$ it follows that

$$|\{\omega_i \in C_l(\omega, \epsilon_l) \,:\, \omega_i \in E(l, \epsilon)\}| \le 7^3 \eta^{aN_l}. \ \Box$$

Lemma 4. *Given $l \in \mathbf{N}$ let $D(l, \epsilon) = \{D_l(\psi_i, \epsilon_l) \,:\, i = 1, \cdots, m''\}$ be an open cover of X such that*

$$\psi_j \notin \bigcup_{i<j} D_l(\psi_i, \epsilon_l) \qquad \text{for all } \ j = 2, \cdots, m''. \tag{13}$$

Then there exists $l_0 \in \mathbf{N}$ such that for all $l \ge l_0$,

$$|D(l, \epsilon)| \le \left[\eta \frac{4N_l}{\epsilon_l^2} \left(\left[\frac{2\pi}{\epsilon_l}\right] + 1 \right)\right]^{aN_l}. \tag{14}$$

Proof. Associate to each prototile p_i a point x_i in its interior $int(p_i)$. Given any tiling of the plane the maximum number of tiles with nonempty intersection with $D_l = B_l \setminus int(B_{l-1})$ is less than aN_l (the calculation is similar to that of (11)).

Construct a grid with regular spacing ϵ_l and covering the annulus D_l. Denote by $C(j,k)$ the cell having lower left hand corner coordinates $(j,k)\epsilon_l$. There are at most $4N_l/\epsilon_l^2$ such cells with nonempty intersection with the annulus.

Define a mapping $\phi : X \to \{\text{matrices over } \mathbf{Z^2}\}$ by $\phi(\psi) = (\psi_{jk})$, where

$$\psi_{jk} = \begin{cases} (0,0) & \text{if } x_i \notin \Pi_{C(j,k)}(\psi) \text{ for all } i = 1, \cdots, \eta \\ \\ (n,i) & \text{if } T^{(j+\delta j, k+\delta k)\epsilon_l} R^{(n+\delta n)\epsilon_l} p_i \subset \psi \text{ for some } 0 \le \delta j, \delta k, \delta n < 1 \end{cases}$$

where $n \in \left\{0, \cdots, \left[\frac{2\pi}{\epsilon_l}\right]\right\}$, $i \in \{1, \cdots, \eta\}$. It should be noted that there exists $l_0 \in \mathbf{N}$ such that ϕ is well defined for all $l \ge l_0$, for then the cells $C(j,k)$ are sufficiently small that they can contain at most one of the x_i's.

Claim. $D_l(\psi_i, \epsilon_l) \neq D_l(\psi_j, \epsilon_l)$ implies $\phi(\psi_i) \neq \phi(\psi_j)$.

Proof of claim. The proof will be by contradiction. Suppose $\phi(\psi_i) = \phi(\psi_j)$, and $j > i$ then $\Pi_{D_l}(\psi_i) = \Pi_{D_l}(T^\alpha R^\beta \psi_j)$, for some $||\alpha||_\infty < \epsilon_l$, $|\beta| < \epsilon_l$. But this implies $\psi_j \in D_l(\psi_i, \epsilon_l)$ which contradicts (13). This proves the claim.

Thus the mapping ϕ is injective on the set $\{\psi_i \,|\, i = 1, \cdots, m''\}$. Observe that for any given $\psi \in X$, $\phi(\psi)$ has at most aN_l nonzero entries amongst a total number of $4N_l\epsilon_l^2$ entries ψ_{jk}. Since there are $\eta\left(\left[\frac{2\pi}{\epsilon_l}\right] + 1\right)$ possible values for each entry it follows immediately that:

$$
\begin{aligned}
|D(l,\epsilon)| &\le |\{(\psi_{jk})\}| \\
&\le C_{aN_l}^{4N_l/\epsilon_l^2}\eta^{aN_l}\left(\left[\frac{2\pi}{\epsilon_l}\right]+1\right)^{aN_l} \\
&\le \left[\eta\frac{4N_l}{\epsilon_l^2}\left(\left[\frac{2\pi}{\epsilon_l}\right]+1\right)\right]^{aN_l}. \ \square
\end{aligned}
$$

Lemma 5. *If $h_{top}(T,\mathbf{R}^2)=\alpha>0$ then there exist $\epsilon>0$ and infinitely many $l_i\in\mathbf{N}$ such that*

$$\frac{|C(l_i,\epsilon)|}{|D(l_i,\epsilon)|}>7^3\eta^{aN_{l_i}}, \tag{15}$$

for all $C(l_i,\epsilon)$ and $D(l_i,\epsilon)$.

Proof. Given $\epsilon>0$ there exists $l^{'}\in\mathbf{N}$ such that

$$\frac{\exp^{\frac{\alpha}{2}N_l^2}}{\left(\eta\frac{4N_l}{\epsilon_l^2}\left(\left[\frac{2\pi}{\epsilon_l}\right]+1\right)\right)^{aN_l}}>7^3\eta^{aN_l} \tag{16}$$

for all $l\ge l^{'}$. By Lemma 2 there exists $\epsilon>0$ and l_i such that

$$|C(l_i,\epsilon)|>\exp^{\frac{\alpha}{2}N_{l_i}^2} \tag{17}$$

for all $l_i>l^{'}$. The conclusion then follows by Lemma 4. $\square$

Lemma 6. *If $h_{top}(T,\mathbf{R}^2)=\alpha>0$, then there exist $l\in\mathbf{N}$ and $\omega\neq\omega'$ such that $\Pi_{D_l}(\omega)=\Pi_{D_l}(\omega')$ but $\Pi_{B_l}(\omega)\neq\Pi_{B_l}(\omega')$.*

Proof. The proof will be by contradiction; assume that for any $\omega, \omega' \in X$ $\Pi_{D_l}(\omega) = \Pi_{D_l}(\omega')$ implies $\Pi_{B_l}(\omega) = \Pi_{B_l}(\omega')$. Choose covers $C(l, \epsilon) = \{C_l(\omega_i, \epsilon_l) \,|\, i = 1, \cdots, m'\}$ and $D(l, \epsilon) = \{D_l(\psi_i, \epsilon_l) \,|\, i = 1, \cdots, m''\}$ such that Lemma 5 holds. By hypothesis it follows that

$$C_l(\psi_i, \epsilon_l) = D_l(\psi_i, \epsilon_l)$$

for all $i = 1, \cdots, m''$. Thus $C'(l, \epsilon) = \{C_l(\psi_i, \epsilon_l) \,|\, i = 1, \cdots, m''\}$ is a cover of X (satisfying (2) by (13)) with $|D(l, \epsilon)| = |C'(l, \epsilon)|$. But by Lemma 5 we must have

$$\frac{|C(l, \epsilon)|}{|C'(l, \epsilon)|} > 7^3 \eta^{aN_l}.$$

This implies that there exists some $C_l(\psi_i, \epsilon_l)$ which contains more than $7^3\eta^{aN_l}$ different ω_i's. This contradicts Lemma 3. □

Now we can complete the proof of the Theorem.

Proof of Theorem. The proof will be by contradiction; assume $h_{top}(T, \mathbf{R}^2) = \alpha > 0$. By restricting our attention to the support of μ we may assume without loss of generality that the space X is minimal; in particular, all open sets in X have nonzero measure. By Lemma 6 there exist $\omega_0 \neq \omega_1$ such that

$$\Pi_{D_l}(\omega_0) = \Pi_{D_l}(\omega_1) \tag{18}$$

but

$$\Pi_{B_l}(\omega_0) \neq \Pi_{B_l}(\omega_1). \tag{19}$$

In other words there exists some patch $P = \Pi_{B_l}(\omega_0)$ the interior to whose boundary, $\Pi_{D_l}(\omega_0)$, may be filled in two different ways: $\Pi_{B_{l-1}}(\omega_0)$ and $\Pi_{B_{l-1}}(\omega_1)$.

By the Local Isomorphism Theorem [16] there exists some $r > 0$ such that an isometric copy of P must occur in any ball of radius r; that is, for each $\omega \in X$, $c \in \mathbf{R^2}$ there exist $\alpha \in \mathbf{R^2}$ and $\beta \in [0, 2\pi)$ such that $T^\alpha R^\beta(P) \subset \Pi_{B_c(r)}(\omega)$ (where $B_c(r)$ denotes the square centered at c of radius r). It follows from this and equations (18) and (19) that given $n \in \mathbf{N}$ there exist at least 2^{n^2} different patches

$$H = \{P_i \subset \omega_i \,|\, i \in I\}$$

such that

$$\Pi_{B_{nr}}(\omega_i) \neq \Pi_{B_{nr}}(\omega_0) \qquad \text{for all } i \in I \tag{20}$$

$$\Pi_{D_{nr}}(\omega_i) = \Pi_{D_{nr}}(\omega_0) \qquad \text{for all } i \in I \tag{21}$$

where I is some index set, $|I| \geq 2^{n^2}$. In particular $\omega_j \notin C_{nr}(\omega_i, \epsilon_{nr})$ for all $j \neq i$; and so Lemma 3 applies, replacing $E(l, \epsilon)$ with $\{\omega_i \,|\, i \in I\}$. Given ϵ sufficiently small choose n such that

$$2^{n^2} > 7^3 \eta^{aN_{nr}} \left(\frac{2nr}{\epsilon_{nr}}\right)^2 \left(\left[\frac{2\pi}{\epsilon_{nr}}\right] + 1\right).$$

Claim. There exists $i \in I$ such that

$$T^t\omega_0 \notin C_{nr+1}(\omega_i, \epsilon_{nr}) \qquad \text{for all } ||t||_\infty \leq nr + 1.$$

Proof of claim. The proof will be by contradiction. Suppose that for each $i \in I$ there exists t_i, $||t_i||_\infty \leq nr+1$ such that $T^{t_i}\omega_0 \in C_{nr+1}(\omega_i, \epsilon_{nr})$. Then by definition there exist $||\alpha_i||_\infty$, $|\beta_i| < \epsilon_{nr}$ such that

$$\Pi_{B_{nr+1}}(T^{\alpha_i}R^{\beta_i}T^{t_i}\omega_0) = \Pi_{B_{nr+1}}(\omega_i)$$

$$\Longrightarrow \Pi_{B_{nr+1}}(R^{\beta_i}T^{\alpha_i + R^{-\beta_i}(\alpha_i)}T^{t_i}\omega_0) = \Pi_{B_{nr+1}}(\omega_i).$$

Choose $(j,k) \in \mathbf{Z^2}$ such that $||\alpha_i + R^{-\beta_i}(\alpha_i) + t_i - (j,k)\epsilon_{nr}||_\infty \leq \epsilon_{nr}/2$ and let $\gamma_i = \alpha_i + R^{-\beta_i}(\alpha_i) + t_i - (j,k)\epsilon_{nr}$. Then for all $i \in I$ we have

$$\Pi_{B_{nr+1}}(R^{\beta_i}T^{\gamma_i}T^{(j,k)\epsilon_{nr}}\omega_0) = \Pi_{B_{nr+1}}(\omega_i)$$

$$\Longrightarrow \Pi_{B_{nr}}(T^{(j,k)\epsilon_{nr}}\omega_0) = \Pi_{B_{nr}}(T^{-\gamma_i}R^{-\beta_i}\omega_i)$$

$$\Longrightarrow \omega_i \in C_{nr}(T^{(j,k)\epsilon_{nr}}\omega_0, \epsilon_{nr}) \tag{22}$$

Clearly $||(j,k)||_\infty \leq \frac{nr+1+2\epsilon_{nr}}{\epsilon_{nr}} < \left[\frac{2nr}{\epsilon_{nr}}\right]$. Consider the cover by open ϵ_{nr}-balls given by

$$\mathcal{O} = \bigcup \left\{ C_{nr}(T^{(j,k)\epsilon_{nr}}R^{n\epsilon_{nr}}\omega_0, \epsilon_{nr}) \mid ||(j,k)||_\infty \leq \left[\frac{2nr}{\epsilon_{nr}}\right],\ 0 \leq n \leq \left[\frac{2\pi}{\epsilon_{nr}}\right] + 1 \right\}.$$

Denote the union of all open sets in $\mathcal{O}$ by O. By (22) $\omega_i \in O$ for all $i \in I$. Now

$$|H| = |I| \geq 2^{n^2} > 7^3\eta^{aN_{nr}}\left(\frac{2nr}{\epsilon_{nr}}\right)^2\left(\left[\frac{2\pi}{\epsilon_{nr}}\right] + 1\right) \geq 7^3\eta^{aN_{nr}}|\mathcal{O}|.$$

This implies that at least one ϵ_{nr}-ball in $\mathcal{O}$ contains more than $7^3\eta^{aN_{nr}}$ different ω_i's. This contradicts Lemma 3. Thus there exists $i_0 \in I$ such that

$$T^t\omega_0 \notin C_{nr+1}(\omega_{i_0}, \epsilon_{nr}) \qquad \text{for all } ||t||_\infty \leq nr+1. \ \square$$

We now construct ω_0' as follows. Choose M sufficiently large so that by a slight change in the proof of the Local Isomorphism Theorem [16] given any tiling in X there exists an isometric copy $T^\alpha R^\beta \Pi_{B_{3(nr+1)}}(\omega_0)$ of $\Pi_{B_{3(nr+1)}}(\omega_0)$ within the interior of any square of area M for some $\alpha \in \mathbf{R^2}$ and $|\beta| < \epsilon_{nr}$. Consider the plane as comprised of a regular lattice of squares of area M. We change ω_0 to ω_0' by replacing an isometric copy of $\Pi_{B_{nr+1}}(\omega_0) \subset \Pi_{B_{3(nr+1)}}(\omega_0)$ contained in each such square with an isometric copy of $\Pi_{B_{nr+1}}(\omega_{i_0})$. It is easy to see that this operation is well defined and consequently ω_0' is a tiling. Let χ be the characteristic function for $C_{nr+1}(\omega_{i_0}, \epsilon_{nr})$. By the above operation and the claim we have the following:

For $l > 0$ large enough we have

$$\frac{1}{N_l^2}\int_{B_l} \chi(T^t\omega_0')dt - \frac{1}{N_l^2}\int_{B_l} \chi(T^t\omega_0)dt \geq \frac{\epsilon_{nr} \cdot \frac{N_l^2}{M}}{N_l^2} = \frac{\epsilon_{nr}^2}{M} > 0.$$

But this contradicts unique ergodicity of (X, T, μ). $\square$

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