Critical indices for the Yukawa$_2$ quantum field theory

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Abstract

The understanding of the Yukawa$_2$ quantum field theory is still incomplete if the fermionic mass is much smaller than the coupling. We analyze the Schwinger functions for small coupling uniformly in the mass and we find that the asymptotic behavior of the two-point Schwinger function is anomalous and described by two critical indices, related to the renormalization of the mass and of the wave function. The indices are explicitly computed by convergent series in the coupling.

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1 Introduction

A considerable effort has been devoted in the last years to understand the properties of $d=2$ interacting quantum field models: they are in fact the simplest examples of interacting quantum fields models but at the same time they have a very rich structure, and their properties may give, hopefully, hints on the intricacies that might be present in higher dimensional models.

Among the $d=2$ quantum field models, one of the more investigated is the Yukawa$_2$ model describing the interaction between a massive boson, described by an *Euclidean Bose field* ϕ_x and a massive fermion, described by two *Euclidean Fermi fields* $\psi^\Lambda_x, \bar\psi^\Lambda_x$, $x=(x_0,\vec x)$ and Λ is an *ultraviolet cut–off*, see below. The theory is *regularized* assuming that Λ is finite and $|\vec x|\le L, |cx_0|\le L$ with the fields having periodic boundary conditions. L is called *infrared cut–off*.

The field ϕ_x has propagator:

$$G(x-y)=\int dk\frac{e^{\frac{ik(x-y)}{\hbar}}}{k^2+M^2c^2} \tag{1}$$

Here and in the following $k\equiv(k_0,\vec k)=\frac{2\hbar\pi}{L}(cn_0,\vec n)$, n_0,n integers and $\int dk\equiv\frac{4\pi^2\hbar^2c}{L^2}\sum_{n_0,\vec n}$.

The two-component spinors $\psi_x^\Lambda, \bar\psi_x^\Lambda$ have propagator

$$g^\Lambda(x-y) = \frac{1}{\hbar}\int dk e^{\frac{-k^2}{\Lambda^2}} e^{\frac{ik(x-y)}{\hbar}} \frac{\not k + mcI}{k^2+m^2c^2} \tag{2}$$

with $\not k = ic^{-1}k_0\gamma_0 + \vec k \gamma_1$

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and I is the identity matrix.

The *regularized fermionic Schwinger functions* are given by the following functional integral:

$$S^{\Lambda,L}(x_1,\ldots,x_n,y_1,\ldots,y_n) = \frac{\int P(d\psi^\Lambda)\int P(d\phi)e^{-V^\Lambda}\bar\psi^\Lambda_{x_1}\ldots\bar\psi^\Lambda_{x_n}\psi^\Lambda_{y_1}\ldots\psi^\Lambda_{y_n}}{\int P(d\psi^\Lambda)\int P(d\phi)e^{-V^\Lambda}} \tag{3}$$

where:

$$V^\Lambda(\psi^\Lambda,\phi) = g\int dx\phi_x\bar\psi^\Lambda_x\psi^\Lambda_x + \frac{g^2\alpha^2_\Lambda}{2}\int dx : \phi_x^2 : -gH_\Lambda\int dx\phi_x$$

and :: is the normal order, g is the *coupling*, $g\alpha_\Lambda$ is the *bosonic mass renormalization* and gH_Λ induces a *fermionic mass renormalization.* The *bosonic integration* $\int P(d\phi)$ is a linear functional and it is defined over the monomials of Bose fields by the commutative Wick rule with propagator eq.(1); in the same way the *fermionic integration* $\int P(\psi^\Lambda)$ is defined over the monomials of Fermi fields by the anticommutative Wick rule with propagator eq.(2). Both the integrations are extended to general functions by linearity. Writing $e^{-V^\Lambda} = \sum_{n=1}^\infty \frac{(-V^\Lambda)^n}{n!}$ and collecting the contributions of the same order in g one can see, by the positive definiteness of the integration over ϕ and the Grahm-Hadamard inequality (which exploits the anticommutativity of the Fermi variables), that the series resulting by this procedure are absolutely convergent, see [1], [2] or [3]; in this case we say that the functional integral eq.(3) is well defined. Of course the hard problem is to prove that, at least for a choice of α^2_Λ, H_Λ, the *fermionic Schwinger functions*

$$\lim_{L\to\infty,\Lambda\to\infty} S^{\Lambda,L}(x_1,\ldots,x_n,y_1,\ldots,y_n) = S(x_1,\ldots,x_n,y_1,\ldots,y_n)$$

are well defined expressions.

In [4] the existence of the Schwinger functions was proved when the ultraviolet cut-off, but not the infrared one, is removed: the idea was to integrate out first, in the functional integral giving the Schwinger functions, the fermionic field and to prove that the resulting expression was integrable with respect to the boson integration by the cluster expansion techniques. The infrared cut-off was removed in [1] , *assuming that the free fermionic mass is quite larger than the coupling* $\frac{|g|\hbar}{mc} << 1$.

The aim of this paper is to study the infrared part of the Yukawa$_2$ model obtaining the Schwinger functions for small values of m and g *uniformly in* m by renormalization group methods, see [5]. In particular the case $\frac{|g|\hbar}{mc} >> 1$ is very interesting as, contrary to the opposite case, the infrared properties of the model, like the large distances decay of the two-point Schwinger function, are *qualitatively different* with respect to the free $g = 0$ case: one speaks in this case of *anomalous behavior*. In other words, the infrared behavior of the theory is qualitatively different in situation of strong coupling $|g|$ respect to the case of small coupling, *where strong or small refers to* $\frac{mc}{\hbar}$.

Our main results are summarized by the following theorem:

Theorem 1 *: There exist an* $\varepsilon > 0$ *and functions* $\alpha_\Lambda, H_\Lambda$ *such that, for* $\frac{m}{M}, \frac{\hbar}{Mc}|g| \leq \varepsilon$ *the two point Schwinger function for the Yukawa*$_2$ *model* $S(x-y) \equiv S(x,y)$ *can be written as* $S(x-y) = S_A(x-y) + \lambda S_B(x-y)$ *with*

$$S_A(x-y) = \frac{1}{\hbar}\int dk \frac{e^{i\frac{k(x-y)}{\hbar}}}{Z(k)} \frac{\not k + m(k)cI}{k^2 + m(k)^2c^2} \tag{4}$$

and $m(k), Z(k)$ *two regular functions such that* $|m(k)-m| = O(m(\frac{g\hbar}{Mc})^2)$, $|Z(k)-1| = O((\frac{g\hbar}{Mc})^2)$ *for* $|k| > \frac{Mc}{\hbar}$ *and*

$$m(0) = \frac{m^{1+\eta_2}}{M^{\eta_2}} \qquad Z(0) = \left(\frac{m}{M}\right)^{-\eta_1}$$

with $\eta_2 = \beta_2(\frac{g\hbar}{Mc})^2 + O((\frac{g\hbar}{Mc})^4)$ *and* $\eta_1 = \beta_1(\frac{g\hbar}{Mc})^4 + O((\frac{g\hbar}{Mc})^6)$, $\beta_1, \beta_2 > 0$.

For $|x-y| > (\frac{\hbar}{m(0)c})$ *the following bound holds*

$$|S(x-y)| \leq \frac{C}{Z(0)} \frac{m(0)c^2}{\hbar} \exp\left(-\frac{\kappa m(0)c|x-y|}{\hbar}\right) \tag{5}$$

where C *and* κ *are constants.*

For $(\frac{\hbar}{Mc}) < |x-y| < (\frac{\hbar}{m(0)c})$ *we find, instead*

$$S_A(x-y) = g_0(x-y)\left(\frac{\hbar}{Mc}\right)^{-\eta_3} |x-y|^{-\eta_3}(1 + A(x-y))$$
$$|S_B(x-y)| \leq C\left(\frac{\hbar}{Mc}\right)^{-\eta_3} |x-y|^{-1-\eta_3} c \tag{6}$$

with $\eta_3 = \eta_1(1+\eta_2)^{-1}$, $|A(x-y)| \leq C\frac{mc|x-y|}{\hbar}$ *and* $g_0(x-y) = \lim_{m\to 0, L\to\infty, \Lambda\to\infty} g^\Lambda(x-y)$.

The existence of all the Schwinger functions is an easy corollary of the theorem derivation. The above results show that the Yukawa$_2$ model, despite its extreme simplicity, have a very rich structure. If $\frac{|g|\hbar}{mc} << 1$, according to [1], the asymptotic behavior of $S(x-y)$ is essentially the same of the free Schwinger function eq.(2) *i.e.* the presence of the interaction does not change the physical properties of the system[1]. However in the opposite situation $\frac{|g|\hbar}{mc} >> 1$ the two-point Schwinger function if $|x-y| \leq \hbar m(0)^{-1}c^{-1}$ has an *anomalous* asymptotic behavior very

[1]in this case in fact one can easily verify from the theorem derivation that $m(0) = m(1+O(g))$, $Z(0) = 1+O(g)$ and that the bounds eq.(5) can be written replacing $m(0), Z(0)$ with $m, 1$ and simply changing the constants C, κ. And in eq.(6) $|x-y|^{-\eta_3}$ has to be replaced by $1 + gA(x-y)$, with $A(x-y)$ bounded. So that in this case $S(x-y)$ has the same infrared behavior of $g(x-y)$

close to the two-point Schwinger function behavior of the exactly solvable *massless Thirring model*, see [6]. On the other hand if $|x-y| \geq \hbar m(0)^{-1} c^{-1}$, $S(x-y)$ obeys to a bound similar to the bound for the free Schwinger function, with the difference that m is replaced by $m(0)$ and $\hbar$ by $\hbar Z(0)^{-1}$.

The Schwinger function is written as a sum of two terms, $S = S_A + \lambda S_B$, with $\lim_{\lambda\to 0} S_A = g^\Lambda$. S_A is formally similar to the free fermionic Schwinger function, but the mass m and the wave function normalization $Z = 1$ are replaced by $m(k)$ and $Z(k)$. One would like to interpret this fact saying that $m(k)$ and $Z(k)$ are "essentially" the *interacting fermionic mass* (or *self-energy*) and the *interacting fermionic wave function* of the Yukawa$_2$ model. In order to prove this fact one needs a detailed knowledge of the analytic properties in the complex plane of the Fourier transform of $S(x-y)$, but this is still lacking. Assuming anyway that this is true, what seems to us very likely, our theorem can be interpreted as saying the somewhat stronger information that, if $\frac{|g|\hbar}{mc} >> 1$, the fermionic mass decreases dramatically between a value close to m (the free mass) for momenta of the order or greater than the boson mass $\frac{Mc}{\hbar}$, and crosses over for momenta close to 0 to a value given by $m(\frac{m^{\eta_2}}{M^{\eta_2}})$. In the same way the wave function normalization increases if $\frac{|g|\hbar}{mc} >> 1$ crossing from a value close to 1 for momenta close to the boson mass over to a value $O((\frac{m}{M})^{-\eta_2})$ for momenta close to 0.

In order to obtain our results we use renormalization group methods, see [5][7], and in particular a set of technical tools which, established for other $d = 1$ models, can used almost without change in this case. The renormalization group allows in fact to extract some universal properties which belong to many different models. We shall not repeat here the proof of all the statements which we use, referring for them to the literature: the possibility of referring to the literature is the reason behind our choice of the renormalization group methods (the alternative equivalent Callan-Symanzik approach has not been applied to derive the results that we need).

For the purpose of comparison, we note that the expressions for the mass and the wave function we find in the Yukawa$_2$ model are similar to what is found in another $d = 1$ quantum field model, the *massive Thirring model*, whose spectrum is studied by the Bethe ansatz directly *i.e.* without using the Schwinger functions (which in this model are unknown), see [8],[9],[10].

2 The ultraviolet problem

In the following we set for simplicity $\hbar = c = M = 1$, $\lambda = g^2$ and we start by studing the denominator of eq.(3).

In this section we summarize the non perturbative solution of the ultraviolet problem, following [11]. It is convenient to decompose ψ_x^Λ as $\psi_x^\Lambda = \psi_x^{(i.r.)} + \psi_x^{(u.v.)}$ where $\psi_x^{(i.r.)}$ and $\psi_x^{(u.v.)}$ are *independent* Grassmanian variables with integrations $P(d\psi^{(i.r.)})$ and $P(d\psi^{(u.v.)})$ with propagators respectively $g^{(i.r)}(x-y)$ and $g^{(u.v)}(x-y)$ where:

$$g^{(i.r)}(x-y) = \int dk e^{-k^2} e^{ik(x-y)} \frac{\not k + mI}{k^2+m^2} \tag{7}$$

$$g^{(u.v.)}(x-y) = \int dk (e^{\frac{-k^2}{\Lambda^2}} - e^{-k^2}) e^{ik(x-y)} \frac{\not k + mI}{k^2+m} \tag{8}$$

A power counting argument, see [11], shows that the theory is *superrinormalizable* and that the counterterms in eq.(3) have to be chosen as: $\alpha_\Lambda^2 = -\int dx \mathrm{Tr}(g^\Lambda(x) g^\Lambda(-x))$ and $H_\Lambda = g^{(u.v.)}(0)$; note that $\alpha_\Lambda, H_\Lambda$ are divergent as $\Lambda \to \infty$. We consider the denominator of eq.(3) and write it ("integrating the boson fields") as:

$$\int P(d\psi^\Lambda) P(d\phi) e^{-V^\Lambda(\psi^\Lambda,\phi)} = \int P(d\psi^\Lambda) e^{-\bar V^\Lambda(\psi^\Lambda)} \tag{9}$$

where

$$\bar V(\psi) = -\frac{\lambda}{2} \int dx dy G_{\alpha_\Lambda}(x-y)(\bar\psi_x \psi_x - H_\Lambda)(\bar\psi_y \psi_y - H_\Lambda) + \lambda K_\Lambda$$

where K_Λ is a suitable constant and

$$G_{\alpha_\Lambda}(x-y) = \int dk \frac{e^{ik(x-y)}}{k^2+M^2+\lambda\alpha_\Lambda^2} \tag{10}$$

The integration over $\psi^{(u.v.)}$ gives

$$\int P(d\psi^{(u.v.)}) e^{-\bar V(\psi^\Lambda)} = e^{-V^{0,\Lambda}(\psi^{(i.r.)})} \tag{11}$$

with[2]:

$$\lim_{\Lambda\to\infty} V^{0,\Lambda}(\psi) \equiv V^0(\psi) = -\frac{\lambda}{2}\int dx dy G_{\alpha_0}(x-y) \bar\psi_x \psi_x \bar\psi_y \psi_y +$$
$$+ \sum_{m\geq 0, \sigma_1,\ldots,\sigma_m} \int dx_1 \ldots dx_m W^m(x_1,\ldots,x_m) \psi_{x_1}^{\sigma_1} \ldots \psi_{x_m}^{\sigma_m} \tag{12}$$

where $G_{\alpha_0}(x-y)$ is given by eq.(10), $\alpha_0^2 = O(\lambda)$, $\psi_x^\sigma = \bar\psi_x$ or ψ_x depending if $\sigma = 1$ or -1 and $W^m(x_1,\ldots,x_m)$ are integrable over any $m-1$ variables. We remind that the two functional integrals in eq.(9)(11) are identities between convergent series obtained by linearity using the Wick rule, as explained after eq.(3).

[2] this statement was proved in [11] with $H_\Lambda = g^\Lambda(0)$ obtaining instead of eq.(12) an analogous formula in which the first term is Wick-ordered. Eq.(12) follows trivially from this results with our choice of H_Λ, which we find more natural as H_Λ admits a bound *uniform* in m, *i.e.* $H_\Lambda \leq C\Lambda$ with C independent on m.

3 The infrared integration

It is convenient to write the fermionic integration in a different way

$$\int P(\psi^{(\leq 0)}) e^{-V^0(\psi^{\leq 0)})} \equiv \tag{13}$$
$$\int \{\mathcal{D}\psi^{(\leq 0)} e^{-\int dk C_0(k) \bar\psi_k^{(\leq 0)}(-\not{k}+m)\psi_k^{(\leq 0)}}\} e^{-V^0(\psi^{(\leq 0)})}$$

where $C_0(k)^{-1} = e^{k^2}$ and $\psi^{(\leq 0)} = \psi^{(i.r.)}$.

Eq.(13) is a definition, as the meaning of the functional integral remains the same explained after eq.(3) *i.e.* it is a series in λ and each term is computed by the Wick rule: however it is rather suggestive of the fact that the above functional integral has some properties of the Gaussian integrals: for instance one can put the terms quadratic in the fields out of the integration

$$\int \{\mathcal{D}\psi^{(\leq 0)} e^{-\int dk C_0(k) \bar\psi_k^{(\leq 0)}(-\not{k}+m)\psi_k^{(\leq 0)}}\} e^{-V^0(\psi^{(i.r.)})} = \tag{14}$$
$$\int \{\mathcal{D}\psi^{(\leq 0)} e^{-\int dk C_0(k) \bar\psi_k^{(\leq 0)}(-\not{k})\psi_k^{(\leq 0)}}\} e^{-m\int dk C_0(k)\bar\psi_k^{(\leq 0)}\psi_k^{(\leq 0)} - V^0(\psi^{(\leq 0)})}$$

where we denote, from now on, with $\int\{\mathcal{D}\psi e^{-\int dk \bar\psi h(k)^{-1}\psi}\}$ a linear functional acting on monomials of field $\psi_x, \bar\psi_x$ by the anticommutative Wick rule with propagator $\int dk e^{ik(x-y)} h(k)$. The identity eq.(14) is an equality between two perturbative series (convergent if the infrared cut-off is not removed) with a massless or massive propagator, *i.e.* the series in the l.h.s of eq.(14) is a *resummation* of the series at the r.h.s. obtained by a change in the propagator. If one introduces another ultraviolet regularization assuming that the fermions are in a lattice so that the number of ψ becomes finite and $\mathcal{D}\psi^{(\leq 0)} = \prod_{\vec k,\vec\omega} d\psi^{+,(\leq 0)}_{k,\vec\omega} d\psi^{-,(\leq 0)}_{k,\vec\omega}$ where $\vec\omega = \pm 1$, $\psi^{-,(\leq 0)}_{k,\vec\omega}$, $\psi^{+,(\leq 0)}_{k,-\vec\omega}$ are the components respectively of the spinor ψ_k and $\bar\psi_k$, then the integration should be written really as an exponential of a quadratic form.

In the following it will be an essential tool of our analysis the fact that the terms quadratic in the fields can be put from the exponent of the integrand to the integration and *vice versa* *i.e.* everything goes as the fermionic integration were really the exponential of a quadratic form in the fields. We use this property just now writing the functional integral we have to study as the r.h.s of eq.(14); this means that we prefer to consider the mass term as a perturbation, as we are looking for results uniform in m.

We decompose the Grassmanian integration $\{\mathcal{D}\psi^{(\leq 0)} e^{-\int dk C_0(k)\bar\psi_k^{(\leq 0)}(-\not{k})\psi_k^{(\leq 0)}}\}$ into a product of independent Grassmanian integrations. This can be done by setting $g^{(i.r.)}(x-y) = \sum_{h=-\infty}^0 g^{(h)}(x-y)$ and by writing $\psi_x^{(\leq 0)} = \sum_{h=-\infty}^0 \psi_x^{(h)}$, with $\psi_x^{(h)}$ being a family of Grassmanian fields with vanishing "cross propagator" (*i.e.* an independent family of variables) and with "self propagator":

$$g^{(h)}(x-y) = \int dk e^{ikx}(e^{-k^2\gamma^{-2h-2}} - e^{-k^2\gamma^{-2h}})\frac{\not{k}}{k^2} \equiv \int e^{ikx} f(k^2\gamma^{-2h})\frac{\not{k}}{k^2} \tag{15}$$

where $\gamma > 1$. The number h is called *scale*.

Note that $g^{(h)}(x-y)$ has good scaling properties *i.e.* $g^{(h)}(x-y) = \gamma^h g^0(\gamma^h(x-y))$ and $|g^{(h)}(x)| \leq A\gamma^h e^{-\kappa\gamma^h|x|}$, if A is a constant matrix and κ is a constant; moreover $|g^{(h)}(k)| \leq A\gamma^{-h}e^{-\kappa\gamma^{-h}|k|}$. We define $\psi_x^{(\leq h)} = \sum_{k=-\infty}^{h}\psi_x^{(k)}$ and we label the spin components of $\psi_x^{(h)}$, $\psi_x^{(\leq h)}$ by the index $\vec{\omega} = \pm 1$.

A naive definition of the *effective potential* is, if $f_h(k) = f(\gamma^{-2k}k^2)$:

$$e^{-V^h(\psi^{(\leq h)})} = \int \prod_{k=h+1}^{0}\{\mathcal{D}\psi^{(k)}e^{-\int dk f_h(k)^{-1}\bar\psi_k^{(k)}(-\not k)\psi_k^{(k)}}\}e^{-\tilde V^0(\psi^{(\leq 0)})}$$

with

$$\tilde V^0(\psi^{(\leq 0)}) \equiv m\int dk C_0(k)\bar\psi_k^{(\leq 0)}\psi_k^{(\leq 0)} + V^0(\psi^{(\leq 0)})$$

where again the meaning is that the r.h.s is given by a perturbative series identical to the one obtained by the l.h.s writing the exponential as a series.

By a standard argument one can verify, see for instance [5], that $V^h(\psi)$ is given by a sum of terms of the form

$$\int dk_1\ldots dk_n f^{n,h}(k_1,\ldots,k_n)\prod_{i=1}^{n}\psi^{\sigma_i}_{k_i,\vec\omega_i}\delta(\sum_{i=1}^{n}\sigma_i k_i)$$

and $f^{n,h}$ are expressed by the sum of the values of suitable *Feynman diagrams*. A k-th order diagram contributing to $f^{n,h}$ can be obtained as usual from k graph elements which represent the addends in eq.(12), formed by vertices with m emerging oriented half-lines with indices k_i, $h_i > h$, symbolizing the fields $\psi_{k_i}^{h_i}$. One pairs the half lines with consistent orientation and same indices and leaves n half-lines not paired. The value of the diagram is obtained by multiplying the propagators $g^{h_i}(k_i)$ associated to each paired line and the kernels in eq.(12) associated to the vertices, and integrating this product over all the momenta k_i associated with paired half-lines.

As usual the connected subgraph formed by lines with scale $\geq h_v$ are called *clusters* with scale h_v and denoted by v; if h'_v is the maximal among the scales of the lines entering in or coming out the cluster one can verify, by a power counting argument, see for instance [5], that the generic k-order diagram is bounded by $C^k\varepsilon^k\prod_v\gamma^{-D_v(h_v-h'_v)}$ where $h'_v < h_v$, C is a constant, $\varepsilon = \max(\lambda, m)$, the product runs over all the clusters v with scale h_v and $D_v > 0$ for each except the clusters with four external lines, for which $D_v = 0$, and the clusters with two external lines, for which $D_v = -1$. Then the graphs with clusters with four or two external lines do not admit a bound *uniform* in h and one has to set a different perturbative expansion. This is done, in the renormalization group framework, see [5], by introducing a *localization operator* in the following way. $\mathcal{L}V_m^h = 0$ if $m > 4$ while

$$\mathcal{L}\int\prod_{i=1}^{4}dk_i\sum_n f^{4,h}(k_1,k_2,k_3,k_4)\delta(\sum_{i=1}^{4}(\sigma_i k_i))\prod_{i=1}^{4}\psi^{\sigma_i,(\leq h)}_{k_i,\vec\omega_i} = \tag{16}$$

$$= f^{4,h}(0,0,0,0)\int\prod_{i=1}^{4}dk_i\delta(k_1+k_2-k_3-k_4)\prod_{i=1}^{4}\psi^{\sigma_i,(\le h)}_{k_i,\vec\omega_i}$$

where $\sigma_1=\sigma_2=-\sigma_3=-\sigma_4=+1$. Moreover if $f^{2,h}(k_1,k_2)\equiv f^{2,h}(k)$:[3]

$$\begin{aligned}&\mathcal{L}\int dk f^{2,h}(k)\psi^{+,(\le h)}_{k,\vec\omega_1}\psi^{-,(\le h)}_{k,\vec\omega_2}\\&=\int dk\left[f^{2,h}(0)+\vec k\tilde\partial_{\vec k}f^{2,h}(0)+k^0\tilde\partial_{k_0}f^{2,h}(0)\right]\psi^{+,(\le h)}_{k,\vec\omega_1}\psi^{-,(\le h)}_{k,\vec\omega_2}\end{aligned}\tag{17}$$

The *relevant part* of $V^h(\psi)$ is then:

$$\mathcal{L}V^h(\psi^{\le h})=z_hF^{\le h}_\zeta+\gamma^h s_hF^{\le h}_\sigma-\lambda_hF^{\le h}_\lambda\tag{18}$$

$$\begin{aligned}F^{\le h}_\zeta&=-\int dk\ \not k\bar\psi^{(\le h)}_k\psi^{(\le h)}_k\\F^{\le h}_\sigma&=\int dk\bar\psi^{(\le h)}_k\psi^{(\le h)}_k\\F^{\le 0}_\lambda&=\int\prod_{i=1}^{4}dk_i(\bar\psi^{(\le h)}_{k_1}\psi^{(\le h)}_{k_2})(\bar\psi^{(\le h)}_{k_3}\psi^{(\le h)}_{k_4})\delta(\sum_i\varepsilon_ik_i)\end{aligned}\tag{19}$$

Note that the *a priori* possible *non covariant* terms produced by the localizations do vanish, in fact, by parity or symmetry arguments.

We then integrate the infrared scales in the following way, if $\mathcal{R}=1-\mathcal{L}$ and $C_h^{-1}(k)=\sum_{k=-\infty}^{h}f(\gamma^{2k}k^2)=e^{-\gamma^{-2h}(\vec k^2+k_0^2)}$:

$$\begin{aligned}&\int\{\mathcal{D}\psi^{(\le 0)}e^{-\int dkC_0(k)\bar\psi^{(\le 0)}_k(-\not k)\psi^{(\le 0)}_k}\}e^{-\mathcal{L}V^0-\mathcal{R}V^0}=\\&\int\{\mathcal{D}\psi^{(\le 1)}e^{-\int dkC_1(k)\bar\psi^{(\le 1)}_k(-\not k)\psi^{(\le 1)}_k}\}e^{-V^{-1}}=\\&\int\{\mathcal{D}\psi^{(\le 1)}e^{-\int dkC_1(k)\bar\psi^{(\le 1)}_k(-\not k)\psi^{(\le 1)}_k}\}e^{-\mathcal{L}V^{-1}-\mathcal{R}V^{-1}}=\dots\end{aligned}\tag{20}$$

i.e. we perform each integration by writing the effective potential V^h as $\mathcal{L}V^h+\mathcal{R}V^h$. In this way V^h is a *renormalized series* of the running coupling constants $v_k=(s_k,z_k,g_k)$, $k>h$ and v_h obeys to a recursive relation, called *Beta function*, $v_h=\beta_h(v_{h+1},\dots,v_0)$.

The effective potential is still given by a sum of Feynman graphs but each cluster v with two or four external lines with value f^{2,h_v} or f^{4,h_v} is replaced by $\mathcal{R}f^{2,h_v}$ and $\mathcal{R}f^{4,h_v}$ and the effect is that the k-th order diagram is bounded by $C^k\varepsilon^k\prod_v\gamma^{-(D_v+d_v)(h_v-h'_v)}$, where everything is again defined as above but $\varepsilon=(\max_{k>h}|v_k|)$ and d_v is 1 for the clusters with four external lines and 2 for the cluster with two external lines. In other words respect to the $\mathcal{R}=\mathrm{Id}$ case there is in the estimates a factor $\gamma^{-(h_v-h'_v)}$ more for the clusters with four external lines and a

[3] by $\tilde\partial$ we mean the discrete derivatives.

factor more $\gamma^{-2(h_v-h'_v)}$ for the clusters with two external lines. It is standard to check that the renormalization produces these factors by writing explicitly the effect of the renormalization, for instance on a graph with four external lines:

$$\mathcal{R}f^{4,h_v}(k_1,k_2,k_3,k_4)=f^{4,h_v}(k_1,k_2,k_3,k_4)-f^{4,h_v}(0,0,0,0)=\int_0^1 dt\frac{\partial f^{4,h_v}(tk_1,tk_2,tk_3,tk_4)}{\partial t}$$

The derivative acting on f^{4,h_v} has the effect that:

1. some propagator $g^i(k)$ in the cluster v with $i\geq h_v$ is derivated and so it is bounded by $A\gamma^{-2i}e^{-\kappa\gamma^{-i}|k|}$, if A is a constant matrix, so there is an extra factor $\gamma^{-i}\leq\gamma^{-h_v}$ in the estimates with respect to the $\mathcal{R}=\mathrm{Id}$ case;

2. a factor k_i, $i=1,...,4$ is produced which has the effect that in the estimates there is an extra (respect to the $\mathcal{R}=\mathrm{Id}$ case) factor $\gamma^{h'_v}$.

Each graph in the renormalized expansion for the effective potential is then bounded uniformly in h. Even more, estimating the fermionic expectations in the renormalized expansion for the effective potential by the *Grahm-Hadamard inequality*, one can prove the analyticity of the effective potential as a function of the running coupling constants in a small domain. This follows from [2] in which a renormalized expansion for the effective potential essentially identical to our one is discussed for the Gross-Neveu model.
However we do not expect that $\max_{k>h}|v_k|$ *is small* as:

1. s_h is a *relevant* coupling in the renormalization group sense as in order to obtain bounds uniform in h in the renormalized expansion eq.(20) the running coupling constants multiplying $F_\sigma^{\leq h}$ have to be multiplied by γ^h. We expect then that s_h grows more and more as $h\to-\infty$.

2. Also the *marginal* couplings give problems. In fact by a second order computation

$$z_{h-1}=z_h+\beta_1\lambda_h^2\qquad\lambda_{h-1}=\lambda_h$$

with $\beta_1>0$, so that the vanishing of the second order beta function for λ_h has the effect that z_h grows more and more as $h\to-\infty$. One could hope that the third order beta function is not vanishing and negative, but indeed an explicit computation shows that it has the effect of increasing λ_h.[4]

4 Anomalous scaling

The fact that, at least by a second order computation, the running coupling constants seem to be unbounded (or at least they grow so large to go beyond the possibility of our non-perturbative

[4]Anyway also if it were $\lambda_{h-1}=\lambda_h-\beta\lambda_h^n$, $\beta\geq 0$, this would mean that $\lambda_h\geq O(|h|^{-\frac{1}{n-1}})$ and the couplings z_h would be anyway *unbounded* at this approximation.

approach) is an indication that the Schwinger function decay for large distances is different in the free or interacting theory. It is possible to extend the methods followed so far to a more general approach which can take into account a possible modification of the Schwinger function behavior with respect to the free case. Such new behavior is called "anomalous scaling" and it was used for the first time in [12] for the ultraviolet problem in the ϕ^4_3 model, and among other applications in [13],[14],[3], [15],[16] for the study of several non relativistic one dimensional Fermi systems and in [5] for the Bose condensation problem.

Due to the possibility, discussed at the beginning of the preceding section, to put the terms quadratic in the fields in the integration, the idea is to integrate over the fields on each scale by inserting the terms of the relevant part of the effective potential, that in the previous approach causes the uncontrollable growth of the running coupling constants, in the integration, so taking them into account by a change in the propagator.

Setting $Z_0 = 1$, $V^0(Z_0\psi^{(\leq 0)}) = \tilde{V}^0(\psi^{(\leq 0)})$, $\sigma_0 = 0$ the anomalous scaling integration is defined *recursively* in the following way. Once the fields $\psi^0, \ldots, \psi^{h+1}$ have been integrated we have to evaluate:

$$\int \{\mathcal{D}\psi^{(\leq h)} e^{-\int dk C_h(k) Z_h \bar{\psi}_k^{(\leq h)}(-\not{k}+\sigma_h(k))\psi_k^{(\leq h)}}\} e^{-V^h(\sqrt{Z_h}\psi^{(\leq h)})} \tag{21}$$

where as in the preceding section we write the effective potential, by the localization operator eq.(16)(17), as the sum of its relevant part $\mathcal{L}V^h(\psi^{(\leq h)}) = s_h F_{\sigma}^{\leq h} + z_h F_{z}^{\leq h} - \lambda_h F_{\lambda}^{\leq h}$ and irrelevant part. Note that, contrary to what was done in the preceding section, we write the running coupling constant multiplying $F_{\sigma}^{\leq h}$ simply as s_h and not as $\gamma^h s_h$.

Before attempting the integration of the fields $\psi^{(h)}$ we extract the terms quadratic in the fields out of $\mathcal{L}V^h$ and we take them into account by changing the propagator. We write eq.(21) as:

$$\int \{\mathcal{D}\psi^{(\leq h)} e^{-\int dk C_h(k) Z_{h-1}(k) \bar{\psi}_k^{(\leq h)}(-\not{k}+\sigma_{h-1}(k))\psi_k^{(\leq h)}}\} e^{\lambda_h F_{\lambda}^{\leq h}(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{R}V^h(\sqrt{Z_h}\psi^{(\leq h)})} \tag{22}$$

where

$$\begin{aligned} Z_{h-1}(k) &= Z_h + C_h^{-1}(k) Z_h z_h \\ Z_{h-1}(k)\sigma_{h-1}(k) &= Z_h \sigma_h(k) + Z_h C_h^{-1}(k) s_h \end{aligned} \tag{23}$$

and $Z_h \equiv Z_h(0)$

The equality between eq.(21) and eq.(22) is, again, a compact form to state the equality between two series, absolutely convergent at finite L. Now one can perform the integration respect to $\psi^{(h)}$ writing eq.(22) as

$$\begin{aligned} &\int \{\mathcal{D}\psi^{(\leq h-1)} e^{-\int dk C_{h-1}(k) Z_{h-1} \bar{\psi}_k^{(\leq h-1)}(-\not{k}+\sigma_{h-1}(k))\psi_k^{(\leq h-1)}}\} \cdot \\ &\int \{\mathcal{D}\psi^{(h)} e^{-\int dk \tilde{f}_h(k)^{-1} Z_{h-1} \bar{\psi}^{(h)}(-\not{k}+\sigma_{h-1}(k))\psi^{(h)}}\} e^{g_h F_{\lambda}^{\leq h}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) - \mathcal{R}V^h(\sqrt{Z_{h-1}}\psi^{(\leq h)})} \end{aligned} \tag{24}$$

where we have "rescaled" the fields in the effective potential, $g_h = (\frac{Z_h}{Z_{h-1}})^2 \lambda_h$ and the second integration has propagator given by

$$g^{(h)}(x-y) = \int dk e^{ik(x-y)} \frac{1}{Z_{h-1}} \frac{\not k + \sigma_{h-1} I}{k^2 + \sigma_{h-1}^2} \tilde{f}_h(k) \tag{25}$$

with

$$\tilde{f}_h(k) = Z_{h-1}[\frac{C_h^{-1}(k)}{Z_{h-1}(k)} - \frac{C_{h-1}^{-1}(k)}{Z_{h-1}}]$$

It is finally the time to perform the integration with respect to the field $\psi^{(h)}$ obtaining

$$\int \{\mathcal{D}\psi^{(\leq h-1)} e^{-\int dk C_{h-1}(k) Z_{h-1} \bar{\psi}_k^{(\leq h-1)}(-\not k + \sigma_{h-1}(k))\psi_k^{(\leq h-1)}}\} e^{-V^h(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})} \tag{26}$$

where $V^h(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})$ is given by the sum of Feynman graphs in which the factor $\sigma_h(k)$, z_h appear only in the propagators eq.(25) and their possible growth does not affect the convergence of the series. Note that eq.(26) as the same form of eq.(21) so that the procedure can be iterated.

It is convenient to write the propagator in the following way (for an analogous statement, and relative proof, see [16]):

$$g^{(h)}(x-y) = g_l^{(h)}(x-y) + C_2^{(h)}(x-y) \tag{27}$$

$$g_l^{(h)}(x-y) = \int dk \frac{e^{ikx}}{Z_h(k)} \frac{\tilde{f}(\gamma^{-2h}k^2)\ \not k}{k^2}$$

where, if κ, A are suitable constants, $|g^{(h)}_{l,\vec{\omega},-\vec{\omega}}(x-y)| \leq A\frac{\gamma^h}{Z_h} e^{-\gamma^h \kappa |x-y|}$ and $|C^{(h)}_{2,\vec{\omega},\vec{\omega}'}(x-y)| \leq A\frac{\gamma^h}{Z_h}\left(\frac{\sigma_h}{\gamma^h}\right)^i e^{-\gamma^h\kappa|x-y|}$ with $i=1$ if $\vec{\omega} = \vec{\omega}'$,$i=2$ if $\vec{\omega} = -\vec{\omega}'$ and $\sigma_h \equiv \sigma_h(0)$.

5 The flow of the renormalization group

The convergence of the series for the anomalous effective potential with $\sigma_k = 0$ for all k was proved in [3] (some technical simplifications are done in [15]). This means that, if

$$\max_{k\geq h} |g_k| \leq \bar{\varepsilon}_h \qquad \max_{\substack{i=0,1,2\\ k\geq h}} \left|\frac{\sigma_k}{\gamma^k}\right|^i \leq \tilde{C}_h \qquad \max_{k\geq h} \frac{Z_k}{Z_{k-1}} \leq e^{\beta_1 \bar{\varepsilon}_h^2} \tag{28}$$

and noting that the propagator, by eq.(27), obeys to the same bound as in the $\sigma_k = 0$ case, the contribution of order k to the effective potential V^{h-1} is bounded by $C^k \tilde{C}_h^k \bar{\varepsilon_h}^k$, if C is a suitable constant; the above bound does not depend on L and we can remove the *infrared cut-off* *i.e.* we can take the limit $L \to -\infty$ on the effective potential. Of course it is possible to find a constant ε_h so that, if $\lambda \leq \varepsilon_h$ then $\bar{\varepsilon}_h \leq C^{-1}\tilde{C}_h^{-1}$ and the series for V^{h-1} are convergent, see [3].

However one is interested to obtain an ε_h *independent* on h but this is not obvious at all and we have to study the beta function equation for the running coupling constants in order to understand their behavior.

The results of the preceding section and a second order beta function analysis imply, as in [3],[15], that the Beta function equations are given by:

$$
\begin{aligned}
g_{h-1} &= g_h + G_\lambda^{1,h}(g_h;\ldots;g_0) + G_\lambda^{2,h}(g_h,\sigma_h;\ldots;g_0,\sigma_0) \\
\sigma_{h-1} &= \sigma_h + G_\sigma^{1,h}(g_h,\sigma_h;\ldots;g_0,\sigma_0) \\
\frac{Z_{h-1}}{Z_h} &= 1 + G_z^1(g_h;\ldots;g_0) + G_z^{2,h}(g_h,\sigma_h;\ldots;g_0,\sigma_0)
\end{aligned}
\tag{29}
$$

where, if eq.(28) holds, $|G_\lambda^{2,h}| \le C_1 \frac{\sigma_h}{\gamma^h} g_h$, $-C_2\sigma_h g_h \le G_\sigma^h \le -C_1\sigma_h g_h$, $C_1 g_h^2 \le G_z \le C_2 g_h^2$ and $|G_z^{2,h}| \le C_1 \frac{\sigma_h}{\gamma^h} g_h$, where $C_2 > C_1$ are suitable positive constants.

In the equations for g_h we have divided, using eq.(27), the beta function into two parts and the term $G_\lambda^{1,h}$ is given by a sum of integrals of products of the part $g^k_{l,\vec\omega}(x-y)$ of the propagator while in the terms $G_\lambda^{2,h}$ at least a $C_2^k(x-y)$ term is involved. In the equation for σ_h we have used that at least a diagonal propagator has to be involved in the term contributing to the beta function. In the last equation again we have divided the beta function into two terms as it was done for g_h.

The Beta function is a *short memory* dynamical system in the sense that is a set of equations of the form $v_{h-1} = \beta_h(v_h, v_{h+1}, \ldots, v_0)$ which behaves "essentially" as a system without memory $v_{h-1} = \beta_h(v_h, v_h, \ldots, v_h)$, (this is a consequence of the convergence of the Beta function as function of its arguments, see [3]). Even more, the convergence of eq.(29) allows to say that the lowest non zero terms determine the evolution of the running coupling constants. All the terms contributing to the Beta function eq.(29) are series starting from the second order except $G_\lambda^{1,h}$. This, however, generates a problem as one cannot exclude *a priori* there there is some non-vanishing term at some large order, and the evolution of the running coupling constants will depend critically on this unknown term. In other words, as the theory is not *asymptotically free* (contrary to the Gross-Neveu model studied in [2]), the convergence of the Beta function and the second order Beta function are not enough to understand the system.

Fortunately it is possible to prove that:

$$
\lim_{h\to-\infty} G_\lambda^{1,h}(\mu;\ldots;\mu) = 0 \tag{30}
$$

i.e. it is vanishing at *all* orders. In fact $G_\lambda^{1,h}$ coincides, up to terms vanishing as $h \to -\infty$ as $O(\gamma^h)$, with the correspondent quantity of the Luttinger model[5] , and, by using the properties of the exact solution [17],[18] one can prove that it is vanishing, see [13], [14], [3], [15][6].

[5]this is not too surprising as the Luttinger model, when the interaction range and the Fermi momentum shrinks to 0, is equivalent to the massless Thirring model, if a suitable wave function normalization is done, see [19]; moreover the massless Thirring model and the massless Yukawa$_2$ have the same infrared beta function up to term $O(\gamma^h)$

[6]Of course it should be possible to prove eq.(30) without using the exact solution but, possibly, exploiting some well known symmetries of the Lagrangian, but they are not so immediate to exploit in this approach as they hold only when the regularizations are removed [20]

By eq.(30), the considerations after eq.(28) and the "short memory properties" of the beta function, see [3], it follows, see [16], that if for $0 \geq k \geq h$ eq.(28) holds then

$$
\begin{aligned}
&-c_1\lambda^2 < |g_{h-1} - g_0| < c_2\lambda^2 \quad \lambda\beta_3 c_3 h \leq \log\left(\frac{|\sigma_{h-1}|}{|\sigma_0|}\right) \leq \lambda\beta_3 c_4 h \\
&-\beta_1 c_3\lambda^2 h \leq \log(|Z_{h-1}|) \leq -\beta_1 c_4 \lambda^2 h
\end{aligned} \tag{31}
$$

i.e. the flow is essentially described by the second order truncation of the beta function, if c_i, β_i are suitable constants.

At the beginning of this section we said that the effective potential V^{h-1} is expressed by a convergent series if $\lambda \leq \varepsilon_h$ where ε_h is such that $\bar{\varepsilon}_h \leq C^{-1}\tilde{C}_h^{-1}$; $\bar{\varepsilon}_h$ and $\tilde{C}_h$ are defined in eq.(30). Unfortunately from eq.(31) we see $\lim_{h\to-\infty} \tilde{C}_h = \infty$ so that $\lim_{h\to-\infty} \varepsilon_h = 0$ and we have proved the summability of the series for the effective potential V^h only for $h \geq h^*$, if h^* is a fixed scale. We will see in a moment that the "natural" way to fix such scale is by requiring that $\tilde{C}_h \leq \tilde{C}$, if $\tilde{C}$ is a proper constant that we will fix below. One finds $\frac{\log_\gamma(\tilde{C}^{-1}m)}{1+\lambda\beta_2 c_3} \leq h^* \leq \frac{\log_\gamma(\tilde{C}^{-1}m)+1}{1+\lambda\beta_2 c_4}$.

It remains to prove the summability of V^h for $h < h^*$. The study of the case $h \geq h^*$ was done in some sense by approximating the anomalous propagator eq.(25) with the propagator of a massless theory by eq.(27). This approximation is of course not reasonable for large $|h|$ corresponding to momenta negligible with respect to the mass term. But for the scales $h < h^*$ the bound

$$
|g^{(\leq h^*)}_{\vec{\omega},\vec{\omega}'}(x-y)| \leq A\frac{\gamma^{h^*}}{Z_{h^*}} e^{-\gamma^{h^*}\kappa|x-y|}
$$

holds. Note in fact that $k^2 + \sigma^2_{h^*}(k) \geq \sigma^2_{h^*}$, as follows from eq.(23). In other words the propagator $g^{(\leq h^*)}_{\vec{\omega},\vec{\omega}'}(x-y)$ obeys to the same bound of $g^{(h)}_{\vec{\omega},\vec{\omega}'}(x-y)$ for $h \geq h^*$; our choice of h^* is done just to obtain this. This means that the integration of the scales between $-\infty$ and h^* is equivalent to the integration of a single scale in a massless theory, so that from [3] [15] also the series expressing

$$
\int \{\mathcal{D}\psi^{(\leq h^*)} e^{-\int dk C_{h^*}(k) Z_{h^*} \bar{\psi}^{(\leq h^*)}_k (-\not{k} + \sigma_{h^*}(k))\psi^{(\leq h^*)}_k}\} e^{-V^{h^*}(\sqrt{Z_{h^*}}\psi^{(\leq h^*)})} \tag{32}
$$

is convergent. The n-th order of the series in the running coupling constants for the effective potential for $h > h^*$ is bounded by $(\bar{\varepsilon}_{h^*+1})^n \tilde{C}^n C_1^n$ while the n-th order term of the series eq.(32) is bounded by $(\bar{\varepsilon}_{h^*})^n \frac{C_2^n}{\tilde{C}^n}$ so that there is a non ambiguous way to fix $\tilde{C}$ so that $\bar{\varepsilon}_{h^*}$ is the largest possible.

We have discussed the convergence of the denominator of eq.(3) when the cut-offs Λ, L are removed; the numerator has a similar expansion, as one can easily check repeating the arguments in sec.(4) for this case, see [3].

The results is:

$$\frac{\int P(d\psi)e^{-V}\bar\psi_x\psi_y}{\int P(d\psi)e^{-V}} = S^{(u.v.)}(x-y) + \sum_{h=h^*}^{0} g^{(h)}(x-y) + \sum_{h=h^*}^{0} \bar g^{(h)}(x-y) \tag{33}$$

with $|S^{(u.v.)}(x-y)| \le Ae^{-\kappa|x-y|}$, $|\bar g^{(h)}_{\vec\omega,\vec\omega'}(x-y)| \le \frac{A\gamma^h}{Z_h}e^{-\kappa\gamma^h|x-y|}$ and we call $g^{(\le h^*)}(x-y)$ simply $g^{(h^*)}(x-y)$.

From the above equation it is easy to state the results in the theorem. In fact the second term in eq.(33) gives eq.(4), defining

$$\eta_1 = -\frac{\log(Z_{h^*})}{\log(m)} \qquad 1+\eta_2 = \frac{\log(\sigma_{h^*})}{\log(m)}$$

The bounds eq.(5) can be obtained by eq.(33), as for $\gamma^{-h_x-1} < |x-y| \le \gamma^{-h_x}$, $h_x > h^*$:

$$\sum_{h=0}^{h^*}|\bar g^{(h)}(x-y)| \le A_1\left[\sum_{h=h^*}^{h_x-1}\frac{\gamma^h}{Z_h} + \sum_{h=h_x}^{0}\frac{\gamma^h}{Z_h}e^{-\kappa\gamma^h|x-y|}\right] \le A_2\gamma^{h_x(1-\eta_3)}$$

while for $|x-y| \ge \sigma_{h^*}^{-1}$:

$$\begin{aligned}\sum_{h=0}^{h^*}|\bar g^{(h)}(x-y)| &\le A_1\frac{\gamma^{h^*}}{Z_{h^*}}e^{-\kappa\gamma^{h^*}|x-y|}\left(1+\sum_{h=1}^{h^*+1}\gamma^{(h-h^*)(1-\eta_3)}(\gamma^{h-h^*}-1)^{-N}\right)\\ &\le A_2\frac{\gamma^{h^*}}{Z_{h^*}}e^{-\kappa\gamma^{h^*}|x-y|}\end{aligned} \tag{34}$$

where $N > 2$ and A_1, A_2 are positive constants.

Finally the asymptotic expression for $S_A(x-y)$ eq.(6) follows from eq.(27).

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6 References

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