POINCARE' RENORMALIZED FORMS

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Summary. In Poincaré Normal Form theory, one considers a series of transformations generated by homogeneous polynomials obtained as solution of the homological equation; such solutions are unique up to terms in the kernel of the homological operator. Careful consideration of the higher order terms generated by polynomials differing for a term in this kernel leads to the possibility of further reducing the Normal Form expansion of a formal power series, in a completely algorithmic way. The algorithm is also applied to planar vector fields whose linear part has eigenvalues $\lambda = \pm i$.

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Introduction.

The theory of Normal Forms, first introduced by Poincaré, is in many aspects central to the study of nonlinear dynamical systems; in particular, its original version – in which we are interested here – deals with expansion of a system of $\mathcal{C}^\infty$ ODEs in R^n (equivalently, of a $\mathcal{C}^\infty$ vector field on a n-dimensional smooth manifold) in the neighbourhood of an equilibrium point, say the origin $x = 0$, and one aims at a classification of these up to formal[1] near-identity changes of coordinates.

It is well known [1-3] that a system of ODEs (having the origin as equilibrium point) can be taken into Normal Form by means of the Poincaré algorithm, i.e. by performing a series of near-identity changes of coordinates (Poincaré transformations) of the form $x' = x + h_k(x)$ where the $h_k(x)$ are homogeneous vector polynomials; these are chosen as solution of a certain equation, the *homological equation*, and such solutions are unique up to terms in the kernel of the relevant *homological operator*.

In this way one manages to eliminate, order by order, all the *nonresonant* terms. However, each of these changes of coordinates produces new terms of higher order; those which are nonresonant will then be disposed of by successive Poincaré transformations, while if resonant terms are generated in this way, we are not able to eliminate them by the Poincaré algorithm.

It should also be stressed that, while two vector polynomials $h_k(x)$ and $h'_k(x)$ which differ only by an element in the kernel of the homological operator produce the same transformation on terms of order k, they will in general give raise to different higher order terms; in particular, we can in this way generate different higher order resonant terms.

It is then clear that, unless we give a prescription concerning the projection of $h_k(x)$ on the kernel of the homological operator $\mathcal{L}_0$ (e.g. that $h_k(x) \in [\mathrm{Ker}(\mathcal{L}_0)]^\perp$), the Normal Form is not uniquely defined.

Here we argue that this should not be seen as a drawback, but as an advantage: indeed, a careful exploitation of the higher order terms generated in each Poincaré transformation can lead to a remarkable simplification of the Normal Form unfolding; this simplification is specially important for systems satisfying resonance relations[2] $\sum_k m_k \lambda_k = \lambda_r$ in a large or even infinite (as e.g. in the classical case where $\lambda_k = \pm i$) number.

In order to present the argument, and the completely defined algorithm which implements it, we need to consider the usual Poincaré scheme, or actually the Lie-Poincaré one [4,5], paying special attention to the higher order terms generated in each change of coordinates. Thus, we will shortly go over the Poincaré and Lie-Poincaré methods providing the relevant

[1] By "formal", it is meant that these are defined by series, and we do not consider the problem of convergence of such series.

[2] Here the λ_k are the eigenvalues of the matrix describing the linearization of the system around the equilibrium point, and m_k are nonnegative integers.

detailed formulas, in sections 1-5. We do have a special advantage in considering Lie-Poincaré transformations, as for these we have closed formulas expressing transformed vector fields in terms of iterated commutators, essentially the Baker-Campbell-Hausdorff formula; although this is a classical result, we reproduce it in the appendix for completeness.

We briefly discuss the problem of non-unicity of Poincaré Normal Forms (in section 6), and in sections 7-9 we implement this discussion by providing an algorithm to reduce a system to a form which amounts essentially to applying iteratively the Poincaré normalization procedure first on the system, then on its normal form, then on this "normalized normal form", and so on (through m normalizations if we want to reduce the terms of order up to m), each time taking into account terms of higher order; due to this, we call the final form thus obtained the **Poincaré renormalized form** for the system. It should be stressed that in actual computation one would prefer to proceed order by order, and this can actually be done; thus, we present our algorithm directly in this form, i.e. we take sequentially the terms of order 1,2,... to their renormalized form.

Finally, in section 10 we apply the algorithm to the case, mentioned above, of planar vector fields having the generator of rigid rotations as linear part: in this case the relevant matrix A has eigenvalues $\lambda_k = \pm i$, and thus an infinite number of satisfied resonance relations and of terms appearing in the Poincaré normal form unfolding, parametrized by two infinite sequences of real numbers. However, the renormalized form unfolding constains only three nonlinear terms – i.e. three free real parameters – at most.

1. General setting

The Poincaré theory of Normal Forms [1-3] for dynamical systems, i.e. for first order autonomous smooth ODEs of the form

$$\dot{x} = f(x) \qquad , \; x \in R^n \; , \; f : R^n \to R^n \tag{1.1}$$

or equivalently for vector fields

$$X = \sum_{i=1}^{n} \; f^i(x) \frac{\partial}{\partial x^i} \; , \tag{1.2}$$

is based on systematically employing near-identity changes of coordinates with homogeneous vector polynomial functions as generator.

One is interested in f being a formal power series, i.e.

$$f(x) = \sum_{k=0}^{\infty} f_k(x) \tag{1.3}$$

with $f_k(x)$ homogeneous of order $(k+1)$ in the x.

We denote by V the set of vector formal power series $f : R^n \to R^n$ which have the origin as a fixed point, and by $V_k \subset V$ the set of polynomial vector functions homogeneous of order $(k+1)$; obviously,

$$V \ = \ \sum_{k=0}^{\infty}{}^{\oplus} \ V_k \ . \tag{1.4}$$

It will be useful to define the bracket $\{.,.\} : V \times V \to V$ given by

$$\{f,g\} = (f \cdot \nabla) g - (g \cdot \nabla) f \equiv f^i \frac{\partial g}{\partial x^i} - g^i \frac{\partial f}{\partial x^i} \ ; \tag{1.5}$$

this expresses the Lie commutator of vector fields when we look at the component of vector fields in the x coordinates; that is, for $X = f^i \partial_i$ and $Y = g^i \partial_i$, we have $[X,Y] = h^i \partial_i$ with $h = \{f,g\}$. Notice that

$$\{.,.\} \ : \ V_k \times V_m \ \to \ V_{k+m} \ . \tag{1.6}$$

The (standard) homological operator $\mathcal{L}_0$ can be defined in terms of this bracket, as $\mathcal{L}_0(.) = \{f_0, .\}$; by (1.6), $\mathcal{L}_0 : V_k \to V_k$.

In the following, we will need (linear) operators acting between the spaces V_k, and in particular we will have to consider the complementary sets of the ranges of such operators; it is thus convenient to introduce a scalar product in V (actually, in each of the V_k), so that we can consider the adjoint operators.

It turns out that the convenient scalar product is defined as follows [6,7]. First of all, we notice that each of the V_k is a finite dimensional vector space. In each of these, we can choose a basis $e_{\mu,j}(x)$ of functions which have components

$$e^i_{\mu,j}(x) = x^\mu \delta_{i,j} = (x^1)^\mu_1 ... (x^n)^\mu_n \delta_{i,j} \ ; \tag{1.7}$$

we define then a scalar product in V_k as

$$(e_{\nu,j}, e_{\mu,i}) \ = < \mu, \nu > \ \ \delta_{i,j} \ , \tag{1.8}$$

where $< .,. >$ is the Bargman [7,8] scalar product

$$< \mu, \nu > = [\partial_\mu x^\nu]_{x=0} = \prod_{i=1}^n (\mu_i !) \delta_{\mu_i, \nu_i} \ . \tag{1.9}$$

The scalar product in V is then naturally defined in terms of these as $(f,g) = \sum_k (f_k, g_k)$.

2. Poincaré transformations.

One considers then near-identity changes of coordinates of the form

$$x = y + h_k(y) \qquad , \qquad h_k \in V_k \ , \tag{2.1}$$

also called Poincaré transformations. We denote by Γ the jacobian of the change of coordinates, i.e. $\Gamma^i{}_j = \partial h^i_k / \partial y^j$. Under the change of coordinates (2.1), our system (1.1) is transformed into

$$\dot{y} = \left[I + \Gamma \right]^{-1} \ f\left(y + h_k(y) \right) \ . \tag{2.2}$$

For y – and therefore x – small enough, $\Lambda = (I+\Gamma)^{-1}$ does surely exist, and we can write it in a power series as $\Lambda \equiv (I+\Gamma)^{-1} = \sum_{m=0}^{\infty} [(-1)^m \, (\Gamma)^m]$.

Similarly, we can expand $f_m(y + h_k(y))$ as a power series; we write $J = (j_1, ..., j_n)$, $|J| = \sum_i j_i$. With this multiindex notation, $\partial_J := \partial_1^{j_1} ... \partial_n^{j_n}$, and similarly $h^J_k := (h^1_k)^{j_1} ... (h^n_k)^{j_n}$. We define the operators $\Phi^r_h = (1/r!) \sum_{|J|=r} \left(h^J \cdot \partial_J \right)$ (representing all the partial derivatives of order $|J|$), and in terms of these the system (2.2) can then be written as

$$\dot{y} \ = \ \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \ \left[(-1)^s \ \Gamma^s \Phi^{r-s}_{h_k} \right] \ f_m(y) \ . \tag{2.3}$$

Thus we see that, in a Poincaré transformation with generator $h_k \in V_k$ each term f_m is transformed into a new $\widetilde{f}_m$ given[3] by

$$\widetilde{f}_m \ = \ f_m \ + \ \sum_{p=1}^{[m/k]} \left[\sum_{s=0}^{p} \ (-1)^s \Gamma^s \Phi^{p-s}_{h_k} \right] f_{m-kp} \ . \tag{2.4}$$

One could, in principle, also obtain explicit formulas for terms of all degrees, but these become quickly too involved to be of practical use. We notice, however, that the terms of degree smaller than k are not changed at all,

$$\widetilde{f}_m = f_m \quad \forall m < k \ , \tag{2.5}$$

and the terms of degree $k \le m < 2k$ are changed according to

$$\widetilde{f}_{k+\nu} = f_{k+\nu} + \left[\Phi_h - \Gamma \right] f_\nu \quad (0 < \nu < k) \ . \tag{2.6}$$

[3] Here the square brackets in $[m/k]$ denotes integer part.

3. Transformation to Poincaré normal form.

The transformation to Poincaré normal form is given by a well known algorithm, which is just the same if we consider the Poincaré or the Lie (see next section) form for the changes of coordinates. Indeed, in both cases we have that for the transformation with generator $h_k \in V_k$ it results $\widetilde{f}_m = f_m$ for $m < k$, and

$$\widetilde{f}_k = f_k + \{h_k, f_0\} = f_k - \mathcal{L}_0(h_k) \; ; \tag{3.1}$$

the expression for higher order terms are (at least for $m \geq 2k$) more involved, and differ slightly in the two approaches.

We consider then sequentially the terms in V_k for $k = 1, 2, 3, ...$ (up to any desired finite order n, or formally for $n \to \infty$), and choose suitable generators h_k; in this way we can normalize sequentially the terms f_k, and successive changes of coordinates of higher order leave these unchanged[4].

The "suitable" generator h_k mentioned here is obtained as solution to the homological equation; we first define the projection operator

$$P_0 : V \to \mathrm{Ran}(\mathcal{L}_0) = \left[\mathrm{Ker}(\mathcal{L}_0^+)\right]^\perp \tag{3.2}$$

(this could be properly defined in each of the V_k considering the restriction $\mathcal{L}_{0,k}$ of $\mathcal{L}_0$ to V_k, and the projection operators $P_0^{(k)} : V_k \to \mathrm{Ran}\,(\mathcal{L}_{0,k})$; with these, $P_0 = \sum^\oplus P_{0,k}$), and then require that h_k solves the homological equation

$$\mathcal{L}_0(h_k) \;=\; P_0\,(f_k) \;\; . \tag{3.3}$$

The solutions to this are given by

$$h_k \;=\; \mathcal{L}_0^+\,(P_0 f_k) \;+\; \ell_k \;, \tag{3.4}$$

where $\ell_k(x)$ is an arbitrary function[5] in $\mathrm{Ker}(\mathcal{L}_0)$.

In this way, $\widetilde{f}_k \in [\mathrm{Ran}(\mathcal{L}_0)]^\perp = \mathrm{Ker}\left(\mathcal{L}_0^+\right)$. As it is well known, proceeding in this way one finally arrives to a system $\dot{x} = \widehat{f}(x)$ which is in normal form (up to any given order n), i.e. such that all the $\widehat{f}_k(x)$ with $k \geq 1$ are in $\mathrm{Ker}(\mathcal{L}_0^+)$, and this means that all the nonlinear terms (up to any given order n) are resonant with the linear part of the system.

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[4] In general, the series of changes of coordinates and of Poincaré transformations defined in this way is only formal (i.e. does not converge in any neighbourhood of the origin), even for finite n; here we will not be concerned by this problem, as we deal with formal power series.

[5] This freedom in choosing ℓ_k, i.e. h_k, does also come into play when we have to take into normal form not a single vector field, but a Lie algebra of vector fields [9].

4. Lie transformations.

A slightly different way of approaching Normal Forms is based on Lie – rather than Poincaré – transformations [4,5]. In this case, the change of coordinates is given by the time-one action[6] of a vector field H_k given by

$$H_k \;=\; h^i_k(x)\frac{\partial}{\partial x^i} \tag{4.1}$$

so that the change of coordinates is written as

$$\widetilde{x} \;:= y \;=\; \left[e^{-\lambda H_k} x\right]_{\lambda=1} \; . \tag{4.2}$$

This has several advantages: first of all, we do not have to worry about the domain of existence of the inverse change of coordinates [4]; second, we are dealing with actions of vector fields and we can use Lie group theory; and finally, we have a representation of the vector field X in the new coordinates which is easier to handle.

In this way, X is transformed into

$$\widetilde{X} \;=\; \left[e^{\lambda H_k}\; X\; e^{-\lambda H_k}\right]_{\lambda=1} \quad , \tag{4.3}$$

and this can be explicitely computed by the Baker-Campbell-Haussdorf formula [5].

We will just recall the final result, i.e. that $\widetilde{X}$ can be written as $\widetilde{X} = \widetilde{f}^i(x)\partial_i$, with

$$\widetilde{f} \;=\; f + \{h,f\} + \frac{1}{2}\{h,\{h,f\}\} + \frac{1}{6}\{h,\{h,\{h,f\}\}\} + ... \quad . \tag{4.4}$$

Full details of the derivation of this formula are given in the appendix.

¿From (4.4) it is easy to derive formulas for the decomposition of $\widetilde{f}$ into homogeneous factors, i.e. for $\widetilde{f} = \sum_m \widetilde{f}_m$. We introduce the notation $\mathcal{H}(.) = \{h,.\}$, and with this we have

$$\widetilde{f}_m \;=\; \sum_{s=0}^{[m/k]} \frac{1}{s!}\mathcal{H}^s\left(f_{m-sk}\right) \; . \tag{4.5}$$

Notice that we have written $[m/k]$ for the integer part of (m/k), and defined $\mathcal{H}^0(f) = f$.

5. The homological operators

We will define a series of homological operators $\mathcal{L}_k$ associated to f in a way slightly different from the customary one; the usual homological operator, which we will denote by $\mathcal{L}_0$, will

[6] That is, the time-one flow for the one-parameter group of diffeomorphisms of R^n generated by the vector field referred to.

however be just the standard one. This definition will suit our way of proceeding, based on Poincaré-Lie transformations and thus on (4.4).

For $f \in V$, $f = \sum f_k$, we define the Lie operator $\mathcal{F} : V \to V$ associated to f as $\mathcal{F} = \{f, .\}$; clearly we can write

$$\mathcal{F} \ = \ \sum_{k=0}^{\infty} \{f_k, .\} \ \equiv \ \sum_{k=0}^{\infty} \mathcal{L}_k \ . \tag{5.1}$$

The operators $\mathcal{L}_k = \{f_k, .\}$ defined in (5.1) are called the series of homological operators associated to f; the operator $\mathcal{L}_0$ coincides with the usual homological operator considered in Poincaré Normal Form theory. Notice that, by (1.6), $\mathcal{L}_k : V_m \to V_{m+k}$. We also denote by $\mathcal{L}_{k,m}$ the restriction of $\mathcal{L}_k$ to V_m.

It should be stressed that the homological operators do not permit to describe (4.4), or (4.5), in full generality: they are only related to the first nontrivial term in (4.5). However, it will turn out that, in the procedure we employ in the following, a suitable choice of the h permits to analyze iterated Poincaré-Lie transformations in terms of the $\mathcal{L}_k$ alone.

6. Non-unicity of Poincaré normal forms.

In the Poincaré procedure[7], shortly described above, one has no need to keep track of the effect of the transformation generated by $h_k \in V_k$ on terms of higher order: indeed, this will generate additional terms in V_s, in principle at all the higher orders $s > k$, but these can then be disposed of by the successive Poincaré transformations with generator h_s.

This point should be considered with some extra care: indeed, while the terms generated by $\mathcal{L}_0(h_s)$ are in $\mathrm{Ran}(\mathcal{L}_0) \cap V_s = \mathrm{Ran}\,(\mathcal{L}_{0,s})$, those appearing as "higher order terms" due to the transformation generated by h_k (with $k < s$) cannot be guaranteed to be (and in general, are not) in the same space; thus, not all of these can then be eliminated by suitably choosing h_s.

Notice in particular that even the action of higher order homological operators (corresponding to the first nontrivial term in the expansion (4.4) – or the equivalent one for standard Poincaré procedure) generate terms which are in $\mathrm{Ran}\,(\mathcal{L}_{s-k}) \cap V_s$. It should be stressed that this is true also if $s - k < k$, i.e. if the relevant homological operator is associated to a term which is already in Poincaré normal form.

However, this feature should not be necessarily seen as a drawback in the Poincaré procedure: if on the one side this shows that we could introduce resonant terms which were not initially present in our system, on the other side we could use the same mechanism to eliminate (some of the) resonant terms initially present, or generated as higher order terms by previous changes of coordinates. More in general, we could use these higher

[7] Here, by this we mean indifferently the usual Poincaré scheme, or the Poincaré-Lie one.

order effects of the Poincaré transformations to further normalize – we will use the term *renormalize* – the Poincaré normal form.

Indeed, it is well known that the Poincaré normal form is by no means unique, i.e. that two different Poincaré normal forms can be conjugated. The idea of further normalization is also not new, and has been considered by several authors [10-15], mainly in the context of Lie algebras filtration; here we mention in particular the work of Broer [13], and the "unique normal forms" studied by Baider [15]. However, such an approach seems (at least to the present author) of difficult implementation for the study of concrete dynamical systems. Here, we want to study the same problem from a direct point of view, i.e. considering the properties of Poincaré transformations and the explicit higher order effects, see (2.4) and (4.5). This allows to give a well definite algorithm, which – once a basis is chosen in each of the V_k, e.g. the one given by the $e_{\mu,j}(x)$ considered in section 4 – only requires linear algebra computations. Moreover, these reduce to consideration of the action of (higher order) homological operators and to solution of relevant (higher order) homological equations, thus representing a natural and straightforward generalization of the Poincaré scheme.

7. Poincaré renormalization – I.

We can use the considerations of the previous section, and the general formulas obtained for the higher order action of a Poincaré transformation, to devise a scheme of Poincaré normalization which includes iterated normalization (renormalization) and leads to a simplified normal form unfolding. We choose to work in the Lie-Poincaré scheme; the reason for this preference will be clear in the following.

We sketch below the construction of such renormalized forms, first for orders two and three (to let the reader grasp an intuitive understanding), and then giving the general algorithm.

We start from a system like (1.1), (1.3), and denote by $A = (Df)(0)$ the matrix corresponding to the linear part of f (which will not be modified by our transformations). We rewrite the system, for further reference, as

$$\dot{x} = \sum_{k=0}^{\infty} f_k^{(0)}(x) \ ; \tag{7.1}$$

the upper index on f_k (and similar ones from now on) refers to the number of Poincaré transformation of order k applied so far.

We can, if we prefer, preliminarly operate a linear change of coordinates to take A into (real or complex) canonical form; however, this would not be of great use: even if A is semisimple, and thus can be diagonalized, our present considerations on higher order homological operators would not benefit from this fact.

We can then, by means of the usual Poincaré transformation with generator $h_1^{(0)}$ chosen as solution to the homological equation for $k = 1$, take the term $f_1^{(0)}$ into NF, i.e. transform it to

$$f_1^{(1)} \;=\; f_1^{(0)} - \mathcal{L}_0 \left(h_1^{(0)} \right) \;\in\; \mathrm{Ker}\left(\mathcal{L}_0^+ \right) \cap V_1 \ . \tag{7.2}$$

Let us now pass to consider the term in V_2 (which is now already changed, but that we still denote by $f_2^{(0)}$); by the usual Poincaré procedure, i.e. choosing $h_2^{(0)}$ as solution to the homological equation at order $k = 2$, this is changed into

$$f_2^{(1)} \;=\; f_2^{(0)} - \mathcal{L}_0 \left(h_2^{(0)} \right) \;\in\; \mathrm{Ker}\left(\mathcal{L}_0^+ \right) \cap V_2 \ . \tag{7.3$'$}$$

However, we know that we can still change this by a transformation with generator $h_1^{(1)} \in V_1$. It is clear that now this $h_1^{(1)}$ cannot be completely arbitrary, if we want to keep the already normalized term $f_1^{(1)}$ in its present – and satisfactory – form; thus, to leave this untouched, we are obliged to require

$$h_1^{(1)} \;\in\; \mathrm{Ker}(\mathcal{L}_0) \cap V_1 \ . \tag{7.3$''$}$$

It is clear that in this way we can only eliminate terms which are in $\mathrm{Ran}(\mathcal{M}_1)$, where we have defined

$$\mathcal{M}_1 = [\mathcal{L}_1]_{\mathrm{Ker}(\mathcal{L}_0)} \quad . \tag{7.4}$$

Indeed, notice that (7.3') implies that in (4.5) we are reduced to $\widetilde{f_2} = f_2 + \{h_1, f_1\}$. It is easy to see that this same remark, suitably generalized, will also hold in general for our proposed procedure: the transformations will be given by action of the series of homological operators. This represents the main advantage, in the present discussion, of considering Lie transformations.

Thus, this second normalization of the term in V_2 will transform $f_2^{(1)}$ into

$$f_2^{(2)} \;=\; f_2^{(1)} - \mathcal{L}_1 \left(h_1^{(1)} \right) \;=\; f_2^{(1)} - \mathcal{M}_1 \left(h_1^{(1)} \right) \ , \tag{7.5}$$

and by suitable choice of $h_2^{(1)}$ we can have

$$f_2^{(2)} \;\in\; \left[\mathrm{Ker}\left(\mathcal{L}_0^+ \right) \cap V_2 \right] \cap \mathrm{Ker}\left(\mathcal{M}_1^+ \right) \ . \tag{7.6$'$}$$

If we denote by P_1 the operator of projection on $\mathrm{Ran}(\mathcal{M}_1)$, this "suitable choice" amounts to

$$h_1^{(1)} \;=\; \mathcal{M}_1^+ \, P_1 \, f_2^{(1)} \quad . \tag{7.6$''$}$$

We could then repeat the same procedure for the term $f_3^{(0)}$, by applying Poincaré transformations with generators $h_3^{(0)}$ (solution of the homological equation at order $k = 3$), $h_2^{(1)}$, and $h_1^{(2)}$; this time, however, we should not only make sure that the $h_2^{(1)}$ does not modify the already established $f_2^{(2)}$, i.e. that

$$h_2^{(1)} \in \mathrm{Ker}\,(\mathcal{L}_1) \ , \tag{7.7$'$}$$

but also that $h_1^{(2)}$ does not modify neither the $f_2^{(2)}$ nor the $f_1^{(1)}$, i.e. that

$$h_1^{(2)} \in \mathrm{Ker}\,(\mathcal{L}_0) \cap \mathrm{Ker}\,(\mathcal{L}_1) \ . \tag{7.7$''$}$$

Again, these requirements will guarantee that in (4.5) only the term $\mathcal{L}_1(h_2^{(1)})$ and $\mathcal{L}_2(h_1^{(2)})$ are effectively giving a contribution to the transformed f_3.

The general scheme of construction should at this point be quite clear, and we can pass to describe it in abstract terms.

8. Functional setting.

We will consider some chains of spaces and operators, together with the chain of spaces $V_k \subset V$ and the chain of operators $\mathcal{L}_k : V \to V$ introduced above. These definitions will be based on a formal power series $f = \sum f_k$ (as it was already the case for the $\mathcal{L}_k$).

We define the spaces $H^{(p)} \subseteq V$ by $H^{(0)} = V$ and, for $p \geq 1$,

$$H^{(p)} = \mathrm{Ker}(\mathcal{L}_0) \cap ... \cap \mathrm{Ker}(\mathcal{L}_{p-1}) = \bigcap_{s=0}^{p-1} \mathrm{Ker}\,(\mathcal{L}_s) \tag{8.1}$$

It is obvious that $H^{(p+1)} \subseteq H^{(p)}$, so that the $H^{(p)}$ realize a filtration of V.

We define then the operators $\mathcal{M}_k$ as the restriction of $\mathcal{L}_k$ to $\mathrm{Ker}(\mathcal{L}_0) \cap ... \cap \mathrm{Ker}(\mathcal{L}_{k-1})$, i.e. $\mathcal{M}_p = \mathcal{L}_p\big|_{H^{(p-1)}}$. Clearly,

$$\mathrm{Ker}\,(\mathcal{M}_p) = \bigcap_{s=0}^{p} \mathrm{Ker}\,(\mathcal{L}_s) = H^{(p+1)} \ . \tag{8.2}$$

Next, we define the spaces $F^{(p)}$ by $F^{(0)} = V$ and

$$F^{(p)} = \bigcap_{k=0}^{p} [\mathrm{Ran}\,(\mathcal{M}_k)]^\perp = \bigcap_{k=0}^{p} \mathrm{Ker}\,\left(\mathcal{M}_k^+\right) \ ; \tag{8.3}$$

hence the $F^{(p)}$'s satisfy $F^{(p+1)} \subseteq F^{(p)}$ and realize a filtration of V, as it was already the case for the $H^{(p)}$'s. It should be noticed that $F^{(p+1)} = F^{(p)} \backslash \left[\mathrm{Ran}\,(\mathcal{M}_p) \cap F^{(p)}\right]$.

We can also define projection operators for each of these spaces; we denote them by:

$$\begin{array}{lll} \pi_s : V \to \mathrm{Ker}(\mathcal{L}_s) & , & \widehat{\pi}_s : V \to \mathrm{Ran}(\mathcal{L}_s^+) \\ \chi_s : V \to \mathrm{Ker}(\mathcal{L}_s^+) & , & \widehat{\chi}_s : V \to \mathrm{Ran}(\mathcal{L}_s) \ . \end{array} \tag{8.4}$$

It will be useful to define the composition of projection operators given by

$$\mu_s = \pi_{s-1} \circ \pi_{s-2} \circ \ldots \circ \pi_0 \qquad (s \geq 1) \tag{8.5}$$

(and $\mu_{s,k}$ as the restriction of this to V_k). Later on we will use the notation μ_0: this should be meant as the identity operator.

We also consider the projection on the range of $\mathcal{M}_s$, i.e.

$$P_s : V \to \mathrm{Ran}\,(\mathcal{M}_s) \tag{8.6}$$

In this way, we can redefine our spaces and operators using the projection operators; in particular we have $H^{(s)} = \mu_s\,[V]$ and $\mathcal{M}_s = \mathcal{L}_s \circ \mu_s$.

`Remark 1.` Notice that we could consider the decomposition of the spaces and operators introduced above according to (1.4), i.e. considering their intersection with, and restriction to, the spaces V_k. As in each of the V_k (considered as vector spaces) we have a finite dimensional basis, the relevant operators – in particular, the $\mathcal{M}_{p,k} = \mathcal{M}_p\big|_{V_k}$ – can be written in matrix form using these bases. $\odot$

9. Poincaré renormalization – II.

We can now, with the notation introduced in the previous section, describe in general and abstract terms the procedure sketched in section 7 above.

Definition. *We say that the dynamical system $\dot{x} = f(x) = \sum f_k(x)$ (the vector field $X = f^i(x)\partial_i = \sum f^i_k(x)\partial_i$; the formal power series $f(x) = \sum f_k(x)$) is in* **Poincaré renormalized form** *up to order n if $f_k \in F^{(k)}$ for all $k \leq n$.*

Proposition. *Any dynamical system (vector field, formal power series) can be brought into Poincaré renormalized form up to any desired order n by means of a formal series of Lie-Poincaré transformations.*

`Proof.` We will prove constructively the above proposition by giving a well defined algorithm for the transformation to Poincaré renormalized form.

We operate sequentially for $k = 1, 2, ..., n$ in the following way. If $f_k^{(0)}$ is the term of order k after performing the required transformation at orders up to $k-1$, we operate then a series of Lie-Poincaré transformations with generators $h_k^{(0)}, h_{k-1}^{(1)}, ..., h_1^{(k-1)}$, where $h_p^{(s)} \in H_p^{(s)}$; this condition guarantees, see (4.5), that at each step the transformation generated by $h_{k-p}^{(p)}$ will transform $f_k^{(p)}$ into

$$f_k^{(p+1)} \ = \ f_k^{(p)} \ - \ \mathcal{M}_p \left(h_{k-p}^{(p)} \right) \ . \tag{9.1}$$

The $h_k^{(0)}$ is chosen as the solution to the standard homological equation (3.3), i.e. as

$$h_k^{(0)} = \mathcal{L}_0^+ \ P_0 \ f_k^{(0)} \equiv \mathcal{M}_0^+ \ P_0 \ f_k^{(0)} \ ; \tag{9.2}$$

the $h_{k-p}^{(p)}$ should be chosen as the projection on $H_{k-p}^{(p)}$ of the solution to the (higher order) homological equations

$$P_p f_k^{(p)} - \mathcal{M}_p \left(h_{k-p}^{(p)} \right) = 0 \ , \tag{9.3}$$

which means, explicitely,

$$h_{k-p}^{(p)} \ = \ \mu_p \circ \mathcal{M}_p^+ \circ P_p \ \left(f_k^{(p)} \right) \tag{9.4}$$

(with $\mu_0 = I$, the (9.2) is included in this formula as well).

Clearly, in this way we arrive in the end – i.e. after applying the procedure to terms of order $k = 2, 3, ...n$ – at a system

$$\dot{x} \ = \ f^*(x) \ \sum_{k=0}^{\infty} f_k^*(x) \tag{9.4}$$

in which, for $k \leq n$,

$$f_k^*(x) \ = \ f_k^{(k)} \ \in \ F_k^{(k)} \ . \tag{9.5}$$

The proof is thus complete. $\odot$

`Remark 2.` We would like to point out that one can also work following a different order; i.e., once we have fixed the order n up to which we want to put the system in Normal Form, we can first proceed to the usual Poincaré normalization, i.e. consider the transformations generated on the $h_k^{(0)}$, and corresponding to the action of $\mathcal{L}_0$, for $k = 1, ..., n$; we can then consider the transformations, generated by $h_k^{(1)}$, corresponding to the action of $\mathcal{L}_1$ (or more precisely of $\mathcal{M}_1$), i.e. "second-normalize the normal form", and so on. $\odot$

`Remark 3.` We would also like to remark explicitely that, although we have preferred to avoid a cumbersome notation, one could consider restriction of operators to the relevant V_k subspaces, and thus transform – by means of bases in each of the V_k – the above equations in algebraic ones. $\odot$

10. Planar vector fields with rotations as linear part.

We will consider, as a meaningful example, the unfolding of Normal Forms for vector fields in R^2 having linear part $f_0(x) = Ax$ with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{10.1}$$

As it is well known, the Poincaré Normal Forms corresponding to this can be written in the form

$$f(x) = Ax + \sum_{k=1}^{\infty} \left(x_1^2 + x_2^2\right)^k [a_k I + b_k A] x , \tag{10.2}$$

where the a_k, b_k are arbitrary real constants. Writing $r^2 = (x_1^2 + x_2^2)$, this reads

$$f_k(x) = \begin{cases} 0 & \text{for } k \text{ odd} \\ r^{2m} [a_m I + b_m A] x & \text{for } k = 2m. \end{cases} \tag{10.3}$$

We will see the the renormalized normal form unfolding is remarkably simpler. In order to do this, and to avoid trivial steps, we will consider the renormalized form of a system which is already in Poincaré normal form.

Thus, let us consider an f which has already be taken into Poincaré normal form, and proceed to its renormalization; we will consider only the nontrivial transformations (but keep the indices notation introduced above).

Thus, let us first consider the term $f_2^{(0)}$; since it is in normal form (and since for the same reason $f_1 = 0$, i.e. $\mathcal{L}_1 \equiv 0$), we cannot modify it by our algorithm, i.e. it will remain in the form given above,

$$f_2^{(2)} = f_2^{(0)} = r^2 (a_1 I + b_1 A) x . \tag{10.4}$$

We then have $f_3 = 0$, and

$$f_4^{(0)} = r^4 (a_2 I + b_2 A) x ; \tag{10.5}$$

the generator $h_2^{(2)}$ of the transformation

$$f_4^{(2)} = f_4^{(0)} - \mathcal{M}_2 \left(h_2^{(2)} \right) \tag{10.6}$$

must be, for $h_2^{(2)} \in \mathrm{Ker}(\mathcal{L}_0)$, of the form

$$h_2^{(2)} = r^2 (\alpha I + \beta A) x , \tag{10.7}$$

and thus

$$\mathcal{M}_2 \left(h_2^{(2)} \right) = 4 \left(a_1 \beta - b_1 \alpha \right) A x ; \tag{10.8}$$

it is then clear that, unless $a_1 = b_1 = 0$, we can always choose α, β so that

$$f_4^{(2)}(x) = f_4^{(4)}(x) = r^4 a_2 x \ . \tag{10.9}$$

Let us now consider f_6; now $h_4^{(2)}(x) = r^4(\alpha I + \beta A)x$ (again for $h \in \mathrm{Ker}(\mathcal{L}_0)$), and

$$\mathcal{L}_2 \left(h_4^{(2)} \right) \ = \ [(-2a_1\alpha)I + (2a_1\beta - 4b_1\alpha)A]\, x \ . \tag{10.10}$$

Thus, if $a_1 \neq 0$, we can eliminate completely f_6 in this way.

It is quite easy to get convinced that, under the same condition $a_1 \neq 0$, the same holds for all the f_{2m}. Indeed, for

$$h_{2(k-1)}^{(3)}(x) = r^{2(k-1)}\, [\alpha I + \beta A]\, x \tag{10.11}$$

(again, $h_{2(k-1)}^{(3)} \in \mathrm{Ker}(\mathcal{L}_0)$ requires this form for h), we have

$$\mathcal{L}_2 \left(h_{2(k-1)}^{(3)} \right) \ = \ - [(2(k-2)a_1\alpha)\, I + (2(k-1)a_1\beta - 2b_1\alpha)\, A]\, x \ ; \tag{10.12}$$

thus, we can eliminate completely all the f_{2m}.

We have thus shown that:

Lemma 1. *If in the Poincaré normal form (10.2) for $f(x)$ the constant a_1 is nonzero, then the corresponding Poincaré renormalized form is given by*

$$f^*(x) \ = \ Ax + r^2\, (a_1 I + b_1 A)\, x + r^4 a_2 x$$

and thus its unfolding depends on three real parameters.

We can also analyze what happens if the nondegeneracy condition $a_1 \neq 0$ is not satisfied. We assume now that $a_1 = 0$ and $b_1 \neq 0$. ¿From (10.8), it appears that, choosing $\alpha = -b_2/(4b_1)$ in $h_2^{(2)}$, we can still reduce f_4 to $r^4 a_2 x$ (no reduction at all would be possible if $a_1 = b_1 = 0$).

When it comes to considering f_6, (10.10) shows that choosing $\alpha = -b_3/(4b_1)$ in $h_4^{(3)}$ we can arrive to $f_6^{(3)} = r^6 a_3 x$. We can then proceed further in the renormalization; $\mathcal{L}_3 \equiv 0$, and thus the next – and last possible – step will be $f_6^{(5)} = f_6^{(3)} - \mathcal{L}_4 \left(h_2^{(4)} \right)$, where $h_2^{(4)} \equiv h = r^2(\alpha I + \beta A)x$, due to the condition $h \in \mathrm{Ker}(\mathcal{L}_0)$. Recall however that we also have to ask $h \in \mathrm{Ker}(\mathcal{L}_2)$: this condition is readily see to be equivalent to $a_1\beta = b_1\alpha$; with our present assumptions, this means that $\alpha = 0$. Thus, we cannot eliminate $f_6^{(3)}$.

We could then check explicitely that the higher order terms, i.e. the f_{2k} with $k \geq 4$, can be completely eliminated.

Rather than going on with discussion of more and more degenerate cases, we will give a general criterion and an inductive proof of it.

Lemma 2. *Let the vector formal power series $f : R^2 \to R^2$ be given by $f(x) = Ax + \sum_{k=1}^\infty r^{2k} [a_k I + b_k A] x$; let μ be the lowest number such that $a_\mu \neq 0$, and ν the lowest number such that $b_\nu \neq 0$, so that $f(x)$ can be written as*

$$f(x) \ = \ Ax \ + \ \sum_{k=\mu}^\infty \ r^{2k} a_k x \ + \ \sum_{k=\nu}^\infty \ r^{2k} b_k Ax \ . \tag{10.13}$$

Then, the Poincaré renormalized form of f up to any given order n is given by

$$f^*(x) \ = \ Ax + r^{2\mu} a_\mu x + r^{2\nu} \beta Ax + r^{4\mu} \alpha x \ , \tag{10.14}$$

where $a_\mu \neq 0$ is the same as in (10.13), and the α, β could (possibly, but not necessarily) vanish. In particular, if $\nu > \mu$, then $\beta = 0$.

`Proof.` To see that this is true, it is convenient to use the vector fields notation, with $X = f^i \partial_i = \sum_k X_k$, and $X_k = f^i_k \partial_i$.

It is useful to consider the vector fields D and R corresponding respectively to dilations and rotations in R^2, i.e.

$$D = x_1 \partial_1 + x_2 \partial_2 \quad , \quad R = -x_2 \partial_1 + x_1 \partial_2 \ ; \tag{10.15}$$

moreover, we consider the vector fields $Z_k = r^k D$ and $Y_k = r^k R$ (for k even); these satisfy

$$[aZ_k + bY_k, \alpha Z_m + \beta Y_m] = (m-k) a\alpha Z_{(m+k)} + (ma\beta - kb\alpha) Y_{(m+k)} \ . \tag{10.16}$$

With the notation introduced above, the effect of $\mathcal{L}_k(h_m)$ with $h_m \in \mathrm{Ker}(\mathcal{L}_0)$ can be computed via

$$[aZ_k + bY_k, \alpha Z_m + \beta Y_m] \ = \ (m-k) a\alpha Z_{k+m} \ + \ (ma\beta - kb\alpha) Y_{k+m} \ . \tag{10.17}$$

First of all, we notice that we can eliminate all the terms $a_p Z_p$ in f, except the one for $p = 2\mu$: indeed, it suffices to choose each time a $h_{p-\mu} = r^{2(p-\mu)} (\alpha I + \beta A) x$ with $\alpha = a_p / ((p - 2\mu) a_\mu)$. Notice that by a suitable choice of β (in particular, $\beta = 0$ if $b_\mu = 0$) we can always manage to do this without modifying the term $b_p Y_p$. Let us then assume we eliminate first all the terms $a_p Z_p$ (except $p = \mu$ and possibly $p = 2\mu$) up to $p = n$.

Let us now look at the terms $b_p Y_p$ with p greater than the smaller of μ and ν: it is clear, again by (10.17), that these can be eliminated via the term $\mathcal{L}_\mu(h_{p-\mu})$ by choosing $\beta = b_p / ((p-\mu) a_\mu)$ (if $\mu < \nu$), or via the term $\mathcal{L}_\nu(h_{p-\nu})$ by choosing $\alpha = -b_p / ((p-\nu) b_\nu)$ (if $\nu < \mu$). Notice that if $\nu \leq \mu$, the term $b_\nu Y_\nu$ cannot be eliminated. $\odot$

Appendix.

In this appendix, we shortly go over the Poincaré-Lie transformation, and the derivation of (4.4); we will follow the discussion given in [5].

We recall that in this case the change of coordinates is given by (4.2), and that this transforms X into $\widetilde{X}$ given by (4.3) [5].

As mentioned in section 4, $\widetilde{X}$ can now be explicitely computed by the Baker-Campbell-Haussdorf formula [5,16], as

$$\widetilde{X} = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} X^{(n)} \tag{A.1}$$

where the $X^{(n)}$ are determined recursively by $X^{(n+1)} = \big[X^{(n)}, H_k\big]$, with $X^{(0)} = X$.

We can thus consider a one-parameter family of vector fields X_λ, where $X_0 = X$ and $X_1 = \widetilde{X}$; this satisfies $dX_\lambda / d\lambda = [H_k, X_\lambda]$. Correspondingly, we write x_λ for $e^{-\lambda H_k} x$ (i.e. the transformed coordinates, see (4.2), corresponding to λ), and $X_\lambda = f^i_\lambda(x_\lambda)(\partial/\partial x^i_\lambda)$; the f^i_λ's satisfy then

$$\frac{df^i_\lambda}{d\lambda} \ = \ \{h_k, f_\lambda\}^i_\lambda \ , \tag{A.2}$$

where $\{.,.\}_\lambda$ is the bracket $\{.,.\}$ in the x_λ coordinates, i.e. $\{f,g\}_\lambda = f^j(\partial g/\partial x^j_\lambda) - g^j(\partial f/\partial x^j_\lambda)$.

If we consider the power series expansion of f, and writing for ease of notation $f(x,\lambda) = f_\lambda(x_\lambda)$ and $\{.,.\}$ for $\{.,.\}_\lambda$, we have

$$\frac{\partial f^i_m(x,\lambda)}{\partial\lambda} \ = \ \{h_k, f_{m-k}\}^i \ . \tag{A.3}$$

The $\widetilde{X} = X_1$ is then written in the $\widetilde{x}$ coordinates as $\widetilde{X} = f^i(x,1)(\partial/\partial\widetilde{x}^i) \equiv \widetilde{f}^i(\widetilde{x})(\partial/\partial\widetilde{x}^i)$; the $\widetilde{f}$ correspond to the solution of (A.2) for $\lambda = 1$.

These can be expressed by means of the BCH formula: indeed, from (A.1) and the recursion relation for $X^{(n)}$, we have immediately that

$$f(x,\lambda) \ = \ \sum_{n=0}^{\infty} \left[\frac{(-1)^n \lambda^n}{n!} \ \varphi^{(n)}(x) \right] \tag{A.4}$$

with $\varphi^{(0)}(x) = f(x,0)$ and $\varphi^{(n+1)} = \{\varphi^{(n)}, h_k\}$.

¿From this, we have indeed, with $\mathcal{H}(.) = \{h,.\}$,

$$f_\lambda \ = \ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \ \mathcal{H}^n(f) \ \ , \tag{A.5}$$

and for $\lambda = 1$, i.e. for $\widetilde{f}(x) = f(x,1)$, this is just (4.4).

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