Discrete Mathematics and Physics on the Planck-Scale exemplified by means of a Class of 'Cellular Network Modells' and their Dynamics

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Abstract

Starting from the hypothesis that both physics, in particular space-time and the physical vacuum, and the corresponding mathematics are discrete on the Planck scale we develop a certain framework in form of a class of '*cellular networks*' consisting of cells (nodes) interacting with each other via bonds according to a certain *'local law'* which governs their evolution. Both the internal states of the cells and the strength/orientation of the bonds are assumed to be dynamical variables. We introduce a couple of candidates of such local laws which, we think, are capable of catalyzing the unfolding of the network towards increasing complexity and pattern formation. In section 3 the basis is laid for a version of '*discrete analysis*' and *'discrete topology/geometry'* which, starting from different, perhaps more physically oriented principles, manages to make contact with the much more abstract machinery of Connes et al. and may complement the latter approach. In section 4 a, as far as we can see, promising concept of '*topological dimension*' in form of a '*degree of connectivity*' for graphs, networks and the like is developed. It is then indicated how this '*dimension*', which for continuous structures or lattices being embedded in a continuous background agrees with the "usual" notion of dimension (i.e. the respective embedding dimension) , may vary dynamically as a result of a '*phase transition like*' change of the '*connectivity*' in the network. A certain (highly) speculative argument, along the lines of statistical mechanics, is supplied in favor of the naturalness of dimension 4 of ordinary (classical) space-time.

1 Introduction

There exists a certain suspicion in parts of the scientific community that nature may be "discrete" on the Planck scale. The point of view held by the majority is however, at least as far as we can see, that quantum theory as we know it holds sway down to arbitrarily small scales as an allembracing general principle, being applied to a sequence of increasingly fine grained effective field theories all the way down up to, say, string field theory. But even on that fundamental level one starts from strings moving in a continuous background. It is then argued that "discreteness" enters somehow through the backdoor via "quantisation".

The possibly most radical and heretical attempt, on the other side, is it to try to generate both gravity and quantum theory as secondary and derived concepts (in fact merely two aspects) of one and the same underlying more primordial theory instead of simply trying to quantise gravity, which is the canonical point of view (see e.g. [1]).

This strategy implies more or less directly that – as gravity is closely linked with the dynamics of (continuous) space-time – the hypothetical underlying more fundamental theory is supposed to live on a substratum which does not support from the outset something like continuous topological or geometrical structures. In our view these continuous structures should emerge as derived concepts via some sort of coarse graining over a relatively large number of "discrete" more elementary building blocks.

This program still leaves us with a lot of possibilities. For various reasons, which may become more plausible in the course of the investigation, we personally favor what we would like to call a "cellular network" as a realisation of this substratum, the precise definitions being given below. Without going into any details at the moment some of our personal motivations are briefly the following:

i) These systems are in a natural way discrete, the local state space at each site being usually finite or at least countable.

ii) Systems like these or their (probably better known) close relatives, the "cellular automata", are known to be capable of socalled "complex behavior", "pattern generation" and "selforganisation" in general while the underlying dynamical laws are frequently strikingly simple (a wellknown example being e.g. Conway's "game of life").

Remark: A beautiful introduction into this fascinating field is e.g. [2]. As a shorter review one may take the contribution of Wolfram (l.c.). More recent material can be found in the proceedings of the Santa Fee Institute, e.g. the article of Kauffman in [3], who investigates slightly different systems ("switching nets").

iii) Some people suspect (as also we do) that physics may be reducible at its very bottom to some sort of "information processing system" (cf. e.g. [4, 5]). Evidently cellular automata and the like are optimally adapted to this purpose.

iv) In "ordinary" field theory phenomena evolving in space-time are typically described by forming a fibre bundle over space-time (being locally homeomorphic to a product). In our view a picture like this can only be an approximate one. It conveys the impression that space-time is kind of an arena or stage being fundamentally different from the various fields and phenomena which evolve and interact in it. In our view these localised attributes, being encoded in the various field values, should rather be attributes of the – in the conventional picture hidden – infinitesimal neighborhoods of space-time points, more properly speaking, neighborhoods in a medium in which space-time is immersed as a lower dimensional "submanifold". To put it in a nutshell: We would prefer a medium in which what we typically regard as irreducible space-time points have an internal structure. To give a simple picture from an entirely different field: take e.g. a classical gas, consider local pressure, temperature etc. as collective coarse grained coordinates with respect to the infinitesimal volume elements, regard then the microscopic degrees of freedom of the particles in this small volume elements as the hidden internal structure of the "points" given by the values of the above collective coordinates (warning: this picture is of course not completely correct as the correspondence between the values of local pressure etc. and volume elements is usually not one-one). It will turn out that a discrete structure as alluded to above is a nice playground for modelling such features.

Remark: Evidently there are close ties between what we have said in iv) and certain foundational investigations in pure mathematics concerning the problem of the 'continuum', a catchword being e.g. "non-standard analysis".

A lot more could be said as to the general physical motivations and a lot more literature could be mentioned as e.g. the work of Finkelstein and many others (see e.g. [6, 7]. For further references cf. also the papers of Dimakis and Müller-Hoissen ([8]). Most similar in spirit is in our view however the approach of 't Hooft ([9]).

2 The Concept of the "Cellular Network"

While our primary interest is the analysis of various partly long standing problems of current physics, which seem to beset physics many orders away from the Planck regime, we nevertheless claim that the understanding of the processes going on in the cellular network at Planck level will provide us with strong clues concerning the phenomena occurring in the "daylight" of "middle-energy-quantum-physics". In fact, as Planck scale physics is – possibly for all times – beyond the reach of experimental confirmation, this sort of serious speculation has to be taken as a substitution for experiments.

To mention some of these urgent problems of present day physics:
i) The unification of quantum theory and gravitation in general, and in our more particular context: both the emergence of "quantum behavior" and gravitation/space-time as two separate but related aspects of the unfolding of the primordial network state,
ii) the origin of the universe, of space-time from "nothing" and its very early period

of existence,
iii) the mystery of the seeming vanishing of the 'cosmological constant', which, in our view, is intimately related to the correct understanding of the nature of vacuum fluctuations,
iv) the primordial nature of the "Higgs mechanism",
v) causality in quantum physics or, put differently, its strongly translocal character,
vi) 'potential' versus 'actual' existence in the quantum world, the ontological status of the wave function (e.g. of the universe)) and the quantum mechanical measurement problem in general.

Most of these topics have been adressed in the thoughtful book of S. Weinberg ([10]) and will be treated by us in much more detail elsewhere in the near future. Therefore we refrain from making more comments as to these fundamental questions at the moment apart from the one remark that our approach will partly be based on the assumption that nature behaves or can be imitated as a cellular network at its very bottom ([11]). It is however crucial for these investigations to have a sufficiently highly developed form of "discrete mathematics" on graphs and networks and the like at ones disposal. Therefore we will concentrate in the following mainly on establishing the necessary (mostly mathematical) prerequisites on which the subsequent physical investigations will be based.

This is the more so necessary because one of our central hypotheses is that most of the hierarchical structure and fundamental building blocks of modern physics come into being via a sequence of *unfolding phase transitions* in this cellular medium. As far as we can see, the study of phase transitions in cellular networks is not yet very far developed, which is understandable given the extreme complexity of the whole field. Therefore a good deal of work should be, to begin with, devoted to an at least qualitative understanding of this intricate subject.

Furthermore discrete mathematics/physics of this kind is an interesting topic as such irrespective of the applications mentioned above which would justify a separate treatment of questions like the following anyhow (cf. e.g. the very interesting paper by Mack, [12], where a complex of ideas is sketched to which we are quite sympathetic)

2.1 Definition(Cellular Automaton): A cellular automaton consists typically of a fixed regular array of cells $\{C_i\}$ sitting on the nodes $\{n_i\}$ of a regular lattice like, say, $\mathbb{Z}^d$ for some d. Each of the cells is characterized by its internal state s_i which can vary over a certain (typically finite) set $\mathcal{S}$ which is usually chosen to be the same for all lattice sites.

Evolution or dynamics take place in discrete steps τ and is given by a certain specific 'local law' ll:

$$s_i(t+\tau) = ll(\{s'_j(t)\} \quad \underline{S}(t+\tau) = LL(\underline{S}(t)) \tag{1}$$

where t denotes a certain "*clock time*" (not necessarily physical time), τ the elementary clock time interval, $\{s'_j\}$ the internal states of the nodes of a certain local neighborhood of the cell C_i, ll a map:

$$ll : \mathcal{S}^n \to \mathcal{S} \tag{2}$$

with n the number of neighbors occurring in (1), $\underline{S}$(t) the global state at "time" t, LL the corresponding global map acting on the total state space $X := \{\underline{S}\}$. LL is called *reversible* if it is a bijective map of X onto itself.

Cellular automata of this type behave generically already very complicated (see [2]). But nevertheless we suspect they are still not complicated enough in order to perform the specific type of complex behavior we want them to do. For one, they are in our view too regular and rigid for our purposes. For another, the occurring regular lattices inherit quasi automatically such a physically important notion like '*dimension*' from the underlying embedding space.

Our intuition is however exactly the other way round. We want to generate something like dimension (among other topological notions) via a dynamical process (of phase transition type) from a more primordial underlying model which, at least initially, is lacking such characteristic properties and features.

There exist a couple of further, perhaps subjective, motivations which will perhaps become more apparent in the following and which result in the choice of the following primordial model system:

2.2 Definition(Cellular Network): In the following we will mainly deal with the class of systems defined below:
i) "Geometrically" they are *graphs*, i.e. they consist of nodes $\{n_i\}$ and bonds $\{b_{ik}\}$ where pictorially the bond b_{ik} connects the nodes n_i and n_k with $n_i \neq n_k$ implied (there are graphs where this is not so), furthermore, to each pair of nodes there exists at most one bond connecting them. In other words the graph is 'simple' (schlicht). There is an intimate relationship between the theory of graphs and the algebra of relations on sets. In this latter context one would call a simple graph a set carrying a homogeneous non-reflexive, (a)symmetric relation.

The graph is assumed to be *connected*, i.e. two arbitrary nodes can be connected by a sequence of consecutive bonds, and *regular*, that is it looks locally the same everywhere. Mathematically this means that the number of bonds being incident with a given node is the same over the graph (*'degree' of a node*). We call the nodes which can be reached from a given node by making one step the *1-order-neighborhood* $\mathcal{U}_1$ and by not more than n steps $\mathcal{U}_n$.
ii) On the graph we implant a class of dynamics in the following way:

2.3 Definition(Dynamics): As for a cellular automaton each node n_i can be in a number of internal states $s_i \in \mathcal{S}$. Each bond b_{ik} carries a corresponding bond state $J_{ik} \in \mathcal{J}$. Then we assume:

$$s_i(t+\tau) = ll_s(\{s'_k(t)\}, \{J'_{kl}(t)\}) \tag{3}$$

$$J_{ik}(t+\tau) = ll_J(\{s'_l(t)\}, \{J'_{lm}(t)\}) \tag{4}$$

$$(\underline{S}, \underline{J})(t+\tau) = LL((\underline{S}, \underline{J}(t)) \tag{5}$$

where ll_s, ll_J are two mappings (being the same all over the graph) from the state space of a local neighborhood of a given fixed node or bond to $\mathcal{S}, \mathcal{J}$, yielding the updated values of s_i and b_{ik}.

Remarks: i) The theory of graphs is developed in e.g. [13, 14]. As to the connections to the algebra of relations see also [15]. There are a lot of concepts in graph theory which are useful in our context, some of which will be introduced below if necessary. On the other hand we do not want to overburden this introductory paper with to much technical machinery.
ii) Synonyma for 'node' and 'bond' are e.g. 'site' and 'link' or 'vertex' and 'edge'.
iii) It may be possible under certain circumstances to replace or rather emulate a cellular network of the above kind by some sort of extended cellular automaton (e.g. by replacing the bonds by additional sites). The description will then however become quite cumbersome and involved.

What is the physical philosophy behind this picture? We assume the primordial substratum from which the physical universe is expected to emerge via a selforganisation process to be devoid of most of the characteristics we are usually accustomed to attribute to something like a manifold or a topological space. What we are prepared to admit is some kind of "*pregeometry*" consisting in this model under discussion of an irregular array of elementary grains and "direct interactions" between them, more specifically, between the members of the various local neighborhoods.

It is an essential ingredient of our approach that the strength of these direct interactions is of a dynamical nature and allowed to vary. In particular it can happen that two nodes or a whole cluster of nodes start to interact very strongly in the course of the evolution and that this type of *collective behavior* persists for a long time or forever (becomes *locked in*) or, on the other extreme, that the interaction between certain nodes becomes weak or vanishes.

It is not an easy task to select from the almost infinity of possible models an appropriate subclass which we think has the potential of displaying some or possibly all of the complex features (typically on length scales far away from the Planck regime) we are confronted with in "ordinary" (middle energy - compared to the Planck scale -) quantum physics, and being, on the other side, sufficiently transparent on, say, its natural primordial scale. We studied in fact a lot of alternatives (we do not mention) and want to present in the following a typical representative of a certain class we are presently favoring.

Our guiding principles have been roughly the following: Most of the cellular automaton rules being in use today (cf. e.g. [2]) are of a pronouncedly dissipative flavor. It is even frequently argued that some kinde of dissipation (or rather: shrinking of 'phase space') is a necessary prerequisite in order to have *'attractors'* and, as a consequence, pattern generation. We are not entirely convinced that the arguments along these lines are really conclusive (for a class of reversible automata see e.g. the book of Toffoli and Margolus, [16])

In any case, as we want our model system to generate "quantum behavior" on a, however, much coarser scale and if being in a certain specific *'phase'*, we consider it to be essential to implant a certain propensity for *'undulation'* in the class of local laws under discussion. Furthermore, it turns out to be extremely useful - in order to tame the horribly large quantum fluctuations occurring on Planck scale, when probing into space-time regions of larger extension - to incorporate a tendency to screen destructive fluctuations. These "boundary conditions" (among other considerations) led us to the following model system, which is only a simple representative of possibly a whole class.

At each site n_i there is sitting a one-dimensional discrete site variable $s_i \in q \cdot \mathbb{Z}$ with q, for the time being, a certain elementary quantum. The bond variables J_{ik} are, in this simple case, assumed to be two-valued, i.e: $J_{ik} \in \{\pm 1\}$.

Remarks: i)For the time being we let the site variables range over the full $\mathbb{Z}$ in order not to complicate the already sufficiently complicated reasoning further. It is of course possible to impose certain boundary conditions (e.g. switching to a subgroup of $\mathbb{Z}$) if it turns out to be sensible.
ii)In an extended model, which we will employ later on in order to catalyze the *'unfolding'* of the network together with the emergence of space-time and gravitation, J_{ik} can also take on the value 0.
iii)In the next section on graphs we will give the graph an *'orientation'*, i.e. the bond b_{ik} is assumed to point from n_i to n_k with n_i initial node, n_k terminal node, b_{ki} denoting the same bond with reverse orientation (see Definition 3.1) As a consequence, in order to be consistent, we assume:

2.4 Consequence:
$$J_{ik} = -J_{ki} \tag{6}$$

The physical idea behind this scheme is the following: If J_{ik} is positive an elementary quantum q is transported in the elementary 'clock-time step' τ from node (cell) n_i to n_k. Then the first half of the local law reads:
$$s_i(t+\tau) - s_i(t) = -q \cdot \sum_k J_{ik} \tag{7}$$
which is sort of a master or continuity equation.

What remains to be specified is the backreaction of the node states onto the bond states. We make the following choice:
$$\text{If} \quad s_i(t) > s_k(t) \quad \text{then} \quad J_{ik}(t+\tau) = +1 \tag{8}$$

$$\text{and hence} \quad J_{ki}(t+\tau) = -J_{ik} = -1 \tag{9}$$

For the borderline case $s_i(t) = s_k(t)$ we have roughly two options B_1, B_2 depending on the admissible state space of J_{ik}, i.e. $\{-1, +1\}$ or $\{-1, 0, +1\}$. In the former case we decree:

$$J_{ik}(t+\tau) = J_{ik}(t) \quad \text{if} \quad s_i(t) = s_k(t) \tag{10}$$

and in the latter case:

$$J_{ik}(t+\tau) = 0 \quad \text{if} \quad s_i(t) = s_k(t) \tag{11}$$

Introducing the signum function sign with

$$sign(x) = 1, 0, -1 \quad \text{if} \quad x > 0, = 0, < 0 \tag{12}$$

we then get:

$$B_1)\ J_{ik}(t+\tau) = -sgn(s_k(t) - s_i(t)) + (1 - |sgn(s_k(t) - s_i(t))|) J_{ik}(t) \tag{13}$$

$$B_2) J_{ik}(t+\tau) = -sgn(s_k(t) - s_i(t)) \tag{14}$$

As already indicated above, in case we want to model the unfolding of our "network universe" beginning with an extremely densly connected initial state (a complete graph or simplex, say) with no genuine physical neighborhood structure – nodes are not experienced by each other as near by or far away – , we intensify the effect implemented in B_2) and simulate what is called in catastrophy theory or in the realm of self organisation a fold (in physics known as hystheresis):

$$C)\ i)\ J_{ik}(t+\tau) = -sgn(s_k(t) - s_i(t)) \quad \text{if} \tag{15}$$

$$|s_k(t) - s_i(t)| \geq \lambda_1 \text{ and } J_{ik}(t) \neq 0 \text{ or } |s_k(t) - s_i(t)| > \lambda_2 \text{ and } J_{ik}(t) = 0 \tag{16}$$

with $\lambda_2 > \lambda_1 > 0$ two critical parameters, indicating the hysteresis interval $I_\lambda = [\lambda_1, \lambda_2]$

$$ii)\ J_{ik}(t+\tau) = 0 \quad \text{if} \quad |s_k(t) - s_i(t)| < \lambda_1 \tag{17}$$

2.5 Class of Local Laws: For our purposes an admissible class of local laws is given by the representatives A) plus B_1, B_2 or C.

Remarks:i)The reason why we do not choose the "current" $q \cdot J_{ik}$ proportional to the "voltage difference" $(s_i - s_k)$ as e.g. in Ohms's law is that we favor a network which is capable of self-excitation and self-organisation rather than self-regulation around a relatively uninteresting equilibrium state. The balance between dissipation and amplification of spontaneous fluctuations has however to be carefully chosen ("complexity at the edge of chaos")
ii)Note that – in contrast to e.g. euclidean lattice field theory – for the time being the socalled '*clock time*'t is not standing on the same footing as, say, potential coordinates

in the network (e.g. curves of bonds). We suppose anyhow that socalled '*physical time*' will emerge as sort of a secondary collective variable in the network, i.e. being different from the clock time (while being of course functionally related to it).

In our view this is consistent with the spirit of relativity. What Einstein was really teaching us is that there is a (dynamical) interdependence between what we experience as space and time, not that they are absolutely identical! In any case the assumption of an overall clock time is at the moment only made just for convenience in order to make the model system not too complicated. If our understanding of the complex behavior of the network dynamics increases, this assumption may be weakened in favor of a possibly local or/and dynamical clock frequency.

As can be seen from the definition of the cellular network it separates quite naturally into two parts of a different mathematical and physical nature. The first one comprises part i) of definition 2.2, the second one part ii) and definition 2.3. The first one is more static and "geometric" in character, the latter one conveys a more dynamical and topological flavor as we shall see in the following.

We begin in section 3 with a representation of what may be called discrete analysis on graphs and networks. This is followed in section 4 by making the first steps into an investigation of certain possible dynamical processes in networks of the defined type which have the character of phase transitions or collective behavior and may induce *dimensional change*. Most importantly we develop a physically appropriate concept of'*dimension*' for such irregular discrete structures which may be of importance in a wider context.

3 Discrete Analysis on Networks

At first glance one would surmise that as an effect of discreteness something like a network will lack sufficient structure for such a discipline to exist, but this is not so. Quite the contrary, there are intimate and subtle relations to various recent branches of pure mathematics as e.g. '*cyclic (co)homology*', '*noncommutative de Rham complexes*', '*noncommutative geometry*' in general and the like (see e.g. [17]-[20], as a beautiful and concise survey we recommend also [21]).

The general aim of these recent developements is it to generate something like a geometrical and differentiable structure within certain mathematical contexts which traditionally are not considered to support such structures. Particularly simple examples are discrete sets of, say, points, e.g. lattices. In a series of papers Dimakis and Müller-Hoissen have applied the general abstract machinery to models like these, having a possible bearing to, say, lattice field theory etc. (see e.g. [8] and further references there).

The fundamental object in these approaches is typically the socalled '*universal differential algebra*' or '*differential envelope*' which can be canonically constructed over any associative algebra and which is considered to be a generalisation or surrogate (depending on the point of view) of a differential structure in the ordinary cases.

As this notion may already indicate, this scheme, paying tribute to its universality and generality, is sometimes relatively far away from the concrete physical models one is perhaps having in mind. In the case of networks, for example, the inevitable starting point is the '*maximally connected*' network or graph (also called a '*complete graph*' or in algebraic topology a '*simplex*'), i.e. any two nodes are directly connected by a bond.

As a consequence, the construction is lacking, at least initially, something which is of tantamount importance in physical models, i.e. a natural and physically motivated neighborhood structure. Typically the interesting physical models are relatively lowly connected, which implies that they usually exhibit a pronounced feeling of what is near by or far away on the network.

One can of course pull this general structure down to the level of the models one may have in mind by imposing '*relations*' between various classes of '*differential forms*' employing a general result that each differential calculus over an algebra is isomorphic to the universal one modulo a certain 'differential ideal' but anyway, given a concrete model this approach is relatively abstract and perhaps not the most transparent and direct one. While being mathematically correct we want nevertheless to make some reservations as to its concrete meaning as far as specific models are concerned (e.g. networks and graphs); see below. Furthermore, it stresses more the global algebraic relations and perhaps not so much the inherent topological/geometrical content of the given model theory.

To put it differently, networks and graphs are only "mildly non-commutative" or rather 'non-local'. On the other hand they convey a lot more of extra structure (as most models do), which is not automatically implemented in the general algebraic scheme but has to be brought to light by scrutinizing the specific model class under discussion.

To put it in a nutshell: one can either go the way "top down", starting from some branch of non-commutative geometry and realize in the course of time that e.g. discrete sets or graphs may serve as certain model systems for this abstract algebraic scheme, or one may start from some concrete physical speculations about the supposed fine structure of the physical vacuum and space-time as dynamical unfolding networks and then make ones way "bottom up" observing that part of the emerging mathematical structure may be viewed as a variant of non-commutative geometry.

We followed this latter route and think the two philosophies may complement each other even if, coming from different directions, one may sometimes end up at formally closely related concepts.

We begin with the introduction of some useful concepts borrowed from algebraic topology and also known from graph theory (as to this we recommend the beautiful book of Lefschetz, [22].

At first we have to give the graph an '*orientation*':

3.1 Definition(Orientation): With the notions defined in definition 2.2 we say the bond b_{ik} points from node n_i to node n_k, the bond b_{ki} from n_k to n_i. We call n_i, n_k initial and terminal node of b_{ik} respectively. We assume the up to now formal relation:

$$b_{ik} = -b_{ki} \tag{18}$$

Remark: Note that orientation in the above (mathematical) sense is different from what is understood in many applications as '*directed bond*' in a network (as e.g. in typical "Kauffman nets", [3]). There a directed bond can typically "transport", say, a message only in one given fixed direction. That is, nets of this type behave, in physical terms, pronouncedly anisotropic locally. The definition 3.1, on the other side, is rather implementing something like the orientation of curves.

3.2 Definition(Chain Complexes): We introduce, to begin with, the two vector spaces C_0, C_1 whose elements, *zero- and one-chains* are defined by up to now formal expressions

$$\underline{c}_0 := \sum f_i n_i \quad \underline{c}_1 := \sum g_{ik} b_{ik} \tag{19}$$

where the f_i's and g_{ik}'s range over a certain given field or ring, of in the simplest cases numbers (i.e.$\mathbb{Z}$,$\mathbb{R}$,$\mathbb{C}$), the n_i's and b_{ik}'s serve as generators of a free abelian group.

Remarks: i) Evidently one could in a next step choose much more general spaces like, say, groups or manifolds.
ii) Furthermore, for the time being, the f_i's and g_{ik}'s should not be confused with the s_i's and J_{ik}'s introduced in section 2. The f_i's and g_{ik}'s are e.g. allowed to vanish outside a certain given cluster of nodes in various calculations or, put differently, it may be convenient to deal only with certain subgraphs.
iii) The spaces C_0, C_1 are in fact only the first two members of a whole sequence of spaces.

3.3 Definition (Boundary): we now define a *boundary operator* by

$$\delta b_{ik} := n_k - n_i \tag{20}$$

which by linearity induces a linear map from C_1 to C_0:

$$\delta : C_1 \ni \sum g_{ik} b_{ik} \to \sum g_{ik}(n_k - n_i) \in C_0 \tag{21}$$

The kernel, Z_1 of this map, the 1-chains without '*boundary*', consist of the '*1-cycles*'. A typical example is a '*loop*', i.e. a sequence of bonds, $\sum_\nu b_{i_\nu k_\nu}$ s.t. $k_\nu = i_{\nu+1}$ and $k_n = i_1$. (However not every cycle is a loop!).

3.4 Definition(Coboundary): Analogously we can define the coboundary operator as a map from C_0 to C_1:

$$dn_i := \sum_k b_{ki} \tag{22}$$

where the sum extends over all bonds having n_i as terminal node, and by linearity:

$$d : \sum_i f_i n_i \to \sum_i f_i \, (\sum_k b_{ki}) \tag{23}$$

Remarks:i)In algebraic topology 'cotheory' is frequently defined on 'dual spaces'. At the moment we do not make this distinction.
ii)To avoid possible formal complications, we always assume the 'degree' of the nodes to be uniformly bounded away from infinity. These matters could however be more appropriately dealt with after the introduction of suitable metrics, norms and related topological concepts.

We will now show that these two operations, well known in algebraic topology, can be fruitfully employed to create something like a discrete calculus. Evidently, the 0-chains can as well be considered as functions over the set of nodes; in this case we abbreviate them by f,g etc. (if necessary, chosen from a certain subclass of 0-chains $\mathcal{A} \subset C_0$, e.g. of '*finite support*', L^1, $L^2 \ldots$). $\mathcal{A}$ is trivially a module over itself (pointwise multiplication) freely generated by the nodes $\{n_i\}$ which can be identified with the '*elementary functions*' $e_i := 1 \cdot n_i$.

With $b_{ik} = -b_{ki}$ we can write the action of d on f differently, thus making more transparent its slightly hidden meaning:
with $b_{ki} = 1/2(b_{ki} - b_{ik})$ we get

$$\sum_i f_i \sum_k b_{ki} = 1/2 \sum_{ik} (f_k - f_i) \, b_{ik} \tag{24}$$

i.e:

3.5 Observation:

$$df = d(\sum_i f_i n_i) = 1/2 \cdot \sum_{ik} (f_k - f_i) \, b_{ik} \tag{25}$$

We have still to show to what extent the operation d defined above has the properties we are expecting from an (exterior) derivation. The really crucial property in the continuum case is the (graded) Leibniz rule. This is in fact a subtle and interesting point. To see this we make a short aside about how discrete differentiation is usually expected to work.

Take the following definition:

3.6 Definition (Partial Forward Derivative and Partial Differential at Node

(i)):

$$\nabla_{ik} f(i) := f(k) - f(i) \tag{26}$$

where n_i, n_k are '*nearest-neighbor-nodes*', i.e. being connected by a bond b_{ik}.

3.7 Observation:

$$\begin{aligned}
\nabla_{ik}(f \cdot g)(i) &= (f \cdot g)(k) - (f \cdot g)(i) \\
&= \nabla_{ik} f(i) \cdot g(i) + f(k) \cdot \nabla_{ik} g(i) &(27)\\
&= \nabla_{ik} f(i) g(i) + f(i) \nabla_{ik} g(i) + \nabla_{ik} f(i) \nabla_{ik} g(i) &(28)
\end{aligned}$$

In other words the "derivation" ∇ does **not** obey the ordinary(!) Leibniz rule. In fact, application of ∇ to, say, higher powers of f becomes increasingly cumbersome (nevertheless there is a certain systematic in it). One gets for example:

$$\nabla_{ik} f^{(n)}(i) = \nabla_{ik} f(i) \cdot \{f^{(n-1)}(k) + f^{(n-2)}(k) f(i) + \ldots + f(k) f^{(n-2)}(i) + f^{(n-1)}(i)\} \tag{29}$$

Due to the discreteness of the formalism and, as a consequence, the inevitable bilocality of the derivative there is no chance to get something as a true Leibniz rule on this level. (That this is impossible has also been stressed from a different point of view in e.g. example 2.1.1 of [18]).

Remark: We will come back to the non-Leibnizean character of ∇ below when establishing a *duality* between d and ∇. It is in fact a rather interesting relation even from a purely algebraic point of view, as it is a structural relation known in algebraic topology as *'Kuntz algebra'* (cf. [21]). Before however doing that we will further clarify the role of d.

In some sense it is considered to be one of the merits of the abstract algebraic framework (mentioned at the beginning of this section) that a graded Leibniz rule holds in that generalized case almost automatically. The concrete network model under investigation offers a good opportunity to test the practical usefulness of concepts like these.

To write down something like a Leibniz rule an important structural element is still missing, i.e. the multiplication of node functions from, say, some $\mathcal{A}$ with the members of C_1, in other words a '*module structure*' over $\mathcal{A}$. One could try to make the following definition:

$$f \cdot b_{ik} := f(i) \cdot b_{ik} \quad b_{ik} \cdot f := f(k) \cdot b_{ik} \tag{30}$$

and extend this by linearity.

Unfortunately this "definition" does not respect the relation $b_{ik} = -b_{ki}$. We have in fact:

$$f(i) b_{ik} = f \cdot b_{ik} = -f \cdot b_{ki} = -f(k) b_{ki} = f(k) b_{ik} \tag{31}$$

which is wrong in general for non-constant f!

Evidently the problem arises from our geometrical intuition which results in the natural condition $b_{ik} = -b_{ki}$, a relation we however want to stick to. On the other side we can extend or embed our formalism algebraically in a way which looses the immediate contact with geometrical evidence but grants us with some additional mathematical structure. This is in fact common mathematical practice and a way to visualize e.g. the *'universal differential algebra'* in *'non-commutatine geometry'* (see e.g. [21]). We want however to complement this more algebraic extension scheme by a, as we think, more geometric one below.

We can define another relation between nodes, calling two nodes related if they are connected by a bond with a built-in direction from the one to the other. We write this in form of a *tensor product* structure. In the general tensor product $C_0 \otimes C_0$ we consider only the subspace $C_0 \hat{\otimes} C_0$ spanned by the elements $n_i \otimes n_k$ with n_i, n_k connected by a bond (i.e. $i \neq k$!) and consider $n_i \otimes n_k$ to be unrelated to $n_k \otimes n_i$, i.e. they are considered to be linearly independent basis elements.

3.8 Observation: There exists an isomorphic embedding of C_1 onto the subspace generated by the antisymmetric elements in $C_0 \hat{\otimes} C_0$, i.e:

$$b_{ik} \to (n_i \otimes n_k - n_k \otimes n_i) =: n_i \wedge n_k \quad (\text{with } i \neq k) \tag{32}$$

generate an isomorphism by linearity between C_1 and the corresponding subspace $C_0 \wedge C_0 \subset C_0 \hat{\otimes} C_0$.

Proof: Both b_{ik} and $n_i \wedge n_k$ are linearly independent in there respective vector spaces.

In contrast to C_1 the larger $C_0 \hat{\otimes} C_0$ now supports a non-trivial and natural bimodule structure:

3.9 Observation/Definition (Bimodule): We can now define

$$f \cdot (n_i \otimes n_k) := f(i)(n_i \otimes n_k) \tag{33}$$

$$(n_i \otimes n_k) \cdot f := f(k)(n_i \otimes n_k) \tag{34}$$

and extend this by linearity to the whole $C_0 \hat{\otimes} C_0$, making it into a bimodule over some $\mathcal{A} \subseteq C_0$.

Remarks:i) Equivalently one could replace $n_i \otimes n_k$ by $e_i \otimes e_k$, the corresponding elementary functions. If one now identifies $e_i \otimes e_k$ with the abstract symbols $e_{ik} := e_i de_k$ employed in [8], one may establish a link to perhaps more abstract but related approaches.
ii) Another case in point is our definition $dn_i = \sum_k b_{ki}$ and the representation $da =$

$\mathbb{I}\otimes a - a\otimes \mathbb{I}$ which is employed within the context of the universal differential algebra. With $\mathbb{I} = \sum_i 1 \cdot n_i$ the close relation becomes apparent.

3.10 Lemma: As a module over $\mathcal{A}$, $C_0 \hat{\otimes} C_0$ is generated by $C_0 \wedge C_0$.

Proof: It suffices to show that every $n_i \otimes n_k$ can be generated this way.

$$n_i \cdot (n_i \otimes n_k - n_k \otimes n_i) = n_i \otimes n_k \tag{35}$$

as $n_i \cdot n_k = 0$ for $i \neq k$.

With the b_{ik} so embedded in a larger space and identified with

$$(n_i \otimes n_k - n_k \otimes n_i) = n_i \wedge n_k \tag{36}$$

we are in the position to derive a graded Leibniz rule on the module (algebra) $\mathcal{A}$. Due to linearity and the structure of the respective spaces it suffices to show this for products of elementary functions $e_i = n_i$. The same relation could of course be directly verified in a slightly more elegant way by regrouping

$$d(f \cdot g) = \sum_i f_i g_i dn_i = \sum_i f_i g_i (\sum_k (n_i \otimes n_k - n_k \otimes n_i)) \tag{37}$$

appropriately, employing **3.9 Observation**. We in fact have:

($i \neq k$ not nearest neighbors):

$$d(n_i \cdot n_k) = 0, \; dn_i \cdot n_k = n_i \cdot dn_k = 0 \tag{38}$$

($i \neq k$ nearest neighbors):

$$d(n_i \cdot n_k) = d(0) = 0 \text{ and} \tag{39}$$

$$\begin{aligned} dn_i \cdot n_k + n_i \cdot dn_k &= (\sum_{k'} b_{ik'}) \cdot n_k + n_i \cdot (\sum_{i'} b_{ki'}) && (40)\\ &= b_{ik} \cdot n_k + n_i \cdot b_{ki} && (41)\\ &= \{(n_i \otimes n_k - n_k \otimes n_i) n_k + n_i (n_k \otimes n_i - n_i \otimes n_k)\} && (42)\\ &= \{n_i \otimes n_k - n_i \otimes n_k\} = 0 && (43) \end{aligned}$$

($i = k$):

$$d(n_i^2) = d(n_i) = \sum_k b_{ik} \text{ and} \tag{44}$$

$$\begin{aligned} dn_i \cdot n_i + n_i \cdot dn_i &= (\sum_k b_{ik})\, n_i + n_i (\sum_k b_{ik}) && (45)\\ &= \sum_k (n_i \otimes n_k - n_k \otimes n_i) = \sum_k b_{ik} = dn_i && (46) \end{aligned}$$

3.11 Conclusion: As a map from the bimodule $\mathcal{A} \subseteq C_0$ to the bimodule $C_0 \hat{\otimes} C_0$ generated by the elements b_{ik} over $\mathcal{A}$ the map d fulfills the Leibniz rule, i.e:

$$d(f \cdot g) = df \cdot g + f \cdot dg \tag{47}$$

¿From the above we see also that functions, i.e. elements from $\mathcal{A}$ and bonds or differentials of functions do no longer commute (more specifically, the two possible ways of imposing a module structure could be considered this way). We have for example:

3.12 Commutation Relations:
($i \neq k$ not nearest neighbors):

$$n_i \cdot dn_k = dn_k \cdot n_i = 0 \tag{48}$$

($i \neq k$ nearest neighbors).

$$\begin{aligned} n_i \cdot dn_k + dn_k \cdot n_i &= 1/2 \sum_{i'} \{ n_i (n_k \otimes n_{i'} - n_{i'} \otimes n_k) & (49) \\ &+ (n_k \otimes n_{i'} - n_{i'} \otimes n_k)\, n_i \} & (50) \\ &= -1/2 (n_i \otimes n_k - n_k \otimes n_i) = -b_{ik} & (51) \end{aligned}$$

($i = k$):

$$n_i \cdot dn_i + dn_i \cdot n_i = \sum_k b_{ik} = dn_i \tag{52}$$

Another important relation we want to mention is the followng: $\delta\, d\, f$ is a map from $C_0 \to C_0$ and reads in detail:

3.13 Observation (Laplacian):

$$\delta\, d\, f = -\sum_i (\sum_k f(k) - n \cdot f(i))\, n_i =: -\Delta\, f \tag{53}$$

with n the number of nearest neighbors of n_i.

Proof:

$$\begin{aligned} \delta\, d\, f &= 1/2 \sum_{ik} (f(k) - f(i))(n_k - n_i) & (54) \\ &= 1/2 \sum_{ik} (f(k)\, n_k + f(i)\, n_i - f(i)\, n_k - f(k)\, n_i) & (55) \\ &= -\sum_i (\sum_k f(k) - n \cdot f(i))\, n_i & (56) \end{aligned}$$

Having now established the first steps in setting up this particular version of discrete calculus one could proceed in various directions. First, one can develop a discrete

Lagrangian variational calculus, derive Euler-Lagrange-equations and Noetherian theorems and the like and compare this approach with other existing schemes in discrete mathematics.

Second, one can continue the above line of reasoning and proceed to more sophisticated geometrical concepts and set them into relation to existing work, e.g. [8]. For the time being we would like to follow this latter route.

The philosophy underlying non-commutative geometry is that e.g. individual points of, say, a manifold have to be dispensed with and replaced by some equivalent of the algebra of functions over the manifold. On the other side networks are, as was already mentioned above, only mildly non-commutative and carry still a pronounced local structure (even the notion of points make still sense), which it may be advisable to implement in the discrete calculus.

As a consequence we will develop, as in the case of ordinary manifolds, differentials and partial derivatives in parallel as dual concepts. This may be, in our opinion, the main difference of our approach to other existing work in the field.

A characteristic feature of network calculus is its non-locality. According to 3.6 Definition the partial derivatives ∇_{ik} act locally, i.e. one can consider them as acting at node n_i. On the other side, this is not so for the b_{ik}; it leads to inconsistencies if one tries to relate them somehow to a definite node. We are however free to introduce the dual concept with respect to the ∇_{ik} and define:

3.14 Definition ((Co)Tangential Space):
i) We call the space spanned by the ∇_{ik} at node n_i the tangential space T_i.
ii) Correspondingly we introduce the space spanned by the d_{ik} at node n_i and call it the cotangential space T_i^* with the d_{ik} acting as linear forms over T_i via:

$$< d_{ik}|\nabla_{ij} >= \delta_{kj} \tag{57}$$

3.15 Definition/Observation: Higher tensor products of differential forms at a node n_i can be defined as *multilinear forms*:

$$< d_{ik_1} \otimes \cdots \otimes d_{ik_n}|(\nabla_{il_1}, \cdots, \nabla_{il_n}) >:= \delta_{k_1 l_1} \times \cdots \times \delta_{k_n l_n} \tag{58}$$

and linear extension.

In a next step we extend these concepts to functions $f \in C_0$ and *'differential operators'* or *'vector fields'* $\sum a_{ik}\nabla_{ik}$ and make the following interpretation:

3.16 Interpretation: We identify b_{ik} with $d_{ik} - d_{ki}$

We have to check whether this is a natural(!) identification.

3.17 Observation: Vector fields $v := \sum a_{ik}\nabla_{ik}$ are assumed to act on functions $f = \sum f_i n_i$ in the following manner:

$$v(f) := \sum a_{ik}(f_k - f_i)n_i \tag{59}$$

i.e. they map $C_0 \to C_0$.

3.18 Corollary: Note that this implies:

$$\nabla_{ik}n_k = n_i \quad \nabla_{ik}n_i = -n_i \tag{60}$$

$$\nabla_{ki}n_k = -n_k \quad \nabla_{ki}n_i = n_k \tag{61}$$

3.19 Observation: *'Differential forms'* $\omega = \sum g_{ik}d_{ik}$ act on vector fields $v = \sum a_{ik}\nabla_{ik}$ according to:

$$< \omega|v >= \sum g_{ik}a_{ik}n_i \tag{62}$$

With these definitions we can calculate $< df|v >$ with

$$df = 1/2\sum(f_k - f_i)b_{ik} = \sum(f_k - f_i)d_{ik} \tag{63}$$

according to 3.16 Interpretation. Hence:

$$< df|v >=< \sum(f_k - f_i)d_{ik}|\sum a_{ik}\nabla_{ik} >= \sum(f_k - f_i)a_{ik}n_i \tag{64}$$

which equals:

$$(\sum a_{ik}\nabla_{ik})(\sum f_i n_i) = v(f) \tag{65}$$

3.20 Consequence: Our geometric interpretation of the algebraic objects reproduces the relation:

i)

$$< df|v >= v(f) \tag{66}$$

known to hold in ordinary differential geometry, as is the case for the following relations, and shows that the definitions made above seem to be natural.

Furthermore vector and covector fields are left modules under the action of $\mathcal{A} \subseteq C_0$:

ii)

$$(\sum f_i n_i)(\sum a_{ik}\nabla_{ik}) := \sum f_i a_{ik}\nabla_{ik} \tag{67}$$

iii)

$$(\sum f_i n_i)(\sum g_{ik}d_{ik}) := \sum f_i g_{ik}d_{ik} \tag{68}$$

iv) As was the case with the $n_i \otimes n_k$ the d_{ik} generate also in a natural way a right module over $\mathcal{A}$.

As was already mentioned above, the partial derivatives ∇_{ik} do not obey the ordinary (graded) Leibniz rule in contrast to the operator d. In the latter case this is however only effected by embedding the "natural" geometric objects in a bigger slightly more abstract space. The structural relation the ∇_{ik} actually are obeying (see 3.7 Definition) is known from what is called by algebraic topologists a *'Kuntz algebra'* (cf. [21]); which however does occur there in a different context. With $q := \nabla_{ik}$ we have:

3.21 Observation (Kuntz algebra):

$$q(f \cdot g) = q(f) \cdot g + f \cdot q(g) + q(f) \cdot q(g) \tag{69}$$

and analogously for vector fields $\sum a_{ik} \nabla_{ik}$.
With $u := 1 + q$ we furthermore get:

$$u(f \cdot g) = u(f) \cdot u(g) \tag{70}$$

and

$$q(f \cdot g) = q(f) \cdot g + u(f) \cdot q(g) \tag{71}$$

i.e. a *'twisted derivation'* with u an endomorphism from $\mathcal{A}$ to $\mathcal{A}$.

As with ∇_{ik} the product rule for higher products can be inferred inductively:

$$q(f_1 \cdots f_n) = \sum_i f_1 \cdots q(f_i) \cdots f_n + \sum_{ij} f_1 \cdots q(f_i) \cdots q(f_j) \cdots f_n + \ldots q(f_1) \cdots q(f_n) \tag{72}$$

It is an interesting task to compare our framework sketched above with the perhaps more algebraic approaches which can be found in the mentioned literature or more recently in [23]. It should however be stressed that the other approaches are typically concerned with energy scales (e.g. standard model) different from our more primordial level. So we prefer to do this elsewhere apart from making some remarks of perhaps more principal interest concerning the appropriate geometric interpretation or meaning of the various abstract and algebraic symbols; among other things about the relation between the socalled universal differential algebra and the highly *'reduced calculi'* occurring typically in physical applications.

In contrast to the universal differential algebra mentioned above, where every two nodes are connected by a bond, this is not so for our *'reduced'* calculus. As a consequence certain operations are straightforward to define in the former approach.

However, descending afterwards to the lower-connected more realistic models is tedious in general and not always particularly transparent. That is, this method does not always really save calculational efforts (for a discussion of certain simple examples see [8]). It even may lead to (in our view) unnatural results as we want to show below.

By the way, the mathematical "triviality " of the differential envelope is reflected by the trivialty of the corresponding '*(co)homology groups*' of the maximally connected graph (simplex). This trivialty is then broken by deleting graphs in the reduction process.

To mention a typical situation: Take e.g. the subgraphs of a graph G consisting of, say, four nodes n_i, n_k, n_l, n_m and all the bonds between them which occur in G. In the case G being a simplex (i.e. non-reduced case) all these subgraphs are geometrically/topologically equivalent. An important consequence of this is that the four nodes can be connected by a *'path'*, i.e. a sequence of consecutive bonds, each being passed only once, the effect being that one can naturally define a multiplication in this scheme via '*concatenation*', e.g:

$$(n_i \otimes n_k) \cdot (n_l \otimes n_m) = n_i \otimes n_k \cdot n_l \otimes n_m \tag{73}$$

with $n_k \cdot n_l = \delta_{kl} \cdot n_k$. Correspondingly, all the higher differentials can be generated by the 1-forms via concatenation. The reason why it is sufficient to concatenate only at the extreme left and right of a '*word*' stems exactly from the simplex-character of each subgraph.

In typical reduced cases all this is no longer the case; the combinatorical topology becomes non-trivial. To give an example: Take as G a regular graph with the degree of the nodes (number of incident bonds) $n = 3$. In the analogous case of 4-node-subgraphs there exists now a kind of subgraph the nodes of which cannot be concatenated in the above way. Take e.g. the subgraph with bonds existing only between, say, $n_1\, n_4$, $n_2\, n_4$, $n_3\, n_4$. I.e., one has the 1-forms $b_{14},\ b_{24},\ b_{34}$ or:

$$n_1 \otimes n_4,\ n_2 \otimes n_4,\ n_3 \otimes n_4 \tag{74}$$

but there is no obvious way to generate the corresponding reduced subgraph by concatenating them sequentially(!) without passing certain bonds twice.

We have developed above two pictures, one of a more algebraic flavor, the other of a perhaps more geometric content. On the one side we identified $C_1 = \{\sum g_{ik} b_{ik}\}$ with the antisymmetric part of a certain subspace $C_0 \hat{\otimes} C_0$ of the tensor product over C_0, see 3.8 Observation. $C_0 \hat{\otimes} C_0$ was built from the elements $n_i \otimes n_k$ being connected by a bond; in physical terms: sites, coupled dynamically, and $b_{ik} = (n_i \otimes n_k - n_k \otimes n_i)$.

Along these lines one can build "higher elements" making the following definition:

3.22 Definition: The space

$$\Omega_{k-1} := C_0 \hat{\otimes} \cdots \hat{\otimes} C_0 \text{ (k-times)} \tag{75}$$

is built from elements

$$n_{i_0} \otimes \cdots \otimes n_{i_k} =: n_{i_1 \ldots i_k} \tag{76}$$

with each pair of consecutive nodes $n_{i_l}, n_{i_{l+1}}$ being connected by a bond. In graph language these elements may be associated with *bond sequences*.
If a pair of consecutive nodes occurs only once in the sequence (in other words: a bond) the bond sequence is called a *path*. If, furthermore, initial and terminal node coincide the bond sequence is called a *closed path* or *loop*. If in a closed path none of the inner nodes occurs twice it is called in graph theory a *circuit*.

Remarks: i) In the following it is always understood that $\otimes$ occurs always between nodes which are connected by a bond.
ii) Note that sequences like e.g. $n_i \otimes n_k \otimes n_i$ are admitted.

Multiplication can now naturally be defined via *'concatenation'*:

3.23 Definition (Concatenation):

$$(n_{i_0} \otimes \cdots \otimes n_{i_k}) \cdot (n_{j_0} \otimes \cdots \otimes n_{j_l}) = n_{i_0} \otimes \cdots \otimes n_{i_k} \cdot n_{j_0} \otimes \cdots \otimes n_{j_l} \tag{77}$$

with the rhs an element of Ω_{k+l}. (Note that $n_{i_k} \cdot n_{j_0}$ means multiplication in C_0; i.e. this yields either 0 or $n_{i_k} = n_{j_0}$).

Remarks:i) Hence, multiplication is basically the pasting together of bond sequences, i.e. a geometric process.
ii) The above shows that the ring which can be formed this way has typically a lot of *'zero divisors'* apart from the special situation where G is a complete graph (simplex).

3.24 Visualisation: Following this philosophy the network or graph G we started from can be identified algebraically with the ring or bimodule with the above multiplication (concatenation) and non-zero elements all the admissible bond sequences defined above.

Remark: Note however that an object like $n_0 \otimes \cdots \otimes n_k \in \Omega_k$ should not be confused with $n_0 \otimes n_1 + n_1 \otimes n_2 + \cdots n_{k-1} \otimes n_k \in \Omega_1$. Both elements can be associated with a bond sequence or path leading from n_0 to n_k, but they are of an entirely different algebraic (and geometric(!)) character. In the former case the pieces of the bond sequence are concatenated multiplicatively in the latter case they are composed additively.

On the other side we can concatenate the *dual forms* d_{ik} non-locally (a process which is not employed in the ordinary continuum differential geometry, which deals typically with local fields, but which could also be performed in the latter context, leading perhaps to new and interesting structures; cf. e.g. [21] where non-commutative calculus is introduced on the commutative algebra of smooth functions over a manifold).

For example one can form the space $T_i^* \hat{\otimes} T_k^*$, its elements acting bilinearly on pairs of vectors v_i, v_k with $v_i \in T_i, v_k \in T_k$.

3.25 Conclusion: From the above we can infer that, whereas $n_i \otimes n_k$ should not be viewed as an element of T_i^* (they are different geometric objects), the whole algebraic structure built over the set $\{n_i \otimes n_k\}$ via concatenation is isomorphic to the corresponding structure built over the set $\{d_{ik}\}$ with e.g:

$$n_{i_0} \otimes \cdots \otimes n_{i_k} \to d_{i_0 i_1} \otimes \cdots \otimes d_{i_{k-1} i_k} \tag{78}$$

The above observations give us the chance to generate higher dimensional geometric objects which may be considered as equivalents of the building blocks of piecewise affine or triangulated smooth manifolds. Following our general philosophy of creating geometric/topological concepts for *'non-standard'* spaces (cf. in particular the next section), one can try to catch the abstract essence of a notion like surface or volume in developing the following scheme which we however only sketch, for the time being, with the help of an example.

The map d mapped a node n_i onto the sum over bonds $\sum b_{ki}$ with endpoint n_i, where the oriented b_{ki} was the antisysmmetric combination $d_{ki} - d_{ik}$ or $n_{ki} - n_{ik}$. In the same sense one can proceed geometrically if the graph has an appropriate structure:

3.26 Definition: A triangle in a graph is a triple of nodes n_1, n_2, n_3 with $n_1\, n_2, n_2\, n_3, n_3\, n_1$ connected by bonds.
In case n_1, n_2, n_3 form a triangle it should be algebraically/geometrically realized (as is the case with the bond b_{ik}) as a geometric object being a totally antisymmetric combination of elements of Ω_2 in the following sense:

$$s_{123} := \sum_{per} \sigma(per) n_{i_1 i_2 i_3} \tag{79}$$

with $\sigma(per)$ the signum of the corresponding permutation of $\{1, 2, 3\}$.

3.27 Observation: With this definition the triangle has exactly two possible orientations , i.e:

$$s_{123} = -s_{321} \tag{80}$$

which corresponds with the two possible orientations of the path around the triangle.

Remarks: i) It is of course possible that n_{12}, n_{23} and therefore n_{123} can be built while $n_1\, n_3$ are not connected s.t. they do not form a triangle. A fortiori a graph need not have triangles.
ii) Following similar lines one can build also higher geometric objects.

3.28 Extension of d: The geometric idea behind the attempt to extend in a next step the range of the map d is that it should relate a bond (edge) with an appropriate combination of the triangles being incident with it, i.e:

$$d(b_{12}) = \sum_i s_{12i} \tag{81}$$

This will cause certain algebraic problems of a general nature for arbitrary, in particular non-regular, graphs (see the above remark) as typical relations known to hold e.g. in *'simplicial cohomology'* like $d \circ d = 0$ will not hold automatically. The necessary preconditions could however be analyzed but we shall not make the attempt to do this here due to lack of space. Instead of that we will close this section by addressing another topic which is in fact closely related to these problems and will make contact with a different approach which starts from the universal differential envelope, i.e., given the class of nodes, from the, in our language, complete graph (simplex); see e.g. [8].

In that extremely regular situation matters are rather smooth and can directly be taken over from the general case of the universal differential algebra $\Omega(\mathcal{A})$ defined over an arbitrary algebra $\mathcal{A}$. In our notation one defines e.g:

$$d_u(n_{1\ldots k}) = \sum_i (\sum_{\nu=0}^{k+1} (-1)^\nu n_{1\ldots i \ldots k}) \tag{82}$$

with ν denoting the place of the insertion of node n_i, beginning with $\nu = 0$, i.e. before n_1 and, as always, consecutive nodes being different understood. One shows immediately that e.g. $d_u \circ d_u = 0$ on Ω.

Remarks:i) Note that each term on the rhs is well defined since all the nodes are connected with each other.
ii) With d_u we denote the *universal* derivation.

We remarked at the beginning of this section that it is a general result that each differential calculus (more specifically: a certain differential algebra) over a given algebra $\mathcal{A}$ is isomorphic to the universal one divided by a certain *'differential ideal'*. This was exploited in [8] to construct a differential calculus on certain simple examples of *'reduced'* (smaller) differential algebras.

In our specific context of networks and graphs we may translate this general result in the following way: Let $\Omega^u = \sum \Omega^u_k$ be the universal differential algebra with Ω^u_k consisting of the $(k+1)$-fold tensor products of arbitrary $(k+1)$-tuples of nodes. We can define a projector Π which projects Ω^u onto $\Omega = \sum \Omega_k$ with Ω_k consisting of the $(k+1)$-fold *admissible* tensor products (bond sequences) of connected nodes in our actual network. We have:

$$\Pi(n_0 \otimes \cdots \otimes n_k) = 0 \tag{83}$$

if $(n_0, \ldots, n_k)$ is not admissible

$$\Pi(n_0 \otimes \cdots \otimes n_k) = n_0 \otimes \cdots \otimes n_k \tag{84}$$

if $(n_0, \ldots, n_k)$ is admissible.

3.29 Consequence: We have

$$\Pi = \Pi^2 \; , \; \Omega^u = \Pi\Omega^u + (\mathbb{1} - \Pi)\Omega^u \tag{85}$$

with $\Pi\Omega^u = \Omega$.
We can define

$$d := \Pi \circ d_u \circ \Pi \tag{86}$$

which leaves Ω invariant but in general $d \circ d \neq 0$ in contrast to $d_u \circ d_u = 0$.

The reason is the following: We can make $Ker(\Pi)$ into a two-sided ideal I consisting of the elements $n_{0\ldots k}$ having at least one pair of consecutive nodes *not* being connected by a bond. This ideal I is *not* invariant under the action of d_u! A closer analysis shows that $d_u(n_{0\ldots k}) \notin I$ if there occur *'insertions'* between non-connected neighbors s.t. non-admissible elements become admissible, i.e. connected.

3.30 Observation: In general there exist elements $n_{0\ldots k} \in Ker(\Pi)$ s.t. $\Pi(d_u(n_{0\ldots k})) \neq 0$, in other words, I is in general not left invariant by d_u.
This is the reason for e.g. the non-vanishing of $d \circ d$.
If one wants to make Ω a real differential algebra one has to enlarge I!

3.31 Consequence: The ideal $I' = I + d_u \circ I$ is invariant under d_u and d defines a differential algebra on the smaller algebra $\Omega^u/I' \subset \Omega$ with $\Omega = \Omega^u/Ker(\Pi)$.

(That I' is an ideal left invariant by d_u is easy to prove with the help of the property $d_u \circ d_u = 0$).

So much so good, but in our view there exists a certain problem. Ω^u/I' is the algebra one automatically arrives at if one defines the homomorphism Φ from Ω^u to the reduced differential algebra in the following canonical way:

$$\Phi : \; n_i \to n_i \quad d_u n_i \to dn_i \tag{87}$$

i.e. under the premise that d defines already another differential algebra. It is in this sense the general result mentioned above has to be understood.

On the other hand this may lead to a host of, at least in our view, unnatural relations in concrete examples as e.g. our network which may already carry a certain physically motivated interpretation going beyond being a mere example of an abstract differential algebra. Note e.g. that in our algebra Ω an element like n_{123} is admissible (i.e. non-zero) if n_1, n_2 and n_2, n_3 are connected. n_{123} may however arise from a differentiation process (i.e. from an insertion) like $d_u(n_{13})$ with n_1, n_3 not(!) connected.

This is exactly the situation discussed above:

$$n_{13} \in I \quad \text{but} \quad d_u(n_{13}) \notin I \tag{88}$$

Dividing now by I' maps $d_u(n_{13})$ onto zero whereas there may be little physical/geometric reason for n_{123} or a certain combination of such admissible elements being zero in our network.

3.32 Conclusion: Given a concrete physical network Ω one has basically two choices. Either one makes it into a full-fledged differential algebra by imposing further relations which may however be unnatural from a physical point of view and very cumbersome for complicated networks. This was the strategy e.g. followed in [8].
Or one considers Ω as the fundamental object and each admissible element in it being non-zero. As a consequence the corresponding algebraic/differential structure on Ω may be less smooth at first glance but on the other side more natural (the *'graded Leibniz-rule'* or $d \circ d = 0$ may only hold up to a certain order).
At the moment we refrain from making a general judgement whereas we would probably prefer the latter choice.

4 Intrinsic Dimension in Networks, Graphs and other Discrete Systems

There exist a variety of concepts in modern mathematics which generalize the notion of '*dimension*' one is accustomed to in e.g. differential topology or linear algebra. In fact, '*topological dimension*' is a notion which seems to be even closer to the underlying intuition (cf. e.g. [24]).

Apart from the purely mathematical concept there is also a physical aspect of something like dimension which has e.g. pronounced effects on the behavior of, say, many-body-systems, especially their microscopic dynamics and, most notably, their possible '*phase transitions*'.

But even in the case of e.g. lattice systems they are usually considered as embedded in an underlying continuous background space (typically euclidean) which supplies the concept of ordinary dimension so that the *'intrinsic dimension'* of the discrete array itself does usually not openly enter the considerations.

Anyway, it is worthwhile even in this relatively transparent situations to have a closer look on where attributes of something like dimension really come into the physical play. Properties of models of, say, statistical mechanics are almost solely derived from the structure of the microscopic interactions of their constituents. This is more or less the only place where dimensional aspects enter the calculations.

Naive reasoning might suggest that it is the number of nearest neighbors (in e.g. lattice systems) which reflects in an obvious way the dimension of the underlying space and influences via that way the dynamics of the system. However, this surmise, as we will show in the following, does not reflect the crucial point which is considerably more subtle.

This holds the more so for systems which cannot be considered as being embedded in a smooth regular background and hence do not get their dimension from the embedding space. A case in point is our primordial network in which Planck-scale-physics is assumed to take place. In our approach it is in fact exactly the other way round: Smooth space-time is assumed to emerge via a *phase transition* or a certain *cooperative behavior* and after some '*coarse graining*' from this more fundamental structure.

4.1 Problem: Formulate an intrinsic notion of dimension for model theories without making recourse to the dimension of some embedding space.

In a first step we will show that graphs and networks as introduced in the preceding sections have a natural metric structure. We have already introduced a certain neighborhood structure in a graph with the help of the minimal number of consecutive bonds connecting two given nodes.

In a connected graph any two nodes can be connected by a sequence of bonds. Without loss of generality one can restrict oneself to '*paths*'. One can then define the length of a path (or sequence of bonds) by the number l of consecutive bonds making up the path.

4.2 Observation/Definition: Among the paths connecting two arbitrary nodes there exists at least one with minimal length which we denote by $d(n_i, n_k)$. This d has the properties of a '*metric*', i.e:

$$d(n_i, n_i) = 0 \tag{89}$$

$$d(n_i, n_k) = d(n_k, n_i) \tag{90}$$

$$d(n_i, n_l) \leq d(n_i, n_k) + d(n_k, n_l) \tag{91}$$

(The proof is more or less evident).

4.3 Corollary: With the help of the metric one gets a natural neighborhood structure around any given node, where $\mathcal{U}_m(n_i)$ comprises all the nodes with $d(n_i, n_k) \leq m$, $\partial\mathcal{U}_m(n_i)$ the nodes with $d(n_i, n_k) = m$.

With the help of the above neighborhood structure we can now develop the concept of an intrinsic dimension on graphs and networks. To this end one has at first to realize what property really matters physically (e.g. dynamically) independently of the model or embedding space.

4.4 Observation: The crucial and characteristic property of, say, a graph or network which may be associated with something like dimension is the number of '*new nodes*' in $\mathcal{U}_{m+1}$ compared to $\mathcal{U}_m$ for m sufficiently large or $m \to \infty$. The deeper meaning of this quantity is that it measures the kind of '*wiring*' or '*connectivity*' in the network and is therefore a '*topological invariant*'.

Remark: In the light of what we have learned in the preceding section it is tempting to relate the number of bonds branching off a node, i.e. the number of nearest neighbors or order of a node, to something like dimension. On the other side there exist quite a few different lattices with a variety of number of nearest neighbors in, say, two- or three- dimensional euclidean space. What however really matters in physics is the embedding dimension of the lattice (e.g. with respect to phase transitions) and only to a much lesser extent the number of nearest neighbors. In contrast to the latter property dimension reflects the degree of connectivity and type of wiring in the network.

In many cases one expects the number of nodes in $\mathcal{U}_m$ to grow like some power D of m for increasing m. By the same token one expects the number of new nodes after an additional step to increase proportional to m^{D-1}. With $|\cdot|$ denoting number of nodes we hence have:

$$|\mathcal{U}_{m+1}| - |\mathcal{U}_m| = |\partial\mathcal{U}_{m+1}| = f(m) \tag{92}$$

with

$$f(m) \sim m^{D-1} \tag{93}$$

for m large.

4.5 Definition: The intrinsic dimension D of a regular (infinite) graph is given by

$$D - 1 := \lim_{m\to\infty} (\ln f(m)/\ln m) \text{ or} \tag{94}$$

$$D := \lim_{m\to\infty} (\ln |\mathcal{U}_m|/\ln m) \tag{95}$$

That this definition is reasonable can be seen by applying it to ordinary cases like regular translation invariant lattices. It is however not evident that such a definition makes sense for arbitrary graphs, in other words, a (unique) limit point may not always exist. It would be tempting to characterize the conditions which entail that such a limit exists. We, however, plan to do this elsewhere.

4.6 Observation For regular lattices D coincides with the dimension of the euclidean embedding space D_E.

Proof: It is instructive to draw a picture of the consecutive series of neighborhoods of a fixed node for e.g. a 2-dimensional Bravais lattice. It is obvious and can also be proved that for m sufficiently large the number of nodes in $\mathcal{U}_m$ goes like a power of m with the exponent being the embedding dimension D_E as the euclidean volume of $\mathcal{U}_m$ grows with the same power.

Remarks:i) For $\mathcal{U}_m$ too small the number of nodes may deviate from an exact power law which in general becomes only correct for sufficiently large m.
ii) The number of nearest neighbors, on the other side, does not(!) influence the exponent but rather enters in the prefactor. In other words, it influences $|\mathcal{U}_m|$ for m small but drops out asymptotically by taking the logarithm. For an ordinary Bravais lattice with N_C the number of nodes in a unit cell one has asymptotically:

$$|\mathcal{U}_m| \sim N_C \cdot m^{D_E} \quad \text{and hence:} \tag{96}$$

$$D = \lim_{m\to\infty}(\ln(N_C \cdot m^{D_E})/\ln m) = D_E + \lim_{m\to\infty}(N_C/\ln m) = D_E \tag{97}$$

independently of N_C.

Before we proceed a remark should be in order concerning related ideas on a concept like dimension occurring in however completely different fields of modern physics:

When we started to work out our own concept we scanned in vain the literature on e.g. graphs accessible to us and consulted various experts working in that field. From this we got the impression that such ideas have not been pursued in that context. (It is however apparent that there exist conceptual relations to the geometry of *'fractals'*.)

Quite some time after we developed the above introduced concept we were kindly informed by Th. Filk that such a concept had been employed in a however quite different context by e.g. A.A. Migdal et al and by himself (see e.g. [25] and [26]). As a consequence one should say that, while a concept like this may perhaps not be widely known for discrete structures like ours, it does, on the other side, not seem to be entirely new. We hope to come back to possible relations between these various highly interesting approaches elsewhere (see our remarks before 4.6 Observation).

Matters become much more interesting and subtle if one studies more general graphs than simple lattices. Note that there exists a general theorem showing that practically every graph can be embedded in $\mathbb{R}^3$ and still quite a few in $\mathbb{R}^2$ (*'planar graphs'*).

The point is however that this embedding is in general not invariant with respect to the euclidean metric. But something like an apriori given euclidean length is unphysical for the models we are after anyhow. This result has the advantage that one can visualize many graphs already in, say, $\mathbb{R}^2$ whereas their intrinsic dimension may be much larger.

An extreme example is a '*tree graph*', i.e. a graph without '*loops*'. In the following we study an infinite, regular tree graph with node degree 3, i.e. 3 bonds branching off each node. The absence of loops means that the '*connectivity*' is extremely low which results in an exceptionally high '*dimension*' as we will see.

Starting from an arbitrary node we can construct the neighborhoods $\mathcal{U}_m$ and count the number of nodes in $\mathcal{U}_m$ or $\partial\mathcal{U}_m$. $\mathcal{U}_1$ contains 3 nodes which are linked with the reference node n_0. There are 2 other bonds branching off each of these nodes. Hence in $\partial\mathcal{U}_2 = \mathcal{U}_2 \backslash \mathcal{U}_1$ we have $3 \cdot 2$ nodes and by induction:

$$|\partial\mathcal{U}_{m+1}| = 3 \cdot 2^m \tag{98}$$

which implies

$$D - 1 := \lim_{m\to\infty}(\ln|\partial\mathcal{U}_{m+1}|/\ln m) = \lim_{m\to\infty}(m \cdot ln2/\ln m + 3/\ln m) = \infty \tag{99}$$

Hence we have:

4.7 Observation(Tree): The intrinsic dimension of an infinite tree is ∞ and the number of new nodes grows exponentially like some $n(n-1)^m$ with m (n being the node degree).

Remark: $D = \infty$ is mainly a result of the absence of loops(!), in other words: there is exactly one path, connecting any two nodes. This is usually not so in other graphs, e.g. lattices, where the number of new nodes grows at a much slower pace (whereas the number of nearest neighbors can nevertheless be large). This is due to the existence of many loops s.t. many of the nodes which can be reached from, say, a node of $\partial\mathcal{U}_m$ by one step are already contained in $\mathcal{U}_m$ itself.

We have seen that for, say, lattices the number of new nodes grows like some fixed power of m while for, say, trees m occurs in the exponent. The borderline can be found as follows:

4.8 Observation: If for $m \to \infty$ the average number of new nodes per node contained in $\partial\mathcal{U}_m$, i.e:

$$|\mathcal{U}_{m+1}|/|\mathcal{U}_m| \geq 1 + \varepsilon \tag{100}$$

for some $\varepsilon \geq 0$ then we have exponential growth, in other words, the intrinsic dimension is ∞.

Proof: If the above estimate holds for all $m \geq m_0$ we have by induction:

$$|\mathcal{U}_m| \geq |\mathcal{U}_{m_0}| \cdot (1+\varepsilon)^{m-m_0} \tag{101}$$

Potential applications of this concept of intrinsic dimension are manifold. Our main goal is it to develop a theory which explains how our classical space-time and what we call the '*physical vacuum*' has emerged from a more primordial and discrete background via some sort of phase transition.

In this context we can also ask in what sense space-time dimension 4 is exceptional, i.e. whether it is merely an accident or whether there is a cogent reason for it.

As the plan of this paper was mainly to introduce and develop the necessary conceptual tools and to pave the ground, the bulk of the investigation in this particular direction shall be presented elsewhere as it is part of a detailed analysis of the (statitical) dynamics on networks as introduced above, their possible phase transitions, selforganisation, emergence of patterns and the like.

In the following we will only supply a speculative and heuristic argument in favor of space-dimension 3. We emphasized in section 2 that also the bond states, modelling the strength of local interactions between neighboring nodes, are in our model theory dynamical variables. In extreme cases these couplings may completely die out and/or become *'locked in'* between certain nodes, depending on the kind of model.

It may now happen that in the course of evolution a local island of 'higher order' (or several of them) emerges via a spontaneous fluctuation in a, on large scales, unordered and erratically fluctuating network in which couplings between nodes are switched on and off more or less randomly.

One important effect of the scenario we have in mind (among others) is that there may occur now a pronounced near order in this island, accompanied by an increase in correlation length and an effective screening of the dangerously large *'quantum fluctuations'* on Planck scale, while the global state outside remains more or less structureless. We assume that this will be effected by a reduction of intrinsic dimension within this island which may become substantially lower than outside, say, finite as compared to (nearly) infinity.

If this '*nucleation center*' is both sufficiently large and its local state '*dynamically favorable*' in a sense to be specified (note that a concept like '*entropy*' or something like that would be of use here) it may start to unfold and trigger a global phase transition.

As a result of this phase transition a relatively smooth and stable submanifold on a certain coarse-grained scale (alluding to the language of synergetics we would like to call it an '*order parameter manifold*') may come into being which displays certain properties we would attribute to space-time.

Under these premises we could now ask what is the probability for such a specific and sufficiently large spontaneous fluctuation? As we are at the moment talking about heuristics and qualitative behavior we make the following thumb-rule-like assumptions:

i) In the primordial network '*correlation lengths*' are supposed to be extremely short (more or less nearest neighbor), i.e. the strengths of the couplings are fluctuating more or less independently.
ii) A large fluctuation of the above type implies in our picture that a substantial

fraction of the couplings in the island passes a certain threshold (cf. the models of section 2) i.e. become sufficiently weak/dead and/or cooperative. The probability per individual bond for this to happen be p. Let L be the diameter of the nucleation center, $const \cdot L^d$ the number of nodes or bonds in this island for some d. The probability for such a fluctuation is then roughly (cf. i)):

$$W_d = const \cdot p^{(L^d)} \tag{102}$$

iii) We know from our experience with phase transitions that there are favorable dimensions, i.e. the nucleation centers may fade away if either they themselves are too small or the dimension of the system is too small. Apart from certain non-generic models $d = 3$ is typically the threshold dimension.
iv) On the other side we can compare the relative probabilities for the occurrence of sufficiently large spontaneous fluctuations for various d's. One has:

$$W_{d+1}/W_d \sim p^{(L^{d+1})}/p^{(L^d)} = p^{L^d(L-1)} \tag{103}$$

Take e.g. $d = 3$, $L = 10$, $p = 1/2$ one gets:

$$W_4/W_3 \sim 2^{-(9 \cdot 10^3)} \tag{104}$$

In other words, provided that this crude estimate has a grain of truth in it, one may at least get a certain clue that space-dimension 3 is both the threshold dimension and, among the class of in principle allowed dimensions (i.e. $d \geq 3$) the one with the dominant probability.

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