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Why Schrödinger's Cat is most likely to be either alive or dead

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Abstract

With appropriate assumptions on the time evolution a sequence of events makes a mixed state pure on the classical quantities.

1 Introduction

Ever since Schrödinger's seminal 1935 paper [S] quantum physicists are wondering why we always find states where classical observables have definite values but never mixtures of them. Mixed states like the canonical state seem to be the rule, we live in a 3^0K universe, but we observe always pure ones on macroscopic observables. The latter can be characterized mathematically as the center of the observable algebra and states which are pure over the center are called factor states. This means they assign a definite c-number to each element of the center, they become dispersion free.

The first hint about what distinguishes physically the factor states is found in [NR] where it is shown that the canonical (= KMS) states which are also factor states enjoy the following property: If one disturbes the Hamiltonian H by any sequence of perturbations h_n such that in the limit $n \to \infty$ $H + h_n$ generate the same time evolution as H then the associated canonical state converges to the one associated to H exactly if the latter is a factor. In one direction this theorem is obvious since if we take the $h_n \neq c\mathbf{1}$ from the center this will not change the time evolution but it does change the canonical state. The converse of this statement is not so trivial.

However this kind of stability is not exactly what one wants: it does not explain a dynamical purification of the state. Ideally one should show that the unavoidable interactions with the surrounding drive the state into a factor state. That this is so if the interactions are measurements and one accepts the usual reduction of the state postulate was shown by one of the authors [N]. In this letter we want first to give detailed derivations of this result within the many histories [G, GH, O] interpretation of quantum mechanics which gives the precise formulation of the reduction postulate. Secondly we will show by a certain model for the measuring device that the same result can be derived by a unitary time evolution with a time dependent Hamiltonian. We consider the simplest case of the mixture of two factor states and show that for almost each history the ratio of the two contributions (or its inverse) decreases exponentially with the number of outside interactions. We say interactions for measurements since our model does not contain an observer (or his friend) who takes note of the result. The observables being measured are a set of complete orthogonal projectors and replace the classical partitioning of phase space. Thus we are dealing with the quantum generalization of symbolic dynamics where the trajectory is replaced by a list of the gates through which the trajectory went at a discrete sequence of times. For some dynamical systems this list if it becomes infinitely long coincides with that of a single trajectory only, corresponding to a state with maximal purification (= sharp initial conditions). Our results cannot be derived for general quantum systems but only for infinite systems where inequivalent representation and a center of the algebra can occur. Furthermore they must be chaotic in as much as we need decaying time correlations. This can be proved for the so-called K-systems, both classical and quantum mechanical.

2 Purification by measurement

In the many histories interpretation of quantum mechanics one assigns probabilities W to histories consisting of a sequence of events. The latter are represented by projection operators $P_k(t) \in \mathcal{A}$, $k = 1 \dots r$, where for the time dependence the Heisenberg representation is used and $\mathcal{A}$ is the algebra of observables. For finite quantum systems a state is given by a density matrix ρ and the key formula for the probabilities is

$$\begin{aligned} W(\underline{\alpha}) &= \mathrm{tr}\ P_{\alpha_n}(t_n) \dots P_{\alpha_1}(t_1) \rho P_{\alpha_1}(t_1) \dots P_{\alpha_n}(t_n) \\ &= \mathrm{tr}\ \sqrt{\rho}\ P_{\alpha_1}(t_1) \dots P_{\alpha_n}(t_n) P_{\alpha_{n-1}}(t_{n-1}) \dots P_{\alpha_1}(t_1) \sqrt{\rho} \end{aligned} \tag{1}$$

where $\underline{\alpha} = (\alpha_1 \dots \alpha_n)$, $\alpha_i \in \{1 \dots r\}$. If the P_k are a complete set of orthogonal projections, $P_k P_{k'} = \delta_{kk'} P_k$, $\sum_{k=1}^r P_k = 1$ the probabilities are correctly normalized

$$W(\underline{\alpha}) \geq 0, \qquad \sum_{\underline{\alpha}} W(\underline{\alpha}) = 1. \tag{2}$$

For the infinte systems we are concerned with, the density matrix ρ will not exist but the notion of a state as positive linear functional ω as generalization of tr $\sqrt{\rho}\ A\ \sqrt{\rho} = \omega(A)$ carries over. For these systems in physical situations the time correlations are expected to decay for an extremal time invariant state ω such that (in [T, NT2] this is proved for the so-called K-systems)

$$W(\underline{\alpha}) \to \prod_i \omega(P_{\alpha_i}) \qquad \text{if } t_{i+1} - t_i \to \infty\ \forall\ i. \tag{3}$$

Extremal invariant means that ω is not a combination

$$\mu\omega_1 + (1-\mu)\omega_2, \qquad 0 < \mu < 1, \tag{4}$$

of two other invariant states and the factorization (3) is lost by convex combinations. Nevertheless consistency of histories is preserved since it can be expressed in shorthand by $\omega(P_{\underline{\alpha}'} P_{\underline{\alpha}}) = \delta_{\underline{\alpha}',\underline{\alpha}}$. It holds for every extremal invariant state and is preserved by convex combinations. Extremal invariance requires that $\mathcal{A}$ does not contain an element constant in time $z \neq c\mathbf{1}$ since with it one could decompose ω into

$$\omega_1(A) = \frac{\omega(z^* A z)}{\omega(z^* z)} \qquad \text{and} \qquad \omega_2(A) = \frac{\omega(\sqrt{1 - z^* z}\ A\ \sqrt{1 - z^* z})}{\omega(1 - z^* z)}$$

(we may assume $z^* z < 1$ by replacing z by $z/\|z\|$). For infinite quantum systems with suitable interactions constant elements belong to the center which is the classical part of the system. On the other hand, for equilibrium (KMS) states all elements of the center are constant. The extremal invariant states are the ones where z is represented by a c-number, thus to the classical quantities one assigns a definite value. In the example of the spin chain to be studied in Section 3 the classical quantities are the mean magnetization

$$\vec{m} = \lim_{N \to \infty} \frac{1}{2N} \sum_{i=-N}^{N} \vec{\sigma}_i$$

and classically pure states are the ones where all spins except a negligible number of them point in the same direction $\vec{m}$. For the mixed states of the form (4) the histories W are also convex combinations

$$W_\omega(\underline{\alpha}) = \mu W_{\omega_1}(\underline{\alpha}) + (1-\mu) W_{\omega_2}(\underline{\alpha}) \tag{5}$$

and what we will show that for long histories $n \to \infty$, $t_{i+1} - t_i \to \infty$ they purify in the sense that in (5) either W_{ω_1} or W_{ω_2} dominates such that $W_{\omega_i}/W_{\omega_j} < \varepsilon$ for arbitrarily small ε. Which one dominates depends on the history. Of course one cannot dominate over the other for all histories since

$$\sum_{\underline{\alpha}} W_{\omega_1}(\underline{\alpha}) = \sum_{\underline{\alpha}} W_{\omega_2}(\underline{\alpha}) = 1.$$

We shall elaborate only on the simplest nontrivial case with two states $\omega_{1,2}$ and two projectors P, $1-P$. The generalization to several states or propositions is easy and shows the same features. Denote $\omega_{1,2}(P) = p_{1,2}$, then for a history with ℓ projectors P and $n-\ell$ projectors $(1-P)$, $0 \le \ell \le n$, we get from (3)

$$W(\underline{\alpha}) = W(n,\ell) = \mu p_1^\ell (1-p_1)^{n-\ell} + (1-\mu) p_2^\ell (1-p_2)^{n-\ell}. \tag{6}$$

The amount of mixing is given by the ratio

$$R = \frac{\mu}{1-\mu} \left(\frac{p_1}{p_2}\right)^\ell \left(\frac{1-p_1}{1-p_2}\right)^{n-\ell}$$

of the two contributions. The mixing is noticeable if for some small number $\varepsilon > 0$ we have $\varepsilon < R < 1/\varepsilon$ or

$$\frac{\varepsilon(1-\mu)}{\mu} < a^\ell b^{n-\ell} < \frac{1-\mu}{\varepsilon\mu} \tag{7}$$

where $a := p_1/p_2$ and $b := (1-p_1)/(1-p_2)$. For definiteness we may assume $a > 1$, thus $b < 1$. If we scale ℓ with n, $\ell = n\lambda$, $0 \le \lambda \le 1$ the condition (7) becomes

$$\frac{1}{\ln a/b}\left(\ln\frac{1}{b} + \frac{1}{n}\ln\frac{1-\mu}{\mu} - \frac{1}{n}\ln\frac{1}{\varepsilon}\right) < \lambda < \frac{1}{\ln a/b}\left(\ln\frac{1}{b} + \frac{1}{n}\ln\frac{1-\mu}{\mu} + \frac{1}{n}\ln\frac{1}{\varepsilon}\right). \tag{8}$$

Thus for the mixed histories λ is in an interval of length $2/(n \ln a/b) \ln 1/\varepsilon$. We are interested in long histories, $n \to \infty$, for which this is a small number and want now to calculate which fraction of the histories are mixed. Since at each event the histories can take two courses there are 2^n histories altogether. Their probabilities (6) depend only on n and ℓ and there are $n!/(\ell!(n-\ell)!)$ histories with this probability. For $n \to \infty$ we calculate with Stirlings"s formula the density d of histories with $\lambda = (1+\delta)/2$ to be

$$d(\delta) = \sqrt{\frac{n}{\pi}}\, e^{-n\delta^2}, \qquad \int_{-1}^{1} d\delta\, d(\delta) = 1 + O(e^{-n}). \tag{9}$$

Then (8) tells us that for $n \to \infty$ the fraction of histories more mixed than ε is less than

$$\frac{4}{\sqrt{\pi n}} \frac{\ln 1/\varepsilon}{\ln a/b} \, e^{-n\delta_m^2}$$

where

$$\delta_m = \left| \frac{\ln 1/b - \ln a}{\ln 1/b + \ln a} \right|$$

is the minimum of $|\delta|$ in the interval defined by (8). We have $\delta_m > 0$ and therefore exponential decrease unless $a = 1/b$ which happens if $p_1 = p_2$ or $1 - p_2$. In the first case $\omega_1 = \omega_2$ for the algebra generated by P and $1 - P$ and there is no mixing. In the second case $\omega_1 = \omega_2$ combined with a reflection and in this special case the fraction decreases only as $1/\sqrt{n}$. In any case for $n \to \infty$ most histories are pure in the sense that within ε only states where the classical quantities have a definite value contribute.

Since the ℓ-dependence of W goes with $(p/(1-p))^\ell$ for the most probable history we have $\ell = 0$ or n depending on whether $p <$ or $> 1/2$. Thus most likely the answer to each experiment is always the most probable one.

3 The spin chain

In this section we shall illustrate the abstract development of Section 2 by a standard model of a measuring apparatus. It is the simplest nontrivial physical example which shows these features. Consider an infinite spin chain $\mathcal{A} = \{\vec{\sigma}_i\}$, the index i ranging over the integers. As discrete evolution we take the shift $\sigma_i \to \sigma_{i+1}$. For the benefit of people working on the interpretation of quantum mechanics who prefer to think in terms of wave functions we shall use now the Schrödinger representation. For each direction $\vec{n}_i$, $\vec{n}_i^2 = 1$ there is a vector $|\vec{n}_i\rangle_i$ in Hilbert space such that $\vec{\sigma}_i$ points in this direction, $\vec{\sigma}_i \cdot \vec{n}_i |\vec{n}_i\rangle_i = |\vec{n}_i\rangle_i$. For all spins together the corresponding vector is the tensor product $\bigotimes_i |\vec{n}_i\rangle_i$. If there were N spins these vectors spanned a 2^N-dimensional space but for $N = \infty$ the space is huge (nonseparable). We shall work in separable subspaces of this monster which are obtained like the Fock space by letting $\mathcal{A}$ act on a reference vector $|\Omega_1\rangle : \mathcal{H}_1 = \mathcal{A}|\Omega_1\rangle$. For $|\Omega_1\rangle$ we chose a polarized state where all spins point in the same direction

$$|\vec{n}\rangle : |\Omega_1\rangle = \bigotimes_{i=-\infty}^{\infty} |\vec{n}\rangle_i. \tag{10}$$

In $\mathcal{H}_1$ we get an irreducible representation π_1 of $\mathcal{A}$. Of course, for individual spins $\pi_1(\sigma_i)$ acts like σ_i but weak limits like the mean magnetization

$$\vec{M} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \pi_1(\vec{\sigma}_i) = \vec{n}\mathbf{1} \tag{11}$$

($\mathbf{1}$ the unit operator) depend on the representation. Had we considered another reference state $|\Omega_2\rangle = \bigotimes_{i=-\infty}^{\infty} |m\rangle_i$ the magnetization would turn out to be $\vec{m}$. Thus the two representations are not unitarily equivalent, $U^{-1}\pi_1(\sigma_i)U = \pi_2(\sigma_i)$ would imply

$$U^{-1}\vec{n}\cdot\mathbf{1}U = \vec{m}\cdot\mathbf{1}$$

which is impossible since U cannot change the unity $\mathbf{1}$. The $|\Omega_{1,2}\rangle$ define states $\omega_{1,2}(\cdot) = \langle\Omega_{1,2}|\cdot|\Omega_{1,2}\rangle$ and the mixed state $\mu\omega_1 + (1-\mu)\omega_2$ is obtained by a vector in the orthogonal sum of π_1 and π_2

$$|\Omega_S\rangle = \sqrt{\mu}\,|\Omega_1\rangle \oplus \sqrt{1-\mu}\,|\Omega_2\rangle \in \mathcal{H}_1 \oplus \mathcal{H}_2 =: \mathcal{H}_S, \qquad \pi = \pi_1 \oplus \pi_2, \tag{12}$$

$\langle\Omega|\pi(\vec{\sigma}_i)|\Omega\rangle = \mu\vec{n} + (1-\mu)\vec{m}$. This representation is reducible, there are two "superselection sectors" [L], the magnetization $\vec{M}$ is in the center and is not a multiple of unity

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{i=-N}^{N}\pi(\vec{\sigma}_i) = \mu\vec{n}\mathbf{1}_1 \oplus (1-\mu)\vec{m}\cdot\mathbf{1}_2. \tag{13}$$

If we identify with a poetic license $\vec{M}$ with Schrödinger's cat and $\vec{n}$ means alive and $\vec{m}$ means dead then the vectors of $\mathcal{H}_1$ ($\mathcal{H}_2$) represent the cat alive (dead) whereas in $|\Omega\rangle$ (with $\mu = 1/2$) it is half dead and half alive. We shall now show how by a succession of measurements $|\Omega\rangle$ purifies in the sense that the dominant components turn into either $\mathcal{H}_1$ or $\mathcal{H}_2$.

First we have to construct for the measuring device a classical system which can store the information contained in $\mathcal{A}$. Since a measurement of σ_i can have only two outcomes pointers with 2 positions suffice. Since also for classical systems the Hilbert space description is useful we represent that state of the device measuring σ_i by a two-dimensional vector $\begin{pmatrix} u_i \\ d_i \end{pmatrix}$, u and d meaning pointer up or down. The measuring array for all spins is again an infinite tensor product $\otimes_i \begin{pmatrix} u_i \\ d_i \end{pmatrix}_i \in \mathcal{H}_A$ and we start with an $|\Omega_A\rangle$ where all u_i are zero.

For the time evolution we take a shift U and then an instantaneous measurement of a direction $\vec{s}$ of a spin. The corresponding proposition is the projector

$$P_k = \frac{1}{2}(1 + \vec{\sigma}_k\cdot\vec{s}) = |s\rangle_k\,{}_k\langle s|.$$

If the answer is one we have the pointer unchanged, if the answer is zero we turn the pointer up. This turning of the pointer is affected by an operator τ,

$$\tau_k \begin{pmatrix} u \\ d \end{pmatrix}_k = \begin{pmatrix} d \\ u \end{pmatrix}_k.$$

Thus the effect of measuring $\vec{\sigma}_1$ is $V_1 = P_1 + (1-P_1)\tau_1$ or written out in full as operators in $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$

$$V_1 = (\pi_1(P_1) \oplus \pi_2(P_1) \otimes \mathbf{1} + (1 - \pi_1(P_1)) \oplus (1 - \pi_2(P_1)) \otimes \tau_1). \tag{14}$$

Note that V_1 is unitary and in $\mathcal{H}_S \otimes \mathcal{H}_A$ there is no reduction of the full state vector. The time evolution U between the measurements shifts by one unit $U\pi_{1,2}(P_k) = \pi_{1,2}(P_{k+1})U$, $U\tau_k = \tau_{k+1}U$, so that the full time evolution of $|\Omega\rangle = |\Omega_S\rangle \otimes |\Omega_A\rangle$ is after n time units

$$|\Omega(n)\rangle = V_1 U V_1 U \dots V_1 U |\Omega\rangle = V_1 V_2 \dots V_n |\Omega\rangle \tag{15}$$

since $U^k|\Omega\rangle = |\Omega\rangle$. The results of the measurements are encoded in the $\mathcal{H}_A$-part of $|\Omega(n)\rangle$ and so we decompose it in an orthogonal basis of $\mathcal{H}_A$

$$|\Omega(n)\rangle = \sum_{\alpha_i=0}^{1} v(\underline{\alpha}) \tau_1^{\alpha_1} \tau_2^{\alpha_2} \dots \tau_n^{\alpha_n} |\Omega_A\rangle. \tag{16}$$

Wherever $\alpha_i = 0$ the corresponding spin was in direction $\vec{s}$, for $\alpha_i = 1$ we had $-\vec{s}$. If we have such a situation the system is left with a wave function $v(\underline{\alpha})$ which has a component in $\mathcal{H}_1$ and one in $\mathcal{H}_2$:

$$v(\underline{\alpha}) = v_1(\underline{\alpha}) \oplus v_2(\underline{\alpha}). \tag{17}$$

To calculate the length of the $v_{1,2}(\underline{\alpha})$ we have to use $P_1|n\rangle_1 = |s\rangle_1\langle s|n\rangle$, $(1-P_1)|n\rangle = |-s\rangle\langle -s|n\rangle$. If we introduce $|\langle s|n\rangle|^2 = p_1$, thus $|\langle -s|n\rangle|^2 = 1-p_1$ and similarly $|\langle s|m\rangle|^2 = p_2$, $|\langle -s|m\rangle|^2 = 1-p_2$ we find that if $\underline{\alpha}$ contains ℓ zeros and $n-\ell$ ones we have

$$\|v_1(\underline{\alpha})\|^2 = \mu p_1^\ell (1-p_1)^{n-\ell}, \qquad \|v_2(\underline{\alpha})\|^2 = (1-\mu) p_2^\ell (1-p_2)^{n-\ell}. \tag{18}$$

Thus with $W(\underline{\alpha}) = \|v(\underline{\alpha})\|^2$ we arrive at exactly the expression (6) and the decomposition (16) displays the 2^n histories. Remember that we had a unitary time evolution, there was no collapse of the wave function after each measurement, only at the end we were reading the configuration of classical pointers.

Remarks

1. In terms of convergence of states our result can be expressed as follows. The vector $|\Omega\rangle$ gives for $\mu \neq 0, 1$ a mixed state over the algebras of system $\otimes$ apparatus since the system is represented reducibly. It evolves by the unitary evolution into the vector $|\Omega(n)\rangle$ such that the state $\omega_n(\cdot) = \langle \Omega(n)| \cdot |\Omega(n)\rangle$ over the system $\otimes$ apparatus stays pure for $\mu = 0$ or 1, otherwise it stays mixed. Reading the pointer in a position $\underline{\alpha}$ changes the state to $\omega_{n,\underline{\alpha}}(\cdot) = \omega_n(P_{\underline{\alpha}} \cdot) W_\omega(\underline{\alpha})^{-1}$ where P_α projects onto the vector $\tau_1^{\alpha_1} \dots \tau_n^{\alpha_n} |\Omega_A\rangle$. For $n \to \infty$ (and making the history $\underline{\alpha}$ infinite) this converges weakly to a pure state. $\lim_{n\to\infty} \omega_n = \sum_{\underline{\alpha}} \omega_{n,\underline{\alpha}} W(\underline{\alpha})$ is a mixed state even for $\mu = 0$ or 1. This is in accordance with the result of K. Hepp who observed that in a similar situation of an infinite quantum system weak limits of pure states may be mixed [H].

2. That the states $\omega_{1,2}$ in the example were pure is irrelevant, for asymptotic abelian systems they only have to be extremely invariant to possess the required cluster properties. If we restrict ourselves to canonical temperature states then only at

phase transition points one has to be aware that they themselves are mixtures of states with the same temperature but different values of a central element, corresponding to the fact that we know that at a transition point space clustering becomes critical (on time clustering no rigorous results are available).

3. When we talk about macroscopic quantum systems we mean many degrees of freedom and not just large in size. So to the proposal of Leggett et al. [LG] our considerations are not applicable. What we really need is that the relaxation time of the system is shorter than the time between measurements.

To summarize we have first to emphasize that we stay within the Copenhagen interpretation of quantum mechanics. The many history interpretation is a systematic formulation of the reduction of the wave function postulate. We consider it as a simple description of the essential effect of a measurement without going into the details of a complicated mechanism (compare [BJ, P]). For one simple model of a measuring apparatus we have shown that actually the unitary evolution of the joint system leads to the same result. Our goal is to see what the special properties of large quantum systems imply for the histories.

To get the optimal knowledge of the state of a finite quantum system one has to repeat the experiment on other members of an ensemble of equally prepared systems to gather enough statistics. For our infinite system one can also redo after an appropriate relaxation time the experiment on the same system and get the same result provided the initial state was pure on the macroscopic part. In all quantum systems even in an optimally refined state some quantities will remain fluctuating. This remains true for infinite quantum systems but we have seen that by repeated measurements the classical observables will assume definite values. Thus f.i. below the phase transition domains of a magnet will be magnetized in a definite direction. Even if nobody looks at it there will be enough "events" (= interactions with the surroundings) to purify the state over the classical part. However, a quantum mechanically pure state over all the microscopic observables will not be obtainable because for these systems all observable projections are infinite dimensional, onedimensional projections in Hilbert space will not belong to the algebra of observables.

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