Geometrical Approach to Inverse Scattering for the Dirac Equation

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Abstract

The high-energy-limit of the scattering operator for multidimensional relativistic dynamics, including a Dirac particle in an electromagnetic field, is investigated by using time-dependent, geometrical methods. This yields a reconstruction formula, by which the field can be obtained uniquely from scattering data.

1 Introduction

For self-adjoint operators H_0 and $H = H_0 + V$ with H_0 having continuous spectrum, the wave operators are defined by $\Omega_\pm = s - \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}$. If they exist on $\mathcal{H}$ and their ranges equal $\mathcal{H}^{ac}(H)$, the scattering system is called complete and the scattering operator $S = \Omega_+^* \Omega_-$ is unitary [10]. The inverse problem is to determine V, given S (and H_0). In [3, 4, 5, 6] Enss and Weder show that for the Schrödinger operator $H_0 = -1/2\,\Delta$ and a translation in momentum space by $\mathbf{v} = v\boldsymbol{\omega}$, $\boldsymbol{\omega} \in S^{\nu-1}$ the high-energy-limit of the scattering operator is given by

$$\Big(\Phi, v\Big(e^{-i\mathbf{v}\mathbf{x}} S e^{i\mathbf{v}\mathbf{x}} - 1\Big)\Psi\Big) \longrightarrow \int_{-\infty}^{+\infty} d\tau\, \Big(\Phi, V(\mathbf{x} + \boldsymbol{\omega}\tau)\Psi\Big) \quad \text{as } v \to \infty \qquad (1)$$

for suitable Φ, Ψ. The short-range potential V, a multiplication operator, can be uniquely reconstructed from this X-ray transform. This approach generalizes to multiparticle systems and long-range potentials.

Following these ideas, we use time-dependent, geometrical methods to study relativistic quantum mechanics, in particular the Dirac equation with the free Hamiltonian $H_0 = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$. The main result is Theorem 3.2:

$$s - \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}_{NW}} S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} = \exp\{-i \int_{-\infty}^{+\infty} (A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega} t)dt\} \qquad (2)$$

from which the electromagnetic field $(A_0, \mathbf{A})$ may be reconstructed. Here $S_\pm$ describe the scattering of positive/negative energy states in the Foldy-Wouthuysen-representation, and $\mathbf{x}_{NW}$ is the Newton-Wigner position operator. The A_i are supposed to be continuous and to decay integrably, i.e. $\int_0^\infty dR\, \sup_{|\mathbf{x}|\geq R} |\mathbf{A}_i(\mathbf{x})| < \infty$. In [7] Ito has given a similar reconstruction formula for the high-energy-limit of the scattering amplitude using stationary methods, for $A_i \in C^2$ satisfying $|A_i(\mathbf{x})| < c|\mathbf{x}|^{-3-\varepsilon}$.

The charge e is incorporated in A_i, furthermore, we let $c = \hbar = 1$. Note that $\int_{-\infty}^{+\infty} (A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}+\boldsymbol{\omega}t)dt$ is the classical action of a particle moving with velocity 1 on a line, as expected in the semi-classical limit. Introducing suitable units and letting $c \to \infty$ in the r.h.s. of (2) yields formally $\exp\{i\int_{-\infty}^{+\infty} \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t)dt\}$, which has been obtained by Arians [1] as the high-energy-limit of S for a Schrödinger particle in an electromagnetic field.

For mathematical quantum mechanics we refer to [8, 9, 10] and for the Dirac equation to [12]. In Section 2 we study $H_0 = \sqrt{\mathbf{p}^2 + m^2}$, which is similar to the Dirac operator, while being easier to handle. In Section 3 we examine the reconstruction formula for the Dirac equation, which is proved in Section 4. Various generalizations are discussed in Section 5.

2 Reconstruction Formula for the Scalar Relativistic Hamiltonian

We consider $\mathcal{H} = L^2(\mathbb{R}^\nu)$ and $H_0 = \sqrt{\mathbf{p}^2 + m^2}$ with $m \geq 0$ and $\mathbf{p} = -i\nabla$. This scalar Hamiltonian H_0 is self-adjoint on the Sobolev-space $H^1(\mathbb{R}^\nu)$ as its domain. It may be considered as a model for relativistic quantum mechanics, since the symbols of the Klein-Gordon- and the Dirac equation have the eigenvalues $\pm\sqrt{\mathbf{p}^2 + m^2}$.

Definition 2.1 (Short-range Potentials) *A symmetric multiplication operator V is called a short-range potential, if it is H_0-bounded with relative bound < 1 and satisfies*

$$\left\| V \sqrt{\mathbf{p}^2 + 1}^{\,-1} F(|\mathbf{x}| > R) \right\| \in L^1\left([0, \infty), dR\right). \tag{3}$$

$F(\ldots)$ denotes multiplication with the characteristic function of the indicated region in $\mathbf{x}$ -space.

This definition corresponds to that of the Schrödinger case [3, 4, 5, 6]. Local singularities of V are possible: If, e.g., $\nu > 1$, $p > \nu$ and $V \in L^p + L^\infty$ with $\|V\chi(|\mathbf{x}| > R)\|_{L^p+L^\infty} \in L^1\left([0, \infty), dR\right)$, then V is short-range. (The norm is defined by $\|f\|_{L^p+L^\infty} := \inf\{\, \|f_1\|_p + \|f_2\|_\infty | \, f = f_1 + f_2\}$.) For $\nu = 3$, a Yukawa-potential is also admitted, if the coupling constant is small. For a short-range potential V and $H = H_0 + V$ the completeness of the scattering system follows from Theorem 2.1 in [11].

For $\Psi \in \mathcal{H}$, the $\mathbf{x}$-representation is given by $\psi(\mathbf{x})$ and the Fourier transform $\hat{\psi}(\mathbf{p})$ yields the momentum representation of Ψ. The positionoperator $\mathbf{x}$ generates tranlations in momentum space, in particular for any $\mathbf{v} = v\boldsymbol{\omega} \in I\!R^\nu$

$$e^{-i\mathbf{v}\mathbf{x}} H\, e^{i\mathbf{v}\mathbf{x}} = e^{-i\mathbf{v}\mathbf{x}} \left(\sqrt{\mathbf{p}^2 + m^2}\; + V(\mathbf{x})\right) e^{i\mathbf{v}\mathbf{x}} = \sqrt{(\mathbf{p}+\mathbf{v})^2 + m^2}\; + V(\mathbf{x})\,.$$

Lemma 2.2 (Integrable Bound) *Let V be a short-range potential. For Ψ with $\hat{\psi} \in C_0^\infty$ there are $v_0 > 0$, $h \in L^1(I\!R)$ such that*

$$\|V e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}}\Psi\| \leq h(t) \quad \textit{for } t \in I\!R, \textit{ uniformly in } v \geq v_0. \tag{4}$$

This v-independent integrable bound will be crucial to apply the dominated convergence theorem in the proof of Theorem 2.3 .

Proof: We first show that there are $c, v_0 > 0$ such that

$$\left| e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \sqrt{\mathbf{p}^2 + 1}\,\Psi \right| (\mathbf{x}) < \frac{c}{(1+|t|)^{\frac{\nu+3}{2}}} \text{ for } |\mathbf{x}| < \frac{|t|}{2}, v > v_0. \tag{5}$$

This follows by a non-stationary phase estimate [12, p.33], [10, p.37] from

$$\left(e^{-i\mathbf{v}\mathbf{x}} e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \sqrt{\mathbf{p}^2+1}\,\Psi \right)(\mathbf{x}) = (2\pi)^{-\frac{\nu}{2}} \int dp\, e^{itf(\mathbf{p};\mathbf{x},t,\mathbf{v})} \sqrt{\mathbf{p}^2+1}\,\hat{\psi}(\mathbf{p}) \tag{6}$$

with $f = \left(\mathbf{p}\cdot\frac{\mathbf{x}}{t} - \sqrt{(\mathbf{p}+\mathbf{v})^2 + m^2}\right)$, since there is a $v_0 > 0$ such that $|\nabla_\mathbf{p} f| > 1/4$ for $|\mathbf{x}/t| < 1/2, v > v_0,\ \mathbf{p} \in \operatorname{supp}(\hat{\psi})$. Also $\partial_\mathbf{p}^\beta f$ is bounded there for $|\beta| > 0$, and $\partial_\mathbf{p}^\gamma \left(\sqrt{\mathbf{p}^2+1}\hat{\psi}(\mathbf{p})\right)$ is bounded for $|\gamma| \geq 0$. Now we consider

$$\begin{aligned}
& \left\| e^{-i\mathbf{v}\mathbf{x}} V e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}}\,\Psi \right\| \\
= & \left\| V \sqrt{\mathbf{p}^2+1}^{\,-1} \left\{ F\Big(|\mathbf{x}| > \frac{|t|}{2}\Big) + F\Big(|\mathbf{x}| < \frac{|t|}{2}\Big) \right\} e^{-it\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}} \sqrt{\mathbf{p}^2+1}\,\Psi \right\| \\
\leq & \left\| V \sqrt{\mathbf{p}^2+1}^{\,-1} F\Big(|\mathbf{x}| > \frac{|t|}{2}\Big) \right\| \cdot \left\| \sqrt{\mathbf{p}^2+1}\,\Psi \right\| \\
+ & \left\| V \sqrt{\mathbf{p}^2+1}^{\,-1} \right\| \cdot \left\| F\Big(|\mathbf{x}| < \frac{|t|}{2}\Big) e^{-iH_0 t} e^{i\mathbf{v}\mathbf{x}} \sqrt{\mathbf{p}^2+1}\Psi \right\|
\end{aligned}$$

The first term is in L^1 by (3) and the second is bounded by $\tilde{c}/(1+|t|)^{\frac{3}{2}}$ by (5). $\Box$

This decomposition is motivated by the geometrical idea that the wave-packet is concentrated around $\boldsymbol{\omega} t$, where the potential is small, and that it is small around the origin, where the potential may be large.

Theorem 2.3 (Reconstruction Formula) *For a short-range potential V the high-energy limit of the scattering operator is given by*

$$s - \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}} S e^{i\mathbf{v}\mathbf{x}} = \exp\{-i \int_{-\infty}^{+\infty} V(\mathbf{x} + \boldsymbol{\omega} t) dt\}, \tag{7}$$

the integral being convergent for a.e. $\mathbf{x} \in I\!R^\nu$.

Proof: It is sufficient to consider the dense set of Ψ with $\hat{\psi} \in C_0^\infty$.

$$\begin{aligned}
e^{-i\mathbf{vx}}\Omega_+ e^{i\mathbf{vx}}\,\Psi &= \Psi + i\int_0^\infty dt\,\frac{d}{dt}e^{-i\mathbf{vx}}e^{iHt}e^{-iH_0t}e^{i\mathbf{vx}}\,\Psi \\
&= \Psi + i\int_0^\infty dt\,e^{i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}+V(\mathbf{x})\right)t}V(\mathbf{x})e^{-i\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}\,t}\Psi \\
&= \Psi + i\int_0^\infty dt\,e^{i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v+V(\mathbf{x})\right)t}V(\mathbf{x})e^{-i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v\right)t}\Psi.
\end{aligned}$$

With $\mathbf{v} = v\boldsymbol{\omega}$ we find $(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2} - v + V) \overset{v\to\infty}{\longrightarrow} \boldsymbol{\omega}\cdot\mathbf{p} + V$ in the strong resolvent sense and the exponential converges strongly [8, Theorems VIII.25,21]. $V\sqrt{\mathbf{p}^2+1}^{\,-1}$ is bounded and

$$e^{-i\left(\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}-v\right)t}\sqrt{\mathbf{p}^2+1}\,\Psi \overset{v\to\infty}{\longrightarrow} e^{-i\boldsymbol{\omega}\mathbf{p}t}\sqrt{\mathbf{p}^2+1}\,\Psi.$$

The integrand is bounded by $h(t)$ independent of $v > v_0$ by Lemma 2.2 . Using the dominated convergence theorem (for the Bochner-integral [2]) we conclude that

$$\begin{aligned}
\lim_{v\to\infty} e^{-i\mathbf{vx}}\Omega_+ e^{i\mathbf{vx}}\Psi &= \Psi + i\int_0^\infty dt\,e^{i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)t}V(\mathbf{x})e^{-i\boldsymbol{\omega}\mathbf{p}t}\Psi \\
&= \lim_{s\to\infty} e^{i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)s}e^{-i\boldsymbol{\omega}\mathbf{p}s}\Psi. \qquad (8)
\end{aligned}$$

In the special case of continuous V with integrable decay , (8) is shown to equal $\exp\{i\int_0^\infty dt\,V(\mathbf{x}+\boldsymbol{\omega}t)\}\Psi$ by considering the family of unitary operators $U(s) = e^{i\boldsymbol{\omega}\mathbf{p}s}e^{-i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)s}$, which satisfies the differential equation

$$i\dot{U}(s) = e^{i\boldsymbol{\omega}\mathbf{p}s}V(\mathbf{x})\,e^{-i\left(\boldsymbol{\omega}\mathbf{p}+V(\mathbf{x})\right)s} = V(\mathbf{x}+\boldsymbol{\omega}s)U(s)$$

and $U(0) = 1$, as does $\exp\{-i\int_0^s dt\,V(\mathbf{x}+\boldsymbol{\omega}t)\}$. For general V we use the decomposition $V = V_+ - V_-$, $V_\pm \geq 0$ and choose $V_{n,\pm} \in C_0^0$ with $V_{n,\pm} \nearrow V_\pm$ a.e.. Let $V_n := V_{n,+} - V_{n,-}$. The estimate of the integral by an integrable function h from Lemma 2.2 also holds for $V_\pm, V_{n,\pm}, V_n$, uniformly in n. Again, by applying the dominated convergence theorem once more, we deduce

$$\begin{aligned}
\lim_{v\to\infty} e^{-i\mathbf{vx}}\Omega_+ e^{i\mathbf{vx}}\Psi &= \Psi + i\lim_{n\to\infty}\int_0^\infty dt\,e^{i(\boldsymbol{\omega}\mathbf{p}+V_n)t}V_n e^{-i\boldsymbol{\omega}\mathbf{p}t}\Psi \\
&= \lim_{n\to\infty}\exp\{i\int_0^\infty dt\,V_n(\mathbf{x}+\boldsymbol{\omega}t)\}\,\Psi.
\end{aligned}$$

Also we see that for a subsequence $\lim_{n\to\infty}\exp\{i\int_0^\infty dt\,V'_{n,\pm}(\mathbf{x}+\boldsymbol{\omega}t)\}$ exists for a.e. $\mathbf{x} \in I\!\!R^\nu$, therefore $\int_0^\infty dt\,V_\pm(\mathbf{x}+\boldsymbol{\omega}t)$ exists for a.e. $\mathbf{x}$ since for a.e. $\mathbf{x}$ the monotone convergence $V'_{n,\pm}(\mathbf{x}+\boldsymbol{\omega}t) \nearrow V_\pm(\mathbf{x}+\boldsymbol{\omega}t)$ holds for a.e. $t \in I\!\!R$. We consider Ω_- similarly and find

$$s - \lim_{v\to\infty} e^{-i\mathbf{vx}}\Omega_\pm e^{i\mathbf{vx}} = \exp\{i\int_0^{\pm\infty} dt\,V(\mathbf{x}+\boldsymbol{\omega}t)\}. \qquad (9)$$

Using $S = \Omega_+^* \Omega_-$ we get the desired reconstruction formula

$$w - \lim_{v \to \infty} e^{-iv\mathbf{x}} S e^{iv\mathbf{x}} = \exp\{-i \int_{-\infty}^{+\infty} V(\mathbf{x} + \boldsymbol{\omega} t) dt\}. \tag{10}$$

The unitarity of $e^{-iv\mathbf{x}} S e^{iv\mathbf{x}}$ and of its weak limit imply the strong convergence. $\square$

The proof suggests the following physical interpretation: If the velocity of a particle approaches 1 (i.e. the speed of light), the spreading of the wave-packet is negligible and the free kinematics reduce to a pure translation, generated by $\boldsymbol{\omega} \cdot \mathbf{p}$.

Theorem 2.4 (Injectivity of the Scattering Map) *Consider* $\nu > 1$ *and* $\mathcal{V} := \{V \in C^0(I\!R^\nu, I\!R) | \, \|V\chi(|\mathbf{x}| > R)\|_\infty \in L^1([0, \infty), dR)\}$. *Then the scattering map:*

$$\begin{array}{rcl} \mathcal{V} & \to & \mathcal{L}(\mathcal{H}) \\ V & \mapsto & S = S(H_0, H_0 + V) \end{array}$$

is injective, i.e. S determines V uniquely.

The case $\nu = 1$ cannot be treated with our methods. The case of more general V shall be a topic of further research, the difficulties arising from the non-injectivity of the exponential function.

Proof: Theorem 2.3 yields $\exp\{-i \int_{-\infty}^{+\infty} V(\mathbf{x} + \boldsymbol{\omega} t) dt\}$ as a continuous function of $\mathbf{x}$, thus giving $\int_{-\infty}^{+\infty} V(\mathbf{x} + \boldsymbol{\omega} t) dt$ up to a fixed multiple of 2π. This X-ray transform is obtained uniquely, since it must vanish as $|\mathbf{x}| \to \infty$ orthogonal to $\boldsymbol{\omega}$. If $\int_{-\infty}^{+\infty} dt \, V(\mathbf{x} + \boldsymbol{\omega} t) = 0$, then considering $\boldsymbol{\omega} t$ as polar coordinates for $I\!R^\nu$ yields

$$0 = \int_{S^{\nu-1}} d\boldsymbol{\omega} \int_{-\infty}^{+\infty} dt \, V(\mathbf{x} + \boldsymbol{\omega} t) = 2 \int d\mathbf{y} \frac{V(\mathbf{x} + \mathbf{y})}{|\mathbf{y}|^{\nu-1}}. \tag{11}$$

To show that V is determined uniquely, we examine the case $\nu = 2$ first. Here $V \in L^2$ and $\frac{1}{|\mathbf{p}|^{1-\varepsilon}} \in L^2_{loc}$ imply $\frac{1}{|\mathbf{p}|^{1-\varepsilon}} \hat{V} \in L^1_{loc}$. As a tempered distribution

$$0 = \mathcal{S}' - \lim_{\varepsilon \to 0} \frac{2}{|\mathbf{x}|^{1+\varepsilon}} * V \, . \tag{12}$$

We conclude that $\mathcal{S}' - \lim_{\varepsilon \to 0} \frac{2}{|\mathbf{p}|^{1-\varepsilon}} \hat{V} = 0$, therefore $\hat{V}(\mathbf{p}) = 0$ a.e. . For $\nu > 2$, this argument shows that the restriction of V to any 2-plane in $I\!R^\nu$ is determined uniquely. $\square$

3 Reconstruction Formula for the Dirac Equation

Let $\mathcal{H} := L^2(I\!R^\nu, \mathbb{C}^\mu)$ and $H_0 := \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$ with $m \geq 0$ and anticommuting, symmetric, unitary matrices $\alpha_1, \ldots, \alpha_\nu, \beta$. The most interesting case is $\nu = 3$,

$\mu = 4$. H_0 is self-adjoint on $H^1(\mathbb{R}^\nu)$. The symbol of H_0 has the eigenvalues $\pm E$, with the abbreviation $E = +\sqrt{\mathbf{p}^2 + m^2}$. V shall be a symmetric-matrix-valued function. Under conditions analogous to Def.2.1 the scattering system is complete [11, 12], but we need more restrictive conditions to prove the following theorem. An electromagnetic field is described by $V = A_0 - \boldsymbol{\alpha}\cdot\mathbf{A}$, where $\mathrm{grad}A_0$ is the electric and $\mathrm{rot}\mathbf{A}$ the magnetic field. For $\boldsymbol{\omega} \in S^{\nu-1}$ we make the decomposition $V = V_{+,\boldsymbol{\omega}} + V_{-,\boldsymbol{\omega}}$ with $V_{\pm,\boldsymbol{\omega}} := 1/2(V \pm \boldsymbol{\alpha}\cdot\boldsymbol{\omega} V \boldsymbol{\alpha}\cdot\boldsymbol{\omega})$, which yields $[V_{+,\boldsymbol{\omega}}, \boldsymbol{\alpha}\cdot\boldsymbol{\omega}] = 0$ and $\{V_{-,\boldsymbol{\omega}}, \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\} = 0$. For the e.m. field we get $V_{+,\boldsymbol{\omega}} = A_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\mathbf{A}$. As operators, the A_i are functions of the standard position operator $\mathbf{x}$, which generates momentum translations in the standard representation. We will discuss the alternative Newton-Wigner position operator $\mathbf{x}_{NW}$ below.

Theorem 3.1 (Reconstruction Formula, Dirac Equation) *Suppose the components of the symmetric-matrix-valued multiplication operator V are continuous with integrable decay, i.e. $\|VF(|\mathbf{x}| > R)\| \in L^1([0,\infty), dR)$, and the matrices $V_{+,\boldsymbol{\omega}}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^\nu$ commute, i.e. $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$ for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^\nu$. Then*

$$s - \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}} S e^{i\mathbf{v}\mathbf{x}} = \exp\{-i\int_{-\infty}^{+\infty} V_{+,\boldsymbol{\omega}}(\mathbf{x} + \boldsymbol{\omega}t)dt\}. \tag{13}$$

For an electromagnetic field this is $\exp\{-i\int_{-\infty}^{+\infty}(A_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x} + \boldsymbol{\omega}t)dt\}$.

If the condition $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$ is violated, the exponential must be replaced by a time-ordered product. Theorem 3.1 will be proved in the next section, after having examined its consequences. Theorem 3.2 gives a modified reconstruction formula, adapted to positive energy states. Theorem 3.3 shows how to obtain the electromagnetic field from scattering data. A general matrix-valued potential cannot be recovered from (13), since for $V = \beta\Phi$, Φ real-valued we have $V_{+,\boldsymbol{\omega}} = 0$.

For $m > 0$ and $\beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ consider the Foldy-Wouthuysen-transform $U(\mathbf{p}) = \sqrt{\frac{E+m}{2E}}\Big(\mathbf{1} + \beta\frac{\boldsymbol{\alpha}\mathbf{p}}{E+m}\Big)$, which diagonalizes H_0: $U(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)U^{-1} = E\beta$. The Foldy-Wouthuysen-representation of Ψ is given by $\hat\psi_{FW}(\mathbf{p}) = U(\mathbf{p})\hat\psi(\mathbf{p})$. S is decomposed to $S_{FW} = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}$, where $S_\pm$ are unitary operators on $L^2(\mathbb{R}^\nu, \mathbb{C}^{\mu/2})$ and describe the scattering of electrons with positive/negative energy. The Newton-Wigner position operator $\mathbf{x}_{NW}$ is the generator of momentum translations in the FW-representation and acts on ψ_{FW} as multiplication with the coordinate function. It is given by $U^*(\mathbf{p})i\nabla_{\mathbf{p}}U(\mathbf{p})$ in the standard momentum representation. (Note that we must fix a representation to employ U.) In contrast to $\mathbf{x}$, the operator $\mathbf{x}_{NW}$ does not mix the states with positive/negative energy. This suggests to investigate $e^{-i\mathbf{v}\mathbf{x}_{NW}} S\, e^{i\mathbf{v}\mathbf{x}_{NW}}$, which is decomposed to $e^{-i\mathbf{v}\mathbf{x}_{NW}} S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}}$, where the restriction of the Newton-Wigner position operator to the positive/negative energy subspaces is also denoted by $\mathbf{x}_{NW}$.

Theorem 3.2 (Reconstruction Formula for $S_\pm$) *Suppose that A_0, $\mathbf{A}$ are continuous with integrable decay. Then*

$$s - \lim_{v\to\infty} e^{-i\mathbf{v}\mathbf{x}_{NW}} S_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} = \exp\{-i\int_{-\infty}^{+\infty} (A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega}t)dt\} \tag{14}$$

Thus the limit of S_+ acts on positive energy states independent of spin.
Proof: In the standard momentum representation, Theorem 3.1 reads

$$s - \lim_{v\to\infty} e^{\mathbf{v}\nabla_\mathbf{p}} S e^{-\mathbf{v}\nabla_\mathbf{p}} = \exp\{-i\int_{-\infty}^{+\infty} (A_0 - \boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\mathbf{A})(i\nabla_\mathbf{p} + \boldsymbol{\omega}t)dt\}$$

and we conclude that

$$\begin{aligned}
e^{-i\mathbf{v}\mathbf{x}_{NW}} S\, e^{i\mathbf{v}\mathbf{x}_{NW}} &= U^*(\mathbf{p})e^{\mathbf{v}\nabla_\mathbf{p}}U(\mathbf{p})S\,U^*(\mathbf{p})e^{-\mathbf{v}\nabla_\mathbf{p}}U(\mathbf{p}) \\
&= U^*(\mathbf{p})U(\mathbf{p}+\mathbf{v})e^{\mathbf{v}\nabla_\mathbf{p}}S\,e^{-\mathbf{v}\nabla_\mathbf{p}}U^*(\mathbf{p}+v)U(\mathbf{p}) \\
&\overset{v\to\infty}{\longrightarrow} U^*(\mathbf{p})\frac{1}{\sqrt{2}}(1 + \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\, e^{-i\int_{-\infty}^{+\infty}\left(A_0 - \boldsymbol{\alpha}\boldsymbol{\omega}\,\boldsymbol{\omega}\cdot\mathbf{A}\right)(i\nabla_\mathbf{p}+\boldsymbol{\omega}t)dt}\frac{1}{\sqrt{2}}(1 - \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega})U(\mathbf{p}) \\
&= U^*(\mathbf{p})\, e^{-i\int_{-\infty}^{+\infty}\left(A_0 - \beta\,\boldsymbol{\omega}\cdot\mathbf{A}\right)(i\nabla_\mathbf{p}+\boldsymbol{\omega}t)dt} U(\mathbf{p})\,,
\end{aligned}$$

where we used $\frac{1}{\sqrt{2}}(1 + \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\,\boldsymbol{\alpha}\cdot\boldsymbol{\omega}\,\frac{1}{\sqrt{2}}(1 - \beta\boldsymbol{\alpha}\cdot\boldsymbol{\omega}) = \beta$. The above expression is equal to $\exp\{-i\int_{-\infty}^{+\infty}(A_0 - \beta\,\boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega}t)dt\}$, which is most easily seen in the FW-representation. The block-structure of β yields the desired result. □

Theorem 3.3 (Injectivity of the Scattering Map) *Consider $\nu > 1$ and $\mathcal{V} := \{(A_0,\mathbf{A}) \mid A_i \in C^0(I\!R^\nu, I\!R), \|A_i\chi(|\mathbf{x}| > R)\|_\infty \in L^1([0,\infty), dR)\,,\ \mathbf{A} \in L^2(I\!R^\nu, I\!R^\nu)\}$. Then the scattering map:*

$$(A_0, \mathbf{A}) \mapsto S_+$$

is injective on $\mathcal{V}$ except for gauge-invariance, i.e. S_+ determines A_0 uniquely and $\mathbf{A}$ up to a gradient. Thus $\mathbf{E} = -gradA_0$, $\mathbf{B} = rot\mathbf{A}$ are determined uniquely.

$\mathbf{A}$ cannot be determined uniquely, for if λ is vanishing at ∞, then S remains unchanged when $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\lambda$. This corresponds to the facts that the phase of a wave-function at a single point has no physical meaning, and that only rot$\mathbf{A}$ is measurable.
Proof: With Theorem 3.2 we get $a(\mathbf{x},\boldsymbol{\omega}) := \int_{-\infty}^{+\infty}(A_0 - \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x} + \boldsymbol{\omega}t)dt$ as in Theorem 2.4. Now $1/2\,(a(\mathbf{x},\boldsymbol{\omega}) + a(\mathbf{x},-\boldsymbol{\omega})) = \int_{-\infty}^{+\infty} A_0(\mathbf{x} + \boldsymbol{\omega}t)dt$, which determines A_0 uniquely, and$1/2\,(a(\mathbf{x},-\boldsymbol{\omega}) - a(\mathbf{x},\boldsymbol{\omega})) = \int_{-\infty}^{+\infty}\boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x} + \boldsymbol{\omega}t)dt$, which determines $\mathbf{A}$ up to a gradient (Lemma 3.4). □
To prove that lemma we need the extra assumption $\mathbf{A} \in L^2$. In [7, Lemma 3.4] a different way to reconstruct $\mathbf{B} =$rot$\mathbf{A}$ is proposed, which does not need $\mathbf{A} \in L^2$ but $\mathbf{A} \in C^1$ with $\mathbf{B}$ decaying integrably.

Lemma 3.4 (Reconstruction of A) *For* $\nu > 1$*, consider* $\mathbf{A} \in C^0 \cap L^2(I\!R^\nu, I\!R^\nu)$ *having integrable decay. Then* $\mathbf{A}$ *is determined up to* $\nabla\lambda$ *by* $\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t)$.

Proof: As in (11) one has $\int_{S^{\nu-1}} d\boldsymbol{\omega}\, \boldsymbol{\omega} \int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\cdot+\boldsymbol{\omega}t) = 2\frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^{\nu+1}} * \mathbf{A}$,which is a bounded, continuous function of $\mathbf{x}$. We will show that $\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t) = 0$ implies $\mathbf{A} = \nabla\lambda$ for a λ vanishing at ∞. In the case $\nu = 3$ one finds $(2\frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^4})^\wedge = \sqrt{\frac{\pi}{2}}\frac{1}{|\mathbf{p}|}(1 - \frac{\mathbf{p}\mathbf{p}^T}{|\mathbf{p}|^2}) \in L^2 + L^\infty$ and $\sqrt{\frac{\pi}{2}}\frac{1}{|\mathbf{p}|}(1 - \frac{\mathbf{p}\mathbf{p}^T}{|\mathbf{p}|^2})\hat{\mathbf{A}} \in L^1 + L^2$, thus

$$2\frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^4} * \mathbf{A} = 2\pi^2\Big(\frac{1}{|\mathbf{p}|}\hat{\mathbf{A}} - \frac{\mathbf{p}}{|\mathbf{p}|^3}\mathbf{p}\cdot\hat{\mathbf{A}}\Big)^\vee \in L^2 + L^\infty. \tag{15}$$

Now $\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t) = 0$ implies $\hat{\mathbf{A}} = \frac{\mathbf{p}}{|\mathbf{p}|^2}\mathbf{p}\cdot\hat{\mathbf{A}}$ a.e. , thus $\mathbf{A} = \nabla\lambda$ for a $\lambda \in L^6_w$. For $\nu > 3$, the proof is similar, but for $\nu = 2$ it must be modified to include a $\mathcal{S}'$-limit as in (12). □
Remark: In the Coulomb-gauge $\mathrm{div}\mathbf{A} = 0$ (in the sense of $\mathcal{S}'$), $\mathbf{A}$ is determined uniquely by the integral transform: For $\nu = 2$, one has

$$\int_{-\infty}^{+\infty} dt\, \mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t) = \boldsymbol{\omega}\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t),$$

and the proof of theorem 2.4 applies to the components A_1, A_2. For $\nu = 3$, (15) implies $\int_{S^2} d\boldsymbol{\omega}\, \boldsymbol{\omega}\int_{-\infty}^{+\infty} dt\, \boldsymbol{\omega}\cdot\mathbf{A}(\mathbf{x}+\boldsymbol{\omega}t) = 2\pi^2(\frac{1}{|\mathbf{p}|}\hat{\mathbf{A}})^\vee$. For $\nu > 3$, the factor $2\pi^2$ must be replaced by $2\pi^{\frac{\nu+1}{2}}/\Gamma(\frac{\nu+1}{2})$.

4 Proof of Theorem 3.1

We first discuss two preparatory lemmata:

Lemma 4.1 (Approximation of S**)** *Let* V *be a matrix-valued multiplication operator, which is* H_0*-bounded with relative bound* < 1 *and satisfies*

$$\|V\sqrt{\mathbf{p}^2+1}^{\,-1}F(|\mathbf{x}| > R)\| \in L^1\left([0,\infty), dR\right).$$

Then for Φ, Ψ *with* $\hat{\phi}, \hat{\psi} \in C_0^\infty$ *there is* $v_0 > 0$ *such that*

$$\lim_{t\to\infty}\Big(\Phi, e^{-i\mathbf{v}\mathbf{x}}e^{iH_0t}e^{-i2Ht}e^{iH_0t}e^{i\mathbf{v}\mathbf{x}}\Psi\Big) = \Big(\Phi, e^{-i\mathbf{v}\mathbf{x}}S\, e^{i\mathbf{v}\mathbf{x}}\Psi\Big) \tag{16}$$

uniformly in $v > v_0$.

Proof: We first note that for $\hat{\phi} \in C_0^\infty$ there are $c, v_0 > 0$ such that

$$\Big|e^{-iH_0t}e^{i\mathbf{v}\mathbf{x}}\sqrt{\mathbf{p}^2+1}\,\Psi\Big|(\mathbf{x}) < \frac{c}{(1+|t|)^{\frac{\nu+3}{2}}} \text{ for } |\mathbf{x}| < \frac{|t|}{2}, v > v_0. \tag{17}$$

This is shown as in Lemma 2.2 by observing that

$$e^{-i\mathbf{vx}}e^{-iH_0t}e^{i\mathbf{vx}} = e^{-i\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}}P_{+,\mathbf{v}} + e^{i\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}}P_{-,\mathbf{v}} \tag{18}$$

with $P_{\pm,\mathbf{v}} = \frac{1}{2}\Big(1 \pm \frac{\boldsymbol{\alpha}(\mathbf{p}+\mathbf{v})+\beta m}{\sqrt{(\mathbf{p}+\mathbf{v})^2+m^2}}\Big)$ and that $\partial^\gamma_{\mathbf{p}}\left(P_{\pm,\mathbf{v}}\sqrt{\mathbf{p}^2+1}\hat{\phi}(\mathbf{p})\right)$ is bounded for $|\gamma| \geq 0$. Using the same decomposition as in the proof of Lemma 2.2, we find $h \in L^1$ such that $\|Ve^{-iH_0t}e^{i\mathbf{vx}}\Phi\| \leq h(t)$ for $t \in I\!R$, $v \geq v_0$. Now

$$\begin{aligned} \|\Omega_+e^{i\mathbf{vx}}\Phi - e^{iHt}e^{-iH_0t}e^{i\mathbf{vx}}\Phi\| &= \|i\int_t^\infty ds\, e^{iHs}Ve^{-iH_0s}e^{i\mathbf{vx}}\Phi\| \\ &\leq \int_t^\infty ds\, h(s) \to 0 \quad \text{as}\, t \to \infty \end{aligned}$$

uniformly in $v > v_0$. We treat Ω_- and Ψ analogously, and the result for S is obtained. □

Lemma 4.2 (Limit for finite t) *Suppose V has bounded, continuous components and satisfies $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$. Then for all $t > 0$*

$$s - \lim_{v\to\infty} e^{-i\mathbf{vx}}e^{iH_0t}e^{-i2Ht}e^{iH_0t}e^{i\mathbf{vx}} = \exp\{-i\int_{-t}^{t} ds\, V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)\}. \tag{19}$$

Proof: According to Theorem X.69 in [9], we have the Dyson-expansion

$$e^{iH_0t}e^{-i2Ht}e^{iH_0t}\Psi = \sum_{n=0}^{\infty}(-i)^n \int\limits_{-t<t_1<\ldots<t_n<t} dt_n \ldots dt_1 V(t_n)\ldots V(t_1)\Psi \quad (t > 0, \Psi \in \mathcal{H}) \tag{20}$$

with $V(s) = e^{iH_0s}V\,e^{-iH_0s}$. In the momentum representation it is easily shown that $s - \lim_{v\to\infty}\left(e^{-i\mathbf{vx}}e^{-iH_0s}e^{i\mathbf{vx}} - e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}\right) = 0$, thus

$$\lim_{v\to\infty}\int_{-t}^{t} ds\left(e^{-i\mathbf{vx}}V(s)e^{i\mathbf{vx}} - e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V\,e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}\right)\Psi = 0.$$

From $\boldsymbol{\alpha}\cdot\boldsymbol{\omega}\, V_{\pm,\boldsymbol{\omega}} = \pm V_{\pm,\boldsymbol{\omega}}\, \boldsymbol{\alpha}\cdot\boldsymbol{\omega}$ we conclude

$$\begin{aligned} & e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V\,e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s} \\ =\ & e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V_{+,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s} + e^{i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s}V_{-,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}(v+\boldsymbol{\omega}\mathbf{p})s} \\ =\ & e^{i\boldsymbol{\alpha\omega}\boldsymbol{\omega}_{\mathbf{p}}s}V_{+,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}\boldsymbol{\omega}_{\mathbf{p}}s} + e^{i2\boldsymbol{\alpha\omega}vs}e^{i\boldsymbol{\alpha\omega}\boldsymbol{\omega}_{\mathbf{p}}s}V_{-,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}\boldsymbol{\omega}_{\mathbf{p}}s}. \end{aligned}$$

The Riemann-Lebesgue Lemma yields

$$\lim_{v\to\infty}\int_{-t}^{t} ds\, e^{i2\boldsymbol{\alpha\omega}vs}e^{i\boldsymbol{\alpha\omega}\boldsymbol{\omega}_{\mathbf{p}}s}V_{-,\boldsymbol{\omega}}e^{-i\boldsymbol{\alpha\omega}\boldsymbol{\omega}_{\mathbf{p}}s}\Psi = 0$$

and thus $\lim_{v\to\infty}\int_{-t}^{t} ds\, e^{-i\mathbf{vx}} V(s) e^{i\mathbf{vx}}\Psi = \int_{-t}^{t} ds\, W(s)\Psi$ with $W(s) = e^{i\boldsymbol{\alpha\omega\omega}\mathbf{p}s} V_{+,\boldsymbol{\omega}} e^{-i\boldsymbol{\alpha\omega\omega}\mathbf{p}s}$. By induction, it is shown that

$$
\begin{aligned}
\lim_{v\to\infty} e^{-i\mathbf{vx}} \int\limits_{-t<t_1<\ldots<t_n<t} dt_n \ldots dt_1 V(t_n)\ldots V(t_1) e^{i\mathbf{vx}}\Psi &= \\
\int\limits_{-t<t_1<\ldots<t_n<t} dt_n \ldots dt_1 W(t_n)\ldots W(t_1)\Psi &= \\
\frac{1}{n!}\int\limits_{[-t,t]^n} dt_n \ldots dt_1 W(t_n)\ldots W(t_1)\Psi &= \frac{1}{n!}\Big(\int\limits_{-t}^{t} ds\, W(s)\Big)^n \Psi .
\end{aligned}
$$

Here the time-ordering in the integral was resolved because of $[W(\mathbf{x}_1), W(\mathbf{x}_2)] = 0$ for $\mathbf{x}_1,\ \mathbf{x}_2 \in I\!R^\nu$, which follows from $[V_{+,\boldsymbol{\omega}}(\mathbf{x}_1), V_{+,\boldsymbol{\omega}}(\mathbf{x}_2)] = 0$ and $[\boldsymbol{\alpha}\cdot\boldsymbol{\omega}, V_{+,\boldsymbol{\omega}}(\mathbf{x})] = 0$, observing that $W(s) = 1/2(1+\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\, V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s) + 1/2(1-\boldsymbol{\alpha}\cdot\boldsymbol{\omega})\, V_{+,\boldsymbol{\omega}}(\mathbf{x}-\boldsymbol{\omega}s)$. This decomposition also yields $\int_{-t}^{t} ds\, W(s) = \int_{-t}^{t} ds\, V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)$. The Dyson-series converges uniformly in v, thus $\lim_{v\to\infty}$ and $\sum_{n=0}^{\infty}$ may be interchanged. □
Proof of Theorem 3.1: The hypotheses of Lemmata 4.1, 4.2 are fulfilled. For $\hat{\phi}, \hat{\psi} \in C_0^\infty$ an $\varepsilon/3$-trick shows: The uniform convergence in (16) entails that the following limits may be interchanged.

$$
\begin{aligned}
\lim_{v\to\infty}\Big(\Phi, e^{-i\mathbf{vx}} S\, e^{i\mathbf{vx}}\Psi\Big) &= \lim_{v\to\infty}\lim_{t\to\infty}\Big(\Phi, e^{-i\mathbf{vx}} e^{iH_0 t} e^{-i2Ht} e^{iH_0 t} e^{i\mathbf{vx}}\Psi\Big) \\
&= \lim_{t\to\infty}\lim_{v\to\infty}\Big(\Phi, e^{-i\mathbf{vx}} e^{iH_0 t} e^{-i2Ht} e^{iH_0 t} e^{i\mathbf{vx}}\Psi\Big) \\
&= \lim_{t\to\infty}\Big(\Phi, \exp\{-i\int_{-t}^{t} ds\, V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)\}\,\Psi\Big) \\
&= \Big(\Phi, \exp\{-i\int_{-\infty}^{\infty} ds\, V_{+,\boldsymbol{\omega}}(\mathbf{x}+\boldsymbol{\omega}s)\}\,\Psi\Big)
\end{aligned}
$$

A density argument yields weak convergence, and the unitarity of $e^{-i\mathbf{vx}} S e^{i\mathbf{vx}}$ and of its weak limit imply the strong convergence. □

5 Summary and Generalizations

For $H_0 = \sqrt{\mathbf{p}^2 + m^2}$ we obtained $\exp\{-i\int_{-\infty}^{+\infty} V(\mathbf{x}+\boldsymbol{\omega}t)dt\}$ from S for very general short-range V, but the reconstruction of V was only accomplished for continuous V with integrable decay.

For $H_0 = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$ we obtained $\exp\{-i\int_{-\infty}^{+\infty} (A_0 - \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}_{NW} + \boldsymbol{\omega}t)dt\}$ from S_+ only for continuous A_i with integrable decay. Lemma 4.1 holds under very general conditions, but the Dyson-expansion in 4.2 demands that V should be bounded. We expect Theorems 3.1, 3.2 to be true for general short-range A_0, but this is not yet proved.

The Aharanov-Bohm-experiment suggests to consider the case $\nu = 2$ with the magnetic field $B = \mathrm{rot}\mathbf{A} \in C_0^0$. This requires the following modifications:

- Given B with $\int B \neq 0$, there is no $\mathbf{A}$ of integrable decay, but there are vectorpotentials with $|\mathbf{A}(\mathbf{x})| < c/|\mathbf{x}|$ and $\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\mathbf{A}(\mathbf{x})$ decaying integrably. If $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\lambda$, then $\Lambda(\mathbf{x}) = \lim_{r\to\infty}\lambda(r\mathbf{x})$ exists and may be $\neq 0$.

- Choosing a special gauge with supp$(\mathbf{A})$ in a cone as in [1], Theorems 3.1, 3.2 are shown to remain valid for this $\mathbf{A}$.

- In a different gauge $\tilde{\mathbf{A}}$ we find $\tilde{S}_\pm = e^{i\Lambda(\pm\mathbf{p})}S_\pm e^{-i\Lambda(\mp\mathbf{p})}$ and conclude that also $e^{-i\mathbf{v}\mathbf{x}_{NW}}\tilde{S}_\pm e^{i\mathbf{v}\mathbf{x}_{NW}} \;\to\; \exp\{-i\int_{-\infty}^{+\infty}(A_0 \mp \boldsymbol{\omega}\cdot\tilde{\mathbf{A}})(\mathbf{x}_{NW}+\boldsymbol{\omega}t)dt\}$.

- The gauge-invariance of S is lost, but we expect all physically measurable quantities to be gauge-independent. Under the idealized assumption that phase-differences are measurable in interference-experiments, the high-energy-limit of S_+ yields $\int_{-\infty}^{+\infty}(A_0 \mp \boldsymbol{\omega}\cdot\tilde{\mathbf{A}})(\mathbf{x}+\boldsymbol{\omega}t)dt$ up to an additive constant, and the variation of this function in the direction of $\boldsymbol{\varpi}$ orthogonal to $\boldsymbol{\omega}$ suffices to obtain B, for

$$\int_{-\infty}^{+\infty} dt\, B(\boldsymbol{\varpi}s + \boldsymbol{\omega}t) = \frac{d}{ds}\int_{-\infty}^{+\infty} dt\,\boldsymbol{\omega}\cdot\mathbf{A}(\boldsymbol{\varpi}s+\boldsymbol{\omega}t) \quad \text{for} \quad \boldsymbol{\varpi} = (\boldsymbol{\omega}_2, -\boldsymbol{\omega}_1)^T. \tag{21}$$

Finally we mention that the Klein-Gordon equation for a charged spin-0-particle can be treated in the same way as the Dirac equation, since the Dyson-expansion also applies to the 2-Hilbertspace-formalism. We find the same result

$$e^{-i\mathbf{v}\mathbf{x}}S_\pm e^{i\mathbf{v}\mathbf{x}} \;\to\; \exp\{-i\int_{-\infty}^{+\infty}(A_0 \mp \boldsymbol{\omega}\cdot\mathbf{A})(\mathbf{x}+\boldsymbol{\omega}t)dt\}. \tag{22}$$

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