Stochastic stability of weakly coupled lattice maps *

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Abstract

We consider a stochastic perturbation of weakly coupled expanding circle maps. We construct the dynamics and its natural invariant measure via a polymer expansion and show the stochastic stability of the system.

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1 Introduction

Coupled map lattices are similar to cellular automata except that the single component state space is continuous. So instead of having e.g. $\{0, 1\}$-valued

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components in binary cellular automata we will use the circle S^1 as single site state space for the coupled map lattices in this paper. This is also one of the reasons why it is sometimes said that coupled map lattices are intermediate between cellular automata and systems of coupled partial differential equations. We refer to [?] for a description of the range of motivations and applications in the study of coupled map lattices. The point of view we take here is that coupled map lattices provide examples of spatially extended dynamical systems for which we can investigate the thermodynamic limit of concepts such as sensitivity to initial conditions, SRB-measures,... well explored for finite dimensional dynamical systems.

Just as in cellular automata, coupled map lattices are discrete time, parallel and deterministic dynamics in which some local interaction is superimposed on the otherwise independent evolution of the individual components. However, the individual evolution may sometimes have very nice though non-trivial (temporal) ergodic properties. [?, ?] and [?] investigated in what sense, for a weak coupling or interaction, the infinite volume system behaves similar to the one-component system, i.e. they proved that for a certain class of "sufficiently nice" single component dynamics and under certain conditions on the imposed coupling, some strong ergodic properties such as mixing are preserved in the thermodynamic limit. These results are sometimes quoted as showing space-time chaos. This terminology may however be slightly misleading in the sense that all the results are about spatio-temporal ergodic properties and it needs to be emphasized that there is yet no rigorous understanding of how to describe a corresponding Lyapounov spectrum or of how this is related to sensitive dependence on initial conditions, i.e. chaotic behavior, for the macroscopic system.

Also this paper is not going to address these last questions but we will discuss another micro-macro transport problem. Another feature of certain "nice" single component maps is that they are stable under stochastic perturbations [?]. Hence the subject of the paper: does this property of stochastic stability extend to the infinite system as well?

Another way of presenting this problem is to refer to the notion of SRB-measures [?]. These are physically relevant states that are asymptotically obtained when starting from random initial data. It is known that (e.g. for so called Axiom A systems) these measures coincide with the zero-noise limit of their stochastic perturbation. So in the attempt to make sense of infinite dimensional SRB-measures, it is natural to also discuss that question. In order to answer it we rederive (in a more direct and simpler way) the results of [?] and [?] and we show that our construction is stable against stochastic perturbations.

In the next section, we introduce the model and we give our main results. Section 3 is devoted to the proof and the last section adds some remarks.

2 Model and Results

To each site $i \in \mathbb{Z}^d$ of the d-dimensional regular lattice is assigned a variable $x(i) \in S^1$ on the unit circle. We write $\Omega_\Lambda = (S^1)^\Lambda$ to denote the state space for the variables $x(i),\ i \in \Lambda$, in volume $\Lambda \subset \mathbb{Z}^d$. Also, $\Omega_{\mathbb{Z}^d} = \Omega$. In what follows Λ is always a finite cube centered around the origin with side length $2l+1$, and $\Lambda \uparrow \mathbb{Z}^d$ implies taking a sequence of such volumes with $l \uparrow \infty$. For arbitrary spatio-temporal boundary conditions we consider a discrete time finite volume (Λ) parallel dynamics. This dynamics is stochastic but all our results will be valid uniformly in the noise-level $\delta > 0$ as $\delta \downarrow 0$, to be specified below. Our main result is indeed that the limits $l \uparrow \infty$ (thermodynamic limit), time $\uparrow \infty$ (asymptotic or stationary regime) and $\delta \downarrow 0$ (deterministic limit) can be mutually interchanged for a class of sufficiently nice dynamics.

The dynamics is specified by a local and translation-invariant synchronous up-dating mechanism

$$x_n \to x_{n+1}(i) \tag{2.1}$$

for $i \in \Lambda, n = 0, 1, 2, \ldots$ In (2.1) $x_{n+1}(i)$ depends only on $x_n(i)$ and the $x_n(j), j \sim i$, for j nearest neighbor to i and we a priori choose all the values for the $x_n(i), i \in \Lambda^c = \mathbb{Z}^d \setminus \Lambda$, $n = 0, 1, 2, \ldots$; say $x_n(i) = \alpha_n(i)$ for some fixed but arbitrary $\alpha \in \Omega^{\mathbb{N}}$. Futhermore, (2.1) consists of two steps: (a stochastic perturbation of) a free evolution and a (deterministic) interaction step. In particular, (2.1) is of the form

$$x_{n+1}(i) = T_{i,n}^{\Lambda,\alpha,\epsilon}\left(\mathcal{S}_{\omega_n}^{\delta}(x_n)\right) \tag{2.2}$$

with free evolution $\mathcal{S}_{\omega_n}^{\delta} : \Omega \to \Omega$:

$$\mathcal{S}_{\omega_n}^{\delta}(x)(j) = \omega_n(j) S\left(x(j)\right) \tag{2.3}$$

for $\left(\omega_n(j); n = 0, 1, 2, \ldots; j \in \mathbb{Z}^d\right)$ a collection of independent and identically distributed random variables with values in the set

$$S_{\delta}^{1} = \left\{z \in S^1 : \quad |z - 1| \leq \delta\right\},\ \delta > 0 \tag{2.4}$$

For the uncoupled, single site evolution S we put ourselves in the framework of [**?**] and require that

1. S is a real analytic function on the unit circle and

2. S is expanding i.e.

$$\exists \gamma > 1 : \quad |S'(z)| \geq \gamma \tag{2.5}$$

The random variable $\omega \in S_{\delta}^{1}$ will be written as $\omega = \exp(2\pi\imath\phi)$, $\imath = \sqrt{-1}$, and the angle ϕ has a C^{∞}-density H^{δ} with supp $\mathrm{H}^{\delta} \subset [-\delta, \delta]$

$$H^{\delta}(\phi) = \frac{1}{\delta} H(\frac{\phi}{\delta}) \tag{2.6}$$

On the level of the angles the free evolution (2.3) is a stochastic perturbation by convolution, see Section 3 of [**?**]. The corresponding Markov operator is

$$P^{\delta} f\left(\exp(2\pi\imath\psi)\right) = \frac{1}{\delta}\int_0^1 d\phi\ H\left(\frac{\psi - \phi}{\delta}\right) Pf\left(\exp(2\pi\imath\phi)\right) \tag{2.7}$$

where P is the Perron-Frobenius operator corresponding to the map S: formally,

$$Pf(z) = \int_{S^1} dz' f(z') \delta(z - Sz') \tag{2.8}$$

P and P^δ are compact operators on the space $\mathcal{H}_\rho$ of bounded analytic functions with domain of analyticity $\mathcal{A}_\rho$

$$\mathcal{A}_\rho = \{z \in \mathbb{C} : 1 - \rho < |z| < 1 + \rho\} \tag{2.9}$$

for some $\rho :\ 0 < \rho < 1$. $\mathcal{H}_\rho$ is a Banach space for the supremum norm

$$\|f\| = \sup_{z \in \mathcal{A}_\rho} |f(z)| \tag{2.10}$$

We will use the following results [**?**] which hold uniformly in δ for $\delta \downarrow 0$:

1. There is a unique density $h^\delta \in \mathcal{H}_{\rho_0}$ that satisfies

$$P^\delta h^\delta = h^\delta \tag{2.11}$$

2. P^δ is compact and can be decomposed as

$$P^\delta = Q^\delta + R^\delta \tag{2.12}$$

where Q^δ is the 1-dimensional projection operator on h^δ,

$$|||(R^\delta)^n||| \leq C\eta^n \tag{2.13}$$

for some $\eta < 1, \quad C < \infty$ and

$$Q^\delta R^\delta = R^\delta Q^\delta = 0 \tag{2.14}$$

3. for every $\rho \leq \rho_0$, P is continuous on $\mathcal{H}_\rho$ and improves the domain of analyticity

$$P: \quad \mathcal{H}_\rho \to \mathcal{H}_{\gamma\rho} \tag{2.15}$$

where $\gamma > 1$ is the expansion coefficient of S.

We will frequently abuse the notation by considering P^δ or P acting on the "lifted" functions. E.g. (2.7) can be written as

$$P^\delta f(\psi) = \int_0^1 d\phi \; H^\delta(\psi - \phi) P f(\phi) \tag{2.16}$$

Finally, the interaction $T_{i,n}^{\epsilon,\Lambda,\alpha}$ in (2.2) is taken of the form

$$T_{i,n}^{\epsilon,\Lambda,\alpha}(x) = x(i) \exp[2\pi\imath\epsilon g(x(i), \{y_n(j)\}_{j\sim i})] \tag{2.17}$$

where $g \in \mathcal{H}_\rho^{2d+1}$ and $y_n(j) = x(j)$ if $j \in \Lambda$, $\quad y_n(j) = \alpha_n(j)$ otherwise. The parameter $\epsilon \geq 0$ is "small" and quantifies the strength of the coupling. Note that, for simplicity, we restrict ourselves to a nearest-neighbor interaction. This is not at all essential but is done to avoid additional notations and less interesting technical complications.

Main Result

1. There is $\epsilon_0 > 0$ so that for all $\epsilon \leq \epsilon_0$ there is a unique invariant probability measure $\mu_{\Lambda,\alpha}^{\epsilon,\delta}$ on $(S^1)^\Lambda = \Omega_\Lambda$ for the dynamics described above which is absolutely continuous with a density $\zeta_\Lambda^{\epsilon,\delta,\alpha} \in \mathcal{H}_\rho^\Lambda$. The

Lebesgue-measure converges exponentially fast under the dynamics to $\mu_{\Lambda,\alpha}^{\epsilon,\delta}$ $(t \uparrow \infty$, weak convergence) with a rate $\lambda = \lambda(\epsilon_0, \delta, \Lambda, \alpha) > 0$ which is bounded away from zero uniformly in $\delta \downarrow 0, \quad \Lambda \uparrow \mathbb{Z}^d$ and $\alpha \in \Omega^{\mathbb{N}}$.

2. The thermodynamic limit $\mu_{\Lambda,\alpha}^{\epsilon,\delta} \to \mu^{\epsilon,\delta}$ (weak convergence, $\Lambda \uparrow \mathbb{Z}^d$) exists and is obtained exponentially fast with a rate $m = m(\epsilon_0, \delta, \alpha) > 0$ bounded away from zero uniformly in $\delta \downarrow 0$ and $\alpha \in \Omega^{\mathbb{N}}$. The infinite volume measure $\mu^{\epsilon,\delta}$ is (a high-temperature Gibbs measure) locally absolutely continuous with respect to the Lebesgue measure.

The mathematical formulation of these results will be stated in the next section as Theorems 3.1 and 3.2.

3 Main Theorems and Proofs

For simplicity we will state our results for the case when the boundary conditions α are fixed in time ($\alpha_n(i) = \alpha(i), \; n = 0, 1, \ldots$). Our finite volume dynamics is then an Ω_Λ-valued time-homogeneous Markov chain x_n with transition probabilities

$$p_\Lambda(dy|x) = \prod_{i \in \Lambda} h^\delta \left(y(i), (Tx)(i)\right) dy(i) \tag{3.1}$$

where $(Tx)(i) = T_i^{\Lambda,\alpha,\epsilon} x$ and

$$h^\delta(z, z') = \frac{1}{\delta} H\left(\frac{\phi - \psi}{\delta}\right) \tag{3.2}$$

for $z = \exp(2\pi\imath\phi)$, $Sz' = \exp(2\pi\imath\psi)$.

We start the dynamics at time $n = 0$ with a probability measure on Ω_Λ with density $\nu \in \mathcal{H}_\rho^\Lambda = \otimes_{i\in\Lambda}\mathcal{H}_\rho$. The expectation with respect to this Markov chain is denoted by $\mathbb{E}_\nu^\Lambda[.]$.

The corresponding Markov operator P_Λ^δ is defined by

$$\left(\mathcal{P}_\Lambda^\delta \nu\right)(x) = \int_{\Omega_\Lambda} dy\ \nu(y) \prod_{i\in\Lambda} h^\delta(x(i), T_i y) \tag{3.3}$$

where we abbreviated $(Ty)(i) = T_i y$ and again $\nu \in \mathcal{H}_\rho^\Lambda$. This operator expresses the evolution of the densities in time. Given the initial density ν at time $n = 0$, the density at time $n = 1$ will be

$$\nu_1 = \mathcal{P}_\Lambda^\delta \nu \tag{3.4}$$

or more generally, for $n = 1, 2, \ldots \quad \nu_n = \mathcal{P}_\Lambda^\delta \nu_{n-1}$ and

$$\mathbb{E}_\nu^\Lambda\left(f(x_n)\right) = \int_{\Omega_\Lambda} f(x)\mathcal{P}_\Lambda^\delta \nu_{n-1}(x)dx \tag{3.5}$$

Theorem 3.1 *For every local function $f \in \mathcal{H}_\rho^F$ there is $c(f) < \infty$ so that*

$$\left|\mathbb{E}_\nu^\Lambda\left(f(x_n)\right) - \int_{\Omega_F} f(x)\zeta_{\Lambda,F}(x)dx\right| \le c(f)e^{-\lambda n} \tag{3.6}$$

*with $\lambda > 0$ uniformly in $\Lambda \uparrow \mathbb{Z}^d$ and $\delta \downarrow 0$ and the boundary conditions $\alpha \in \Omega$. The invariant density $\zeta_\Lambda = \mathcal{P}_\Lambda^\delta \zeta_\Lambda \in \mathcal{H}_\rho^\Lambda$ has restriction $\zeta_{\Lambda,F} \in \mathcal{H}_\rho^F$ to the set F as expressed via the expansion (**??**).*

Theorem 3.2 *For every local function $f \in \mathcal{H}_\rho^F$ there is $c(f) < \infty$ so that*

$$\left| \int_{\Omega_\Lambda} f(x) \zeta_{\Lambda,F}(x) dx - \int_{\Omega_\Lambda} f(x) \zeta_{\mathbb{Z}^d,F}(x) dx \right| \le c(f) e^{-ml} \tag{3.7}$$

*where $\zeta_{\Lambda,F}(x)$ is as in Theorem 3.1 for $\zeta_{\mathbb{Z}^d,F}(x)$ given by the expansion (**??**) with Λ replaced by $\mathbb{Z}^d$, with rate $m > 0$ uniformly in the boundary conditions $\alpha \in \Omega$ and $\delta \downarrow 0$.*

3.1 The time-one expansion

Before we discuss the operator (3.3)-(3.5), we decompose the updating-kernel as

$$h^\delta\left(x(i), T_i y\right) = h^\delta(x(i)) + r^\delta(x(i), y(i)) + \beta_i^\delta(x(i), y) \tag{3.8}$$

where $h^\delta \in \mathcal{H}_{\rho_0}$ is the unique solution to (2.11) $P^\delta h^\delta = h^\delta$ and $\beta_i^\delta(x(i), y) = h^\delta(x(i), T_i y) - h^\delta(x(i), y(i))$. The first two terms in (3.8) correspond to the decomposition (2.12)

$$P^\delta = Q^\delta + R^\delta$$

We now substitute (3.8) into (3.3) and expand the product over $i \in \Lambda$. For better organization we can graphically represent the terms we get on two spatial layers ($\mathbb{Z}^d \times \mathbb{Z}^d$ corresponding to times $n-1$ and n). We therefore introduce two types of monomers connecting points $(i, n-1)$ with (j, n) $\quad (i, j \in \Lambda)$:

- a line from $(i, n-1)$ to (i, n) corresponds to the term $r^\delta(x_n(i), x_{n-1}(i))$ (or $r^\delta(x(i), y(i))$ in (3.8)).

- a triangle with base points $\{(i, n-1), (j, n-1) \quad j \sim i\}$ and apex (i, n) corresponds to the term $\beta_i^\delta(x_n(i), x_{n-1})$ (or $\beta_i^\delta(x(i), y)$ in (3.8)).

We call such a connected collection of "triangles" and "lines" a polymer (of time-length 1). Connection is by overlapping points. Two polymers are disjoint if they have no points in common. Such a polymer γ is completely determined if we specify the set

$$A(\gamma) = \{i \in \Lambda : \ (i, n) \text{ is an apex in } \gamma \ \} \tag{3.9}$$

and the set $L(\gamma)$ of its line-end points,

$$L(\gamma) = \{i \in \Lambda : \ (i, n) \text{ is a line-end point of } \gamma \ \} \tag{3.10}$$

Moreover, to every γ there corresponds a set

$$\gamma^- = \{i \in \Lambda : \ (i, n-1) \in \gamma\} \tag{3.11}$$

For example, such γ and γ' are disjoint if $\gamma^- \cap \gamma'^-$ is empty. Finally, there is the set

$$\gamma^+ = \{i \in \Lambda : \ (i, n) \in \gamma\} \tag{3.12}$$

of endpoints. Of course, $|\gamma^+| = |A(\gamma)| + |L(\gamma)|$.

Lemma 3.1 *The operator*

$$W(\gamma)f(x) = \int_{\Omega_{\gamma^-}} dy f(y) \prod_{i\in A(\gamma)} \beta_i^\delta(x(i), y) \prod_{i\in L(\gamma)} r^\delta(x(i), y(i)) \tag{3.13}$$

is defined from $\mathcal{H}_\rho^{\gamma^-}$ to $\mathcal{H}_\rho^{\gamma^+}$ with norm

$$|||W(\gamma)||| \leq C\tilde{\epsilon}^{|A(\gamma)|}\eta^{|L(\gamma)|} \tag{3.14}$$

(with $\tilde{\epsilon} \to 0$ as $\epsilon \to 0$) uniformly in $\delta \downarrow 0$ and the boundary conditions α (implicit in the β_i^δ).

Proof of Lemma 1: We first treat the case where $L(\gamma)$ is empty in (3.13). We use the identity

$$\begin{aligned} \beta_i^\delta(x(i), y) &= h^\delta(x(i), T_i y) - h^\delta(x(i), y(i)) \\ &= \int_0^\epsilon d\epsilon_i \frac{\partial}{\partial \epsilon_i} h^\delta(x(i), T_i^{\epsilon_i} y) \end{aligned} \tag{3.15}$$

where $T_i^{\epsilon_i} y = T_i^{\epsilon_i,\Lambda,\alpha} y$, to write for $x \in \Omega_{A_\gamma}, \quad y \in \Omega_{\gamma^- \setminus A_\gamma}$,

$$\begin{aligned} &W(\gamma, y)f(x) \\ &\equiv \int_{\Omega_{A_\gamma}} \prod_{i\in A_\gamma} dv(i) f(v, y) \prod_{i\in A_\gamma} \beta_i^\delta(x(i); v, y) \\ &= \int_{\Omega_{A_\gamma}} dz \prod_{A(\gamma)} h^\delta(x(i), z(i)) \int_{[0,\epsilon]^{A_\gamma}} \prod_{A(\gamma)} d\epsilon_i \frac{\partial}{\partial \epsilon_i} \Phi_\gamma^{\epsilon, y} f(z) \end{aligned} \tag{3.16}$$

where

$$\Phi_\gamma^{\epsilon,y} f(z) = \int_{\Omega_{A_\gamma}} dv f(v, y) \prod_{i\in A(\gamma)} \delta(z_i - T_i^{\epsilon_i}(v, y)) \tag{3.17}$$

is the Perron-Frobenius operator corresponding to the interaction step. The extra-dependence of $W(\gamma, y)$ and $\Phi_\gamma^{\epsilon,y}$ on y is because in (3.16)-(3.17) we keep fixed the variables $y(j),\ j \in \gamma^- \setminus A_\gamma$. In particular, $f(v,y) = f(x)$ for $x(i) = v(i),\ i \in A_\gamma, = y(i),\ i \in \gamma^- \setminus A_\gamma$. ¿From (2.17) we know that

$$|T_i^{\epsilon,\Lambda,\alpha}(x) - x(i)| \leq C_1\epsilon \tag{3.18}$$

and

$$|DT_i^{\epsilon,\Lambda,\alpha}(x)_{ij} - \delta_{ij}| \leq C_2\epsilon \tag{3.19}$$

This implies that $T_i^{\epsilon,\Lambda,\alpha}$ is invertible provided ϵ is small enough and

$$\left(T_i^{\epsilon,\Lambda,\alpha}\right)^{-1} \in \mathcal{H}_{\rho-a}^{\Lambda} \tag{3.20}$$

for $a > C_3\epsilon$. Now define the operator

$$T_\gamma = \otimes_{i\in A_\gamma} T_i^{\epsilon_i, A_\gamma, (y,\alpha)}$$

where we put boundary conditions y on $\gamma^- \setminus A_\gamma$ and α on Λ^c. Then T_γ is a diffeomorfism from $\mathcal{A}_\rho^{A_\gamma}$ onto a set in $\mathcal{A}_{\rho-a}^{A_\gamma}$ and

$$\Phi_\gamma^{\epsilon,y} :\ \mathcal{H}_\rho^{A_\gamma} \to \mathcal{H}_{\rho-a}^{A_\gamma} \tag{3.21}$$

is a holomorphic family of operators. The inequalities (3.18) and (3.19) imply that

$$\| \det D\left(T_\gamma\right)^{-1} \| \leq \exp(C\epsilon|A_\gamma|) \tag{3.22}$$

for some $C,\ 0 < C < \infty$. Then from a Cauchy estimate

$$\begin{aligned}\left|\int_{[0,\epsilon]^{A_\gamma}} \prod_i d\epsilon_i \frac{\partial}{\partial \epsilon_i} \Phi_\gamma^{\epsilon,y} f(z)\right| &\leq \epsilon^{|A(\gamma)|} C^{|A(\gamma)|} \\ &= \tilde{\epsilon}^{|A(\gamma)|} \end{aligned} \tag{3.23}$$

where $\tilde{\epsilon} \to 0$ if $\epsilon \to 0$. Hence

$$|||W(\gamma, y)||| \leq C \tilde{\epsilon}^{|A(\gamma)|} \tag{3.24}$$

and $W(\gamma, y)f$ can be seen as a function of the $y(j),\ j \in \gamma^- \setminus A_\gamma : W(\gamma, \cdot)f \in \mathcal{H}_\rho^{\gamma^- \setminus A_\gamma} \otimes \mathcal{H}_\rho^{A_\gamma}$. The general case (3.13) follows from integrating

$$W(\gamma)f(x) = \int_{\Omega_{\gamma^- \setminus A_\gamma}} dy \left(W(\gamma, y)f\right)(x(i),\ i \in A_\gamma) \prod_{i \in L(\gamma)} r^\delta(x(i), y(i)) \tag{3.25}$$

which gives rise to the bound (3.14).

To finish the proof, we argue that applying $W(\gamma)$ doesn't change the domain of analyticity, i.e.

$$W(\gamma): \quad \mathcal{H}_\rho^{\gamma^-} \to \mathcal{H}_\rho^{\gamma^+}$$

Since $\Phi_\gamma^{\epsilon,y} f(z) \in \mathcal{H}_{\rho-a}^{A_\gamma}$ for every ϵ and y, also

$$G^{y,\epsilon}(z) \equiv \int_{[0,\epsilon]^{A_\gamma}} \prod_i d\epsilon_i \frac{\partial}{\partial \epsilon_i} \Phi_\gamma^{\epsilon,y} f(z) \tag{3.26}$$

will be in $\mathcal{H}_{\rho-a}^{A_\gamma}$. Then

$$\begin{aligned} W(\gamma, y)f(x) &= \int dz \prod_{i \in A_\gamma} h^\delta(x(i), z(i)) G^{y,\epsilon}(z) \\ &= \mathcal{P}_{A_\gamma}^\delta\ G^{y,\epsilon}(x) \\ &\in \mathcal{H}_{\gamma(\rho-a)}^{A_\gamma} \end{aligned}$$

Thus

$$W(\gamma,\cdot): \ \mathcal{H}_\rho^{\gamma^-} \to \mathcal{H}_\rho^{\gamma^-\setminus A_\gamma} \otimes \mathcal{H}_{\gamma(\rho-a)}^{A_\gamma} \subset \mathcal{H}_\rho^{\gamma^-}$$

for $\epsilon(a)$ small enough to have $\gamma(\rho - a) \geq \rho$. In the total expression

$$W(\gamma)f(x) = \int_{\Omega_{\gamma^-\setminus A_\gamma}} dy \left(W(\gamma,y)f\right)(x(i),\ i\in A_\gamma) \prod_{i\in L(\gamma)} r^\delta(x(i),y(i)) \quad (3.27)$$

the integral over $y(i),\ i \in L(\gamma)$ corresponds to the operator R^δ which improves the domain of analyticity ($R^\delta = P^\delta - Q^\delta \Rightarrow R^\delta : \mathcal{H}_\rho \to \mathcal{H}_{\min(\gamma\rho,\rho_0)} \quad \rho < \gamma\rho;\ \rho \leq \rho_0$). Therefore

$$W(\gamma): \ \mathcal{H}_\rho^{\gamma^-} \to \mathcal{H}_\rho^{\gamma^+}$$

■

In this formalism, the complete expression for $\mathcal{P}_\Lambda^\delta$ as a polymerexpansion reads

$$\left(\mathcal{P}_\Lambda^\delta \nu\right)(x) = \sum_\Gamma \left(\otimes_{\gamma\in\Gamma} W(\gamma) \otimes Q_{\Lambda\setminus\Gamma^+}^{\Lambda\setminus\Gamma^-}\right)\ \nu\ (x) \quad (3.28)$$

The sum runs over all sets Γ of disjoint polymers γ in Λ and

$$Q_B^A: \ \mathcal{H}_\rho^A \to \mathcal{H}_\rho^B$$

$$(Q_B^A f)(x) = \int_{\Omega_A} dy f(y) \left(\otimes_{i\in B} h^\delta\right)(x) \quad (3.29)$$

In a next step we will iterate this operation to get a clusterexpansion for $\left(\mathcal{P}_\Lambda^\delta\right)^n$. This is possible because $\mathcal{P}_\Lambda^\delta$ respects the domain of analyticity.

3.2 Polymer expansion

The decomposition (3.8) can be made at every timestep. We can thus write

$$\left(\mathcal{P}_\Lambda^\delta\right)^n \nu(x) = \int_{\Omega_\Lambda} dy_0 \ldots \int_{\Omega_\Lambda} dy_{n-1}\ \nu(y_0)$$
$$\prod_{t=1}^{n} \prod_{i\in\Lambda} \left[h^\delta(y_t(i)) + r^\delta(y_t(i), y_{t-1}(i)) + \beta_i(y_t(i), y_{t-1})\right] \tag{3.30}$$

where $y_n = x \in \Omega_\Lambda$. We now introduce polymers that are extended not only in space (Λ) but also in time. Just as before, a polymer is a connected collection of "triangles" and "lines". Connection is again by overlapping points and two polymers are disjoint if they have no (space-time) points in common. With "length" we will refer to the time direction while "width" stands for the spatial size. A disjoint collection of polymers in $\Lambda \times \{0, \ldots, n\}$ is denoted by Γ_n. For a polymer γ we define

$$t^- = \min\{k \in \mathbb{Z} :\ (i,k) \in \gamma \text{ for some } i \in \Lambda\}$$

$$t^+ = \max\{k \in \mathbb{Z} :\ (i,k) \in \gamma \text{ for some } i \in \Lambda\}$$

The definitions (3.11) and (3.12) are then generalized as

$$\gamma^- = \{i \in \Lambda :\ (i, t^-) \in \gamma\}$$

$$\gamma^+ = \{i \in \Lambda :\ (i, t^+) \in \gamma\}$$

while (3.9), (3.10) become

$$A(\gamma) = \{(i,n) \in \Lambda \times \mathbb{Z} :\ (i,n) \text{ is an apex in } \gamma\}$$