ON THE STATISTICAL INDEPENDENCE OF ALGEBRAS OF OBSERVABLES

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May 20, 1996

Abstract. We re-examine various notions of statistical independence presently in use in algebraic quantum theory, establishing alternative characterizations for such independence, some of which are also valid without assuming that the observable algebras mutually commute. In addition, in the context which holds in concrete applications to quantum theory, the equivalence of three major notions of statistical independence is proven.

Introduction

The notions of "independence of two systems" are legion in quantum theory. This is quite understandable, since the physical concept is central in many aspects of quantum theory, and the various formalizations of independence capture different qualitative and quantitative aspects for possibly different ends.[1] Those notions which have appeared in the literature and have formulations in algebraic quantum theory have been extensively reviewed in [24], where their logical interrelationships have been discussed. However, some issues of logical relation were left open in that review. One of the goals of this paper is to settle the few remaining conjectures in [24].

Representing the algebras of observables associated to the two subsystems by $\mathcal{A}$ and $\mathcal{B}$, respectively, one of the most commonly used expressions of independence is the requirement that the algebras mutually commute elementwise. Another is expressed heuristically in the condition that each system can be prepared independently of the other. It is known [10][18][5][16][6] that this latter notion of statistical independence is logically independent of the requirement that the algebras $\mathcal{A}$ and $\mathcal{B}$ commute. In [24] the various versions of statistical independence utilized in the literature were discussed almost exclusively in the context of commuting pairs of algebras. Another purpose of this paper is to provide further information about statistical independence in the more general circumstance that the observable algebras do not necessarily commute.

[1]See [8][12][20] for recent applications of these notions.

We shall provide alternative characterizations of C^*- and W^*-independence (see below for definitions), which are also valid if the algebras do not necessarily commute. In addition, we shall prove that in the category of commuting pairs of von Neumann algebras on separable Hilbert spaces, where all three notions are applicable, C^*-independence, strict locality and W^*-independence are equivalent. This is precisely the setting most often met in applications to quantum theory. Moreover, we shall furnish some new results about the notion of C^*-independence in the product sense, shall show that W^*-independence is strictly weaker than W^*-independence in the product sense, and shall close with some comments concerning the question of additional conditions sufficient to conclude the mutual commutativity of a pair of C^*-independent algebras.

Throughout this paper a few assumptions will be made tacitly, since they obtain in the applications to theoretical physics known to us. All algebras are assumed to have identities (represented by 1), and subalgebras considered here will always contain the identity of the original algebra. The symbol $\mathcal{B}(\mathcal{H})$ will be used to denote the algebra of all bounded operators on the Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is a subset of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}'$ represents the von Neumann algebra of all elements of $\mathcal{B}(\mathcal{H})$ which commute with every element of $\mathcal{A}$. If ϕ is a state on $\mathcal{C}$ and $\mathcal{A} \subset \mathcal{C}$ is a subalgebra, then the restriction of ϕ to $\mathcal{A}$ will be represented by $\phi \mid \mathcal{A}$. In addition, if $\mathcal{A}$ and $\mathcal{B}$ are two C^*- , resp. W^*-, subalgebras, then $\mathcal{A} \vee \mathcal{B}$ will denote the C^*- , resp. W^*-, algebra they generate.

Statistical Independence

In quantum theory, observables (or effects) are represented by selfadjoint operators and preparations by states on the *-algebra generated by the observables. Basic observables are represented in algebraic quantum theory by positive, selfadjoint elements of a C^*-algebra with norm bounded by 1, of which projection operators (or decision effects) are only a special case. More general observables are assembled from such basic observables, usually in the form of measures taking such operators as values, of which the projection-valued measures found in the spectral theorem for normal operators are a special case (see, *e.g.* [4][2]). The probability of measuring "yes" for the basic observable F after the preparation represented by the state ϕ is $\phi(F)$, while the probability for measuring "no" under the same conditions is $\phi(1 - F)$.

In [11] Haag and Kastler introduced a notion they called statistical independence. If $\mathcal{A}$ and $\mathcal{B}$ represent the algebras generated by the observables associated with two quantum subsystems, the statistical independence of $\mathcal{A}$ and $\mathcal{B}$ can be loosely construed as follows: *any* two partial states on the two subsystems can be realized by the *same* preparation procedure (assuming that arbitrary states on the algebra of observables for the entire quantum system can actually be prepared); or alternatively, no choice of a state prepared on one subsystem can prevent the preparation of any state on the other subsystem. There are different mathematical formulations of this notion. We begin with the most natural version in the category of C^*-algebras. At many points in the sequel we shall utilize without further comment a basic fact about C^*-algebras: If $a \in \mathbb{C}$ is any spectral value of a selfadjoint element A of a C^*-algebra $\mathcal{A}$, there exists a state φ on $\mathcal{A}$ such that $\varphi(A) = a$; hence for any normal element $A \in \mathcal{A}$ there exists a state φ on $\mathcal{A}$ such that $|\varphi(A)| = \|A\|$.

Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of a C^*-algebra $\mathcal{C}$. The pair $(\mathcal{A}, \mathcal{B})$ is (or $\mathcal{A}$ and $\mathcal{B}$ are) said to be C^*-independent if for every state ϕ_1 on $\mathcal{A}$ and every state ϕ_2 on $\mathcal{B}$ there exists a state ϕ on $\mathcal{C}$ such that $\phi \mid \mathcal{A} = \phi_1$ and $\phi \mid \mathcal{B} = \phi_2$.*

The following characterization of C^*-independent pairs of *commuting* algebras was proven by Roos [21]. The algebraic tensor product of two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is represented by $\mathcal{A} \odot \mathcal{B}$, while the minimal C^*-tensor product is denoted by $\mathcal{A} \otimes \mathcal{B}$.

Theorem 1. [21] *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting subalgebras of the C^*-algebra $\mathcal{C}$. Then the following are equivalent.*

(i) $\mathcal{A}$ and $\mathcal{B}$ are C^*-independent.

(ii) $0 \neq A \in \mathcal{A}$ and $0 \neq B \in \mathcal{B}$ imply that $AB \neq 0$.

(iii) The map $\eta : \mathcal{AB} \to \mathcal{A} \odot \mathcal{B}$ defined by $\eta(AB) = A \otimes B$, $A \in \mathcal{A}$, $B \in \mathcal{B}$ is an isomorphism continuous in the minimal C^*-cross norm on $\mathcal{A} \odot \mathcal{B}$ and can therefore be continuously extended to a surjective homomorphism $\overline{\eta} : \mathcal{A} \vee \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$.

(iv) For every state ϕ_1 on $\mathcal{A}$ and every state ϕ_2 on $\mathcal{B}$ there exists a state ϕ on $\mathcal{C}$ such that $\phi(AB) = \phi_1(A)\, \phi_2(B) = \phi(A)\phi(B)$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$.

Condition (ii) in Theorem 1 is called the Schlieder property by algebraic quantum field theorists. We recall that Murray and von Neumann [19] showed that if $\mathcal{A}$ is a von Neumann factor on a Hilbert space $\mathcal{H}$, then the pair $(\mathcal{A}, \mathcal{A}')$ satisfies the Schlieder condition and is therefore C^*-independent. States ϕ on $\mathcal{C}$ with the property that $\phi(AB) = \phi(A)\, \phi(B)$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ are called product states. Roos' theorem therefore establishes that for C^*-independent pairs $(\mathcal{A}, \mathcal{B})$, arbitrary states on $\mathcal{A}$ and $\mathcal{B}$ can be simultaneously extended to a *product* state. This suggests an alternative characterization of C^*-independent pairs of commuting algebras.

Proposition 2. *If $\mathcal{A}$ and $\mathcal{B}$ are commuting subalgebras of a C^*-algebra $\mathcal{C}$, then the following are equivalent.*

(i) $(\mathcal{A}, \mathcal{B})$ is C^*-independent.

(ii) $\|AB\| = \|A\|\|B\|$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proof. $(i) \Rightarrow (ii)$: By Roos [21], if $(\mathcal{A}, \mathcal{B})$ is C^*-independent, then $\mathcal{A}$ and $\mathcal{B}$ are algebraically independent, so by Turumaru [27], the *-algebra generated by $\mathcal{A}$ and $\mathcal{B}$ is isomorphic to $\mathcal{A} \odot \mathcal{B}$. Hence the norm on $\mathcal{C}$ induces a C^*-norm α on $\mathcal{A} \odot \mathcal{B}$ by $\alpha(\sum_{i=1}^{n} A_i \otimes B_i) = \| \sum_{i=1}^{n} A_i B_i \|$. By Corollary 11.3.10 in [14], every C^*-norm on $\mathcal{A} \odot \mathcal{B}$ is a cross norm, i.e. $\alpha(A \otimes B) = \|A\|\|B\|$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Hence, assertion (ii) follows.

$(ii) \Rightarrow (i)$: This condition trivially implies the Schlieder property and thus the C^*-independence of $(\mathcal{A}, \mathcal{B})$. $\square$

However, the notion of C^*-independence as defined above is not restricted to commuting pairs of C^*-algebras, and it is of interest to master this notion also in the noncommuting case. Note that if $\mathcal{A}$ and $\mathcal{B}$ do not commute, we can appeal to none of the above-cited results. Nonetheless, one can still prove that conditions (i) and (ii) in Proposition 2 are equivalent. With $A \in \mathcal{A}$, $\sigma(A)$ will denote the spectrum of A in $\mathcal{A}$.

Proposition 3. *If $\mathcal{A}$ and $\mathcal{B}$ are (not necessarily commuting) subalgebras of a C^*-algebra $\mathcal{C}$, then the following are equivalent.*
(i) $(\mathcal{A}, \mathcal{B})$ is C^*-independent.
(ii) $\|AB\| = \|A\|\|B\|$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proof. $(i) \Rightarrow (ii)$: Assume that $(\mathcal{A}, \mathcal{B})$ is C^*-independent, and let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The hypothesis of C^*-independence implies the existence of a state φ on $\mathcal{A} \vee \mathcal{B}$ such that $\varphi(|A|) = \|A\|$ and $\varphi(|B|) = \|B\|$. Then one observes that the Cauchy-Schwarz inequality implies

$$\|A\|^2 = \varphi(|A|)^2 \leq \varphi(1)\varphi(|A|^2) \leq \||A|^2\| = \|A\|^2 \,.$$

In other words, one has the equality $\varphi(|A|^2) = \|A\|^2$. This entails that for any $C \in \mathcal{A} \vee \mathcal{B}$,

$$\begin{aligned} |\varphi((|A| - \|A\| \cdot 1)C)|^2 &\leq \varphi((|A| - \|A\| \cdot 1)^2)\varphi(C^*C) \\ &= \{\varphi(|A|^2) - 2\|A\|\varphi(|A|) + \|A\|^2\}\varphi(C^*C) = 0 \,. \end{aligned}$$

Hence one has $\varphi(|A|C) = \|A\|\varphi(C)$ for all $C \in \mathcal{A} \vee \mathcal{B}$. Thus, it follows that $\varphi(|A||B|) = \|A\|\varphi(|B|) = \|A\|\|B\|$. One therefore can conclude that

$$\begin{aligned} \|AB\| &= \|B^*A^*AB\|^{\frac{1}{2}} = \|B^*|A|^2B\|^{\frac{1}{2}} = \|\,|A|B\| \\ &= \|\,|A|B(|A|B)^*\|^{\frac{1}{2}} = \|\,|A||B^*|^2|A|\,\|^{\frac{1}{2}} \\ &= \|\,|A||B^*|\,\| = \|A\|\|B^*\| \\ &= \|A\|\|B\| \,. \end{aligned}$$

$(ii) \Rightarrow (i)$: Assume that $\|AB\| = \|A\|\|B\|$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $A = A^* \in \mathcal{A}$ and $B = B^* \in \mathcal{B}$ be arbitrary. The first step is to show that for any $\lambda \in [\min \sigma(A), \max \sigma(A)]$ and any $\mu \in [\min \sigma(B), \max \sigma(B)]$, there exists a state φ on $\mathcal{C}$ such that $\varphi(A) = \lambda$ and $\varphi(B) = \mu$. Assume without loss of generality (by adding suitable multiples of the identity to A and B) that $A \geq 0$, $0 \in \sigma(A)$, $B \geq 0$ and $0 \in \sigma(B)$. By taking suitable convex combinations of states, it will also suffice to take $\lambda = \|A\|$ and $\mu = \|B\|$.

There exists a state φ on $\mathcal{C}$ with

$$\varphi((A^{\frac{1}{2}}B^{\frac{1}{2}})^*A^{\frac{1}{2}}B^{\frac{1}{2}}) = \|A^{\frac{1}{2}}B^{\frac{1}{2}}\|^2 = (\|A^{\frac{1}{2}}\|\|B^{\frac{1}{2}}\|)^2 = \|A\|\|B\| \,,$$

where the second equality uses the hypothesis and the third standard properties of C^*-norms. Squaring both sides of this equality, the Cauchy-Schwarz inequality then entails that one has

$$\begin{aligned}\|A\|^2\|B\|^2 = \varphi((A^{\frac{1}{2}}B^{\frac{1}{2}})^*A^{\frac{1}{2}}B^{\frac{1}{2}})^2 &= \varphi((B^{\frac{1}{2}}AB^{\frac{1}{4}})B^{\frac{1}{4}})^2 \\ &\leq \varphi(B^{\frac{1}{2}}AB^{\frac{1}{2}}AB^{\frac{1}{2}})\varphi(B^{\frac{1}{2}}) \\ &\leq \|A\|^2\|B\|^{\frac{3}{2}}\varphi(B^{\frac{1}{2}}) \\ &\leq \|A\|^2\|B\|^2 .\end{aligned}$$

But this entails $\varphi(B^{\frac{1}{2}}) = \|B^{\frac{1}{2}}\|$, and squaring the above inequalities also yields $\varphi(B) = \|B\|$, since $\varphi(1 \cdot B^{\frac{1}{2}})^2 \leq \varphi(B)$. Moreover,

$$\begin{aligned}|\varphi((\|B\|^{\frac{1}{2}} \cdot 1 - B^{\frac{1}{2}})C)|^2 &\leq \varphi((\|B\|^{\frac{1}{2}} \cdot 1 - B^{\frac{1}{2}})^2)\varphi(C^*C) \\ &= \varphi(\|B\| \cdot 1 - 2\|B\|^{\frac{1}{2}}B^{\frac{1}{2}} + B)\varphi(C^*C) \\ &= 0 ,\end{aligned}$$

for arbitrary $C \in \mathcal{C}$, which entails $\varphi(B^{\frac{1}{2}}C) = \|B\|^{\frac{1}{2}}\varphi(C)$. One therefore observes that

$$\varphi(B^{\frac{1}{2}}AB^{\frac{1}{2}}) = \|B\|\varphi(A) \leq \|A\|\|B\| ,$$

so that $\varphi(A) = \|A\|$ and $\varphi(B) = \|B\|$.

As the next step, consider, for fixed $B = B^* \in \mathcal{B}$ and $\mu \in [\min\sigma(B), \max\sigma(B)]$, the set of states ($\mathcal{S}(\mathcal{A})$ represents the set of all states on the algebra $\mathcal{A}$)

$$\mathcal{V} \equiv \{\psi \in \mathcal{S}(\mathcal{A}) \mid \psi = \varphi \mid \mathcal{A}\} ,$$

for any $\varphi \in \mathcal{S}(\mathcal{C})$ satisfying $\varphi(B) = \mu$. Due to the weak*-compactness of $\mathcal{S}(\mathcal{C})$, every net of states φ_ν on $\mathcal{C}$ whose restrictions to $\mathcal{A}$ are Cauchy in the weak*-topology and satisfy $\varphi_\nu(B) = \mu$ has a convergent subnet, the limit of which when restricted to $\mathcal{A}$ is an element of $\mathcal{V}$. Hence, $\mathcal{V}$ is weak*-closed and convex. If there exists an element $\xi \in \mathcal{S}(\mathcal{A})$ which is not contained in $\mathcal{V}$, then, by the Hahn-Banach Theorem, there exists a selfadjoint element $A \in \mathcal{A}$ such that $\xi(A) \neq \psi(A)$ for all $\psi \in \mathcal{V}$. But this contradicts the first step of this proof, since, necessarily, $\xi(A) \in [\min\sigma(A), \max\sigma(A)]$. Hence, one must have $\mathcal{V} = \mathcal{S}(\mathcal{A})$. Repeating similar arguments for B, the stated conclusion follows easily. □

If we have subalgebras of a W^*-algebra, which are replete with projections, indeed are generated by their projections, then we can establish the following equivalences. We denote the set of all projections of an algebra $\mathcal{A}$ by $\mathrm{Proj}(\mathcal{A})$.

Proposition 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be (not necessarily commuting) subalgebras of a W^*-algebra $\mathcal{C}$. The following are equivalent.*

(i) $(\mathcal{A}, \mathcal{B})$ is C^*-independent.

(ii) For any nontrivial projections $P \in \mathcal{A}$, $Q \in \mathcal{B}$, and $\lambda, \mu \in [0, 1]$, there exists a state φ on $\mathcal{C}$ such that $\varphi(P) = \lambda$ and $\varphi(Q) = \mu$.

(iii) For any nonzero projection $P \in \mathcal{A}$ and every state φ on $\mathcal{B}$, there exists a state ϕ on $\mathcal{C}$ such that $\phi(P) = 1$ and $\phi \mid \mathcal{B} = \varphi$.

Proof. The implication $(i) \Rightarrow (iii)$ is trivial. To see the validity of the implication $(iii) \Rightarrow (ii)$, simply choose states ϕ_0, ϕ_1 on $\mathcal{C}$ such that $\phi_0(1-P) = 1 = \phi_1(P)$ and $\phi_0(Q) = \mu = \phi_1(Q)$. Then set $\varphi = (1-\lambda)\phi_0 + \lambda\phi_1$.

Hence, proof of the implication $(ii) \Rightarrow (i)$ will suffice to verify Prop. 4. Note that choosing $\lambda = \mu = 1$, the resulting equalities $\varphi(P) = 1 = \varphi(Q)$ entail

$$\begin{aligned}\|PQ\| \geq |\varphi(PQ)| &= |\varphi(Q) - \varphi((1-P)Q)| \\ &\geq 1 - \varphi(1-P)^{\frac{1}{2}}\varphi(Q)^{\frac{1}{2}} = 1\,,\end{aligned}$$

and therefore one has $\|PQ\| = 1$ for any choice of nontrivial projections $P \in \mathrm{Proj}(\mathcal{A})$ and $Q \in \mathrm{Proj}(\mathcal{B})$.

Let $A \neq 0 \neq B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By considering the spectral projections of $|A|$ and $|B^*|$, one can conclude that for any sufficiently small $\epsilon > 0$ there exist nonzero projections $R \in \mathrm{Proj}(\mathcal{A})$ and $S \in \mathrm{Proj}(\mathcal{B})$ such that

$$R \leq (\|A\|^2 - \epsilon)^{-1} A^* A \quad \text{and} \quad S \leq (\|B\|^2 - \epsilon)^{-1} B B^*\,.$$

Then one has

$$\begin{aligned}1 &= \|RS\| = \|S^* R^* R S\|^{\frac{1}{2}} = \|SRS\|^{\frac{1}{2}} \leq (\|A\|^2 - \epsilon)^{-\frac{1}{2}} \|S A^* A S\|^{\frac{1}{2}} \\ &= (\|A\|^2 - \epsilon)^{-\frac{1}{2}} \|AS\| = (\|A\|^2 - \epsilon)^{-\frac{1}{2}} \|A S A^*\|^{\frac{1}{2}} \\ &\leq (\|A\|^2 - \epsilon)^{-\frac{1}{2}} (\|B\|^2 - \epsilon)^{-\frac{1}{2}} \|A B B^* A^*\|^{\frac{1}{2}} \\ &= (\|A\|^2 - \epsilon)^{-\frac{1}{2}} (\|B\|^2 - \epsilon)^{-\frac{1}{2}} \|AB\|\,.\end{aligned}$$

But this entails $\|AB\| = \|A\|\|B\|$, and appeal to Prop. 3 completes the proof. □

Of course, this argument also is valid for those C^*-algebras $\mathcal{C}$ which are not W^*-algebras but which are generated by their projections - each selfadjoint element of $\mathcal{C}$ has a spectral resolution in $\mathcal{C}$ (and there do exist such algebras - see [7][13]). A natural formulation of statistical independence in the category of W^*-algebras is discussed next.

Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of a W^*-algebra $\mathcal{C}$. The pair $(\mathcal{A}, \mathcal{B})$ is (or $\mathcal{A}$ and $\mathcal{B}$ are) said to be W^*-independent if for every normal state ϕ_1 on $\mathcal{A}$ and every normal state ϕ_2 on $\mathcal{B}$ there exists a normal state ϕ on $\mathcal{C}$ such that $\phi \mid \mathcal{A} = \phi_1$ and $\phi \mid \mathcal{B} = \phi_2$.*

There are some related and strictly stronger conditions in use in the literature - see [24] for an overview. In particular, in the light of Roos' product state characterization of C^*-independence of commuting subalgebras, it would be natural to ask if the same is true of W^*-independence. This is, however, not the case. In fact, if the normal extension ϕ in the above definition is a product state over the commuting pair $(\mathcal{A}, \mathcal{B})$, then[2] $\mathcal{A} \vee \mathcal{B}$ is isomorphic to the (unique) W^*-product $\mathcal{A}\overline{\otimes}\mathcal{B}$ [25][3]. This property was called W^*-independence in the product sense in [24]. We shall see below that W^*-independence of commuting subalgebras is strictly weaker than W^*-independence in the product sense.

And of course the notion of C^*-independence is also applicable to this setting of a pair of subalgebras of a W^*-algebra. It was established in Corollary 3.5 of [24] that the W^*-independence of commuting subalgebras of a W^*-algebra $\mathcal{C}$ implies that they are also C^*-independent. (It shall be shown below that this is also true if the subalgebras do not mutually commute.) Here we wish to show that these properties are actually equivalent if $\mathcal{C}$ can be realized as a von Neumann algebra on a separable Hilbert space, in other words, if $\mathcal{C}$ is σ-finite.[3] We prepare the way with the following technical remark, which we shall use more than once in the sequel.

Lemma 5. *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and let $\Omega \in \mathcal{H}$ be an arbitrary nonzero vector. Let ω be the functional defined on $\mathcal{A}$ by $\omega(A) = (\Omega, A\Omega)$ for all $A \in \mathcal{A}$. Then for any linear functional φ on $\mathcal{A}$ such that $0 \le \varphi \le c\omega$, for some positive real c, there exists a positive element $A' \in \mathcal{A}'$ such that $\varphi(A) = \omega(A'A) = (A'^{\frac{1}{2}}\Omega, AA'^{\frac{1}{2}}\Omega)$ for all $A \in \mathcal{A}$. Moreover, for any positive element $A' \in \mathcal{A}'$, the linear functional defined by $\varphi(A) \equiv \omega(A'A) = (A'^{\frac{1}{2}}\Omega, AA'^{\frac{1}{2}}\Omega)$ for all $A \in \mathcal{A}$ is positive and satisfies $\varphi \le \|A'\|\omega$.*

Proof. Let φ be a linear functional on $\mathcal{A}$ such that $0 \le \varphi \le c\omega$, for some positive real c, and let $P' \in \mathcal{A}'$ be the projection onto the closure of $\mathcal{A}\Omega$. Moreover, let $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ be the GNS-representation of $(\mathcal{A}, \omega)$. According to Theorem 29.2 [9], there then exists a positive $T \in \pi_\omega(\mathcal{A})' = (\mathcal{A}_{P'})' = (\mathcal{A}')_{P'}$ such that $\varphi(A) = \omega(TA)$ for all $A \in \mathcal{A}$. This entails that $T = P'A_1'P' \mid_{P'\mathcal{H}}$ for some positive $A_1' \in \mathcal{A}'$, so, in fact, one has $\varphi(A) = \omega(P'A_1{}'P'A)$ for all $A \in \mathcal{A}$. The desired element is therefore $A' = P'A_1'P'$. The second assertion is demonstrated in the proof of Theorem 29.2 in [9]. □

The next step is an alternative characterization of W^*-independent subalgebras of a W^*-algebra.

Lemma 6. *Let $\mathcal{A}$ and $\mathcal{B}$ be (not necessarily commuting) subalgebras of a W^*-algebra $\mathcal{C}$ acting on a Hilbert space $\mathcal{H}$. The following are equivalent.*

(i) $(\mathcal{A}, \mathcal{B})$ is W^*-independent.

(ii) For any nonzero vectors $\Phi, \Psi \in \mathcal{H}$, there exist $A' \in \mathcal{A}'$ and $B' \in \mathcal{B}'$ such that $A'\Phi = B'\Psi \neq 0$.

Proof. *(i)* ⇒ *(ii)*: Without loss of generality one may assume $\|\Phi\| = \|\Psi\| = 1$, and let ϕ and ψ be the corresponding normal states on $\mathcal{A}$ and $\mathcal{B}$, respectively, *i.e.* $\phi(A) = (\Phi, A\Phi)$ and $\psi(B) = (\Psi, B\Psi)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By assumption, there exists a normal state ω on $\mathcal{C}$ such that $\omega \mid \mathcal{A} = \phi$ and $\omega \mid \mathcal{B} = \psi$. Choose a suitable nonzero $\Xi \in \mathcal{H}$ such that the corresponding positive linear functional ξ on $\mathcal{C}$, defined by $\xi(C) = (\Xi, C\Xi)$ for all $C \in \mathcal{C}$, satisfies the inequality $\xi \le \omega$.

[2] if ϕ has central support 1 in $\mathcal{A} \vee \mathcal{B}$

[3] The term countably decomposable is used synonymously in the literature.

By Lemma 5 there exist positive elements $A'_1 \in \mathcal{A}'$ and $B'_1 \in \mathcal{B}'$ such that $\xi(A) = \phi(A'_1 A) = (A'^{\frac{1}{2}}_1 \Phi, A A'^{\frac{1}{2}}_1 \Phi)$ and $\xi(B) = \psi(B'_1 B) = (B'^{\frac{1}{2}}_1 \Psi, B B'^{\frac{1}{2}}_1 \Psi)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $P, Q \in \mathcal{A}'$ be the projections onto the closures of $\mathcal{A} A'^{\frac{1}{2}}_1 \Phi$ and $\mathcal{A}\Xi$, respectively. Since the GNS representation induced by a state is unique up to unitary equivalence, there exists a partial isometry $U = QUP \in \mathcal{B}(\mathcal{H})$ such that $U A'^{\frac{1}{2}}_1 \Phi = \Xi$, $U^* U = P$, $UU^* = Q$ and

$$A\tilde{A}\Xi = UA\tilde{A}A_1^{\frac{1}{2}}\Phi = UAP\tilde{A}A_1^{\frac{1}{2}}\Phi = UAU^*U\tilde{A}A_1^{\frac{1}{2}}\Phi = (UAU^*)\tilde{A}\Xi \, ,$$

for all $A, \tilde{A} \in \mathcal{A}$. One concludes $U^* A = AU^*$, in other words, one has $U \in \mathcal{A}'$. Similarly, one obtains the existence of a partial isometry $V \in \mathcal{B}'$ such that $V B'^{\frac{1}{2}}_1 \Psi = \Xi = U A'^{\frac{1}{2}}_1 \Phi$. So $A' \equiv U A'^{\frac{1}{2}}_1$ and $B' \equiv V B'^{\frac{1}{2}}_1$ are the desired elements.

$(ii) \Rightarrow (i)$: Let ϕ and ψ be normal states on $\mathcal{A}$ and $\mathcal{B}$, respectively. Consider the set

$$\mathcal{U} \equiv \{\{\varphi_\alpha\} \mid \sum_\alpha \varphi_\alpha \in \mathcal{C}_*, \varphi_\alpha \geq 0, (\sum_\alpha \varphi_\alpha \mid \mathcal{A}) \leq \phi, (\sum_\alpha \varphi_\alpha \mid \mathcal{B}) \leq \psi\} \, ,$$

which can be partially ordered by set inclusion. It will be shown that an arbitrary linearly ordered subset $\{\mathcal{G}_\beta = \{\varphi_\alpha \mid \alpha \in \mathcal{F}_\beta\}\}$ of $\mathcal{U}$ has an upper bound in $\mathcal{U}$. (Note that for arbitrary indices β_1 and β_2 entering into this linearly ordered subset, one must have either $\mathcal{F}_{\beta_1} \subset \mathcal{F}_{\beta_2}$ or $\mathcal{F}_{\beta_2} \subset \mathcal{F}_{\beta_1}$.) For any $n \in \mathbb{N}$ there exists only a finite number of states $\varphi_\alpha \in \cup_\beta \mathcal{G}_\beta$ satisfying $\varphi_\alpha(1) > \frac{1}{n}$, since otherwise there would exist a β_0 with $\sum_{\alpha \in \mathcal{F}_{\beta_0}} \varphi_\alpha(1) > 1 = \phi(1)$. Hence, the set $\{\varphi_\alpha \mid \alpha \in \cup_\beta \mathcal{F}_\beta\}$ is countable. But $\| \sum_{\alpha=n}^m \varphi_\alpha \| = \sum_{\alpha=n}^m \varphi_\alpha(1)$ and $\sum_{\alpha \leq n} \varphi_\alpha(1) \leq 1$ then imply the norm convergence of $\sum_\alpha \varphi_\alpha$, and also the normality of the limit, since $\mathcal{C}_*$ is norm-closed. Therefore, $\{\varphi_\alpha \mid \alpha \in \cup_\beta \mathcal{F}_\beta\}$ is an element of $\mathcal{U}$ and an upper bound for $\{\mathcal{G}_\beta = \{\varphi_\alpha \mid \alpha \in \mathcal{F}_\beta\}\}$ in $\mathcal{U}$. One may now employ Zorn's Lemma to conclude the existence of a maximal element $\{\chi_\alpha\}$ in $\mathcal{U}$. Let $\chi \equiv \sum_\alpha \chi_\alpha$. By construction, χ is a normal positive linear functional on $\mathcal{C}$ whose restriction to $\mathcal{A}$, resp. $\mathcal{B}$, is bounded by ϕ, resp. ψ.

Assume that $\chi \mid \mathcal{A} < \phi$ and $\chi \mid \mathcal{B} < \psi$. The normality of $\phi - \chi > 0$, resp. $\psi - \chi > 0$, entails the existence of a nonzero vector $\Xi_1 \in \mathcal{H}$, resp. $\Xi_2 \in \mathcal{H}$, such that the corresponding positive linear functional $\omega_\mathcal{A}$, resp. $\omega_\mathcal{B}$, defined on $\mathcal{A}$, resp. $\mathcal{B}$, by $\omega_\mathcal{A}(A) = (\Xi_1, A\Xi_1)$ for all $A \in \mathcal{A}$, resp. $\omega_\mathcal{B}(B) = (\Xi_2, B\Xi_2)$ for all $B \in \mathcal{B}$, satisfies the inequality $\omega_\mathcal{A} + \chi \mid \mathcal{A} \leq \phi$, resp. $\omega_\mathcal{B} + \chi \mid \mathcal{B} \leq \psi$. By assumption, there exist $A' \in \mathcal{A}'$ and $B' \in \mathcal{B}'$ such that $0 \neq A'\Xi_1 = B'\Xi_2$. Define a positive linear functional ω on $\mathcal{C}$ by $\omega(C) = (A'\Xi_1, CA'\Xi_1) = (B'\Xi_2, CB'\Xi_2)$ for all $C \in \mathcal{C}$. Then $\omega(A) = (A'^* A'\Xi_1, A\Xi_1)$, for all $A \in \mathcal{A}$. From Lemma 5 one sees that this entails the inequality $\omega \mid \mathcal{A} \leq \|A'^* A'\| \omega_\mathcal{A}$. Similarly, one proves that $\omega \mid \mathcal{B} \leq \|B'^* B'\| \omega_\mathcal{B}$. Let μ be the larger of the numbers $\|A'^* A'\|$, $\|B'^* B'\|$. Then $\{\chi_\alpha\} \cup \{c\omega\}$ is an element of $\mathcal{U}$ for any $0 < c \leq \frac{1}{\mu}$. Since $\{\chi_\alpha\}$ is countable, the maximality of $\{\chi_\alpha\}$ is contradicted for some choice of c as indicated (note ω is nonzero).

Thus one can conclude that either $\chi \mid \mathcal{A} = \phi$ or $\chi \mid \mathcal{B} = \psi$. Assume without loss of generality that the former obtains. Then $\psi \geq \chi \mid \mathcal{B}$ and $\psi(1) = 1 = \phi(1)$ imply the equalities

$$\|\psi - (\chi \mid \mathcal{B})\| = \psi(1) - (\chi \mid \mathcal{B})(1) = \phi(1) - (\chi \mid \mathcal{A})(1) = 0\,.$$

□

We need one more technical lemma.

Lemma 7. *Let $\mathcal{A}$ be a von Neumann algebra in standard form on the Hilbert space $\mathcal{H}$ (in other words, $\mathcal{A}$ has a cyclic and separating vector $\Omega \in \mathcal{H}$). Then for every nonzero vector $\Phi \in \mathcal{H}$ there exist elements $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$ such that $A'\Phi = A\Omega \neq 0$.*

Proof. Consider the linear form l on $\mathcal{A}'$ defined by $l(A') = (\Omega, A'\Phi)$ for $A' \in \mathcal{A}'$. Since Ω is cyclic with respect to $\mathcal{A}'$, the linear functional l is nonzero. From the polar decomposition of normal functionals (see, *e.g.* Theorem III.4.2 of [26]) one may conclude the existence of a partial isometry $U' \in \mathcal{A}'$ such that the functional on $\mathcal{A}'$ given by $(\Omega, A'U'\Phi) = (A'^*\Omega, U'\Phi)$ is positive on $\mathcal{A}'$ and nontrivial. By Prop. 2.5.27 (1) of [1] there exists a positive, selfadjoint operator Q affiliated with $\mathcal{A}$ such that $U'\Phi = Q\Omega$.

So consider the positive linear functional on $\mathcal{A}$ defined by this vector: $(U'\Phi, AU'\Phi) = (Q\Omega, AQ\Omega)$ for all $A \in \mathcal{A}$. By Theorem 2.5.31 in [1], this normal positive linear functional is implemented on $\mathcal{A}$ by a (unique) vector Ξ in the natural positive cone associated to $(\mathcal{A}, \Omega)$, *i.e.* $(U'\Phi, AU'\Phi) = (Q\Omega, AQ\Omega) = (\Xi, A\Xi)$ for all $A \in \mathcal{A}$. As in the proof of Lemma 6, since the corresponding GNS-representation is unique up to unitary equivalence, there exists a partial isometry $V' \in \mathcal{A}'$ such that $V'U'\Phi = V'Q\Omega = \Xi$ is contained in the natural positive cone associated to $(\mathcal{A}, \Omega)$ and such that $V'^*V'Q\Omega = Q\Omega$. Then since the modular conjugation J associated with $(\mathcal{A}, \Omega)$ leaves the natural positive cone pointwise invariant, one has $V'U'\Phi = JV'U'\Phi = JV'Q\Omega = (JV'J)(JQJ)\Omega$.

By Lemma 2.5.8 in [1] one can choose a suitable spectral projection $P' \in \mathcal{A}'$ of the selfadjoint JQJ (which is affiliated with $\mathcal{A}'$) such that $P'(JQJ) \in \mathcal{A}'$ and $P'(JQJ)\Omega \neq 0$. Hence one sees that $P'V'U'\Phi = P'(JV'J)(JQJ)\Omega = (JV'J)(P'(JQJ))\Omega = JV'(JP'J)Q\Omega \neq 0$, utilizing the facts that $JV'J \in \mathcal{A}$ and $V' : \overline{\mathcal{A}Q\Omega} \mapsto \overline{\mathcal{A}\Xi}$ is bijective. Using this vector to induce a normal positive linear functional on $\mathcal{A}$, the same argument again implies the existence of a partial isometry $W' \in \mathcal{A}'$ such that $0 \neq W'P'V'U'\Phi$ is contained in the natural positive cone. Hence one finds

$$0 \neq W'P'V'U'\Phi = JW'(P'V'U'\Phi) = JW'(JV'J)P'(JQJ)\Omega = V'(JW'J)(JP'J)Q\Omega\,,$$

since $JW'J \in \mathcal{A}$. Noting that $(JW'J)(JP'J)Q\Omega \in \mathcal{A}Q\Omega$, one therefore has

$$\begin{aligned} 0 \neq V'^*W'P'V'U'\Phi &= V'^*V'(JW'J)(JP'J)Q\Omega \\ &= (JW'J)(JP'J)V'^*V'Q\Omega = (JW'J)(JP'J)Q\Omega\,. \end{aligned}$$

The claim then follows after noting that $A' \equiv V'^*W'P'V'U' \in \mathcal{A}'$ and $A \equiv (JW'J)(JP'J)Q = (JW'J)J(P'JQJ)J \in \mathcal{A}$.

□

We may now prove that C^*- and W^*-independence are equivalent in a context which arises quite commonly in quantum physics. In particular, the hypothesis of the next proposition obtains when $\mathcal{A}$ and $\mathcal{B}$ are commuting von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$.

Proposition 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting subalgebras of a σ-finite W^*-algebra $\mathcal{C}$. The following are equivalent.*

(i) $(\mathcal{A}, \mathcal{B})$ is C^*-independent.

(ii) $(\mathcal{A}, \mathcal{B})$ is W^*-independent.

Proof. The implication $(ii) \Rightarrow (i)$ was proven in Corollary 3.5 in [24]. So let $(\mathcal{A}, \mathcal{B})$ be C^*-independent. $\mathcal{C}$ can be realized as a von Neumann algebra in standard form on a Hilbert space $\mathcal{H}$ (see, *e.g.* Prop. 2.5.6 in [1]). Let $\Omega \in \mathcal{H}$ be the corresponding cyclic and separating vector. In addition, let $P' \in \mathcal{A}'$ be the projection onto the closure of $\mathcal{A}\Omega$. Since Ω is separating for $\mathcal{A}$, the von Neumann algebra $\mathcal{A}_{P'}$ is isomorphic to $\mathcal{A}$. Hence, one may consider normal states on $\mathcal{A}_{P'}$ without loss of generality. Similarly, one may consider normal states on $\mathcal{B}_{Q'}$, where $Q' \in \mathcal{B}'$ is the projection onto the closure of $\mathcal{B}\Omega$. Note that $\mathcal{A}_{P'}$, resp. $\mathcal{B}_{Q'}$, has Ω as a cyclic and separating vector in the Hilbert space $P'\mathcal{H}$, resp. $Q'\mathcal{H}$. Since every normal positive linear functional on a von Neumann algebra in standard form can be realized as the vector functional induced by a vector in the natural positive cone of the algebra (see *e.g.* Theorem 2.5.31 in [1]), it is no loss of generality to take the nonzero vector Φ, resp. Ψ, in Lemma 6 to lie in $P'\mathcal{H}$, resp. $Q'\mathcal{H}$.

In the one case, Lemma 7 entails the existence of $A \in \mathcal{A}$ and $A' = P'A'P' \in \mathcal{A}'$ with $0 \neq A\Omega = P'AP'\Omega = P'A'P'\Phi = A'\Phi$, with similar results in the other case. So consider the vectors $0 \neq A\Omega = A'\Phi$ in the one case and $0 \neq B\Omega = B'\Psi$, for some $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$, in the other. Since $A \neq 0 \neq B$, the hypothesis and the equivalence of C^*-independence and the Schlieder property entail that $AB \neq 0$. But then, since $\overline{\mathcal{C}'\Omega} = \mathcal{H}$, there exists an element $C' \in \mathcal{C}'$ such that $ABC'\Omega \neq 0$. Then $BC'A'\Phi = BC'A\Omega = ABC'\Omega = AC'B\Omega = AC'B'\Psi \neq 0$ and $BC'A' \in \mathcal{A}'$ with $AC'B' \in \mathcal{B}'$ complete the proof, by Lemma 6. □

A further notion of statistical independence suitable for the category of W^*-algebras is called strict locality and was introduced in [15] on physical grounds (see also [17] for an earlier version).

Definition. *Let $(\mathcal{A}, \mathcal{B})$ be an ordered pair of subalgebras of the W^*-algebra $\mathcal{C}$. $(\mathcal{A}, \mathcal{B})$ is said to be strictly local if for any nonzero projection $P \in \mathcal{A}$ and any state $\varphi \in \mathcal{B}_*$ there exists a state $\phi \in (\mathcal{A} \vee \mathcal{B})_*$ such that $\phi(P) = 1$ and $\phi \mid \mathcal{B} = \varphi$.*

Interpreting $P \in \mathrm{Proj}(\mathcal{A})$ as a decision effect, we see that if the pair $(\mathcal{A}, \mathcal{B})$ is strictly local, no preparation on subsystem $\mathcal{B}$ can exclude the occurrence of any probability that P is true in subsystem $\mathcal{A}$. Indeed, for $1 - P \in \mathrm{Proj}(\mathcal{A})$ there also exists a state $\tilde{\phi} \in (\mathcal{A} \vee \mathcal{B})_*$ such that $\tilde{\phi} \mid \mathcal{B} = \varphi$ and $\tilde{\phi}(1 - P) = 1$. Hence for any $\lambda \in [0, 1]$ the state $\phi_\lambda \equiv \lambda\phi + (1 - \lambda)\tilde{\phi}$ satisfies $\phi_\lambda \mid \mathcal{B} = \varphi$ and $\phi_\lambda(P) = \lambda$. See [15][24] for a further discussion of this property.

It is obvious that if $(\mathcal{A}, \mathcal{B})$ is W^*-independent, then it is also strictly local. It was shown in [24] that the strict locality of a pair of *commuting* subalgebras of a W^*-algebra implies the C^*-independence of the pair. Here we show that, in fact, the converse is also true. Hence, at least for commuting pairs of W^*-subalgebras, if $(\mathcal{A},\mathcal{B})$ is strictly local, then $(\mathcal{B},\mathcal{A})$ is also strictly local.

Proposition 9. *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting subalgebras of a W^*-algebra $\mathcal{C}$. The pair $(\mathcal{A},\mathcal{B})$ is strictly local if and only if it is C^*-independent.*

Proof. The implication $\Rightarrow$ was proven in Prop. 4.2 in [24]. So let $(\mathcal{A},\mathcal{B})$ be C^*-independent, and let $P \in \mathcal{A}$ be a nonzero projection and $\varphi \in \mathcal{B}_*$ be a state. Moreover, let $\mathcal{C}$ be realized as a von Neumann algebra acting upon a Hilbert space $\mathcal{H}$. Since by assumption one has $P \in \mathcal{B}'$, the canonical *-homomorphism $\tau : \mathcal{B} \mapsto \mathcal{B}_P = P\mathcal{B}P \mid P\mathcal{H}$ given by $\tau(B) = PBP = BP$ is well-defined, and by the Schlieder property is a *-isomorphism onto a von Neumann algebra. τ is therefore σ-weakly bicontinuous, so that $\varphi \circ \tau^{-1}$ is a normal state on $\tau(B)$. There must therefore exist a density matrix ρ on $\mathcal{H}$ such that $\rho = P\rho P$ implements the state $\varphi \circ \tau^{-1}$. Hence, $\psi(C) \equiv \mathrm{tr}(\rho C)$ defines a normal state on $\mathcal{C}$ satisfying $\psi(P) = 1$ and

$$\psi(B) = \psi(BP) = \mathrm{tr}(\rho BP) = (\varphi \circ \tau^{-1})(\tau(B)) = \varphi(B)\,,$$

for any $B \in \mathcal{B}$. □

We note that strict locality implies C^*-independence even if the pair of subalgebras of the W^*-algebra $\mathcal{C}$ does not necessarily commute.

Corollary 10. *Let $\mathcal{A}$ and $\mathcal{B}$ be (not necessarily commuting) subalgebras of a W^*-algebra $\mathcal{C}$. If $(\mathcal{A},\mathcal{B})$ is strictly local, then it is also C^*-independent.*

Proof. Observe that the argument $(iii) \Rightarrow (ii) \Rightarrow (i)$ in the proof of Prop. 4 is also valid if the states in (iii) and (ii) are required to be normal. □

We have therefore proven the following theorem, establishing the equivalence of three standard notions of statistical independence in a context where they can be compared and which commonly occurs in physical applications, and providing alternative characterizations of these properties. However, as shown in [24], there are other versions of statistical independence which are not equivalent to these.

Theorem 11. *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting subalgebras of a σ-finite W^*-algebra $\mathcal{C}$. The following are equivalent.*

(i) $(\mathcal{A},\mathcal{B})$ is C^*-independent.

(ii) $(\mathcal{A},\mathcal{B})$ is strictly local.

(iii) $(\mathcal{A},\mathcal{B})$ is W^*-independent.

(iv) For every state ϕ_1 on $\mathcal{A}$ and every state ϕ_2 on $\mathcal{B}$ there exists a state ϕ on $\mathcal{C}$ such that $\phi(AB) = \phi_1(A)\,\phi_2(B)$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$.

(v) $\|AB\| = \|A\|\|B\|$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(vi) $0 \neq A \in \mathcal{A}$ and $0 \neq B \in \mathcal{B}$ imply that $AB \neq 0$.

(vii) The map $\eta : \mathcal{AB} \to \mathcal{A}\odot\mathcal{B}$ defined by $\eta(AB) = A\otimes B$, $A \in \mathcal{A}$, $B \in \mathcal{B}$ is an isomorphism continuous in the minimal C^*-cross norm on $\mathcal{A}\odot\mathcal{B}$ and can therefore be continuously extended to a surjective homomorphism $\overline{\eta} : \mathcal{A}\vee\mathcal{B} \to \mathcal{A}\otimes\mathcal{B}$.

(viii) For any nonzero vectors $\Phi, \Psi \in \mathcal{H}$, there exist $A' \in \mathcal{A}'$ and $B' \in \mathcal{B}'$ such that $A'\Phi = B'\Psi \neq 0$.

(ix) For any nontrivial projections $P \in \mathcal{A}$, $Q \in \mathcal{B}$, and $\lambda, \mu \in [0,1]$, there exists a state φ on $\mathcal{C}$ such that $\varphi(P) = \lambda$ and $\varphi(Q) = \mu$.

(x) For any nonzero projection $P \in \mathcal{A}$ and every state φ on $\mathcal{B}$, there exists a state ϕ on $\mathcal{C}$ such that $\phi(P) = 1$ and $\phi \mid \mathcal{B} = \varphi$.

We note, in particular, that it follows that the algebras of observables associated to complementary wedges in irreducible vacuum representations in quantum field theory are W^*-independent. Hence, such pairs of algebras are W^*-independent, but not W^*-independent in the product sense. It was previously known that they were C^*-independent and strictly local, but this is the first proof that they are also W^*-independent. Such tangent wedge algebras provide examples of algebras of observables satisfying W^*-independence but still maximally violating Bell's inequalities in every normal state (see [22][23][24] for further explanation).

These examples therefore demonstrate that W^*-independence is strictly weaker than W^*-independence in the product sense. Without using the rather involved arguments concerning Bell's inequalities, we point out also that if $\mathcal{A}$ is a type III factor acting on a separable Hilbert space, then $(\mathcal{A}, \mathcal{A}')$ is a C^*-independent, hence W^*-independent pair of W^*-algebras which is not W^*-independent in the product sense. It is remarkable that for such pairs all normal partial states have normal extensions, *none* of which is allowed to be a product state, and also all partial states have extensions to product states, *none* of which can be normal.

It was pointed out in [24] that a property called there C^*-independence in the product sense, namely the property that $\mathcal{A} \vee \mathcal{B}$ is isomorphic to the minimal C^*-product $\mathcal{A} \otimes \mathcal{B}$, is strictly stronger than C^*-independence. We present some further results concerning this notion. We first show that the existence of a faithful product state over the pair $(\mathcal{A}, \mathcal{B})$ implies that $(\mathcal{A}, \mathcal{B})$ is C^*-independent in the product sense.

Proposition 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting subalgebras of a C^*-algebra $\mathcal{C}$. If for some state ϕ_1 on $\mathcal{A}$ and some state ϕ_2 on $\mathcal{B}$ there exists a faithful state ϕ on $\mathcal{A} \vee \mathcal{B}$ such that $\phi(AB) = \phi_1(A)\phi_2(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $(\mathcal{A}, \mathcal{B})$ is C^*-independent in the product sense.*

Proof. For any states ϕ_1 on $\mathcal{A}$ and ϕ_2 on $\mathcal{B}$ there exists a state ψ on $\mathcal{A} \otimes \mathcal{B}$ such that $\psi(A \otimes B) = \phi_1(A)\phi_2(B) = \phi(AB)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. It was shown in Corollary 3.3 in [24] that the existence of a faithful product state implies that the pair $(\mathcal{A}, \mathcal{B})$ is C^*-independent. Hence, using the map $\overline{\eta} : \mathcal{A} \vee \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ from Theorem 1 (or Theorem 11) above, one has $\psi \circ \overline{\eta}(AB) = \psi(A \otimes B) = \phi(AB)$. The faithfulness of ϕ entails that $\overline{\eta}$ is an isomorphism. □

It is next shown that in the context most commonly met in quantum theory, the converse of Prop. 12 is also true.

Proposition 13. *Let $\mathcal{A}$ and $\mathcal{B}$ be commuting C^*-algebras acting on a separable Hilbert space $\mathcal{H}$. If $(\mathcal{A}, \mathcal{B})$ is C^*-independent in the product sense, then there exists a faithful product state ϕ on $\mathcal{A} \vee \mathcal{B}$.*

Proof. Assume that $\mathcal{A} \vee \mathcal{B}$ is isomorphic to $\mathcal{A} \otimes \mathcal{B}$. By Prop. II.3.19 in [26] there are faithful normal states on $\mathcal{A}''$ and $\mathcal{B}''$. The corresponding product state on

$\mathcal{A}''\overline{\otimes}\mathcal{B}''$ is also faithful (Corollary IV.5.12 in [26]). Since $\mathcal{A}\otimes\mathcal{B}$ is isomorphic to a subalgebra of $\mathcal{A}''\overline{\otimes}\mathcal{B}''$ and, by assumption, $\mathcal{A}\vee\mathcal{B}$ is itself isomorphic to $\mathcal{A}\otimes\mathcal{B}$, the assertion follows. $\square$

We comment that if $\mathcal{A}$ is a hyperfinite type III factor, then the pair $(\mathcal{A},\mathcal{A}')$ is C^*-independent in the product sense (since $\mathcal{A}$ is semidiscrete), but it is not W^*-independent in the product sense. On the other hand, W^*-independence in the product sense implies C^*-independence in the product sense [24].

Remarks on Statistical and Kinematical Independence

As mentioned earlier, the various notions of the statistical independence of the pair $(\mathcal{A},\mathcal{B})$ are logically independent of the requirement that the algebras $\mathcal{A}$ and $\mathcal{B}$ commute elementwise, which was called kinematical independence in [24]. In [24] a conjecture due to Werner was mentioned: if there exists a faithful product state across the C^*-independent pair $(\mathcal{A},\mathcal{B})$, then $\mathcal{A}$ and $\mathcal{B}$ commute. The following example shows that this conjecture is false. Consider the projections E and F on $\mathcal{H}=\mathbb{C}^6$:

$$E=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\end{pmatrix}\quad,\quad F=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&\frac{1}{2}&\frac{1}{2}\\0&0&0&0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

The operators E and F clearly do not commute, so the algebras, $\mathcal{A}$ and $\mathcal{B}$, they respectively generate do not commute. Moreover, any state on $\mathcal{A}$, resp. $\mathcal{B}$, is uniquely determined by its value on E, resp. F. Noting that any such value must lie between 0 and 1, if we take $a\equiv\phi_1(E)$ and $b\equiv\phi_2(F)$ then the following matrix

$$\rho=\begin{pmatrix}\frac{a(b+b^2)}{2}&0&0&0&0&0\\0&\frac{(1-a)(b+b^2)}{2})&0&0&0&0\\0&0&\frac{a(2-3b+b^2)}{2})&0&0&0\\0&0&0&\frac{(1-a)(2-3b+b^2)}{2})&0&0\\0&0&0&0&a(b-b^2)&0\\0&0&0&0&0&(1-a)(b-b^2)\end{pmatrix}$$

is a density matrix determining a state on $\mathcal{C}$ which is a common extension of the states ϕ_1 and ϕ_2. Hence $\mathcal{A}$ and $\mathcal{B}$ are C^*-independent. Moreover, this state is a faithful product state as long as ϕ_1 and ϕ_2 are chosen so that $\phi_1(E),\phi_2(F)\notin\{0,1\}$.

Some conditions sufficient for C^*-independence of $(\mathcal{A},\mathcal{B})$ to imply that $\mathcal{A}$ and $\mathcal{B}$ commute were proven in [18][16]. These conditions are fairly unmanageable, and we would like to emphasize that it would be of interest to find more useful sufficient conditions. We point out, however, that in the light of Prop. 2, this is not obvious, since if P and Q are projections on a Hilbert space $\mathcal{H}$ of at least six dimensions such that the restriction of P to a four-dimensional subspace $\mathcal{K}$ is diagonal with entries $1, 1, 0, 0$, say, and the restriction of Q to the same subspace is diagonal with entries $1, 0, 1, 0$, then the algebras respectively generated by these projections are C^*-independent no matter what the projections P and Q are doing on the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$.

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