Bounded Bose fields

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Abstract. Examples for bounded Bose fields in two dimensions are presented.

1. Introduction

Scalar free fields give rise to unbounded smeared field operators $\varphi(f) = \int f(\underline{x})\varphi(\underline{x})\, d^dx$. In contrast [1], free Fermi fields give rise to bounded smeared field operators $\psi(f) = \int f(\underline{x})\psi(\underline{x})\, d^dx$. These familiar facts are due to the unlimited occupation number in every excitation mode of the symmetric Fock space, and to the Pauli exclusion principle in the anti-symmetric Fock space, respectively.

The boundedness or unboundedness being a probe for the structure of the state space (in a sense suggested by the above examples), the question is of interest whether unboundedness of smeared field operators is a necessary feature of Bose fields. Indeed, Baumann [2] has proven that the spectrum condition entails that *chiral* Bose fields in two dimensions are necessarily unbounded. On the other hand, Buchholz [3] has observed that a very simple Bose field in two dimensions which is the tensor product of two chiral free Fermi fields, that is

$$\varphi(f) = \int d^2x\, f(t,x)\, \psi(t+x) \otimes \psi(t-x)\ , \tag{1}$$

is bounded. Clearly, this field commutes with itself not only at space-like distance but also at time-like distance. In view of Huygens' principle, this feature is usually interpreted as the absence of interaction for massless fields.

In the present letter we construct bounded Bose fields in two dimensions which commute with themselves at space-like distance but not at time-like distance.

All our examples are based on various factorizations of chiral Fermi fields into chiral vertex operators.

2. The construction

The starting point is the bosonization formula [4], which represents chiral free Fermi fields as vertex operators of unit charge. Using the boundedness of smeared free Fermi fields, we shall infer the boundedness of smeared chiral vertex operators with charge below unity. In a second step, we pass to two-dimensional local fields.

We briefly recall the relevant definitions. Chiral vertex operators $E_\alpha(x)$ are defined by

$$E_\alpha(x) = \mu^{\frac{1}{2}\alpha^2} :e^{i\alpha\phi(x)}: \equiv \mu^{\frac{1}{2}\alpha^2} e^{i\alpha\phi_+(x)} e^{i\alpha\phi_-(x)} = E_{-\alpha}(x)^* \tag{2}$$

where $\phi = \phi_+ + \phi_-$ with $\phi_+ = (\phi_-)^*$, satisfying the commutation relations

$$[\phi_-(x), \phi_+(y)] = -\log i\mu(x - y - i\epsilon) \tag{3}$$

while $[\phi_-, \phi_-] = [\phi_+, \phi_+] = 0$. The variable $x \in \mathbb{R}$ is a chiral light-cone coordinate. The regulating parameter $\mu > 0$ for the highly singular field ϕ will ultimately be sent to zero. Expressions involving $i\epsilon$ are always understood as boundary values as $\epsilon \searrow 0$.

The vertex operators (2) satisfy commutation relations

$$E_\alpha(x) E_\beta(y) = e^{\pm i\pi\alpha\beta} E_\beta(y) E_\alpha(x) \tag{4}$$

with the $+$ sign if $x > y$ and the $-$ sign if $x < y$.

By definition, the vacuum vector is annihilated by the negative frequency part,

$$\phi_-\Omega = 0 \qquad \Leftrightarrow \qquad e^{i\alpha\phi_-}\Omega = \Omega\,. \tag{5}$$

This property determines the correlation functions among $E_\alpha(x)$:

$$(\Omega, E_{\alpha_1}(x_1) \cdots E_{\alpha_r}(x_r)\Omega) = \mu^{\frac{1}{2}(\sum \alpha_i)^2} \prod_{i<j} \Delta(x_i - x_j)^{-\alpha_i\alpha_j} \tag{6}$$

where

$$\Delta(x) = \frac{-i}{x - i\epsilon} = \int_0^\infty dk e^{-ikx}\,. \tag{7}$$

In the limit $\mu \searrow 0$, the only correlations (6) which survive are those with total charge $\sum_i \alpha_i = 0$. By inspection of the correlation functions, one verifies that in this limit

$$\psi(x) = E_{+1}(x) \qquad \text{and} \qquad \psi^*(x) = E_{-1}(x) \tag{8}$$

are chiral free Fermi fields satisfying the canonical anti-commutation relations

$$\{\psi^*(x), \psi(y)\} = 2\pi\,\delta(x - y), \qquad \{\psi(x), \psi(y)\} = 0\,. \tag{9}$$

The derivative of the original field ϕ is related to the current $j = \,:\psi^*\psi:$ by

$$\partial\phi(x) = 2\pi j(x)\ . \tag{10}$$

The charge operator $Q = \int j(x)dx = \frac{1}{2\pi}(\phi(\infty) - \phi(-\infty))$ measures the total charge $\sum_i \alpha_i$:

$$Q \prod E_{\alpha_i}(x_i)\Omega = (\textstyle\sum_i \alpha_i) \prod E_{\alpha_i}(x_i)\Omega \tag{11}$$

provided $\mu \searrow 0$. Only in this limit, the vacuum is invariant under the $U(1)$ gauge symmetry e^{itQ}.

Unlike the singular field ϕ, the vertex operators $E_\alpha(x)$ are operator valued distributions on a Hilbert space with positive metric. They are non-local conformally covariant fields with chiral scaling dimension $h = \frac{1}{2}\alpha^2$.

All the previous holds true if one starts from a multi-component field $\underline{\phi}$, that is

$$[\phi^a_-(x), \phi^b_+(y)] = -\delta^{ab} \log i\mu(x-y-i\epsilon) \qquad (a,b = 1,\dots n)\ . \tag{12}$$

One only has to read quadratic expressions such as α^2 or $\alpha\phi$ as scalar products $\underline{\alpha}^2 = \sum_a (\alpha^a)^2$ or $\underline{\alpha}\underline{\phi} = \sum_a \alpha^a\phi^a$, etc. $E_{\underline{\alpha}}$ is a chiral free Fermi field whenever $\underline{\alpha}^2 = 1$, but different such fields do not anti-commute with each other. They commute, actually, when $\underline{\alpha}'\underline{\alpha} = 0$.

2.1. Factorization

For our purpose, it is sufficient to consider vertex operators with two-component charges $\underline{\alpha} = (\alpha^1, \alpha^2) = \alpha^1\underline{e}^1 + \alpha^2\underline{e}^2$. Since ϕ^1 and ϕ^2 decouple, we may write

$$E_{\underline{\alpha}}(x) \equiv \mu^{\frac{1}{2}\underline{\alpha}^2} :e^{i\underline{\alpha}\underline{\phi}(x)}: \, = E_{\alpha^1\underline{e}^1}(x) \cdot E_{\alpha^2\underline{e}^2}(x) \cong E_{\alpha^1}(x) \otimes E_{\alpha^2}(x)\ . \tag{13}$$

Namely, the cyclic Hilbert space $\mathcal{H}$ for all vertex operators with two-component charges decomposes as a tensor product of the cyclic Hilbert spaces $\mathcal{H}^a$ for the vertex operators with charges $\alpha\underline{e}^a$ in the basis directions $(a = 1, 2)$, and the latter are isomorphic to vertex operators with scalar charges α.

Note that the distributional meaning of the tensor product here is different from the one in the introduction: eq. (13) is the pointwise product of distributions in the same chiral variable which is well defined due to spectral (or analytic) properties of the correlation functions (6), while eq. (1) corresponds to the product of distributions in two independent chiral variables and yields a distribution over $\mathbb{R}^2$.

Now we take $\underline{\alpha}^2 = 1$, $\underline{\alpha} = (\cos\tau, \sin\tau)$ such that $\psi \equiv E_{\underline{\alpha}}$ is a free Fermi field. The latter is not only defined on its own cyclic Hilbert space $\mathcal{H}_\psi$ but on the much larger Hilbert space $\mathcal{H}$ into which $\mathcal{H}_\psi$ is embedded.

We evaluate ψ in a partial state $(\mathrm{id}\otimes\omega)$ where ω is some vector state $\omega = (\Phi, \cdot\,\Phi)$ of $\mathcal{H}^2$. (For an operator A on $\mathcal{H}$, $(\mathrm{id}\otimes\omega)(A)$ is defined by its matrix elements $(\Psi_1\otimes\Phi, A\Psi_2\otimes\Phi)$.) We obtain an operator valued distribution on $\mathcal{H}^1$:

$$(\mathrm{id}\otimes\omega)(\psi(x)) = E_{\cos\tau}(x)\cdot\omega(E_{\sin\tau}(x)) \ . \tag{14}$$

Solving for $\psi_\tau(x) \equiv E_{\cos\tau}(x)$ on $\mathcal{H}^1$ and smearing with a test function g, we find that this field can be represented by the free Fermi field evaluated in a partial state,

$$\psi_\tau(g) = (\mathrm{id}\otimes\omega)(\psi(m_\omega g)) \ , \tag{15}$$

where $m_\omega(x) = \omega(E_{\sin\tau}(x))^{-1}$. We have to convince ourselves that for an appropriate choice of the state ω this reciprocal exists.

A convenient choice is given by the vector $\Phi = \Omega + \Omega_{\sin\tau}$ where $\Omega_{\sin\tau}$ is the ground state for the conformal Hamiltonian L_0 (with 'energy' $h = \frac{1}{2}\sin^2\tau$) within the sector of total charge $Q = \sin\tau$. With this choice, $\omega(E_{\sin\tau}(x)) = (\Omega_{\sin\tau}, E_{\sin\tau}(x)\Omega) \sim (1-ix)^{-\sin^2\tau}$, so the reciprocal function

$$m_\omega(x) \equiv \omega(E_{\sin\tau}(x))^{-1} = const.\cdot(1-ix)^{\sin^2\tau} \tag{16}$$

is well defined.

2.2. Boundedness

Chiral free Fermi fields, and hence the vertex operators with charge $\underline{\alpha}^2 = 1$ satisfy the L^2-bound

$$\|\psi(f)\| = \|E_{\underline{\alpha}}(f)\| \le const.\cdot\|f\|_2 \ . \tag{17}$$

This bound is algebraically determined [1] by the canonical anti-commutation relations (9), and therefore holds not only on the cyclic Hilbert space $\mathcal{H}_\psi$ but on the entire Hilbert space $\mathcal{H}$.

It is elementary to show that a partial state $(\mathrm{id}\otimes\omega)$ takes bounded operators on $\mathcal{H}^1\otimes\mathcal{H}^2$ into bounded operators on $\mathcal{H}^1$,

$$\|(\mathrm{id}\otimes\omega)(A)\| \le \|\omega\|\,\|A\| \ . \tag{18}$$

Hence by eq. (15), $\psi_\tau(g)$ is bounded by

$$\|\psi_\tau(g)\| \leq \|\omega\| \, \|\psi(m_\omega g)\| \leq const. \cdot \|m_\omega g\|_2 \tag{19}$$

whenever the test function g is in the domain of the multiplication operator $(m_\omega g)(x) = m_\omega(x)g(x)$.

For the choice of ω as above, m_ω has dense domain in $L^2(\mathbb{R})$. Clearly, all Schwartz functions are in the domain of m_ω. Hence we have established the L^2-bound (19) for vertex operators with charge below unity smeared with any test function of sufficiently rapid decay.

We note that the multiplication operator m_ω is absent in the 'compact' description of chiral fields. Namely, the conformal vertex operators $E^c_\alpha(z)$ on the compactified light-cone S^1 (where $z = \frac{1+ix}{1-ix} \in S^1$) given by

$$E^c_\alpha(z) = (dz/dx)^{-\frac{1}{2}\alpha^2} E_\alpha(x) = const. \cdot (1-ix)^{\alpha^2} E_\alpha(x) \tag{20}$$

acquire a weight factor $\sim (1-ix)^{\alpha^2}$. The different weight factors pertaining to ψ^c_τ (with $\alpha^2 = \cos^2\tau$)) and ψ^c (with $\underline{\alpha}^2 = 1$), respectively, precisely cancel the multiplication operator m_ω given by (16), and one has $\psi^c_\tau(z) = const. \cdot (\mathrm{id} \otimes \omega)(\psi^c(z))$. Thus, ψ^c_τ satisfies the same $L^2(S^1)$-bound as the chiral free Fermi field on the circle.

2.3. Locality of tensor products

Due to eq. (4), the bounded operators $\psi_\tau(g)$ satisfy the non-local commutation relations

$$\psi_\tau(g_1)\psi_\tau(g_2) = e^{\pm i\pi\cos^2\tau}\psi_\tau(g_2)\psi_\tau(g_1) \tag{21}$$

whenever $\mathrm{supp}\, g_1 > \mathrm{supp}\, g_2$ resp. $\mathrm{supp}\, g_2 > \mathrm{supp}\, g_1$.

In order to obtain Bose fields, we proceed to the tensor product of two non-local chiral fields ψ_τ

$$\varphi_\tau(g \otimes h) := \psi_\tau(g) \otimes \psi_\tau(h) \tag{22}$$

for test functions $(g \otimes h)(t,x) = g(t+x)h(t-x)$ on $\mathbb{R}^2$. This tensor product is understood to extend by linearity, as in eq. (1), to the two-dimensional field

$$\varphi_\tau(f) = \int d^2x \, f(t,x) \, \psi_\tau(t+x) \otimes \psi_\tau(t-x) \; . \tag{23}$$

If two test functions $f_i = g_i \otimes h_i$ on $\mathbb{R}^2$ $(i = 1,2)$ have supports at space-like distance, then either $\mathrm{supp}\, g_1 < \mathrm{supp}\, g_2$ and $\mathrm{supp}\, h_1 > \mathrm{supp}\, h_2$, or $\mathrm{supp}\, g_1 > \mathrm{supp}\, g_2$ and $\mathrm{supp}\, h_1 <$

$\operatorname{supp} h_2$. In either case, the commutation relations (21) produce two opposite phase factors, and consequently $\varphi_\tau(f_1)$ and $\varphi_\tau(f_2)$ commute, whenever f_i have supports at space-like distance. On the other hand, if f_i have supports at time-like distance, then the phase factors add rather than cancel, and consequently φ_τ does not satisfy Huygens' principle. Choosing $\cos^2\tau = \frac{1}{2}$ one may even have time-like anti-commutation. (This possibility was also previously discovered by Buchholz [3].)

By linearity, the local commutativity extends to tensor product fields smeared with arbitrary test functions on $\mathbb{R}^2$.

2.4. Boundedness of tensor products

By Schwartz' nuclear theorem, the extension of the tensor product of two (operator-valued) distributions in one variable to a functional in two variables is again a distribution, and the chiral bound (19) implies bounds for the two-dimensional fields (23) in terms of some Schwartz norms of the test function f. The following estimates due to Buchholz [3] improve the bounds due to the theorem.

We rewrite eq. (23) in the form

$$\varphi_\tau(f) = \int dp\, dq\; P(i\partial_p)P(i\partial_q)\tilde{f}(p,q) \;\int d^2x \frac{e^{ipx_+}}{P(x_+)}\psi_\tau(x_+) \otimes \frac{e^{iqx_-}}{P(x_-)}\psi_\tau(x_-)\;, \qquad (24)$$

where $\tilde{f}(p,q)$ is the Fourier transform of f with respect to the chiral coordinates $x_\pm = t \pm x$, and P is a polynomial in one chiral coordinate such that $1/P$ is in the domain of m_ω. The operator-valued integral $\int d^2x \ldots$ in eq. (24) is of the form (22), and is hence of norm less than $const. \cdot \|m_\omega/P\|_2^2$. Thus

$$\|\varphi_\tau(f)\| \leq const. \cdot \|P(i\partial_p)P(i\partial_q)\tilde{f}\|_1 \;. \qquad (25)$$

Estimating the L^1-norm by the Cauchy-Schwarz inequality $\|f\|_1 \leq \|1/Q\|_2\|Qf\|_2$ provided $1/Q$ is square integrable, and using the Plancherel equality of the L^2-norms of a function and its Fourier transform, we also have

$$\|\varphi_\tau(f)\| \leq const. \cdot \|Q \cdot P(i\partial_p)P(i\partial_q)\tilde{f}\|_2 = const. \cdot \|Q(-i\partial_+, -i\partial_-)(P \otimes P)f\|_2 \qquad (26)$$

for appropriate polynomials $P(x_\pm)$ and $Q(p,q)$ as qualified before, e.g., $P(x) = 1 + x^2$ and $Q(p,q) = (1-ip)(1-iq)$. The bound (26) holds for all test functions f which are sufficiently smooth and of sufficiently rapid decay, and in particular for all Schwartz functions. The constant depends on τ, on the state ω, and on P and Q.

We conclude that φ_τ are bounded Bose fields with non-trivial time-like commutators. As the parameter τ varies between 0 and $\frac{\pi}{2}$, the fields φ_τ interpolate between Buchholz' example (1) and the identity operator.

2.5. Further examples

The above construction has a non-abelian generalization. It was shown by Wassermann and by Loke, with essentially the same method as ours, that – when passing from level k to level $k+1$ – the coset model factorization of chiral free $SU(N)$ Fermi fields produces L^2-*bounded* chiral exchange fields (non-abelian vertex operators) which are primary for the level $k+1$ current algebra [5], and for the coset Virasoro algebra (for $N=2$) [6], respectively. The $SU(2)$ coset Virasoro algebra has central charge in the discrete series $c<1$, and the primary chiral fields thus obtained are, in the standard nomenclature, the fields ϕ_{12} and ϕ_{22} with chiral scaling dimension $h_{12}=\frac{k-1}{4(k+2)}$ and $h_{22}=\frac{3}{4(k+1)(k+2)}$. The primary fields for the current algebra are those corresponding to the vector and conjugate vector representations of $SU(N)$.

On the other hand, it is well known (see, e.g., [7]) that tensor products of chiral exchange fields in rational theories yield *local* two-dimensional fields of the form

$$\varphi(t,x)=\sum_{e\bar{e}}\zeta_{e\bar{e}}\,\phi_e(t+x)\otimes\phi_{\bar{e}}(t-x) \tag{27}$$

with appropriate numerical coefficients $\zeta_{e\bar{e}}$; the labels e and $\bar{e}$ run over the pairs of initial and final sectors connected by the primary field ϕ and its conjugate, respectively. This result relies only on a 'CPT' symmetry due to pentagon and hexagon identities of fusion and braiding matrices.

Taken together, these two results show that φ in eq. (27) are bounded two-dimensional Bose fields if the primary field ϕ is one of the fields produced by the coset factorization of chiral free Fermi fields. These fields are the fields φ_{12} and φ_{22} in the minimal models [8], and the basic matrix fields g_{ab} of the two-dimensional $SU(N)$ Wess-Zumino-Witten models [9], respectively.

3. Conclusion

We have presented a variety of bounded Bose fields in two dimensions. The family of fields (23) for various parameters τ are actually defined on a common Hilbert space and are relatively local among each other.

All the bounded Bose fields we have constructed here are conformally covariant fields with scaling dimension $d\equiv h_+ + h_- \leq 1$. To remove this upper bound for the scaling dimension, one may consider (non-primary) derivative fields

$$\partial_{\mu_r}\dots\partial_{\mu_1}\varphi(f)=(-1)^r\varphi(\partial_{\mu_r}\dots\partial_{\mu_1}f) \tag{28}$$

which are again bounded Bose fields but with scaling dimensions $r \leq d \leq r+1$.

We conclude that in two dimensions, bounded Bose fields are quite abundant. We still do not know whether bounded Bose fields exist in more than two dimensions.

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