Invariant manifolds associated to non-resonant spectral subspaces.

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Abstract. We show that, if the linearization of a map at a fixed point leaves invariant a spectral subspace which satisfies certain non-resonance conditions, the map leaves invariant a smooth manifold tangent to this subspace. This manifold is as smooth as the map, but is unique among C^L invariant manifolds, where L depends only on the spectrum of the linearization. We show that if the non-resonance conditions are not satisfied, a smooth invariant manifold need not exist and also establish smooth dependence on parameters. We also discuss some applications of these invariant manifolds and briefly survey related work.

1. Introduction

Besides their intrinsic appeal, invariant manifold theorems are interesting in Dynamics because they provide landmarks which organize the long–time behavior.

¿From this point of view, having more invariant manifolds is quite desirable, since it means having more tools for the analysis of the dynamical system. In particular, it is often the case, that associated to invariant manifolds in the tangent bundle, one can associate other invariant structures in the manifold itself. For example one of the standard constructions of stable and unstable foliations [HP] includes as one of the steps to apply the stable manifold theorem to the operator f_* acting on vector fields by: $[f_* v](x) = Df(f^{-1}(x))v(f^{-1}(x)$. The invariant manifolds constructed here, will also lead to invariant structures on the manifold. Nevertheless, they are not foliations, as shown in [JLP]. (These structures seem to have been considered first in [Pe]) We will discuss some of their uses later. For example, they play a role in rigidity theory. Another motivation for this paper is the study of renormalization group transformations. Even if a precise analytical definition of a renormalization group operator is fraught with technical difficulties (See e.g [EFS], [MO]) it is fruitful to study maps that are well defined as caricatures of the real situation. (See [Be] Chap. 3 [LMS], Sec 5., Appendix E) In that picture, different ways to approach the fixed point correspond to different properties of finite size scale fluctuations. An example of physical properties that can be described by properties of the approach to the fixed point can be found in [GKT]. See also [LMS] for the analogy of some of the manifolds constructed here with the beta functions or renormalization group theory. The renormalization group in dynamical systems – especially in the period doubling case – is much better behaved that the renormalization group in Statistical Mechanics and in that case, it is sometimes possible to write well defined renormalization maps that are analytic in an appropriate space and which have with a compact derivative (See e.g. [CEK], [CEL], [VSK]) (These motivations are the reason why we have chosen to prove our results in the generality of Banach spaces.)

If the system were linear, a very natural invariant set would be an spectral subspace. For a non-linear system close to a fixed point – and hence approximable by the derivative at the fixed point one can ask if there are analogues of the spectral subspaces of the derivative which are invariant for the full non-linear system.

The classical theory of invariant manifolds establishes the existence of invariant manifolds associated with spectral subsets which are disks around the origin or complements of disks around the origin. Usually the manifold associated to $\{z \in \mathbb{C} \big| |z| \leq \rho < 1\}$ is called the strong stable manifold, that associated to $\{z \in \mathbb{C} \big| |z| < 1\}$ is called the stable manifold, those associated to sets of the form $\{z \in \mathbb{C} \big| |z| \leq 1\}$ are called center stable manifolds and those associated to sets $\{z \in \mathbb{C} \big| |z| \leq \rho > 1\}$ are called pseudostable manifolds. We have used "is" or "are" on purpose to indicate whether uniqueness under local assumptions holds or not. We refer to Section 7. about other results which also include uniqueness under global assumptions.

On the other hand, for some finite dimensional systems, one can apply the Sternberg linearization theorem and conclude that the system is equivalent to the linearization expressed in another smooth system of coordinates. Of course, since the spectral subspaces are invariant under the linearization, their image under the change of variables that linearizes the map will be invariant under the full map. Hence, the non-linear map may leave invariant some smooth manifolds that correspond to spectral subspaces.

Even if the above argument indicates that the classical theory of invariant manifolds is not as general as possible, the theory based on the the Sternberg linearization theorem is not very satisfactory either. For example, for infinite dimensional systems the non-resonance conditions become harder to verify – or even false if the spectrum includes open sets –. Even in finite dimensions, the conditions of Sternberg theorem are non-open. Moreover, since the linearizing changes of variables provided by the Sternberg linearization theorem are highly non-unique, it seems that the smooth invariant manifolds produced this way are also not unique.

In this paper we try to obtain a compromise between the Sternberg linearization theorem and the classical invariant manifold theory. The proofs will start as in the Sternberg linearization theorem, using non-resonance assumptions to eliminate undesired terms, but we will switch as soon as possible to the – much easier than linearization – invariant manifold theory.

This will allow to prove some invariant manifold theorems for spectral subspaces satisfying some mild non-resonance conditions that persist for open sets of problems. Moreover, there will be local conditions that guarantee uniqueness. Once that this uniqueness is established, it makes sense to study the dependence with respect to parameters. We show that, indeed these manifolds depend smoothly with respect to parameters and compute explicit formulas for the derivatives. These formulas can be used to justify some perturbative calculations of beta functions in renormalization group.

We will also provide examples that show that if those non-resonance conditions are not met, the conclusions are false.

The ideas presented above are closely related to partial normal forms. That is, showing certain terms are irrelevant for the dynamics since they can be eliminated just by switching to appropriate systems of coordinates. From the renormalization group point of view, the discussion of when a term cannot be eliminated is quite interesting.

Note that if we express the dynamics in a system of coordinates where some terms in the map are not present, we can sometimes obtain uniform estimates about all the iterates of the map in this system of coordinates. Then, these estimates can be read off in the original system. This type of argument is often used in dynamical systems to obtain very uniform control of iterates. It is sometimes also used in renormalization group arguments as a way to control the approach of surfaces defining an interesting phenomenon to critical surfaces.

In Section 6 we just quote some results on these problems of partial linearizations. We point out that, compared to the method of graph transforms developed in Section 4, they have the advantage that they can produce invariant manifolds whose spectral subset straddles the unit circle. Nevertheless the manifolds produced using this method are much less differentiable than the map.

Finally, in Section 7 we review briefly other works that study invariant manifolds other than the classical stable, strong stable etc. and in Section 8, we sketch some applications of the results presented here.

2. Acknowledgments.

This paper would not have been written without the friendly prodding of A. Sokal, who raised the problem in 1987, explained the statistical mechanics motivation and has kept encouraging me to write the answer. Independently of that, I was driven to the same problems by an effort to construct more invariant foliations to be used in rigidity questions. Y. Yomdin pointed to me the paper [Pe] which uses similar ideas. (See also [JLP].) I am also grateful to H. Rosenthal for bringing to my attention [LeW],[K]. Also to C. Pugh for encouragement and pointing out that C^r regularity was also possible. This research has been supported by NSF grants and a Centennial Fellowship from AMS and a URI from Univ. of Texas. We also thank an anonymous referee that improved greatly the exposition.

3. Notation and statement of the main results.

In this paper X will be Banach space (not necessarily separable), over the reals or over the complex. f will be a C^r, $1 \le r \le \infty, \omega$, mapping from X to itself such that it has a fixed point, which we will place at the origin (i.e. $f(0) = 0$). We will call $Df(0) = A$ and write $f(x) = Ax + N(x)$.

The question we will address is the existence of "invariant manifolds" passing though the fixed point. That is, submanifolds W of X such that $f(W) = W$, $0 \in W$.

In the cases that we will consider, the problem is equivalent to a local version so that we just need to assume f is defined in a small neighborhood of the origin.

The guiding idea behind the results presented here is that, in small scales, f is a perturbation of A and the invariant manifolds of f should be very similar to those invariant under A.

The most natural invariant manifolds for a linear operator (they are not the only ones!) are invariant subspaces associated to spectral projections. If X is a complex space, to each closed subset of the spectrum bounded away from the rest of the spectrum we can associate a spectral projection whose range is a closed subspace. (If X is real, the subspaces are associated to subsets of the spectrum as above which are also invariant under complex conjugation). (See [RS] [Ka].)

The invariant manifolds we will construct are perturbations of these spaces. As it is often the case with perturbation theory, it is very advantageous to try to remove some terms with an appropriate change of variables. Notice that if we could eliminate all of them as in the Sternberg linearization theorem, the result would follow: the invariant manifolds would be the images under the linearizing map of the invariant subspaces. The problem with applying the Sternberg linearization theorem in infinite dimensional Banach spaces, is that the non-resonance conditions fail in open sets of maps. (The spectrum of the linearization may have non-empty interior.)

One can observe that the best known results on existence of invariant manifolds : the strong stable, stable, center stable, center, center unstable pseudostable manifolds ([Ha], [Ke], [Fe1], [Ir1], [Ir2], [Ir3], [HPS],) amount to considering subsets of the spectrum obtained by intersecting it with disks, complements of disks or circles.

We will be able to generalize those results in that we will be able to consider more general sets. (At the end of the paper, we will list other results that go in this direction)

We will not be able to associate invariant manifolds to all the subsets of the spectrum of A, but will have to impose "non-resonance" conditions so that some eliminations can be performed. We will, furthermore show that, if these non-resonance conditions fail, there are counterexamples to the theorems considered here (or slightly stronger versions).

The technique we will use is the "graph transform" coupled with some manipulations standard in "normal form" theory. These manipulations, even if they simplify the proof, are not really necessary and it is possible to construct a proof without using them, (we will give an sketch of these alternative methods).

Since the paper [La1] (parts of it reproduced in [MMc]) contains not only an excellent exposition of the graph transform method for invariant manifolds but also an exposition of normal forms, we will refer the reader there for some basic results.

Notation

Given a splitting $X = E^S \oplus E^U$ we will denote by Π_S, Π_U the corresponding projection.

Given a function $N : X \mapsto X$ we will define N_S (resp. N_U) by $N_{S,U}(\Pi_S x, \Pi_U x) = \Pi_{S,U} N(x)$.

If the splitting is invariant under a linear operator A, we will call A_S, A_U the restriction of A to those subspaces. This is slightly inconsistent with the previous convention but is also customary and will not lead to confusion.

Finally, we will call $B^S(\ell)$ the ball of radius ℓ around 0 in the space E^S, and analogously for all spaces. If some of the indexes are clear from the context, we will omit them.

Our main theorem is:

Theorem 3.1. *Let X be a real or complex Banach space, f a C^r mapping, $r \in \mathbb{N} \cup \{\infty, \omega\}$, $r \geq 1$, $f(0) = 0$, $Df(0) = A$. Assume that there is a decomposition $X = E^S \oplus E^U$ such that:*

i) The splitting is invariant under A (Use the notation A_S, A_U for the restrictions.)

ii) $\sigma(A_S)$ bounded away from $\sigma(A_U)$

iii) $\sigma(A_U)$ bounded away from zero

iv) $\sigma(A_S)$ contained strictly inside the unit disk.

We will call L a integer big enough so that

$$(\sup\{|t| \mid t \in \sigma(A_S)\})^L (\sup\{|t|^{-1} \mid t \in \sigma(A_U)\}) < 1$$

And we will assume that

v) $L < r$

vi) If $i < L$ then $[\sigma(A_S)]^i \cap \sigma(A_U) = \emptyset$

where $[\sigma(A_S)]^i = \{x_1 \cdot x_2 \cdot \ldots \cdot x_i \mid x_j \in \sigma(A_S) \quad 1 \leq j \leq i\}$.

vii) The function $N(x) = f(x) - A(x)$ has sufficiently small C^{L+1} norm when restricted to a ball of radius 1 around 0. (The smallness conditions depend only on the *spectrum of A).*

Then, there is a function $w : B^S(1) \longrightarrow U$ such that:

a) *The graph of w is invariant under f.*

b) *w is $C^{r-1+Lipschitz}$*

Moreover, w is unique among the C^L functions satisfying a).

Remark. Given any function $f : X \mapsto X$, $f(0) = 0$ and $Df(0)$ satisfying the assumptions $i)$–$vii)$, $f_\lambda(x) \equiv \frac{1}{\lambda} f(\lambda x)$ will verify the smallness hypothesis for λ small enough. Indeed, recall that the smallness conditions only depend on $Df_\lambda(0)$, which does not depend on λ. On the other hand, N_λ gets smaller in the C^r sense as λ gets small. If w_λ is the function whose graph is invariant under f_λ, $w \equiv \lambda w(x/\lambda)$ will have a graph invariant under f. Note that $f_\lambda = A + N_\lambda$ and that N_λ converges to 0 in C^r in the ball as λ tends to zero. Therefore, assuming smallness conditions in N and considering only a small neighborhood are equivalent.

Remark. It will be important later that the smallness conditions we impose to N are only in C^L and not in C^r. If we want to consider C^∞ f's, we will have to do a different proof for every finite r. It will be important that we can choose the same λ in all cases so that the function w corresponding to different r's will be defined in the same domain.

Remark. Condition $vi)$ will be referred to as the "non-resonance" condition. Its interpretation is obvious when the X is finite dimensional, it just means that the product of any set of less than L eigenvalues of A_S are not eigenvalues of A_U.

Remark. We will derive later stronger uniqueness properties than those claimed in the theorem. In particular, Theorem 5.2 implies that the solution is unique among $C^{r_0+\epsilon}$ with $r_0 = \log \|A_U^{-1}\| / \log \|A_S\|$. Note that $L = [r_0] + 1$.

Remark. Observe that if the spectral subset we consider is given by $\sigma(A) \cap \{z \mid |z| < \alpha\}$ $\alpha < 1$ Theorem 3.1 reduces to the α-stable manifold theorem characterized as the set of points x such that $\alpha^{-n} f^n(x)$ remains bounded. In this case, the non-resonance conditions are obviously satisfied.

Remark. Note that by the definition of L, $[\sigma(A_S)]^i \cap \sigma(A_U) = \emptyset$ when $i \geq L$. Hence, with $vi)$, the intersection is empty for all $i \in \mathbb{N}$. ¿From the presentation in the text, it is clear that the condition holds in an open set of mappings.

Remark. The conclusion $b)$ of Theorem 3.1 can be improved to C^r. See the remarks after the proof for details

There are many equivalent norms we can use in X. Given an operator A we can choose a norm in such a way that $\|A\| \leq \sup\{|t| \mid t \in \sigma(A)\} + \varepsilon$ for any $\varepsilon > 0$. Moreover, we could also chose the norm in such a way that the splittings associated to a finite number of closed subsets of the spectrum have projections of norm 1. We will, henceforth assume such a norm, with a sufficiently small ε, has been defined so that the L introduced in the assumption $iv)$ of the theorem also satisfies

$$\|A_S\|^L \|A_U^{-1}\| < 1.$$

Note that all the smallness assumptions etc. are to be understood in this norm. Since it is equivalent to the original one, all "sufficiently small" requirements in this norm are implied by "sufficiently small" in the original one.

Theorem 3.1 will be derived from the following theorem, which we will prove first:

Theorem 3.2. *In the same set up of Theorem 3.1, do not assume iv), but assume instead iv′)* $N_U(s,u) = 0\,(\|u\|^{L+1},\ \|s\|^2)$ *near* 0. *Then, the same conclusions as in Theorem 3.1 hold but we moreover have.*

c)$||w(s)|| = \mathcal{O}(||s||^{L+1})$

Remark. Note that, in the conditions of Theorem 3.2, denoting by (s,u) the projections on the stable and unstable component, we have:

$$f(s,u) = (A_S s + N_S(s,u), A_U u) + \mathcal{O}(||u||^{L+1}, ||s||^2) \tag{3.1}$$

Observe that if we ignore the high order terms, the set corresponding to points with $u = 0$ is invariant under the dynamics. In a neighborhood of this set, the terms ignored are a very small perturbation and, we will be able to construct the invariant manifold as a perturbation of the set $\{(s,0)\}$.

4. Proof of Theorem 3.1 and Theorem 3.2

As in [La1], the proof will consist in formulating the invariant manifold as the graph of a function $w : B^S(1) \longrightarrow U$ The fact that the manifold is invariant will be equivalent to the fact that the function w is a fixed point of an operator $\mathcal{T}$ acting on an appropriate space of functions

First, we will prove the theorem when $r < \infty$. We will take $\mathcal{T}$ to be

$$\mathcal{T}[w](x) = A_U^{-1}(w(A_S x + N_S(x, w(x))) - N_U(x, w(x))). \tag{4.1}$$

This operator was used in [La2] – not in [La1]. It is not exactly the graph transform, but is somewhat more manageable. The reason to define the operator is that if we compute f on a point of the graph of the w, we have:

$$f(x, w(x)) = \big(A_S x + N_S(x, w(x)),\ A_U w(x) + N_U(x, w(x))\big) \tag{4.2}$$

This transformed point belongs to the graph of w if and only if the second coordinate is the result of applying w to the first. That is,

$$w(A_S x + N_S(x, w(x)) = \ A_U w(x) + N_U(x, w(x))$$

which, provided that all the compositions can be defined, is equivalent to w being a fixed point of $\mathcal{T}$.

The operator $\mathcal{T}$ is well defined if $\|A_S\| + \|N_S\|_{L^\infty} < 1$. We will assume that N is small enough that indeed $\mathcal{T}$ is defined on functions on the unit ball of S.

We will consider $\mathcal{T}$ as acting on the following spaces

$$\begin{aligned}\chi^r_{\varepsilon_1 \ldots \varepsilon_{r-L}} = \Big\{ w : B^S(1) \mapsto U \text{ such that} \\ &\text{a)}\quad w \in C^r \\ &\text{b)}\quad D^k w(0) = 0, 0 \le k \le L \\ &\text{c)}\quad \sup_{x \in B^S(1)} \|D^k w(x)\| \le 1, \quad 0 \le k \le L \\ &\text{d)}\quad \sup_{x \in B^S(1)} \|D^{L+i} w(x)\| \le \varepsilon_i, \quad 1 \le i \le r - L \Big\}\end{aligned}$$

Note that, because of condition b) the $w \in \chi$ is determined uniquely by $D^L w$. We will therefore consider χ endowed with the topology given by the norm $||w|| = ||D^L w||_{L^\infty(B^S(1))}$. We also point out that by condition b) we also have $||D^i w||_{L^\infty(B^S(1)} \leq (1/(L-i)!)||D^L w||_{L^\infty(B^S(1))}$.

For the sake of notation we will suppress the $B^S(1)$ from the spaces, but L^∞ is to be understood always to be on unit ball.

Proposition 4.1. *Given some smallness assumptions in $\|D^k N_S\|_{L^\infty}$, $0 \leq k \leq L$, it is possible to find $\varepsilon_1, \dots, \varepsilon_{r-L} > 0$ such a way that*

$$\mathcal{T}(\chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}) \subset \chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}.$$

These ε's can be found in a recursive consistent way: that is if $\bar{r} > r$ and we have found $\varepsilon_1 \dots \varepsilon_{r-L}$, we can find $\varepsilon_{r-L+1} \cdots \varepsilon_{\bar{r}-L+1}$ so that

$$\mathcal{T}(\chi^{\bar{r}}_{\varepsilon_1,\dots,\varepsilon_{r-L},\varepsilon_{r-L+1},\dots,\varepsilon_{r-L}}) \subset \chi^{\bar{r}}_{\varepsilon_1,\dots,\varepsilon_{r-L},\varepsilon_{r-L+1},\dots\varepsilon_{\bar{r}-L}}.$$

Proof. The fact that $\mathcal{T}[w]$ satisfies a), b) if w does is quite easy.

Since for functions satisfying a), b) we have for $k < L$

$$\|D^k w(x)\| \leq \sup_{y \in B^S(\|x\|)} \|D^L w(y)\| \quad \frac{\|x\|^{L-k}}{(L-k)!},$$

to check c), it suffices to check it for $k = L$.

Taking L derivatives of (4.1) we obtain:

$$\begin{aligned} D^L \mathcal{T}[w](x) =& A_U^{-1} D^L w(A_S x + w(x))(A_S + D_1 N_S(x, w(x))^{\otimes L} \\ &+ \ D_2 N_S(x, w(x)) Dw(x))^{\otimes L} \\ &+ R \end{aligned} \tag{4.3}$$

Where the Remainder R is a polynomial all of whose terms contain a factor which is a derivative of N and derivatives of w up to order $L - 1$.

Since all the derivatives of w of order up to L are bounded by 1, we can make this remainder as small as we please by making smallness assumptions on $||N||_{C^L}$

The first term can be bounded by $\|A_U^{-1}\|(\|A_S\|+2\|N\|_{C^1})^L$ and, by the assumption iv) this can be made strictly smaller than 1 by assuming smallness conditions in $\|N\|_{C^1}$.

In order to adjust conditions d) we will not make further smallness assumptions for N but rather adjust the ε's. This is the recursion, alluded to above, that allows to determine the ε's.

Taking $L+i$ derivatives we have, in a way similar to (4.3)

$$\begin{aligned} D^{L+i}\mathcal{T}[w](x) =& A_U^{-1}D^{L+i}w(A_Sx+w(x))(A_S+D_1N_S(x,w(x))^{\otimes L+i} \\ &+\ D_2N_S(x,w(x))Dw(x))^{\otimes L+i} \\ &+R \end{aligned} \tag{4.4}$$

Where R is an polynomial expression that involves only derivatives of w up to order $L+i-1$ and derivatives of N. Hence, if N is fixed, and w satisfies the inequalities in d) up to $i-1$, we can bound:

$$\|D^{L+i}\mathcal{T}[w](x)\| \le \|A_U^{-1}\|(\|A_S\|+2\|N\|_{C^1})^{L+1}\varepsilon_i + R(\varepsilon_{i-1}\dots\varepsilon_1).$$

Since the factor in front of ε_i in the right hand side above is smaller than 1, assuming we have fixed $\varepsilon_{i-1}\dots\varepsilon_1$, we can choose ε_i big enough so that the right hand side of the previous inequality is smaller than ε_i.

$\blacksquare$

Remark. We emphasize that the choices of ε_i are always possible provided that $||N||_{C^1}$ is sufficiently small. (The smallness assumptions on $\|N\|_{C^L}$ are just to adjust that the derivatives of w to order L are less than 1. Even if we will not use it much in this paper, we point out that it is possible to choose ε_i small if C^r are small. Nevertheless, for us, the main point is that the ε_i can be found in such a way that $\chi^r_{\varepsilon_1,\cdots,\varepsilon_{r-L}}$ are mapped onto itself by $\mathcal{T}$.

The following propositions, consequences of more general results whose proofs can be found in the references indicated, will allow to study fixed points in χ by a small modification of the contraction mapping theorem.

We recall Lemma 2.5 in [La1]

Proposition 4.2. *If we give The χ's defined before the topology of the distance* $d(w,w') = \sup_{x\in B^S(1)} \|D^L w(x) - D^L w'(x)\|$, *the sequence closure of* $\chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}$ *is contained in*

$$\Big\{ w : B^S(1) \mapsto U, \text{ such that}$$

$$\begin{aligned}
&\text{a)}\quad w \in C^{r-1+Lipschitz}\\
&\text{b)}\quad D^k w(0) = 0, 0 \le k \le L\\
&\text{c)}\quad \sup_{x\in B^S(1)} \|D^k w(x)\| \le 1\\
&\text{d)}\quad \sup_{x\in B^S(1)} \|D^{L+i} w(x)\| \le \varepsilon_i \qquad 1 \le i \le r-L-1\\
&\text{e)}\quad Lip(D^r w) \le \varepsilon_{r-L} \Big\}.
\end{aligned}$$

Proof.

This is a particular case of Lemma 2.5 of [La1]. We refer to this paper for the astute proof. We call attention to two subtle points of the proof:

1) We are not considering the closure of the set χ, only the closure under sequences. In a non-separable Hilbert space they could be different. Looking ahead to the argument, we note that, for the contraction mapping principle, what matters is that the sets that get mapped into themselves are sequentially closed.

2) Even if this looks very similar to the Ascoli-Arzelá theorem, it is not necessary to assume that the space has a countable base.

(Of course, the proof can be greatly simplified if the space X is assumed to be separable and, if the space is finite-dimensional, the proof reduces to the Ascoli-Arzelá theorem.)

■

Proposition [Ir1] **4.3.** *The mapping $\mathcal{T}$ considered as a mapping from $C^L(S,U)$ to itself is differentiable in the Frechet sense at all the points of χ^r (Recall that we assumed $r > L$ and both integers. (In the finite dimensional case, it is a Gateaux differential)*

Its differential is:

$$
\begin{aligned}
[D\mathcal{T}(w)]\eta(x) = A_U^{-1}[&\eta(A_S x + N_S(x, w(x))) \\
&+ Dw(A_S x + N_S(x, w(x)))D_2 N_S(x, w(x))\eta(x) \\
&- \; D_2 N_U(x, w(x))\eta(x)].
\end{aligned}
\tag{4.5}
$$

Proof.

Since the operator $\mathcal{T}$ is obtained as a repeated application of the composition operator, the result can be proved by the same methods as those in [Ir1].

For every point x we have the formula

(4.6)

$$
\begin{aligned}
\mathcal{T}[w + \eta](x) = \mathcal{T}[w](x) + &\int_0^1 A_U^{-1}\, \eta(A_S x + N_S(x, (w + \lambda\eta)(x))) \\
+ &\int_0^1 A_U^{-1} Dw(A_S x + N_S(x, (w + \lambda\eta)(x)))\eta(x) D_2 N_S(x, (w + \lambda\eta)(x))\eta(x)\, d\lambda \\
- &\int_0^1 A_U^{-1} D_2 N_U(x, (w + \lambda\eta)(x))\eta(x)\, d\lambda
\end{aligned}
$$

The desired result follows from interpreting the above formula as a formula in Banach space of functions and estimating the continuity of the remainders. This is the same method used in [Ir1] and we refer to this paper for details.

We note that the composition operator, considered as an operator in C^L is differentiable in the Gateaux sense only at functions which are at least C^{L+1} and with a uniform continuity of the derivatives of high order. Other than that, it is only Frechet.

$\blacksquare$

Proposition 4.4. *Assume that N is C^r and that A satisfies the assumptions of Theorem 3.2. Denote by $\chi^r_{\varepsilon_1,\ldots,\varepsilon_{r-L}}$ a set of the type produced in Proposition 4.1. Assume that $\|N\|_{C^L}$, is sufficiently small where the smallness assumptions depend only on ε_1 (and, therefore, only on $|D^{L+1}\|_{L^\infty}$) Then, the derivative in (4.5) is a contraction in $\chi^r_{\varepsilon_1,\cdots,\varepsilon_{r-L}}$ in the norm of the supremum of $D^L w$.*

We emphasize that the smallness assumptions on $\|N\|_{C^L}$ needed for the conclusion of the theorem. are independent on all the derivatives of order higher than $L+1$. This is reasonable because the norm only involves the first L derivatives and we need one more to establish that $\mathcal{T}$ is Lipschitz. The reason we chose the wording of the theorem in this way is that we obtain uniqueness in $\chi^r_{\varepsilon_1,\dots,\varepsilon_{r-L}}$, hence it is somewhat interesting to let the $\varepsilon_1,\dots,\varepsilon_{r-L}$ to be as unrestricted as possible.

We also emphasize that we do not need that $\|D^{L+1}N\|$ is small but rather that the smallness assumptions that we need on $\|N\|_{C^L}$ depend on the bounds that we have for it.

In the applications, however, by restricting our attention to an small enough ball we can assume, as we have discussed, that $\|N\|_{C^{L+1}}$ is small so that for the sake of applications it would have been enough to prove Proposition 4.4 under the assumption that $\|N\|_{C^{L+1}}$ is small.

Proof. In the formula (4.5) for the differential of $\mathcal{T}$ take L derivatives with respect to x, expand using the chain rule and the rules for sums and products (tensor products) of derivatives as often as possible. We get $D^L_x[(D_w\mathcal{T})[w]\eta](x)$ as a sum of terms. The only term in this sum that does not contain a derivative of N is:

$$A_U^{-1}\, D^L\eta(A_S x + N_S(x+w(x))A_S^{\otimes L}. \tag{4.7}$$

All the other terms contain a factor which is a derivative of N of order not bigger than L. The factors other than derivatives of N of order up to L that appear in the other terms term are either derivatives of η of order not bigger than L or derivatives of w of order not bigger than $L+1$.

The only terms that include a derivative of w of order $L+1$ are those resulting form expanding

$$\begin{aligned} &A_U^{-1}(D^{L+1}w)(\ A_S x + N_S(x,w(x))\ )\cdot \\ &\quad\cdot[A_S + (D_1N_S(x,w(x)) + D_2N_S(x,w(x))Dw(x))]^{\otimes L} D_2N_S(x,w(x))\eta(x)) \end{aligned} \tag{4.8}$$

Therefore, except from (4.7) and (4.8), all the other terms involve derivatives of η of order not more than L, derivatives of w of order not more than L and at least one factor which is a derivative of N or order not more than L.

The term (4.7) can be bounded by $\lambda\|D^L\eta\|_{L^\infty}$ where $0<\lambda\equiv\|A_S\|^L\|A_U^{-1}\|<1$ by assumption iv). Hence, the linear operator defined in (4.7) has a norm which is strictly smaller than 1.

To bound (4.8), we note that the $L+1$ derivative of w is bounded by ε_1 and all the other derivatives of w are bounded by 1. Recalling that, since $D^i\eta(0)=0$ for $0\le i\le L-1$ we can bound the supremum of η by $1/L!\|D^L\eta||_{L^\infty}$. Therefore, we can bound the norm of the linear operator in (4.8) by

$$(1/L!)\varepsilon_1||A_U^{-1}||(||A_S||+2|DN\|_{L^\infty})\|DN\|_{L^\infty}.$$

This, clearly, can be made arbitrarily small by assuming that $\|DN\|_{L^\infty}$ is sufficiently small.

Since all the other terms include at least a factor which is a derivative of N of order not bigger than L and derivatives of w of order not larger than L and derivatives of η up to order L the norm of all the other terms can be bounded by $\rho\|\eta\|_{C^L}$ where ρ can be made as close to zero as desired by making $\|N\|_{C^L}$ sufficiently small. That is, we can estimate the norm of the derivative by a number, which is strictly smaller than 1 and a finite number of other terms that can be made arbitrarily small by assuming that $||N||_{C^L}$ is sufficiently small.

Hence, by the contraction mapping theorem, there is a unique fixed point in the sequence closure of χ^r. This finishes the proof of the Theorem 3.2 except for the C^∞, C^ω cases. (Notice that this method automatically produces the uniqueness statement claimed in the theorem).

To prove the C^∞ result, the key observation (standard in the theory, see [La1], [HPS]) is that all the fixed points in different χ^r have to agree.

Using only smallness assumptions in $||N||_{C^L}$, for any r we have the choices of $\varepsilon_1,\ldots,\varepsilon_{r-L}$ in such a way that $\chi^r_{\varepsilon_1,\ldots,\varepsilon_{r-L}}$ is mapped into itself and $\mathcal{T}$ is a contraction. We have that if $r'>r$, then $\chi^{r'}_{\varepsilon_1,\ldots,\varepsilon_{r'-L}}\subset\chi^r_{\varepsilon_1,\ldots,\varepsilon_{r-L}}$ The fixed points produced for every r have to be the same.

We emphasize that here we use essentially that S is contained in the unit circle. Indeed, there are examples in which the spectrum of A_S contains the unit circle and in which there is no C^∞ invariant manifold.

The proof of C^ω regularity is simpler. We have to consider $\mathcal{T}$ acting on a space of analytic functions vanishing up to order L at the origin with the C^L norm in a complex neighborhood of the space. The same argument used here shows it is a contraction (the properties of the absolute value are the same be it real or complex) and the uniform limit of analytic functions is analytic ([Ka] ch. 7).

$\blacksquare$

Proposition 4.5. *If S, S' are spectral subsets both of which satisfy the assumption of the theorem and $S \subset S'$, then $W^S \subset W^{S'}$.*

Proof. If we consider the derivative at zero of $f|_{W^{S'}}$ we see that its spectrum is precisely S' and, clearly S satisfies the non-resonance assumptions. Hence, we can find a C^r manifold associated to it in the restriction to $W^{S'}$.

Now, this manifold can be considered as a submanifold of X, and it fulfills the assumptions of the uniqueness theorem. So it is W^S, hence W^S is contained in W^{S^1}.

Remark. When X is finite dimensional the conclusion $b)$ of Theorem 3.1 can be improved to C^r using the same proof. The idea of the proof is showing that the sequence of functions $\{D^r \mathcal{T}^n[w](x)\}_{n=0}^\infty$ are equicontinuous and equibounded in n. The proof is not very difficult given the bounds we have already developed and a similar calculation is in [La1]. Unfortunately, this argument does not work in infinite dimensional spaces because uniform continuity on the unit ball does not follow from continuity. So, another argument is needed. For the stable manifold case, C^r regularity even in infinite dimensional Banach spaces is proved by a different argument in [HP]. It seems that this argument can be adapted to our case, but since this borderline regularity in infinite dimensional spaces seems specialized, we postpone the discussion of this point.

The following proposition – whose proof is well known, will show that Theorem 3.1 follows from Theorem 3.2.

Proposition 4.6. *Given C^∞ function f satisfying assumptions $ii), iii), v), vi)$ of Theorem 3.1 , there is a C^∞ map ϕ with a C^∞ local inverse such that* *i)* $\phi(0) = 0$

ii) $D\phi(0) = \mathrm{Id}$

iii) $\phi^{-1} \circ f \circ \phi$ *verifies the assumptions of Theorem 3.2.*

Remark. We emphasize that Proposition 4.6 does not require assumption *iv)* of Theorem 3.1. That is, we do not require that $\sigma(A_S)$ is contained inside the unit disk. This will become crucial when we discuss pseudostable non-resonant sets.

Proof. (See any of [La], [St], [Ne], [BM] among many others for very similar computations.)

We try to write $\phi = \phi^2 \circ \ldots \circ \phi^L$ where each of the ϕ^i can be written as $\phi^i = Id + \phi^i_U$ and $\phi^i_U(x, y)$ only depends on the first argument and is multilinear of order i.

The implicit function theorem shows that ϕ^i has a local inverse $(\phi^i)^{-1}(x, y) = (x, y) - \phi^i_U(x) + \mathcal{O}(\|x\|^{i+1})$.

Our goal is to determine ϕ^i_i so that $\Pi_U(\phi^i)^{-1} \circ \ldots \circ (\phi^i)^{-1} \circ f \circ \phi^i \ldots \phi^1(x, y) = \Pi_U f(y) + \mathcal{O}(\|x\|^{i+1})$.

This can be achieved by recursively finding ϕ^i_U recursively If we assume that $\phi^l_U, l < i$ are already known . Substituting in the equation in Proposition 4.6, we see that ϕ^i_U should satisfy an equation of the form of the form:

$$\phi^i_U(A_S(x)) - A_U\phi^i(x) = h^i(x)$$

where h^i is a multilinear function of degree i that can be computed out of the previously known ones.

These equations can be solved because the operator induced by A_S on the multilinear functions has spectrum contained in the ith set product of the spectrum of A_S.

In the case X is finite dimensional and A diagonalizable one can choose a basis for multilinear functions as the monomials of degree i in the coordinates in a basis of eigenvectors. The result, however can be proved easily even if A is not diagonalizable and X is infinite dimensional. See e.g. [Ne]

$\blacksquare$

Remark. We notice for future reference that the ϕ^i and the ϕ can be chosen in such a way that they depend in a C^ω fashion on N, A_S, A_U.

Remark. The introduction of the intermediate reduction using a change of variables is not really necessary. Alternatively, one could write

$$w(x) = \sum^{L} \tfrac{1}{i!} D^i w(0) x^{\otimes i} + w^{[>L]}(x)$$

and, matching powers derive equations for the $D^i w(0)$, which can be solved because of the non-resonance conditions.

It is easy to see that if the $D^i w(0)$ solve these equations

$$\mathcal{T}(\sum^{L} \tfrac{1}{i!} D^i w(0) x^{\otimes i} + w^{[>L]}(x) = \sum^{L} \tfrac{1}{i} D^i w(0) x^{\otimes i} + \widetilde{\mathcal{T}}(w^{[>L]}(x)).$$

The operator $\widetilde{\mathcal{T}}$ can be studied by the same methods $\mathcal{T}$ was studied here. But the computations are much more cumbersome.

5. Dependence of the manifolds on the map and uniqueness results.

We can think of Theorem 3.2 as defining a mapping that, given any N produces w. Since our point of view was thinking of these results as perturbations of $N \equiv 0$, it is quite natural to investigate the dependence of w on N.

Theorem 5.1. *Assume the conditions of Theorem 3.2 as well as smallness assumptions in $\|N\|_{C^r}$. If we give the w's the C^{L+k} norm and the N's the C^r norm the mapping $N \longrightarrow w(N)$ is $C^{r-(L+k+3)}$ provided $r > L+k+3$. An analogous result holds for the invariant manifolds constructed in Theorem 3.1.*

Proof. If we were going to prove that the mapping was differentiable the most natural to thing to do would be to write down explicitly the N dependence in $\mathcal{T}$ and apply the implicit function theorem to the functional equation $\mathcal{T}(w, N) = w$.

Unfortunately, this fails because in order to compute $D_w \mathcal{T}$, the first derivative of w enters. (See (4.3))

However, to prove differentiability on parameters it is possible to follow the strategy of the proof of the implicit function theorem observing that at the solutions, the mapping $\mathcal{T}$ is differentiable with respect to w and, moreover

$$D\mathcal{T} : C^{L+k} \longrightarrow C^{L+k}$$

is a contraction.

(This follow from C^{L+k+1} smallness assumptions in N and in w, the later are implied by C^{L+k+2} smallness assumptions in N.)

Consequently, the mapping $N \mapsto w(N)$, which,in principle, is only continuous has a candidate W' for a derivative at the solutions of $\mathcal{T}(w) = w$)

$$W'(N) = \sum_k^\infty [D_w \mathcal{T}]^k \, D_N \mathcal{T}$$

when w is a solution.

(This follows by repeating the argument that lead to conclude that $D_w \mathcal{T}$ is a contraction in $C^L \longrightarrow C^L$. The assumptions needed to repeat the argument are C^{L+k+1} smallness in N and in w, but the later are implied by C^{L+k+2} smallness in N.)

Since the sum $\sum_k [D_w \mathcal{T}]^k$ converges, this map is well defined and is a continuous function.

The fact that this is the true derivative comes from the fact that it is continuous and, if we integrate back,

$$\widetilde{w}(N_\lambda) = w(N_0 + \int_0^\lambda \; w'(N_0 + t(N - N_0)) \, dt$$

satisfies

$$\begin{aligned}
&\mathcal{T}(\widetilde{w}(N_0), N_0) - \widetilde{w}(N_0) = 0 \\
&\tfrac{d}{d\lambda}[\mathcal{T}(\widetilde{w}(N_\lambda), N_\lambda) - \widetilde{w}(N_\lambda)] = 0
\end{aligned}$$

so that, by the uniqueness properties established in theorem 2, we have that $\widetilde{w} \equiv w$ or that w' is a bona fide derivative of w.

Once we have that w is differentiable, the expression for the derivative we just need to observe that, working on a segment, we have estimates for the remainder of the first order Taylor formula, which are uniform in the segment – here we use one derivative more in f – so that indeed w' is a true derivative.

The existence of higher derivatives can be established in the same way or invoking the " tangent functor trick" of [AR]. We leave the details to the reader.

This is, of course the crucial step in the proof of smooth dependence of the manifolds on the map. To finish the proof we only have to verify that the other step in the reduction – choosing coordinates along the spectral subspaces – depends smoothly on the map. But the smooth dependence of the spectral spaces on the linear map is an standard result in functional analysis (see e.g. [Ka].)

The uniqueness part of theorem 2 can be considerably strengthened. Since this may be useful for other developments, we will formulate it precisely.

Denote by r_0 a number – not necessarily an integer – in this section such that

$$\|A_U^{-1}\| \, \|A_S\|^{r_0} < 1.$$

(That is, $r_0 < \log \|A_U^{-1}\|/\|A_S\|$) Then set:

$$\chi^\delta = \Big\{ w : B^S(1) \longrightarrow U | w(0) = 0; \qquad w\text{Lipschitz}; \\ \frac{\mathrm{Lip}(w|_{B^S(r)})}{r^{(r_0-1)}} \le \delta \ r > 0 \Big\}$$

where Lip denotes the Lipschitz constant and, for the sake of typography are suppressing the dependence of χ^δ on r_0.

We observe that then, for all w in this space

$$\big| |w| \big| = \sup_{x \neq 0} \frac{\|w(x)\|}{\|x\|^{r_0}}$$

is finite and, if we topologize χ^σ with this norm it is complete.

Theorem 5.2. *In the assumptions of Theorem 3.2, there exist a δ such that $\mathcal{T}(\chi^\sigma) \subset \chi^\sigma$ and $\mathcal{T}$ is a contraction there. Hence, the w of theorem 2, which is actually smooth, is the only one function in χ^δ satisfying $\mathcal{T}w = w$.*

Proof. The proof is a quite straightforward calculation. We first show $\mathcal{T}$ is a contraction

$$
\begin{aligned}
[\mathcal{T}u](x) - [\mathcal{T}w](x) = & A_U^{-1}\big[(u(A_S x + N_S(x, u(x))) - w(A_S x + N_S(x, u(x))))\\
& + (w(A_S x + N_S(x, u(x))) - w(A_S x + N_S(x, w(x)))\\
& + (N_u(x, w(x))) - N_u(x, u(x)))\big].
\end{aligned}
$$

Taking norms and dividing by $\|x\|^{r_0}$, the first term can be estimated by inserting in the numerator and denominator $\|A_S x + N_S(x, u(x))\|^{r_0}$; the second one uses that $\|N_S(x, u(x)) - N_S(x, w(x))\| \le (\text{Lip } N_S(x, u))\, \|u(x) - w(x)\|$ and that this factor can be made as small as we wish by assumption and a similar argument works for the last.

The proof that $\mathcal{T}(\chi^\delta) \subset \chi^\delta$ follows easily from the chain rule for Lipschitz constants we have:

$$
\text{Lip } \mathcal{T}[w]\big|_{B^S(r)} \le \ \|A_U^{-1}\|(\|A_S\| + \text{Lip } N_S(1+\delta))\,\text{Lip } w|_{B^S(r^*)} + \text{Lip } N_U(r_0 + \delta)
$$

where $r^* \ge \sup\limits_{|x|>r} \|A_S x + N_S(x, w(x))\|$ which up to errors arbitrarily small by suitable smallness assumptions is just $\|A_S\| r$.

If we now divide both sides by $r^{r_0 - 1}$ we get

$$
\begin{aligned}
\frac{\text{Lip } \mathcal{T}[w]\big|_{B^S(r)}}{r^{r_0-1}} \quad & \le \quad \|A_U^{-1}\|(\|A_S\| + \gamma)\frac{\text{Lip } w|_{B^S(r^*)}}{r^{*\,r_0-1}} \quad \frac{r^{*\,r_0-1}}{r^{r_0-1}}\\
& \le \quad \|A_U^{-1}\|(\|A_S\| + \gamma)^{r_0} \quad \frac{\text{Lip } w|_{B^S(r^*)}}{r^{*\,r_0-1}}
\end{aligned}
$$

where γ denotes a number which can be made arbitrarily small by assuming smallness conditions on Lip N.

$\blacksquare$

The following characterization of invariant manifolds is more restrictive in the conditions we impose in the spectrum, but, on the other hand does not make any regularity assumptions.

This characterization roughly says that if we only consider orbits for which the components are bounded by a power of the S component, the S component determines all of them. In other words, restricted to these "parabolic region" the set of points that converge is a graph.

This is quite analogue to the usual proof of the fact that the stable manifold (characterized only by topological properties) is indeed the graph of a function.

Theorem 5.3. *Let f, A, S, U be as in Theorem 3.1. Call*

$$\Gamma_{\rho,C,\ell} = \{x \in B^\ell \,|\, \|\Pi_U x\| \le C\|\Pi_S x\|^\rho\}.$$

If ρ satisfies $\rho < \frac{\ln \|A_U^{-1}\|^{-1}}{\ln \|A_S\|}$ we can find $\ell^*(\rho)$ in such a way that if two points x_1, x_2 satisfy $\Pi_S x_1 = \Pi_S x_2$ and $\{f^n(x_i)\}_{n=0}^\infty \subset \Gamma_{\rho,C,\ell}$ (any $C \ge \ell < \ell^*(\rho)$) then $x_1 = x_2$.

Proof. We will denote by $\varepsilon = \sup_{x \in B^\ell} \|DM(x)\|$. We will show that if ε is small enough, which amounts to ℓ small enough, we get the conclusions of the theorem. We have if $x, y \in B^\ell$

$$\|\Pi_U f(x) - \Pi_U f(y)\| \ge \|A_U^{-1}\|^{-1} \|\Pi_U(x-y)\| - \varepsilon\|\Pi_S(x-y)\|$$
$$\|\Pi_S f(x) - \Pi_S(f_y)\| \le \|A_S\| \, \|\Pi_S(x-y)\| + \varepsilon\|\Pi_U(x-y)\|.$$

If we consider

$$\begin{pmatrix} \|A_S\| & \varepsilon \\ -\varepsilon & \|A_U^{-1}\| \end{pmatrix}^n \quad \begin{pmatrix} \|\Pi_S(x-y)\| \\ \|\Pi_U(x-y)\| \end{pmatrix}$$

the first component gives an upper bound for $\|\Pi_S f^n x - f^n(y)\|$ and the second a lower bound for $\|\Pi_U i(f^n(x) - f^n(y))\|$.

If we diagonalize the matrix we can see that

$$\|\Pi_S f^n(x) - f^n(y)\| \le (\|A_S\| + \varepsilon')^n \; (\|\Pi_S(x-y)\| + \varepsilon'\|\Pi_U(x-y)\|)$$

$$\|\Pi_U f^n(x) - f^n(y)\| \ge (\|A_U^{-1}\| - \varepsilon')^n \; (\|\Pi_U(x-y)\| - \varepsilon'\|\Pi_S(x-y)\|).$$

(ε' depends only on ε and is as small as we wish with ε.)

If we now apply this result taking $x = x_i$, $y = 0$, we get $\|\Pi_S f^n(x_i)\| \leq (\|A_S\| + \varepsilon')^n \|\Pi_S x_i\|$ and, if we apply it with $x = x_1$, $y = x_2$ we obtain:

$$\|\Pi_U(f^n(x_1) - f^n(x_2))\| \geq (\|A_U^{-1}\|^{-1} - \varepsilon)^n \|\Pi_U(x - y)\|.$$

Unless $\|\Pi_U(x - y)\| = 0$, this is a contradiction with the assumption about the orbits of x_1, x_2 satisfying $\|\Pi_S f(x_i)\| \leq C\|\Pi_S f(x_i)\|^\rho$.

Remark. We note that if we have a map satisfying the conditions of Theorem 3.2 and it is $C^{r_0+\epsilon}$, in a sufficiently small ball it has to be be in χ^δ. The reason is because, by the non-resonance argument that we had before, all the derivatives up to order $[r_0]$ have to vanish. Then, the fact that the map is in $C^{r_0+\epsilon}$ implies that the remainder of the Taylor expansion of the derivative has to make it be in χ^δ.

The existence and uniqueness results developed so far can be counterpointed with the following examples:

Example 1

The mapping $(x, y) \mapsto (1/2x, 1/4y)$, besides the spectral subspaces has (x, x^2) as an invariant manifold. This manifold is clearly analytic.

Therefore, in general there could be other invariant manifolds besides the ones we consider here; notice how this example violates the assumptions of both of our uniqueness theorems. This example shows that the parameters appearing in our uniqueness theorems cannot be lowered.

Example 2 Let A be diagonalizable.

If the relation between eigenvalues $\lambda_{i_1} \ldots \lambda_{i_r} = \lambda_1$ holds, the map

$$x \to Ax + \underline{\ell}_1 x_{i_1} \ldots x_{i_r}$$

where $\underline{\ell}_1$ denotes the eigenvector corresponding to λ_1 and x_{i_j} denote the coordinates along the directions of the eigenvectors corresponding to λ_{i_j} does not have a C^r invariant manifold tangent to the invariant subspace spanned by the eigenvectors of $\lambda_{i_1} \ldots \lambda_{i_r}$. (We are not assuming $\lambda_{i_1} \ldots \lambda_{i_r}$ different.)

Proof. Since in a sufficiently small neighborhood we would have that the manifold would have to be a graph, it suffices to show there is no C^r solution of $\mathcal{T}w = w$ (as in Theorem 3.1).

If this w had a Taylor expansion of order r, we could match powers. Substituting the definitions we see this is impossible.

The construction can be easily modified to produce a similar counterexample when the matrix is not diagonalizable. So, the non-resonance assumptions of our theorem are sharp.

6. Partial linearizations and pseudo-stable manifolds.

The following result is proved in [BLW]. (Even if the statement of Theorem 1.1 is only for $\mathbb{R}^n$, the remarks along the proof make it clear that the result is true also for a general Banach space which admits smooth cut-off functions.

We recall that a cut-off function is a function that takes the value one on a ball and the value zero outside a bigger ball. For finite dimensional spaces, the existence of smooth cut-off functions can be proved very easily. On the other hand, for infinite dimensional Banach spaces, it is a non-trivial assumption on the space. For example, the space of continuous functions on the interval does not admit a C^1 cut-off function [K]. More generally, a separable Banach space admits a Frechet differentiable cut-off function if and only if its dual is separable [LeW]. Any Hilbert space admits smooth cut-off functions.

Theorem 6.1. *Let f, g be C^r $r \in \mathbb{N}$ diffeomorphisms of a Banach space, admitting smooth cut-off functions. $f(0) = g(0) = 0$, and let A and B be numbers computed explicitly in the proof which depend only on the spectrum of $Df(0)$.*

Assume

i) $D^i f(0) = D^i g(0) \quad i = 0, \ldots, k < r - 1,$

ii) $\operatorname{Spec} Df(0) \subset \{z \in \mathbb{C} \mid \lambda_-^{-1} \le |z| \le \lambda_+\} \cup \{z \in \mathbb{C} \mid \mu_-^{-1} \le |z| \le \mu_+\}$

for some $0 < \lambda_-^{-1} < \lambda_+ < 1 < \mu_-^{-1} < \mu_+$.

Then, provided that $1 \le \ell < kA - B$, *for some integer* ℓ, *we can find a* C^ℓ *diffeomorphism* h *such that*

$$h^{-1} \circ f \circ h = g$$

on a neighborhood of the origin, $h(0) = 0$, $Dh(0) = Id$.

This paper also contains explicit expressions for the numbers A and B in terms of $\lambda_\pm$, $\mu_\pm$, which undoubtedly are not optimal – indeed [BLW] sketches the proof of some better number for finite-dimensional spaces – but there are examples that show that one cannot get the conjugating map to be as smooth as the order of tangency.

The way that these cut-off functions enter in the proof is in the observation that, if ϕ is a cut-off function, setting

$$\tilde{f}(x) = \phi(x)f(x) + (1 - \phi(x))(N(x) - f(x))$$

the function $\tilde{f}$ is identical with f in a neighborhood of the origin and is globally close to N. In particular, when f is tangent to N to a high order in the origin, $\tilde{f}$ is tangent to a high order to N in the origin and globally close to N.

In the case that we have considered in this paper, the graph transform operator mapped functions defined in a ball into functions defined in a bigger set. If A was not strictly contractive this would not be the case and, then, we would have to deal with functions defined everywhere. But then, it is necessary to have global proximity assumptions

The meaning of Theorem 6.1 is that, if we get maps which are hyperbolic and tangent to one another to a high enough order, we can make a change of variables that is moderately smooth in such a way that they become exactly the same.

Recall that given the non resonance conditions Proposition 4.6 allowed us to make a change of variables in our original map in such a way that it had the form (3.1)

$$f(s,u) = (A_S s + N_S(s,u), A_S u + N_U(s,u)) + O(|u|^{L+1}, |s|^2)$$

We can apply Theorem 6.1 to f and to g defined by:

$$g(s,u) = (A_S s + N_S(s,u), A_U u + N_U(s,u))$$

Notice that, as already noted in the remark after Theorem 3.2, the map g leaves invariant the manifold $u = 0$. Hence, the manifold $W = h(\{(s, 0)\})$ is invariant under f.

Note that the method of partial linearization has the advantages that it gives more detailed dynamical information and that it can deal with non-resonant subsets that have components on both sides of the unit circle.

On the other hand, note that the partial linearization method requires that the map we consider is hyperbolic, the manifolds thus considered are not unique under natural hypotheses, and the regularity of the invariant manifolds is only a fraction of the regularity of the map.

As we have shown the regularity conclusions can be considerably improved in the case the non-resonant spectral subsets are inside the unit circle. In the case that the spectral subset straddles the unit circle, presumably the regularity of the invariant manifolds is not any better than a fraction of that of the original map, even if the exact value of the fraction is, better than that of the partial linearization – and undoubtedly better than that in Theorem 6.1 –.

7. Other results.

Finally let us mention other results that are also concerned with obtaining invariant manifolds on spectral subspaces that are not disks or complements of disks.

The paper [CF] contains a proof of the result Theorem 3.1 using a very different method when the invariant subspace is one-dimensional. The method used there gives not only the invariant manifold but also a very useful parameterization of it. The method presented in this paper can be readily implemented in a computer for analytic mappings.

The paper [Pö] considers finite dimensional analytic maps and sets S of eigenvalues that satisfy a Diophantine condition.

$$|\lambda_{i_1} \cdots \lambda_{i_n} - \lambda_j| \geq \Omega(n), \quad i_1, \cdots, i_n \in S, j \in [0, d] \tag{7.1}$$

where Ω is a decreasing function satisfying the so-called Brjuno condition

$$\sum_{n=1}^{\infty} 2^{-n} \ln \Omega(2^n) < \infty.$$

(For example, $\Omega(n) = Kn^{-\tau}$, the standard condition of Diophantine approximation, satisfies the Brjuno condition.) The conclusions are not only that there exists an invariant manifold, but also that the motion on it is conjugate to its linear part. Note that the above result applies even when the eigenvalues in S lie on the unit circle. In that case, condition (7.1) amounts to a Diophantine condition among the angles of the rotation. It also applies to situations when some eigenvalues λ are inside the unit circle and others outside. In the case that all the eigenvalues are inside the unit circle, the conclusions are stronger than those of Theorem 3.1 – it also concludes equivalence to the linear part – but so are the hypotheses.

Sometimes one can even get invariant manifolds corresponding to eigenspaces of eigenvalues 1. Since 1 leads to resonances, the example 2, shows that one cannot have a general theorem concluding the existence of an invariant manifold for all such maps. On the other hand, if the system contains several parameters of there is an internal symmetry, one can sometimes have results for some values of the parameter. (They are harder to prove since one does not have that the linear part is contractive) Such extra parameter problems or symmetries appear naturally in celestial mechanics. For example in the study of "invariant manifolds at infinity". With these motivations, the case of a nilpotent block corresponding to an eigenvalue one is considered in [CFN]. The paper [F] finds necessary and sufficient conditions for the existence of analytic invariant manifolds tangent to eigenvalues equal to 1 in Henon-like mappings. Again they are finite codimension. It seems that similar phenomena appear in the applications of renormalization group in field theory. In the language of field theory, the eigenvalues with modulus 1 are called "marginal". Very often they are precisely 1. (See e.g [GKT] for more details about this problem in field theory.)

8. Some Applications.

In this section we describe three applications of these non-resonant manifolds.

Two of them (beta functions of renormalization group and intermediate foliations) are explained in the literature. The third one — lack of smoothness of invariant circles in Hopf bifurcation — is presumably known to experts. Hence, we are only somewhat sketchy.

8.1. Renormalization group in field theory.

As a first application we call attention to the paper [LMS] where some applications to field theory/statistical mechanics are described (as a matter of fact, the present paper was partly motivated by this application).

Even if the rigorous definition of renormalization group operators as a bona fide differentiable maps in appropriate Banach spaces is not so clear except in some special models, this simplified picture is a heuristic guide.

Strictly speaking our results do not apply to the infinite dimensional models that appear in renormalization, since for some of these infinite dimensional spaces, the A_U are compact and, hence, not invertible. Nevertheless, since the RG philosophy is that field theory phenomena can be described by finite number of parameters, it seems interesting investigate these descriptions. Anyhow, since the main point in this subsection will be to call attention to some potential problems – identified in [LMS] – it seems that identifying them in a simplified model is legitimate. Moreover, it seems that an extention of the results here to the case that A_U is compact can be carried out (in collaboration with C.E. Wayne).

In [LMS] it is explained how the invariant manifolds tangent to the intermediate eigenvalues can be identified with the so-called β-functions.

In this language, the results of this paper show that C^L β-functions are uniquely determined. In particular, there is no reason why these smooth β functions connect the non-trivial fixed point with the trivial one.

Conversely, the trajectories of the renormalization group flow connecting two fixed points should not be expected to be smooth curves.

8.2. Lack of smoothness of invariant circles with periodic orbits.

As a second application, we consider invariant circles in the plane consisting of heteroclinic orbits.

These circles appear naturally after a Hopf bifurcation. When the rotation number of the invariant circle is rational, there is at least one periodic orbit and, indeed, one expects to have two periodic orbits, one stable and one unstable. This will be the situation that we consider.

We will show that, using the theory developed in this paper, we should expect this circles to be only finitely differentiable and that indeed there are finite calculations – that can be carried out either perturbatively or with a finite precision computer – that exclude certain regularities

More precisely, we will consider a circle will consist of two periodic orbits $P = \{p_1, \ldots, p_n\}$, $Q = \{q_1, \ldots, q_n\}$ and manifolds connecting them as described below.

We will have $f(p_i) = p_{i+1}$, $f(q_i) = q_{i+1}$ — hence $f^n(p_i) = p_i$; $f^n(q_i) = q_i$. We will assume that the orbit P has one stable direction and an unstable one while the orbit Q is completely stable.

$$\mathrm{Spec}(Df^n(p_i)) = \{\lambda^p, \mu^p\}$$
$$\mathrm{Spec}(Df^n(q_i)) = \{\lambda^q, \mu^q\}$$

We will assume

$$|\lambda^p|, |\lambda^q| < 1 \ , \qquad |\mu^p| < 1 \quad |\mu^q| > 1 \ , \qquad |\lambda^p| \ |\mu|^q < 1$$

To avoid unnecessary complications, we will also assume that the map f is C^∞.

We will furthermore assume that the unstable manifold of Q is contained in the stable manifold of P.

Such situations arise after a Hopf bifurcation. In this case, we have

$$\begin{aligned} &\lambda^p = 1 - c_1\sqrt{\varepsilon} + o(\varepsilon) \ ; \quad \lambda^q = 1 - c_2\sqrt{\varepsilon} + o(\varepsilon) \\ &|\mu^{p,q} - 1| = o(\varepsilon^{N/2}) \end{aligned}$$

where ε is the bifurcation parameter N is the order of the smallest resonance in the bifurcation (it is at least 4) and c_1, c_2 are positive constants. Hence, the assumption $|\lambda^p| \, |\mu^q| < 1$ is certainly true for small values of the perturbation parameter.

The standard theory of normally hyperbolic manifolds [HPS], [Fe], [Wi] shows that this situation is structurally stable. In particular, the invariant circles continue to exist. Moreover, the invariant circle described before is $C^{r_0 - \delta}$ $\forall \ \delta > 0$, $r_0 = \log|\lambda^p| / \log|\mu^p|$. (This can be seen by observing that, by taking sufficiently high iterates we can get the exponent of normal contraction to be not bigger than $\lambda^p| + \delta$ outside of a neighborhood of the point Q. Also the exponent of contraction outside this neighborhood is not smaller than $\log|\mu^p| - \delta$. We can arrange that the circle minus this neighborhood gets mapped into itself. The standard theory gives us $C^{r_0-\delta}$ in this set. However, the theory of unstable manifolds shows the circle near the orbit Q — it is a piece of the unstable manifold — is C^∞.

We want to show how the theory of non-resonant manifolds developed in this paper allows us to obtain computable conditions that show that this manifold is not $C^{r_0+\delta}$.

Since our goal is to exclude that the circle produced by Hopf bifurcation is $C^{r_0+\varepsilon}$ in concrete cases, we proceed by contradiction.

We assume that it is C^{r_0} and then derive numerical facts that should hold. Given a concrete map, these numerical facts can be refuted by a finite precision calculation. We can also show that they hold only in sets of infinite codimension.

We will distinguish two cases $\ln|\lambda^p|/\ln|\mu^p| \notin \mathbb{N}$ and $\ln|\lambda^p|/\ln|\mu^p| \in \mathbb{N}$. The first case is the generic one. We will refer to it as the non-resonant case.

The linear map $Df^n(p_1)$ has exactly two invariant subspaces. One of them associated to λ^p and another one to μ^p. In the non-resonant case, there are two one dimensional invariant subspace for Df^n. Each of them has a one dimensional non-resonant invariant manifold.

Of course the manifold associated to λ^p is the well known strong stable manifold. We also note that in the circles that appear after the Hopf bifurcation, the unstable manifold of Q does not agree with the strong stable manifold of P.

As we argued in Theorem 3.1 these are the only C^L invariant manifolds in a neighborhood of P. To show that the circle is not C^L we just need to show that these two non-resonant manifolds of P do not agree with the unstable manifold for Q.

For the point of view of numerical applications we just point out that the non-resonant manifolds and the unstable ones are numerically computable with high accuracy with a finite calculation and it is possible to show that they do not agree with a finite calculation. ¿From the theoretical point of view we compute the derivatives with respect to parameters and it is easy to show that, these derivatives depend on the values of the perturbation — and its derivatives — evaluated at different points (roughly the perturbation of the unstable manifold of Q depends on the values of the perturbations near Q and that of the non-resonant manifold at P on the values of the perturbation at P) hence, their agreement is an infinite codimension phenomenon. Even if the manifolds happen to agree at one point a generic perturbation would destroy the agreement.

The resonant case can be handled similarly, but one needs to distinguish different possibilities. The strong stable manifold of P still exists and is smooth but it is possible to check that it does not agree with the unstable manifold of Q. Unless there is certain combination of derivatives that vanish, there is no C^{r_0} intermediate invariant manifold and, of course we are done showing that the circle is not $C^{r_0+\delta}$. If this combination of derivatives vanishes, then, as we showed, there is an intermediate non-resonant manifold but, the same arguments as in the non-resonant case may be used to exclude that it agrees with the unstable manifold of Q and, as before, this leads to the circle not being $C^{r_0+\delta}$.

8.3. Non-resonant invariant foliations.

As a third application we discuss the possibility of extending these results to invariant foliations. Given a diffeomorphism f on a compact manifold it an standard construction [HP], [HPS], [Sh] to consider the operator $\tilde{f}$ acting on C^0 vector fields by

$$(\tilde{f}v)(x) = \exp_x^{-1} f(\exp_{f^{-1}(x)}(v(f^{-1}(x)))$$

where $\exp_x v(x)$ denotes the differential geometry exponential map obtained by flowing a unit of time the geodesic with initial conditions x, $v(x)$. (It is useful to think of $\exp_x v(x)$ as $x + v(x)$. Indeed this is what it amounts to in Euclidean space.)

It is not difficult to check that if $\|v\|_{C^0}$ is sufficiently small $\tilde{f}$ is well defined.

Moreover,

$$[D\tilde{f}(0)] = f^*$$

where f^* is the push forward

$$[f^*v](x) = Df(f^{-1}x)v(f^{-1}(x))$$

The spectrum of f^* in the complexification of C^0 vector fields (to study spectral properties, it is much better to have a complex space) has been intensively studied since [Ma]. In that paper it is shown that if f is an Anosov system the spectrum consists of annuli. Moreover, quite remarkably, the spectral projections associated to each of the annuli correspond to projections over a subbundle. As a corollary of this last property, we obtain that the number of annuli is at most the dimension of the space.

Hence, if a subset of these annuli satisfies the non-resonance conditions of Theorem 3.1 we can obtain non-resonant invariant manifolds for $\tilde{f}$. For the case of an annulus, this non-resonant invariant manifold is constructed directly in [Pe]. In the case where the non-resonant set is the whole stable component (resp. the annuli within a ball of radius $\rho < 1$) this is the way that stable (resp. ρ stable) foliations are constructed in [HP], [HPS].

Unfortunately, the geometric interpretation of the non-resonant invariant manifolds for $\tilde{f}$ is more complicated than that of the stable (or ρ stable ones).

The non-resonant invariant manifolds for $\tilde{f}$ correspond to "*invariant leaf fields*". That is, maps that to each point x associate a leaf L_x – a diffeomorphic image of a disk – in such a way that

$$\begin{aligned} f(L_x) &\subset L_{f(x)} \\ T_x L_x &= E^s_x \end{aligned} \tag{8.1}$$

(The proof consists in walking through the proof of the non-resonant manifold checking that all the steps are bundle maps. Fuller details can be found in [Pe] for the one annulus case or in [JLP] or in lecture notes by the author. In any case, these details can be more or less found in [Sh])

The regularity and uniqueness statements in Theorem 3.1 carry through to show that the leaf fields are characterized uniquely by (8.1) and having C^L leaves. The result in Theorem 3.1 implies that the leaves L_x are $C^{r-1+\mathrm{Lip}}$ if the map is C^r. (It can be improved to C^r.)

In the case of the stable manifold, (ρ-stable manifold) it is possible to show that these leafs are a foliation because

$$y \in \bigcup_{n\geq 0} f^{-n}(L_x) \Leftrightarrow d(f^n(x), f^n(y)) \leq K_{x,y}\lambda^n$$

(resp. $\leq K_{x,y}\rho^n$) and this is clearly an equivalence relation.

Unfortunately, this argument does not carry through for general non-resonant manifolds and indeed the conclusions are false. In [JLP] there are examples where these leaf fields fail to be a foliation in a very strong sense. For a generic map f, any neighborhood contains intersections of leaves.

There is another twist to the discussion of invariant foliations corresponding to these invariant sets. There is another construction of invariant manifolds that correspond to spectral subsets. For example [Ir] (A more modern version with several extensions is [LW]) describes method to construct invariant manifolds (usually called pseudo-stable) associated to spectral sets of the form $\{z \in \mathbb{C} \mid |z| \leq \rho\}$ with $\rho > 1$. By taking intersections of these manifolds with strong stable manifolds of the inverse it is possible to obtain invariant manifolds associated to spectral subsets of the form $\{z \in \mathbb{C} \mid \rho_- \leq |z| \leq \rho_+\}$. We emphasize that the construction of Irwin manifolds does not involve non-resonance conditions.

The somewhat surprising fact is that these Irwin manifolds are not the same as those constructed in this paper even in the case where the manifolds in these paper can be defined. In [LW] one can find examples where these Irwin manifold are not smooth and, therefore do not coincide with the smooth non-resonant manifolds constructed in this paper.

The Irwin construction can be lifted to maps on manifolds. This seems to require extra properties of the manifold such as having $\mathbb{R}^n$ as universal cover and that the map is globally close to linear, in contrast with the situation described here, which can be carried out in any manifold and for any map that whose Mather spectrum satisfies the non-resonance conditions. An example when all these conditions occur is perturbations of linear automorphism of the tori.

When the Irwin construction can be carried out for maps on a manifold, it leads to foliations (It turns out that $y \in W_x^{s,\mathrm{Irwin}} \iff d(f^n(x) - f^n(y)) \leq C_{x,y}\rho_+^n$, $n \geq 0$, with f in the universal cover which is an equivalence relation, so that it indeed is a foliation.) but the leaves may be significantly less smooth than the map – the degree of differentiability is related to the gaps of the Mather spectrum –.

There are also uniqueness statements for these Irwin foliations based on asymptotic behavior or, in the fact that they are foliations. We refer to [LW] for proofs of the results on Irwin pseudostable manifolds.

In summary, the construction of non-resonant invariant manifolds in this paper can be lifted to all manifolds, it produces leaf fields of smooth leaves that, in a generic case, fail to be foliations. Moreover there is another natural construction (Irwin's) that only works on certain manifolds and for certain maps but which produces foliations with leaves that are not smooth.

9. References

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