Macroscopic Quantum Electrodynamics of a Plasma Model:

Derivation of the Vlasov Kinetics

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Abstract.

We derive the large-scale Vlasov kinetics of a plasma model from its underlying quantum electrodynamics. The model comprises a system of non-relativistic electrons, coupled to a quantised electromagnetic field and to a passive, positively charged, neutralising background.

1. Introduction.

This note is concerned with the extraction of the large-scale kinetics of a plasma model from its underlying quantum electrodynamics. The model consists of a system of non-relativistic electrons, coupled to a quantised electromagnetic field and to a passive, neutralising, positively charged background. Its specific form is obtained by coupling the purely electrostatic plasma of [1,2] to a radiation field. Our reason for favouring a quantum, rather than classical, model is simply that quantum mechanics is quintessential both to the stability of matter under electromagnetic interactions [3,4] and to the very specific properties of the plasmas that arise in condensed matter physics [5].

Our principal result is that the macroscopic dynamics of this quantum model corresponds to a *classical* Vlasov-cum-Maxwell hydro-electrodynamics. Its new feature, at least at the level of mathematical physics, is the incorporation of the magnetic and transverse electric fields into the theory: these are certainly of cardinal importance for plasma physics (cf. [6]).

We shall present our material as follows. In Section 2, we shall formulate our model as an interacting quantum system of N electrons and a radiation field in a box of side L, which provides a background of uniformly distributed, neutralising, positive charge. In Section 3, we shall reformulate it on a 'macroscopic' scale, with length unit L, and then simplify it by a short distance cut-off. The resultant model reduces to a mean field theoretic form. In Section 4, we shall present our principal results, namely Theorems 4.1 and 4.2, which assert that the large scale dynamics of the model conforms to the classical Vlasov-cum-Maxwell kinetics in a limit where N tends to infinity and the density, N/L^3, remains fixed and finite. There, we shall observe that, as in [2], this kinetics supports non-equilibrium phase transitions from deterministic (Eulerian) to stochastic flows. Section 5 will be devoted to obtaining key estimates, which we shall employ in Section 6 to extend the Vlasov methodology of [7-10] and [1,2] and thereby to prove the above-mentioned theorems. We shall conclude in Section 7 with some brief comments about outstanding problems.

2. The Model.

This is a system, $\Sigma^{(N,L)}$, consisting of N non- relativistic electrons and their radiation field in a three-dimensional periodic cube, $\Omega^{(L)}$, of side L, which provides a background of passive, uniformly distributed, neutralising positive charge. Thus, $\Omega^{(L)} = (\mathbf{R}/L\mathbf{Z})^3$, and the particle number density, n_0, and classical plasma frequency, ω_p, of the model are given by the formulae

$$n_0 = N/L^3 \tag{2.1}$$

and

$$\omega_p = (n_0\epsilon^2/m)^{1/2}, \tag{2.2}$$

where m and $-\epsilon$ are the electronic mass and charge, respectively. We denote points in $\Omega^{(L)}$ by X, sometimes with indices j or k, and the gradient operator in this space by $\nabla^{(L)}$. Components of vectors in $\mathbf{R}^3$ will generally be indicated by suffixes μ or ν.

We represent the positions and momenta of the electrons by the standard multiplicative and differential operators $\{X_j, P_j = -i\hbar\nabla^{(L)} | j = 1,..,N\}$, acting on the Hilbert space, $\mathcal{H}_{el}^{(N,L)}$, of antisymmetric, square integrable functions on $(\Omega^{(L)})^N$. For notational simplicity, we take the particles to be spinless; though, in view of the stability properties established in [4], the electron spin would cause no difficulties. The radiation is formulated in terms of a transversely gauged vector potential, A, and the transverse electric field, F, which are both Hermitian distribution-valued operators in a Fock-Hilbert space, $\mathcal{H}_{rad}^{(L)}$, and are defined by the following conditions.

(1) A and F satisfy the canonical commutation relations

$$[A_\mu(X), F_\nu(X')] = i\hbar D_{\mu\nu}^{(L)}(X - X'), \tag{2.3}$$

where $D^{(L)}$ is the divergence-free part of the product of the unit tensor in $\mathbf{R}^3$ and the Dirac distribution in $\Omega^{(L)}$, i.e., its Fourier coefficients are

$$\hat{D}_{\mu\nu}^{(L)}(Q) = \int_{\Omega^{(L)}} dX D_{\mu\nu}^{(L)}(X)\exp(-2\pi i Q.X) = \delta_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}(1 - \delta_{Q,0}) \ \forall Q \in (\frac{2\pi\mathbf{Z}}{L})^3.$$

(2) $\mathcal{H}_{rad}^{(L)}$ is the vacuum sector of the free transverse electromagnetic field with Hamiltonian

$$\frac{1}{2}\int_{\Omega^{(L)}} : (F(X))^2 + c^2(\nabla^{(L)}\times A(X))^2 : dX,$$

the colons denoting Wick ordering.

We define $\mathcal{H}^{(N,L)} := \mathcal{H}_{el}^{(N,L)} \otimes \mathcal{H}_{rad}^{(L)}$, and canonically identify operators R, in $\mathcal{H}_{el}^{(N,L)}$, and S, in $\mathcal{H}_{rad}^{(L)}$, with $R \otimes I$ and $I \otimes S$, respectively.

We assume that the interactions are the standard electromagnetic ones, and thus that the Hamiltonian of the model takes the following form (cf [11]).

$$H^{(N,L)} = \sum_{j=1}^{N} \frac{1}{2m}(P_j + \epsilon(A_\kappa(X_j))^2 + \epsilon^2 \sum_{k,l(>k)=1}^{N} U^{(L)}(X_k - X_l) +$$
$$\frac{1}{2}\int_{\Omega^{(L)}} : (F(X))^2 + c^2(\nabla^{(L)} \times A(X))^2 : dX, \tag{2.4}$$

where the replacement of A by A_κ in the first term represents the cut-off obtained by removing from A its Fourier components whose wave-vectors have magnitude greater than $\kappa := \hbar/mc$, and where $U^{(L)}$, the two-body Coulomb interaction between electrons in the presence of the neutralising background is given by (cf. [12])

$$U^{(L)}(X) = L^{-3} \sum_{Q \in (2\pi L^{-1}\mathbf{Z})^3 \backslash \{0\}} \frac{\exp(iQ.X)}{Q^2}. \tag{2.5}$$

Our aim is to investigate the dynamics of the model on the length scale L, in a limit where L and N tend to infinity and the particle density, n_0, remains fixed and finite.

3. The Rescaled Description.

We take our macroscopic description of the model to be a 'large' scale one, where the unit of length is L. Since we know from phenomenological considerations that the corresponding time scale is ω_p^{-1}, we effect this description by rescaling the variables of $\Sigma^{(N,L)}$ so that its units of mass, length and time are m, L and ω_p^{-1}, respectively. In this scaling, Planck's constant is

$$\hbar_N = \frac{\hbar}{mL^2\omega_p} \equiv \frac{\hbar}{m\omega_p}(\frac{n_0}{N})^{2/3}, \tag{3.1}$$

and the speed of light is

$$c_0 = \frac{c}{L\omega_p} \equiv \frac{c}{\omega_p}(\frac{n_0}{N})^{1/3}. \tag{3.2}$$

We formulate our macroscopic description of $\Sigma^{(N,L)}$ by mapping it onto a system, $\Sigma^{(N)}$, of N particles and its radiation field in the unit periodic cube $\Omega := \Omega^{(1)} \equiv (\mathbf{R}/\mathbf{Z})^3$. Thus, we define $\mathcal{H}^{(N)}$ to be $\mathcal{H}^{(N,1)}$ and V to be the canonical isometry of $\mathcal{H}^{(N,L)}$ onto $\mathcal{H}^{(N)}$, corresponding to the mapping $X \to x := X/L$ of $\Omega^{(L)}$ onto Ω. We then define the particle positions and momenta, (x_j, p_j), and the radiation field, (a, f), by the following formulae.

$$x_j := L^{-1} V X_j V^{-1};\ p_j := (mL\omega_p)^{-1} V P_j V^{-1} \equiv -i\hbar_N \nabla_{x_j}, \tag{3.3}$$

where ∇ (or ∇_x) is the gradient operator in Ω, and

$$a(x) := \frac{\epsilon}{mL\omega_p} V A(Lx) V^{-1};\ f(x) := \frac{\epsilon}{mL\omega_p^2} V F(Lx) V^{-1}. \tag{3.4}$$

Likewise, we define the rescaled Hamiltonian, representing the macroscopic description, to be

$$H^{(N)} = (mL^2\omega_p^2)^{-1} V H^{(N,L)} V^{-1}$$

$$= \frac{1}{2}\sum_{j=1}^{N} (p_j + a_\kappa(x_j))^2 + N^{-1}\sum_{j,k(>j)=1}^{N} U(x_j - x_k) +$$

$$\frac{N}{2}\int_\Omega : (f(x))^2 + c_0^2(\nabla \times a(x))^2 : dx, \tag{3.5}$$

where

$$U(x) = \sum_{q\in(2\pi\mathbf{Z})^3\backslash\{0\}} \frac{\exp(iq.x)}{q^2} \tag{3.6}$$

and a_κ is the regularised version of a, obtained by discarding the Fourier components of this field with wave-numbers greater than $\kappa_N = \kappa L \equiv \kappa(N/n_0)^{1/3}$. The magnetic field vector is

$$b = \nabla \times a. \tag{3.7}$$

Further, by equns. (2.1)-(2.3) and (3.4), the fields a and f satisfy the CCR

$$[a_\mu(x), f_\nu(x')] = iN^{-1}\hbar_N D_{\mu\nu}(x - x'), \tag{3.8}$$

where $D \equiv D^{(1)}$ is the transverse part of the product of the unit tensor in $\mathbf{R}^3$ and the Dirac distribution in Ω.

We now regularise the interactions by replacing U and a_κ by their respective convolutions with a positive, $\mathcal{D}-$ class, $L-$independent function g, whose integral over Ω is unity. Thus, in view of the above specification of a_κ, following equn. (3.6), we replace U and a_κ by U_g and $a_g^{(N)}$, respectively, where

$$U_g = g * U;\ a_g^{(N)} = g^{(N)} * a \tag{3.9}$$

and $g^{(N)}$ is the truncated form of g obtained by removal of its Fourier components of wave-vector lying outside the ball of radius $\kappa(N/n_0)^{1/3}$. Evidently, $g^{(N)}$ converges to g in the $\mathcal{D}-$topology, as $N \to \infty$. We define

$$b_g^{(N)} = g^{(N)} * b;\ f_g^{(N)} = g^{(N)} * f. \tag{3.10}$$

Note. The regularisation of the interactions by convolution with g (or $g^{(N)}$) is radically different from that involved in the definition of A_κ, since it corresponds to a *macroscopic* cut-off, at distance proportional to L, when referred back to $\Sigma^{(N,L)}$.

It follows from our specifications that the Hamiltonian of the modified model, $\Sigma_g^{(N)}$, is

$$H_g^{(N)} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 + N^{-1}\sum_{j,k(>j)=1}^{N} U_g(x_j - x_k) + Nh_{rad}, \tag{3.11}$$

where

$$v_j = p_j + a_g^{(N)}(x_j) \tag{3.12}$$

is the velocity of the j'th particle and

$$h_{rad} = \frac{1}{2}\int : (f(x))^2 + c_0^2(b(x))^2 : dx \tag{3.13}$$

is the radiative energy, as measured in units of N. Note that $H_g^{(N)}$ is expressed in terms of gauge invariant operators only.

Fixing of c_0. The physical demand that the particle speeds of $\Sigma_g^{(N)}$ cannot exceed the speed of light for the model implies that c_0 must be at least of the order of unity. In order to meet this demand, we shall treat c_0, rather than c, as a constant parameter, *even when passing to the limit* $N\to\infty$. This is appropriate for a treatment of $\Sigma_g^{(N)}$, where, on the one hand, $N >> 1$, so that this finite system is 'close' to its hydro- thermodynamic limit; while, on the other hand, N is small enough to ensure that c_0, as given by equn. (3.2), is at least of the order of unity. One may easily check that both these requirements can be fulfilled in realistic situations, e.g. with $L = 1$ cm. and $N\simeq 10^{13}$.

To express the fields $b,\ f$ as distribution-valued operators, we introduce the Schwartz space $\mathcal{D}_{tr}$ of infinitely differentiable, divergence-free vector fields in Ω, whose Fourier transforms are fast-decreasing functions on $(2\pi\mathbf{Z})^3$, and we define the 'smeared fields'

$$b(\phi) := \int_\Omega dx b(x).\phi(x);\ f(\psi) := \int_\Omega f(x).\psi(x)\ \forall\phi,\ \psi\in\mathcal{D}_{tr}. \tag{3.14}$$

Thus, $b,\ f$ are maps from $\mathcal{D}_{tr}$ into the self- adjoint operators in $\mathcal{H}$, and the CCR (3.8) yields the following one for these distributions.

$$[b(\phi), f(\psi)] = iN^{-1}\hbar_N(\phi, \nabla\times\psi)\ \forall\phi,\psi\in\mathcal{D}_{tr}, \tag{3.15}$$

where $(.,.)$ denotes the L^2 inner product. Further, by equns. (3.8), (3.12), and (3.15), the only other non-zero commutators between the positions, momenta and fields of the model are the following.

$$[x_{j,\mu}, v_{k,\nu}] = i\hbar_N\delta_{jk}\delta_{\mu\nu}I;\ [v_{j,\mu}, v_{k,\nu}] = i\hbar_N\epsilon_{\mu\nu\sigma}b_{g,\sigma}^{(N)}(x_j)\delta_{jk};$$

$$and\ [v_j, f(\psi)] = i\hbar_N N^{-1}\int dx g^{(N)}(x)\psi(x), \tag{3.16}$$

where ϵ is the alternate tensor, i.e., $\epsilon_{\mu\nu\sigma} = 1$ $(resp.\ -1)$ if (μ,ν,σ) is an even (resp. odd) permutation of $(1,2,3)$, and is otherwise zero.

Thus, the algebra generated by the operators $\{x_j, v_j, b(\phi), f(\psi)|j = 1,..,N;\ \phi,\psi\in\mathcal{D}_{tr}\}$ is closed w.r.t. commutation. We take the Hermitian elements of this algebra to be the observables of $\Sigma_g^{(N)}$ and the density matrices in $\mathcal{H}^{(N)}$ to represent its states, so that the expectation value of an observable, A, in the state, $\rho^{(N)}$, is $\rho^{(N)}(A) := Tr(\rho^{(N)}(A))$.

The dynamics of $\Sigma_g^{(N)}$, in the Schrödinger representation, is given by the unitary transformations $\rho^{(N)}\rightarrow\rho_t^{(N)}$ of its states, with

$$\rho_t^{(N)}=\exp(-iH_g^{(N)}t/\hbar_N)\rho^{(N)}\exp(iH_g^{(N)}t/\hbar_N)\ \forall t\in\mathbf{R}. \tag{3.17}$$

The time-derivatives of the observables are determined, in the Heisenberg picture, by the action on them of the derivation

$$\Lambda_g^{(N)}=\frac{i}{\hbar_N}[H_g^{(N)},.]\ . \tag{3.18}$$

In particular, we see from equations (3.11), (3.15) and (3.16) that this action is given by

$$\Lambda_g^{(N)}x_j=v_j;\ \Lambda_g^{(N)}v_j=-f_g^{(N)}(x_j)-(v_j\times b_g^{(N)}(x_j))_{sym}+N^{-1}\sum\nolimits_{k\neq j}\nabla U_g(x_j-x_k) \tag{3.19}$$

and

$$\Lambda_g^{(N)}b=-\nabla\times f;\ \Lambda_g^{(N)}f=c_0^2\nabla\times b_g+N^{-1}\sum\nolimits_{k=1}^{N}(v_k.D_g^{(N)}(x-x_k))_{sym}, \tag{3.20}$$

where

$$D_g^{(N)}=g^{(N)}*D, \tag{3.21}$$

$(v.D_g^{(N)})_\mu:=\sum_\nu v_\nu D_{g,\mu\nu}^{(N)}$ and the subscript $'sym'$ denotes symmetrised product. Thus, the last term in equn. (3.20) represents the transverse part of the regularised current density.

Comment. The model $\Sigma_g^{(N)}$, which carries the macrodynamics of the original one, $\Sigma^{(N,L)}$, exhibits the hallmarks of a *classical mean field theory.* For, on the one hand, it follows from equns. (3.1), (3.15) and (3.16) that the effective Planck constant, $\hbar_N$, vanishes and that the observables $x_j,\ p_j,\ b,\ f$ all intercommute in the limit $N\rightarrow\infty$; while, on the other hand, the last terms in equns. (3.19) and (3.20) are of typical mean field theoretic form, namely arithmetic means of N copies of single-particle observables.

The following definitions are designed to accommodate the anticipated classical structures of the macroscopic dynamics of the model.

Definition 3.1. (1) We define the 'classical single- particle phase space' $K:=\Omega\times\mathbf{R}^3$ and its dual $\hat{K}:=(2\pi\mathbf{Z})^3\times\mathbf{R}^3$.

(2) We define $\mathcal{P}(K)$ to be the space of probability measures on K, with the vague topology.

(3) We define the Weyl maps, $\{W^{(N,n)}:\hat{K}^n\times(\mathcal{D}_{tr})^2\rightarrow\mathcal{U}(\mathcal{H}^{(N)})|n=0,.\,.,N\}$, by the formula

$$W^{(N,n)}(\xi_1,\eta_1;.\,.;\xi_n,\eta_n;\phi,\psi)=$$

$$\exp(\frac{i}{2}(b(\phi)+f(\psi)))(\Pi_{j=1}^n\exp(\frac{1}{2}\eta_j.v_j)\exp(i\xi_j.x_j)\exp(\frac{1}{2}\eta_j.v_j))\exp(\frac{i}{2}(b(\phi)+f(\psi))). \tag{3.22}$$

(4) We define the quantum characteristic functions $\{C_t^{(N,n)}:\hat{K}^n\times(\mathcal{D}_{tr})^2\rightarrow\mathbf{C}|n=0,..,N\}$ by the formula

$$C_t^{(N,n)}=\rho_t^{(N)}\circ W^{(N,n)}. \tag{3.23}$$

Thus, the characteristic functions $\{C_t^{(N,n)}|n=0,1,..,N\}$ faithfully represent the state $\rho_t^{(N)}$.

4. The Classical Vlasov Limit.

In order to treat the dynamics of $\Sigma_g^{(N)}$ in the limit $N\rightarrow\infty$, we shall need to formulate the evolution of the family of systems $\{\Sigma_g^{(N)}|N\in\mathbf{N}\}$. We assume the following initial conditions.

(I.1). The expectation value of the energy per particle of $\Sigma_g^{(N)}$, in its initial state, $\rho^{(N)}$, is bounded, uniformly w.r.t. N, i.e.,

$$\rho^{(N)}(H_g^{(N)})<\gamma N\ \forall N\in\mathbf{N}, \tag{4.1}$$

where γ is a finite constant. Since the system is conservative, this inequality is equivalent to

$$\rho_t^{(N)}(H_g^{(N)})<\gamma N\ \forall N\in\mathbf{N},\ t\in\mathbf{R}. \tag*{(4.1)$'$}$$

This energy bound for $\Sigma_g^{(N)}$ corresponds to one proportional to $N^{5/3}$ for the original system $\Sigma^{(N,L)}$, and is chosen to represent the situation where the latter is prepared in a state where the charge density and current densities are of the form $\sigma(X/L)$ and $Lu(X/L)$, respectively, with σ and u smooth and $L-$independent. For then, both the particle kinetic energy and the electromagnetic field energy are proportional to $N^{5/3}$.

(I.2) The characteristic functions $C_0^{(N,n)}$ factorise, in the limit $N\rightarrow\infty$, according to the formula

$$\lim_{N\rightarrow\infty}[C_0^{(N,n)}(\xi_1,\eta_1;..,\xi_n,\eta_n;\phi,\psi)-(\Pi_{j=1}^nC_0^{(N,1)}(\xi_j,\eta_j;0,0))C_0^{(N,0)}(\phi,\psi)]=0. \tag{4.2}$$

This condition represents the situation where $\Sigma^{(N,L)}$ is prepared in a pure phase, carrying correlations only of short range, which scale down to zero range for $\Sigma_g^{(N)}$ in the limit $N\rightarrow\infty$.

(I.3) The initial state of the radiation field is macroscopically coherent, in that its fluctuations reduce to zero, on the $\Sigma_g^{(N)}$ scale, in the limit $N\rightarrow\infty$, i.e.,

$$\lim_{N\rightarrow\infty}C_0^{(N,0)}(\phi,\psi)=\mathrm{exp}i(b_0(\phi)+f_0(\psi)), \tag{4.3}$$

where b_0 and f_0 are classical fields.

$(I.4)$ These latter fields are continuous functions on X.

Theorem 4.1. *(1) Under the assumption (I.1), the quantum characteristic functions* $C_{\cdot}^{(N,n)}$ *tend pointwise and subsequentially, as* $N \rightarrow \infty$, *to classical ones,* $C_{\cdot}^{(n)}$, *the Fourier transforms of probability measures* $M_{\cdot}^{(n)}$ *on* $K^n \times (\mathcal{D}'_{tr})^2$, *i.e.,*

$$\lim_{N\rightarrow\infty} C_t^{(N,n)}(\xi_1,\eta_1;\,.\,.;\xi_n,\eta_n;\phi,\psi) =$$

$$\int dM_t^{(n)}(x_1,.\,.,x_n;v_1,.\,.,v_n;b,f)\mathrm{expi}[\sum_{j=1}^n(\xi_j.x_j+\eta_j.v_j)+b(\phi)+f(\psi)]. \tag{4.4}$$

(2) The set $\{M_t^{(n)} | n \in \mathbf{N}\}$ *canonically defines a probability measure* M_t *on* $K^{\mathbf{N}} \otimes (\mathcal{D}'_{tr})^2$, *that is symmetric w.r.t. the* $K-$*components and satisfies the conditions*

$$\int dM_t v_j^2 \leq \gamma_1 \ \forall j \in \mathbf{N}, \ t \in \mathbf{R} \tag{4.5}$$

and

$$\int dM_t(b(\phi))^2 \leq c_0^{-2}\gamma_1 \|\phi\|^2; \ \int dM_t(f(\psi))^2 < \gamma_1 \|\psi\|^2 \ \forall \phi,\psi \in \mathcal{D}_{tr}, \ t \in \mathbf{R}, \tag{4.6}$$

where γ_1 *is a finite constant.*

Note. Here, x_j, v_j, b and f denote classical quantities, whereas previously they represented quantum observables. In the theory that follows, it should be clear, in each specific context, whether they are to be interpreted as classical or quantum variables.

Theorem 4.2. *Under the initial conditions (I.1)-(I.4),*

(1) the convergence of the characteristic functions in Theorem 4.1 becomes fully sequential;

(2) M_t *takes the form*

$$M_t = m_t^{\otimes \mathbf{N}} \otimes \delta_{b_t,f_t} \ \forall t \in \mathbf{R}, \tag{4.7}$$

where m_t *is a probability measure on* K, *which satisfies the condition*

$$\int dm_t v^2 \leq \gamma_1 \ \forall t \in \mathbf{R} \tag{4.8}$$

and δ_{b_t,f_t} *is the Dirac measure on* $(\mathcal{D}'_{tr})^2$, *with support at a point* (b_t, f_t); *and*

(3) (m_t, b_t, f_t) *is the unique solution of the following classical Vlasov-Maxwell equations, subject to the condition (4.8):-*

$$\frac{d}{dt}\int dm_t h = \int dm_t (v.\nabla_x h - (e_{g,t} + v \times b_{g,t}).\nabla_v h) \ \forall h \in C_0^{(1,1)}(K), \tag{4.9}$$

$$\frac{\partial b_t}{\partial t} = -\nabla \times f_t \tag{4.10}$$

and

$$\frac{\partial f_t}{\partial t} = c_0^2 \nabla \times b_t + \int dm_t * v.D_g, \tag{4.11}$$

where $*$ *denotes convulution w.r.t. the position variable* x *only,*

$$e_{g,t} = f_{g,t} + \int dm_t * \nabla U_g \tag{4.12}$$

and

$$b_{g,t} = g * b_t;\ f_{g,t} = g * f_t;\ D_g = g * D. \tag{4.13}$$

Thus, these equations define a one-parameter group $\{\tau_t | t \in \mathbf{R}\}$ *of transformations of the space* $\mathcal{P}(K) \times (\mathcal{D}'_{tr})^2$*, according to the formula*

$$\tau_t(m_0, b_0, f_0) = (m_t, b_t, f_t). \tag{4.14}$$

Note on Hydrodynamical Phase Transitions.

In a previous work [2], we showed that the electrostatic model supports transitions from Eulerian deterministic to stochastic flow when the initial conditions are sufficiently non-uniform. The present model must exhibit these same transitions, since the Vlasov-Maxwell equations (4.9)-(4.13) harbour the solutions of those of the electrostatic model, with $b = f = 0$. The interesting question, of course, concerns what other hydrodynamical phase transitions the present model may support.

5. Key Estimates.

Lemma 5.1 *Under the assumption (I.1), there is a finite constant,* γ_1*, such that*

$$\rho_t^{(N)}(v_j^2) < \gamma_1,\ \forall t \in \mathbf{R},\ N \in \mathbf{N},\ j = 1, .., N \tag{5.1}$$

and

$$\rho_t^{(N)}(f(\phi)^2 + c_0^2 b(\phi)^2) < \gamma_1 \|\phi\|^2 + N^{-1}\hbar_N c_0 \|\phi\|\ \|\nabla \times \phi\|\ \forall t \in \mathbf{R},\ N \in \mathbf{N},\ \phi \in \mathcal{D}_{tr}, \tag{5.2}$$

where $\|.\|$ *is the* L^2 *norm.*

Proof. Since U_g is bounded, it follows from equns. (3.11), (3.13) and (4.1)$'$, together with the Fermi statistics of the electrons, that the expectation values of v_j^2 and h_{rad}, for the state $\rho_t^{(N)}$, are both majorised by some finite constant, γ_1. This establishes the estimate (5.1) and the inequality

$$\rho_t^{(N)}(h_{rad}) < \gamma_1. \tag{5.3}$$

To obtain the estimate (5.2), we Fourier analyse the test function ϕ and the fields b and f, noting that, for each wave-vector k $(\in (2\pi\mathbf{Z})^3 \backslash \{0\})$, we may choose two unit vectors $u_{k,-1},\ u_{k,1}$, such that the orthonormal vectors $(k^{(1)}\ (:= k/|k|), u_{k,-1}, u_{k,1})$ form a right-handed triad in $\mathbf{R}^3$, i.e.,

$$k^{(1)} \times u_{k,s} = -s u_{k,-s};\ \textit{and}\ u_{k,-1} \times u_{k,1} = k^{(1)}. \tag{5.4}$$

Thus, in view of the CCR (3.15), the Fourier decompositions of ϕ, f and b take the form

$$\phi(x) = \sum\nolimits_{k,s} \hat{\phi}_{k,s} u_{k,s} \mathrm{exp}(ik.x), \tag{5.5}$$

$$f(x) = \sum\nolimits_{k,s} (\frac{1}{2} N^{-1} \hbar_N c_0 |k|)^{\frac{1}{2}} (\alpha_{k,s} + \alpha^{\star}_{-k,s}) u_{k,s} \mathrm{exp}(ik.x) \tag{5.6}$$

and

$$c_0 b(x) = -\sum\nolimits_{k,s} (\frac{1}{2} N^{-1} \hbar_N c_0 |k|)^{\frac{1}{2}} (\alpha_{k,s} - \alpha^{\star}_{-k,s}) s u_{k,-s} \mathrm{exp}(ik.x), \tag{5.7}$$

where k and s run over $(2\pi\mathbf{Z})^3 \backslash \{0\}$ and $\{-1, 1\}$, respectively, and the $\alpha^{\star}$'s and α's are creation and annihilation operators satisfying the CCR

$$[\alpha_{k,s}, \alpha^{\star}_{k',s'}] = \delta_{kk'} \delta_{ss'};\ [\alpha_{k,s}, \alpha_{k',s'}] = 0. \tag{5.8}$$

Hence,

$$f(\phi) = \sum\nolimits_{k,s} (\frac{1}{2} N^{-1} \hbar_N c_0 |k|)^{\frac{1}{2}} (\alpha_{k,s} + \alpha^{\star}_{-k,s}) \hat{\phi}_{-k,s} \tag{5.9}$$

and

$$c_0 b(\phi) = -i\sum\nolimits_{k,s} (\frac{1}{2} N^{-1} \hbar_N c_0 |k|)^{\frac{1}{2}} (\alpha_{k,s} - \alpha^{\star}_{-k,s}) s \hat{\phi}_{-k,-s}. \tag{5.10}$$

It follows from these formulae and the Schwartz inequality that

$$f(\phi)^2 + c_0^2 b(\phi)^2 - (: f(\phi)^2 + c_0^2 b(\phi)^2 :) = N^{-1} \hbar_N \sum\nolimits_{k,s} c_0 |k|\ |\hat{\phi}_{k,s}|^2 \leq$$

$$N^{-1} \hbar_N \sum\nolimits_{k,s} c_0 \|\phi\ \|\ \|\nabla\times\phi\|, \tag{5.11}$$

and also, by equns. (3.13) and (5.3), that

$$\rho_t^{(N)}(: f(\phi)^2 + c_0^2 b(\phi)^2 :) \leq \rho_t^{(N)}(h_{rad}) \|\phi\|^2. \tag{5.12}$$

The estimate (5.2) follows immediately from these last two inequalities and equn. (5.3).

Definition 5.2. (1) For each $R \in \mathbf{N}$, we define $\mathcal{D}_{tr}^{(R)}$ to be the subspace of $\mathcal{D}_{tr}$ whose Fourier coefficients vanish outside the ball of radius $2\pi R$; and, for $\phi \in \mathcal{D}_{tr}$, we define $\phi^{(R)}$ to be the element of $\mathcal{D}_{tr}^{(R)}$ whose Fourier coefficients coincide with those of ϕ inside this ball.

(2) We define the truncated characteristic functions $C_t^{(N,n,R)} : \hat{K}^n \times (\mathcal{D}_{tr})^2 \rightarrow \mathbf{C}$ by the formula

$$C_t^{(N,n,R)}(\xi_1, \eta_1; . .; \xi_n, \eta_n : \phi, \psi) \equiv C_t^{(N,n)}(\xi_1, \eta_1; . .; \xi_n, \eta_n : \phi^{(R)}, \psi^{(R)}). \tag{5.13}$$

Lemma 5.3. *$C_{\cdot}^{(N,n,R)}$ converges pointwise to $C_{\cdot}^{(N,n)}$ as $R \rightarrow \infty$, the convergence being uniform w.r.t. N, t and the ξ's and η's.*

Proof. By equn. (3.22),

$$W^{(N,n)}(\xi_1,\eta_1;\,.\,.;\xi_n,\eta_n;2\phi,2\psi)\equiv$$

$$W^{(N,0)}(\phi,\psi)W^{(N,n)}(\xi_1,\eta_1;\,.\,.;\xi_n,\eta_n;0,0)W^{(N,0)}(\phi,\psi). \tag{5.14}$$

Further, by the CCR (3.15) and equn. (3.22),

$$W^{(N,0)}(\phi,\psi)-W^{(N,0)}(\phi^{(R)},\psi^{(R)})=W^{(N,0)}(\phi,\psi)(I+F^{(N,R)}+G^{(N,R)}), \tag{5.15}$$

where

$$F^{(N,R)}=W^{(N,0)}(\phi-\phi^{(R)},\psi-\psi^{(R)})-I \tag{5.16}$$

and

$$G^{(N,R)}=W^{(N,0)}((\phi-\phi^{(R)}),(\psi-\psi^{(R)}))[\exp(\frac{i}{2}N^{-1}\hbar_N((\psi^{(R)},\phi)-(\phi^{(R)},\psi)))-1], \tag{5.17}$$

$(.,.)$ being the L^2 inner product. Hence, by equns. (3.22), (5.16) and (5.17), together with the inequality $|1-exp(ix)|\leq|x|$, whether x is a real number or Hermitian operator,

$$\rho_t^{(N)}(F^{(N,R)\star}F^{(N,R)})\leq 2\rho_t^{(N)}((b(\phi^{(R)}-\phi))^2+(f(\psi^{(R)}-\psi))^2) \tag{5.18}$$

and

$$\|G^{(N,R)}\|\leq\frac{1}{2}N^{-1}\hbar_N(\|\phi^{(R)}-\phi\|\;\|\psi\|+\|\psi^{(R)}-\psi\|\;\|\phi\|), \tag{5.19}$$

this latter estimate being an upper bound of the modulus of the exponent in equn. (5.17). It follows from the last two inequalities and Def. 5.2 that both $\rho_t^{(N)}(F^{(N,R)\star}F^{(N,R)})$ and $\|G^{(N,R)}\|$ tend to zero, uniformly w.r.t. N and all other arguments, as $R\rightarrow\infty$; and by equns. (3.23) and (5.13)-(5.15), together with the uniform boundedness of $W^{(N,n)}$, this implies the required result.

6. Proof of Theorems.

Proof of Theorem 4.1. This is based on an extension of the method employed in [9] and [1] to obtain the corresponding result in the absence of the quantum field (b,f).

(1) We start by noting that $C_t^{(N,n,R)}$, as defined in Def. 5.2(2), may be canonically identified with the restriction of $C_t^{(N,n)}$ to $\hat{K}^n\times(\mathcal{D}_{tr}^{(R)})^2$. We next observe that, by equns. (3.17), (3.18), (3.22) and (3.23),

$$\frac{d}{dt}C_t^{(N,n)}=\rho_t^{(N)}\circ\Lambda_g^{(N)}W^{(N,n)}, \tag{6.1}$$

the r.h.s. of this equation being well-defined, by Lemma 5.1 and the fact that, by equns. (3.11), (3.15), (3.16) and (3.18), $\Lambda_g^{(N)}W^{(N,n)}$ is of the form $W^{(N,n)}\mathcal{Q}$, where $\mathcal{Q}$ is a linear combination of velocities, smeared fields and bounded operators.

It follows now from equns. (3.22), (3.23) and (6.1), as restricted to $\hat{K}^n\times(\mathcal{D}_{tr}^{(R)})^2$, together with the commutation relations (3.15) and (3.16), that the derivatives of $C_{\cdot}^{(N,n,R)}$ w.r.t. the continuous variables t, η_j and the Fourier coefficients of $\phi^{(R)}, \psi^{(R)}$, of which there are but a finite number, are bounded, uniformly on the compacts. Consequently (cf. [9]), it may be inferred from the Arzela-Ascoli theorem that $C_{\cdot}^{(N,n,R)}$ converges subsequentially and pointwise, over the full range of these arguments, as $N\rightarrow\infty$. Hence, by Lemma 5.1, the same is true for $C_{\cdot}^{(N,n)}$. We denote its limit by $C_{\cdot}^{(n)}$.

In view of the asymptotic commutativity of the observables $x_j, v_j, b(\phi), f(\psi)$, as $N\rightarrow\infty$ (cf. equns. (3.15) and (3.16)), it is now a simple matter to employ Bochner's theorem, as in [9], to prove that $C_t^{(n)}$ is indeed the characteristic function of a classical probability measure $M_t^{(n)}$.

(2) By Def. 3.1(4) and equn. (4.4), $\{M_t^{(n)}|n\in\mathbf{N}\}$ canonically induces a probability measure M_t on $K^{\mathbf{N}}\otimes(\mathcal{D}'_{tr})^2$, according to the specification that $M_t^{(n)}$ is the restriction of M_t to $K^n\otimes(\mathcal{D}'_{tr})^2$. The symmetry of M_t w.r.t. the $K-$components is ensured by the Fermi statistics of the electrons.

To establish the estimate (4.5), we employ equn. (4.4) for the case where $n = 1$ and $\xi_1,\ \phi,\ \psi$ and all but the μ'th component of η_1 vanish. On integrating the resultant equation against $\hat{F}(\eta_{1,\mu})$, where $\hat{F}$ is the Fourier transform of a $\mathcal{D}-$class function F on $\mathbf{R}$, we see from Def. 3.1(4) that

$$\lim_{N\rightarrow\infty}\rho_t^{(N)}(F(v_{1,\mu})) = \int dM_tF(v_{1,\mu}). \tag{6.2}$$

We now choose F to be of the form

$$F(y) = y^2G_R(y), \tag{6.3}$$

where G_R and $1 - G_R$ are positive valued functions, with G_R taking the value unity on the interval $[-R, R]$ and vanishing outside $[-R-1, R+1]$. Thus, defining χ_R to be the index function for $[-R, R]$,It follows from these specifications and equations (4.2), (6.2) and (6.3) that

$$\sum_{\mu=1}^3\int dM_tv_{1,\mu}^2\chi_R(v_{1,\mu})\leq\sum_{\mu=1}^3\int dM_tv_{1,\mu}^2G_R(v_{1,\mu}) =$$

$$\lim_{N\rightarrow\infty}\sum_{\mu=1}^3\rho_t^{(N)}(v_{1,\mu}^2G_R(v_{1,\mu})\leq\gamma_1.$$

Since R may chosen arbitrarily, this estimate implies the validity of equn. (4.5) for $j = 1$, and hence, by symmetry, for all j. The proof of the estimate (4.6) may be obtained analogously.

Definition 6.1. (1) We define $\mathbf{C}_{0,cyl}^{(n)}$ (resp. $\mathbf{C}_{0,cyl}^{(n,1)}$) to be the space of real-valued functions on $K^n\times(\mathcal{D}'_{tr})^2$, whose elements, $F^{(n)}$, are of the form

$$F^{(n)}(x_1, v_1, ;..; x_n, v_n; b, f) = \tilde{F}^{(n)}(x_1, v_1, ;..; x_n, v_n; b(\phi_1), .., b(\phi_m); f(\psi_1), .., f(\psi_m)), \tag{6.4}$$

where $\tilde{F}^{(n)}$ is a continuous (resp. continuously differentiable) function on $K^n \times \mathbf{R}^{2m}$, with compact support, and the ϕ's and ψ's are elements of $\mathcal{D}_{tr}$, with $m < \infty$.

(2) Under the canonical identification of $\mathbf{C}_{0,cyl}^{(n)}$ with a space of functions on $K^{\mathbf{N}} \times (\mathcal{D}'_{tr})^2$, we define $\mathbf{C}_{0,cyl}$ (*resp.* $\mathbf{C}_{0,cyl}^1$) to be $\cup_{n \in \mathbf{N}} \mathbf{C}_{0,cyl}^{(n)}$ (*resp.* $\cup_{n \in \mathbf{N}} \mathbf{C}_{0,cyl}^{(n,1)}$).

(3) For $F^{(n)} \in \mathbf{C}_{0,cyl}^{(n)}$, we define $\hat{F}$ to be the Fourier transform of $\tilde{F}^{(n)}$, i.e.,

$$\hat{F}^{(n)}(\xi_1, \eta_1; . .; \xi_n, \eta_n; q_1, ., q_m; r_1, . ., r_m) =$$

$$\int \tilde{F}^{(n)}(x_1, v_1, ; . .; x_n, v_n; y_1, . ., y_m; z_1, . z_m) \times$$

$$\exp - i[\sum_{j=1}^{n} (x_j . \xi_j + v_j . \eta_j) + \sum_{k=1}^{m} (y_j q_j + z_j r_j)]$$

$$\forall \xi_j \in (2\pi \mathbf{Z})^3, \ \eta_j \in \mathbf{R}^3, \ q_k, r_k \in \mathbf{R}; \ j = 0, . ., n; k = 1, . ., m. \tag{6.5}$$

(4) We define differentiation of the $\mathbf{C}_{0,cyl}^{(n,1)}$-class functions w.r.t. the fields b, f, by the formula

$$\frac{\partial \tilde{F}^{(n)}}{\partial b} = \sum_{k=1}^{m} \frac{\partial \tilde{F}^{(n)}}{\partial b(\phi_j)} \phi_j; \ \frac{\partial \tilde{F}^{(n)}}{\partial f} = \sum_{k=1}^{m} \frac{\partial \tilde{F}^{(n)}}{\partial f(\psi_k)} \psi_k, \tag{6.6}$$

where $\tilde{F}^{(n)}$ is the representative of $F^{(n)}$ given by equn. (6.4).

Definition 6.2. (1) Let $\mathcal{D}_{tr}^{(N)\prime}$ be the subspace of $\mathcal{D}'_{tr}$ consisting of elements whose Fourier transforms have support in the ball of radius κ_N, and let $q \to q^{(N)}$ be the canonical projection of $\mathcal{D}'_{tr}$ onto $\mathcal{D}_{tr}^{(N)\prime}$. We define $\Sigma_{g,c}^{(N)}$ to be the classical system, whose phase space in $K^N \times (\mathcal{D}_{tr}^{(N)\prime})^2$ and whose equations of motion for its time-dependent phase points $(x_{1,t}, v_{1,t}; . .; x_{N,t}, v_{N,t}; b_t^{(N)}, f_t^{(N)})$ are

$$\frac{dx_{j,t}}{dt} = v_{j,t}; \ \frac{dv_{j,t}}{dt} = -(f_{g,t}^{(N)}(x_{j,t}) + v_{j,t} \times b_{g,t}^{(N)}(x_{j,t})) + N^{-1} \sum_{k \neq j} \nabla U_g(x_{j,t} - x_{k,t}) \tag{6.7}$$

and

$$\frac{\partial b_t^{(N)}}{\partial t} = -\nabla \times f_t^{(N)}; \ \frac{\partial f_t^{(N)}(x)}{\partial t} = c_0^2 \nabla \times b_t^{(N)}(x) + N^{-1} \sum_{k=1}^{N} v_{k,t} . D_g^{(N)}(x - x_{k,t}). \tag{6.8}$$

These are equations of motion for a system of a finite number of degrees of freedom, as represented by the x's and v's and the Fourier coefficients of $b^{(N)}$ and $f^{(N)}$; and it follows from standard fixed point methods that they have a unique global solution. We define $T_t^{(N)}$ to be the transformation of $K^N \times (\mathcal{D}_{tr}^{(N)\prime})^2$ that takes $(x_{1,0}, v_{1,0}; . .; x_{N,0}, v_{N,0}; b_0^{(N)}, f_0^{(N)})$ to $(x_{1,t}, v_{1,t}; . .; x_{N,t}, v_{N,t}; b_t^{(N)}, f_t^{(N)})$.

(2) We define $m_t^{(N)}$ is the probability measure on K given by

$$m_t^{(N)} = \sum_{j=1}^{N} \delta_{x_{j,t}, v_{j,t}}. \tag{6.9}$$

Proof of Theorem 4.2. This is a consequence of Theorem 4.1 and the following three theorems, together with the observation that equns. (4.5) and (4.7) imply equn. (4.8).

Theorem 6.3. *Under the assumption (I.1), the probability measure M_t satisfies the Vlasov hierarchy*

$$\frac{d}{dt}\int dM_t F = \int dM_t \mathcal{L}F \ \forall F\in\mathbf{C}^1_{0,cyl}, \tag{6.10}$$

where the restriction $\mathcal{L}^{(n)}$ of L to $\mathbf{C}^{(n,1)}_{0,cyl}$ is given by

$$\mathcal{L}^{(n)} = \sum_{k=1}^{n}[v_k.\nabla_{x_k} - (f_g(x_k) + (v_k\times b_g(x_k) - (\nabla U_g(x_k - x_{n+1})).\nabla_{v_k})]$$
$$-(\nabla\times f).\frac{\partial}{\partial b} + [c_0^2\nabla\times b + (v_k.D_g)(x - x_k)].\frac{\partial}{\partial f}. \tag{6.11}$$

Theorem 6.4. *(1) Assuming the estimate (4.8) and initial condition (I.4), the Vlasov equations (4.9)-(4.13) have a unique global solution.*

(2) This solution (m_t, b_t, f_t) is the limit, as $N\to\infty$, of that $(m_t^{(N)}, a_t^{(N)}, f_t^{(N)})$, as specified in Def. 6.2 for the classical system $\Sigma_{g,c}^{(N)}$, subject to the condition that the initial configuration of the latter system is chosen so that m_0 is the limit of $m_0^{(N)}$.

Theorem 6.5. *Assuming Theorem 4.1(2) and the conditions (I.2) and (I.3), the Vlasov hierarchy (6.10) has the unique solution*

$$M_t = m_t^{\otimes\mathbf{N}}\otimes\delta_{b_t,f_t} \ \forall t\in\mathbf{R},\ n\in\mathbf{N}, \tag{6.12}$$

where (m_t, b_t, f_t) is the solution of the Vlasov equations (4.9)-(4.13).

Note. It is the uniqueness of the solutions both of the latter equations and of the hierarchy (6.10) that renders the subsequential convergence of $C_{\cdot}^{(N,n)}$ fully sequential.

Proof of Theorem 6.3. This is based on an extension of the method employed in [1] for the purely electrostatic case. Thus, we let $F^{(n)}$ be an element of $\mathbf{C}^{(n,1)}_{0,cyl}(K)$, for which $\hat{F}^{(n)}$ is of the Schwartz class $\mathcal{S}$, i.e. fast-decreasing and infinitely differentiable w.r.t. its continuous arguments. On multiplying equn. (6.1) by $\hat{F}^{(n)}$, integrating (or summing) over all arguments, and passing to the limit $N\to\infty$, we infer from equns. (3.15), (3.16), (3.18)-(3.20), Theorem 4.1 and Lemma 5.1, after some straightforward, but rather tedious, manipulations, that M_t satisfies the Vlasov hierarchy (6.10), as restricted to the case where $\hat{F}^{(n)}$ is of class $\mathcal{S}$. The removal of this restriction is then achieved by continuity.

Proof of Theorem 6.4. This is based on an extension of the method devised in [7,8] for the case where there is no radiation field (b, f).

We start by solving equns. (4.10) and (4.11) for (b_t, f_t) in terms of their initial values (b_0, f_0) and the measure $m_{.}$, by means of elementary Green function techniques. Thus,

$$b_t = \dot{\kappa}_t * b_0 - \nabla\times\kappa_t * f_0 + \int_0^t ds\int dm_s * \nabla\times\kappa_{t-s} * v.D_g \tag{6.13}$$

and

$$f_t = c_0^2 \nabla \times \kappa_t * b_0 + \dot{\kappa}_t * f_0 - \int_0^t ds \int dm_s * \dot{\kappa}_{t-s} * v.D_g, \tag{6.14}$$

where

$$\kappa_t(x) = \sum\nolimits_{k \in (2\pi \mathbf{Z})^3} \cos(k.x) \frac{\sin(c_0|k|t)}{c_0|k|t} \tag{6.15}$$

and $\dot{\kappa}_t := \partial \kappa_t / \partial t$.

We now consider equn. (4.9) as an equation of motion for m_t, in which $e_{g,t}, b_{g,t}$ are expressed in terms of this measure via the equns. (4.12), (4.13), (6.13) and (6.14). The auxilliary equations of motion for a particle, for whose phase point m_t is the probability measure, are then (cf. [8])

$$\frac{dx_t}{dt} = v_t; \ \frac{dv_t}{dt} = -(e_{g,t}(x_t) + v_t \times b_{g,t}(x_t)), \tag{6.16}$$

i.e, defining

$$\kappa_{g,t} = g * \kappa_t, \tag{6.17}$$

$$\frac{dx_t}{dt} = v_t; \ \frac{dv_t}{dt} = V_1(x_t, v_t|t) + V_2(x_t|m_t) + \int_0^t ds V_3(x_t, v_t|m_s, t-s), \qquad (6.18))$$

where

$$V_1(x, v|t) = -[c_0^2 \nabla \times \kappa_{g,t} * b_0](x) + v \times [\dot{\kappa}_{g,t} * b_0 - \nabla \times \kappa_{g,t} * f_0](x), \tag{6.19}$$

$$V_2(x|m) = [\int dm * \nabla U_g](x) \tag{6.20}$$

and

$$V_3(x, v|m, t) = [\int dm * \dot{\kappa}_{g,t} * v.D_g](x) - v \times [\int dm * \nabla \times \kappa_{g,t} * v.D_g](x). \tag{6.21}$$

It follows now from the $\mathcal{D}$ property of g and the assumed continuity of (b_0, f_0) that the V's satisfy Lipschitz conditions of the form

$$|V_j(x, v|.) - V_j(x', v'|.)| < B(|x - x'| + |v - v'|) \ for \ j = 1, 2, 3, \tag{6.22}$$

where B is a finite constant, independent of $m_.$ and t.

In view of these conditions, it is a simple matter to adapt Neunzert's treatment [8] of Vlasov dynamics to the present circumstances to infer the existence and uniqueness of the solution of equn. (4.9), when (b_t, f_t) are expressed in terms of (b_0, f_0) by equns. (6.13) and (6.14). The latter equations then provide the solutions for b_t and f_t. This establishes Part (1) of the theorem. Part (2) may similarly be confirmed on the basis of the Lipshitz conditions (6.22).

Proof of Theorem 6.5. This is based on an extension of Spohn's treatment [10] in the absence of the radiation field.

We first note that, by conditions (I.2) and (I.3), the factorisation condition (6.12) holds at $t = 0$, i.e.,

$$M_0 = m_0^{\otimes\mathbf{N}} \otimes \delta_{b_0, f_0}. \tag{6.23}$$

Further, by the symmetry of M_t w.r.t the $K-$components of Z, it follows from the Hewitt-Savage theorem [13] that there is a unique probability measure μ_t on $\mathcal{P}(K)\times(\mathcal{D}'_{tr})^2$, that implements the decomposition of M_t according to the formula

$$M_t = \int d\mu_t(m, b, f) P_{(m,b,f)}, \tag{6.24}$$

where

$$P_{(m,b,f)} := m^{\otimes\mathbf{N}} \otimes \delta_{b,f}. \tag{6.25}$$

Following Spohn [10], we now employ equation (6.10) with F replaced by $F \circ T_s^{(N)}$, for arbitrary $s \in \mathbf{R}$, where $T^{(N)}$ is as defined following equn. (6.8). A straightforward extension of Spohn's argument then yields the following result on passing to the limit $N \to \infty$.

$$(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}) \int d\mu_t(m, b, f) \int dP_{\tau_s(m,b,f)} F = 0 \ \forall F \in \mathbf{C}^1_{0,cyl}, \tag{6.26}$$

where τ is defined by equn. (4.14). Hence,

$$\int d\mu_t P_{(m,b,f)} = \int d\mu_0 P_{\tau_t(m,b,f)}.$$

Consequently, since, by equations (6.23)-(6.25),

$$\mu_0 = \delta_{(m_0, b_0, f_0)},$$

where $\delta_.$ is again the Dirac measure, it follows that

$$\mu_t = \delta_{\tau_t(m_0, b_0, f_0)} \equiv \delta_{(m_t, b_t, f_t)},$$

and therefore, by equn. (6.24), M_t is given by equn. (6.12), as required.

7. Concluding Remarks.

Our derivation of the macroscopic Vlasov electrodynamics of the plasma model from its underlying quantum mechanics depended heavily on

(a) the scaling properties of the model, which led to a simple $N-$ dependence of its Hamiltonian in the large-scale description; and

(b) the truncation of the electromagnetic interactions according to the specifications of equn. (3.9). Here, the cut-off is essentially macroscopic, when referred to the original system $\Sigma^{(N,L)}$ (cf. the Note following equn. (3.10)).

Now, in the simpler case of purely electrostatic interactions, we have argued [1,2] that (b) should not be essential to the large-scale Vlasov dynamics, since the combined effects of the Heisenberg and Pauli principles may be expected blunt the singularity in the true Coulomb forces and thus effectively truncate them. We envisage that this should also be the case for the present model, though the retardation of the interactions mediated by the quantum field raises serious problems.

A major deficiency of the present treatment is that it misses any viscosity that the model might possess. There are two reasons for this. Firstly, the cut-off of (b) serves to eliminate the 'close collisions' that could lead to viscosity; and secondly, the employment of the large-scale limit, with or without that cut- off, scales away any Navier-Stokes friction term in the hydrodynamic description of the model (cf [1]).

We conclude, therefore, that two outstanding problems are those of the removal of the cut-off of (b), and the treatment of the kinetics on a smaller scale, where viscous effects can be revealed.

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