Recent Developments in Macroscopic Quantum Electrodynamics*

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Abstract

We review two recent developments in constructive macroscopic quantum electrodynamics. These concern the derivation of the large-scale dynamical properties of plasma and laser models from their underlying quantum structures. In the case of the plasma model, consisting of a non-relativistic system of electrons coupled to a quantised electromagnetic field, we show that the macroscopic dynamics is governed by classical Vlasov-Maxwell equations and supports transitions from deterministic to stochastic flows. In the case of the laser model, which is a new version of that of Hepp and Lieb [HL], recast within the framework of quantum dynamical semigroups, we obtain a generalisation of the HL theory and show that it supports optically chaotic phases, as well as the usual quiescent and coherent ones.

1. Introduction

This article will be devoted to a review of two recent developments in constructive macroscopic quantum electrodynamics. These are concerned with the large-scale dynamical properties of models of a plasma and of a laser.

The plasma model is a system of non-relativistic electrons, coupled via regularised interactions to a quantised electromagnetic field and a passive neutralising positively charged background. Our treatment of this model, which we present in Section 2, is based on an extension of the methods devised previously [Se1,2] for the case where the interactions were purely electrostatic. As in that case, we establish that, in a suitable large-scale limit, the dynamics of the plasma is governed by classical Vlasov-cum-Maxwell equations, and exhibits a hydrodynamical phase transition from a deterministic Eulerian flow to a stochastic one, when the initial conditions become sufficiently non-uniform.

The laser model is a new version [AS] of that of Dicke [Di] and Hepp and Lieb [HL], recast within the framework of quantum dynamical semigroups and stochastic processes. Thus, whereas the HL model is a conservative system, consisting of matter, radiation and reservoirs (pumps and sinks), ours is an open dissipative system of matter and radiation, whose dynamics is governed by a one-parameter semigroup, which incorporates the action of the reservoirs. The recasting of the model in this way enables us to take advantage of the theory of quantum Markov processes [AFL, HP, Ku], and thereby to gain some new

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perspectives on the theory. Our treatment of the model, which we present in Section 3, leads to a generalisation of the HL results. In particular, it yields a macroscopic dynamics that supports optically chaotic phases, as well as the coherent and quiescent ones.

Both of the models presented here reduce to mean field theories. In Section 4, we shall briefly discuss the common features of our treatments of them, as well as the possible extension of our methods to more realistic models.

2. The Plasma Model

2.1 The Model.

This is a system, $\Sigma^{(N,L)}$, consisting of N non- relativistic electrons and their radiation field in a three-dimensional periodic cube, $\Omega^{(L)}$, of side L, which carries a fixed, uniform, neutralising charge background. Thus, $\Omega^{(L)} = (\mathbf{R}/L\mathbf{Z})^3$, and the particle number density, n_0, and classical plasma frequency, ω_p, of the model are given by the formulae

$$n_0 = N/L^3, \tag{2.1}$$

and

$$\omega_p = (n_0\epsilon^2/m)^{1/2} \tag{2.2}$$

where m and ϵ are the electronic mass and charge, respectively. We denote points in $\Omega^{(L)}$ by X, sometimes with indices j or k, and the gradient operator in this space by $\nabla^{(L)}$. Components of $\mathbf{R}^3$ vectors will generally be indicated by suffixes μ or ν.

We represent the positions and momenta of the electrons by the standard multiplicative and differential operators $\{X_j, P_j = -i\hbar\nabla^{(L)} | j = 1, . ., N\}$, acting on the Hilbert space, $\mathcal{H}_{el}^{(N,L)}$, of antisymmetric, square integrable functions on $(\Omega^{(L)})^N$.

The interactions are assumed to be the standard electromagnetic ones. These comprise the electrostatic two-body potential, $\epsilon^2 U^{(L)}$, and the quantum field of a transversely gauged vector potential, A, corresponding to the magnetic field B :

$$B = \nabla^{(L)} \times A \tag{2.3}$$

In view of the uniform charge background, the Coulomb potential $U^{(L)}$ may be expressed in the form [BP]

$$U^{(L)}(X) = L^{-3} \sum_{Q \in (2\pi L^{-1}\mathbf{Z})^3 \backslash \{0\}} \frac{\exp(iQ.X)}{Q^2}, \tag{2.4}$$

while A and its canonical conjugate, F, the transverse electric field, are defined as Hermitian distribution-valued operators in a Fock-Hilbert space, $\mathcal{H}_{rad}^{(L)}$, by the following conditions.

(1) A and F satisfy the canonical commutation relations

$$[A_\mu(X), F_\nu(X')] = i\hbar D_{\mu\nu}^{(L)}(X - X') \tag{2.5}$$

where $D^{(L)}$ is the divergence-free part of the product of the unit tensor in $\mathbf{R}^3$ and the Dirac distribution in $\Omega^{(L)}$, i.e., its Fourier coefficients are

$$\hat{D}^{(L)}_{\mu\nu}(Q)=\int_{\Omega^{(L)}}dXD^{(L)}_{\mu\nu}(X)\exp(2\pi iQ.X)=\delta_{\mu\nu}-\frac{Q_\mu Q_\nu}{Q^2}(1-\delta_{Q,0})\ \forall Q\in(\frac{2\pi\mathbf{Z}}{L})^3$$

(2) $\mathcal{H}^{(L)}_{rad}$ is the vacuum sector of the free transverse electromagnetic field with Hamiltonian

$$\frac{1}{2}\int_{\Omega^{(L)}}:(F(X))^2+c^2(\nabla^{(L)}\times A(X))^2):dX,$$

the colons denoting Wick ordering.

Thus, we represent the electronic and radiative field observables by the self-adjoint operators in $\mathcal{H}^{(N,L)}_{el}$ and $\mathcal{H}^{(L)}_{rad}$, respectively. We now define $\mathcal{H}^{(N,L)}:=\mathcal{H}^{(N,L)}_{el}\otimes\mathcal{H}^{(L)}_{rad}$, and canonically identify operators R, in $\mathcal{H}^{(N,L)}_{el}$, and S, in $\mathcal{H}^{(L)}_{rad}$, with $R\otimes I$ and $I\otimes S$, respectively. We take the observables and states of whole system $\Sigma^{(N,L)}$ to be the self-adjoint operators and density matrices, repectively, in $\mathcal{H}^{(N,L)}$.

We assume that the Hamiltonian, $H^{(N,L)}$, for the model is of the standard form for non-relativistic particles with the above electromagnetic interactions. Thus [BFS],

$$H^{(N,L)}=\sum_{j=1}^N\frac{1}{2m}(P_j-\epsilon(A_\kappa(X_j)^2)+\epsilon^2\sum_{k,l(>k)=1}^NU^{(L)}(X_k-X_l)+$$

$$\frac{1}{2}\int_{\Omega^{(L)}}dX:(F(X))^2+c^2(\nabla^{(L)}\times A(X))^2: \tag{2.6}$$

where the replacement of A by A_κ in the first term represents a cut-off obtained by removing from A its Fourier components whose wave-vectors have magnitude greater than $\kappa:=\hbar/mc$.

Our aim now is to investigate the dynamics of the model on the length scale L, in a limit where L and N tend to infinity and the particle density n_0 remains fixed and finite.

2.2. The Rescaled Description.

We take our macroscopic description of the model to be the 'large' scale one, where the unit of length is L. Since we know from phenomenological considerations that the corresponding time scale is ω_p^{-1}, we effect this description by rescaling the variables of $\Sigma^{(N,L)}$ so that its units of mass, length and time are m, L and ω_p^{-1}, respectively. In this scaling, Planck's constant is

$$\hbar_N=\frac{\hbar}{mL^2\omega_p}\equiv\frac{\hbar}{m\omega_p}(\frac{n_0}{N})^{2/3} \tag{2.7}$$

and the speed of light is

$$c_0=c/L\omega_p \tag{2.8}$$

We formulate our macroscopic description of $\Sigma^{(N,L)}$ by mapping it onto a system $\Sigma^{(N)}$ of N particles and its radiation field in the unit periodic cube $\Omega := (\mathbf{R}/\mathbf{Z})^3$. Thus, we define $\mathcal{H}^{(N)}$ to be the Hilbert space of square integrable, antisymmetric functions on Ω^N, and V to be the canonical isometry of $\mathcal{H}^{(N,L)}$ onto $\mathcal{H}^{(N)}$, corresponding to the mapping $X \rightarrow x := X/L$ of $\Omega^{(L)}$ onto Ω. We then define the particle positions and momenta, (x_j, p_j), and the radiation field, (a, f), to the macroscopic description, by the following formulae.

$$x_j := L^{-1} V X_j V^{-1};\ p_j := (mL\omega_p)^{-1} V P_j V^{-1} \equiv -i\hbar_N \nabla_{x_j} \tag{2.9}$$

where ∇ (or ∇_x) is the gradient operator in Ω,

$$a(x) := \frac{\epsilon}{mL\omega_p} V A(Lx) V^{-1};\ f(x) := \frac{\epsilon}{mL\omega_p^2} V F(Lx) V^{-1} \tag{2.10}$$

and

$$H^{(N)} = (mL^2\omega_p)^{-1} V H^{(N,L)} V^{-1}$$

$$= \frac{1}{2}\sum_{j=1}^{N} (p_j - a_\kappa(x_j))^2 + N^{-1}\sum_{j,k(>j)=1}^{N} U(x_j - x_k) +$$

$$\frac{N}{2}\int_\Omega dx : (f(x)^2 + c_0^2(\nabla \times a(x))^2) : \tag{2.11}$$

where

$$U(x) = \sum_{q \in (2\pi\mathbf{Z})^3 \backslash \{0\}} \exp(iq.x)/q^2 \tag{2.12}$$

and a_κ is the regularised version of a, obtained by discarding the Fourier components of this field with wave-numbers greater than $\kappa_N = \kappa L \equiv \kappa(N/n_0)^{1/3}$. The magnetic field vector is

$$b = \nabla \times a \tag{2.13}$$

Thus, in the rescaled description, the system corresponds to a model $\Sigma^{(N)}$, whose observables and normal states are the self-adjoint operators and density matrices, respectively, in $\mathcal{H}^{(N)}$, and whose dynamics is governed by the Hamiltonian $H^{(N)}$. Further, by equns. (2.1), (2.2), (2.5) and (2.10), the fields a and f satisfy the CCR

$$[a_\mu(x), f_\nu(x')] = iN^{-1}\hbar_N D_{\mu\nu}(x - x') \tag{2.14}$$

where $D \equiv D^{(1)}$ is the transverse part of the product of the unit tensor and the Dirac distribution in Ω.

To express the fields $a,\ f$ as distribution-valued operators, we introduce the Schwartz space $\mathcal{D}_{tr}$ of infinitely differentiable, divergence-free vector fields in Ω, whose Fourier transforms are fast-decreasing functions on $(2\pi\mathbf{Z})^3$, and we define the 'smeared fields'

$$a(\phi) := \int_\Omega dx a(x).\phi(x);\ f(\psi) := \int_\Omega f(x).\psi(x)\ \forall \phi,\ \psi \in \mathcal{D}_{tr} \tag{2.15}$$

Thus, $a,\ f$ are maps from $\mathcal{D}_{tr}$ into the self- adjoint operators in $\mathcal{H}$, and the CCR (2.14) may be re- expressed as

$$[a(\phi), f(\psi)] = iN^{-1}\hbar_N \int_\Omega dx\phi(x).\psi(x)\ \forall\phi, \psi\in\mathcal{D}_{tr} \tag{2.14'}$$

Note on c_0. In view of the definition (2.8) of c_0, the physical demand that the particle speeds in $\Sigma^{(N,L)}$ cannot exceed c signifies that c_0 must always be at least of the order of unity. In order to meet this demand, we shall treat c_0, rather than c, as a constant parameter when we pass to the limit $N\to\infty$. In physical terms, this is tantamount to assuming that, in the finite model, $\Sigma^{(N,L)}$, under consideration, the length $c\omega_p^{-1}$ is at least of the order of L and also that $N >> 1$. One may easily check that these requirements may both be fulfilled in realistic situations [Se1].

We now regularise the interactions by replacing U and a_κ by their respective convolutions with a positive, $\mathcal{D}-$ class, $L-$independent function g, whose integral over Ω is unity. Thus, in view of the above specification of a_κ, following equn. (2.12), we replace U and a_κ by U_g and $a_g^{(N)}$, respectively, where

$$U_g = g * U;\ a_g^{(N)} = g^{(N)} * a \tag{2.16}$$

and $g^{(N)}$ is the truncated form of g obtained by removal of its Fourier components of wave-vector lying outside the ball of radius $\kappa(N/n_0)^{1/3}$. Evidently, $g^{(N)}$ converges to g in the $\mathcal{D}$ topology, as $N\to\infty$.

Note. This regularisation is quite different from that involved in the definition of A_κ, since it corresponds to a *macroscopic* cut-off, at distance proportional to L, when referred back to $\Sigma^{(N,L)}$.

It follows from our specifications that the Hamiltonian of the modified model, $\Sigma_g^{(N)}$, is

$$H_g^{(N)} = \frac{1}{2}\sum_{j=1}^N v_j^2 + N^{-1}\sum_{j,k(>j)=1}^N U_g(x_j - x_k) + Nh_{rad} \tag{2.17}$$

where

$$v_j = p_j - a_g^{(N)}(x_j) \tag{2.18}$$

is the velocity of the j'th particle, and

$$h_{rad} = \frac{1}{2}\int dx : ((f(x))^2 + (\nabla\times a(x))^2) : \tag{2.19}$$

is the radiative energy, as measured in units of N.

The algebraic structure of $\Sigma_g^{(N)}$ is governed by the commutation relations between its position, velocity and field observables. In fact, it follows easily from our definitions that the only non-zero commutators between these operators are those given by equn. (2.14) and the following ones.

$$[x_{j,\mu}, v_{k,\nu}] = i\hbar_N \delta_{jk}\delta_{\mu\nu} I;\ [v_{j,\mu}, v_{k,\nu}] = i\hbar_N \epsilon_{\mu\nu\ sigma} b_{g,\sigma}^{(N)}(x_j)\delta_{jk};$$

$$and\ [v_j, f(\psi)] = -i\hbar_N N^{-1}\int dx g^{(N)}(x)\psi(x) \tag{2.20}$$

where ϵ is the alternate tensor, i.e., $\epsilon_{\mu\nu\sigma} = 1\ (resp.\ -1)$ if (μ, ν, σ) is an even (resp. odd) permutaion of $(1, 2, 3)$, and is otherwise zero.

The time-derivatives of the observables are determined by the action on them of the derivation

$$\Lambda_g^{(N)} = \frac{i}{\hbar_N}[H_g^{(N)}, .] \tag{2.21}$$

In particular, we see from equations (2.14), (2.17) and (2.20) that this action is given by

$$\Lambda_g^{(N)} x_j = v_j;\ \Lambda_g^{(N)} v_j = f_g^{(N)}(x_j) - (v_j \times b_g^{(N)}(x_j))_{sym} - N^{-1}\sum\nolimits_{k\neq j} \nabla U_g(x_j - x_k) \tag{2.22}$$

and

$$\Lambda_g^{(N)} a(x) = f(x);\ \Lambda_g^{(N)} f(x) = c_0^2 \Delta a(x) + N^{-1}\Sigma_{k=1}^N (v_k g^{(N)}(x - x_k))_{sym} \tag{2.23}$$

where

$$b_g^{(N)} = g^{(N)} * b\ and\ f_g^{(N)} = g^{(N)} * f \tag{2.24}$$

are the regularised magnetic and transverse electric fields, respectively and the subscript (sym) denotes symmetrised product.

Comment. The model $\Sigma_g^{(N)}$, which carries the macrodynamics of the original one, $\Sigma^{(N,L)}$, exhibits the hallmarks of a *classical mean field theory.* For, on the one hand, it follows from equns. (2.7), (2.14) and (2.20) that the effective Planck constant, $\hbar_N$, vanishes, and that the observables $x_j,\ p_j,\ a,\ f$ all intercommute in the limit $N\to\infty$: while, on the other hand, the last terms in the formulae (2.22) and (2.23) are of typical mean field theoretic form, namely $N^{-1}\sum_{k=1}^N Q_k$, with the Q_k's copies of the same single-particle observable.

2.3. Dynamics of $\Sigma_g^{(N)}$.

We shall now formulate the evolution of this system from an initial, possibly pure, state, represented by a density matrix, $\rho_0^{(N)}$, in $\mathcal{H}^{(N)}$. The evolute of this state at time t is thus

$$\rho_t^{(N)} = \exp(-iH_g^{(N)} t/\hbar_N)\rho_0^{(N)} \exp(iH_g^{(N)} t/\hbar_N) \tag{2.25}$$

The following definition provides a phase space for the anticipated classical limit of the macroscopic dynamics.

Definition 2.1. We define K to be the classical, one- particle phase space $\Omega\times\mathbf{R}^3$ and $\hat{K}$ to be its dual, $(2\pi\mathbf{Z})^3\times\mathbf{R}^3$.

We now represent the state $\rho_t^{(N)}$, by a family of quantum characteristic functions (QCF's)

$$\{C_t^{(N,n)}:\hat{K}^n\times(\mathcal{D}_{tr})^2\to\mathbf{C}|n=0,1,..,N\},$$

as defined by the formula

$$C_t^{(N,n)}(\xi_1,\eta_1;..;\xi_n,\eta_n;\phi,\psi)=$$

$$\langle\rho_t^{(N)};\exp(\frac{i}{2}(a(\phi)+f(\psi))(\Pi_{j=1}^n\exp(\frac{1}{2}\eta_j.v_j)\exp(i\xi_j.x_j)\exp(\frac{1}{2}\eta_j.v_j))\exp(\frac{i}{2}(a(\phi)+f(\psi))\rangle$$

$$\forall(\xi_j,\eta_j)\in\hat{K},\ j=1,..,n\ \phi,\psi\in\mathcal{D}_{tr} \tag{2.26}$$

where $\langle\rho;A\rangle\equiv Tr(\rho A)$. Thus, but for the non-commutativity of the observables, $C_t^{(N,n)}$ would be the characteristic function of a classical probability measure.

We assume the following initial conditions.

$(I.1)$ The expectation value of the energy per particle of $\Sigma^{(N)}$ is bounded, uniformly w.r.t. N, i.e.

$$\rho_0^{(N)}(H_g^{(N)})<\gamma N,\ \forall N\in\mathbf{N} \tag{2.27}$$

where γ is a constant. This energy bound corresponds to one proportional to $N^{5/3}$ for $\Sigma^{(N,L)}$, and is chosen to represent the situation where the latter system is prepared in a state where the charge density and current densities are of the form $\sigma(X/L)$ and $Lu(X/L)$, respectively, with σ and u smooth. For then, both the particle kinetic energy and the electromagnetic field energy are proportional to $N^{5/3}$.

Note. Since $\Sigma_g^{(N)}$ is a conservative system, the condition (2.27) remains valid if $\rho_0^{(N)}$ is replaced by $\rho_t^{(N)}$. Hence, as this state is invariant w.r.t. permutations of the coordinates x_j, it follows easily from equns. (2.17)-(2.19) that we can find a finite constant γ_1, such that

$$\langle\rho_t^{(N)};v_j^2\rangle<\gamma_1;\ \langle\rho_t^{(N)};(a(\nabla\times\phi))^2\rangle<\gamma_1\|\phi\|_2^2;\ and\ \langle\rho_t^{(N)};(f(\psi))^2\rangle<\gamma_1\|\psi\|_2^2;$$

$$\forall N\in\mathbf{N},\ t\in\mathbf{R},\ j=1,..,n,\ \phi,\psi\in\mathcal{D}_{tr} \tag{2.28}$$

where $\|.\|_2$ is the L^2 norm.

$(I.2)$ The characteristic functions $C_0^{(N,n)}$ factorise, in the limit $N\to\infty$, according to the formula

$$\lim_{N\to\infty}[C_0^{(N,n)}(\xi_1,\eta_1;..;\xi_n,\eta_n;\phi,\psi)-(\Pi_{j=1}^n C_0^{(N,1)}(\xi_j,\eta_j;0,0))C_0^{(N,0)}(\phi,\psi)]=0 \tag{2.29}$$

This condition represents the situation where $\Sigma^{(N,L)}$ is prepared in a pure phase, carrying correlations only of short range correlations, which scale down to zero range for $\Sigma_g^{(N)}$ in the limit $N \to \infty$.

$(I.3)$ The initial state of the radiation field is macroscopically coherent, in that it fluctuations reducing to zero on the $\Sigma_g^{(N)}$ scale in the limit $N \to \infty$. Hence,

$$\lim_{N\to\infty} C_0^{(N,0)}(\phi, \psi) = \exp i(a_0(\phi) + f_0(\psi)) \tag{2.30}$$

where a_0 and f_0 are classical fields.

$(I.4)$ These latter fields are continuous functions on X.

2.4. The Vlasov Dynamics.

The following theorem represents our main result, concerning the large-scale classical electrodynamics of the quantum plasma model.

Theorem 2.2 *(1) Under the assumption (I.1-4), and for each* $n \in \mathbf{N}$, $C_t^{(N,n)}$ *converges pointwise, as* $N \to \infty$, *to the characteristic functions of a classical probability measure* $M_t^{(n)}$ *on* $K^{(n)} \times (\mathcal{D}'_{tr})^2$, *i.e.,*

$$\lim_{N\to\infty} C_t^{(N,n)}(\xi_1, \eta_1; . .; \xi_n, \eta_n; \phi, \psi) =$$

$$\int dM_t^{(n)}(x_1, v_1; . .; x_n, v_n; a, f) \exp i[\sum_{j=1}^n (\xi_j . v_j + \eta_j . x_j) + a(\phi) + f(\psi)] \tag{2.31}$$

Furthermore, the set $\{M_t^{(n)} | n \in \mathbf{N}\}$ *are invariant under permutations of the* $(x_j, v_j)'s$ *and canonically defines a probability measure* M_t *on the space* $K^n \times (\mathcal{D}'_{tr})^2$.

(2) M_t *factorises into the form* $m_t^{\otimes \mathbf{N}} \otimes \delta_{a_t, f_t}$, *where* m_t *is a probability measure on* K *and* δ_{a_t, f_t} *is the Dirac measure on* $(\mathcal{D}'_{tr})^2)$ *with support at classical fields* (a_t, f_t).

(3) (m_t, a_t, f_t) *evolves according to the following Vlasov-Maxwell equations, and these have a unique global solution.*

$$\frac{d}{dt} \int dm_t h =$$

$$\int dm_t [v . \nabla_x h + \{f_{g,t} + v \times b_{g,t} - \int dm_t(x', v) \nabla (U_g(x - x')\} . \nabla_v h] \; \forall h \in \mathcal{C}_0^{(1)}(K) \tag{2.32}$$

and

$$c_0^2 \Delta a_t - \frac{\partial^2 a_t}{\partial t^2} = \int dm_t(x', v) v g(x - x') \tag{2.33}$$

where

$$f_t = \frac{\partial a_t}{\partial t}; \; b_t = \nabla \times a_t \tag{2.34}$$

and

$$b_{g,t} = g * a_{g,t}; \; f_{g,t} = g * f_t \tag{2.35}$$

The proof of this theorem, which we shall provide elsewhere [Se3], is based on a generalisation of the methods employed in the purely electrostatic case. The main steps in this proof are the following.

(a) We extend the method of [NS] to infer part (1) of the theorem, as restricted to a subsequence of N's, from the equations of motion for $\Sigma_g^{(N)}$, together with the commutation relations (2.14) and (2.20) and the condition (2.28) (i.e. $(I.1)$).

(b) We show that conditions (I.2,3) imply the specified factorisation property of the resultant M_t at $t = 0$.

(c) We derive a hierarchy of equations of motion for the measures $M_t^{(n)}$ from the corresponding ones for the QCF's $C_t^{(N,n)}$ by passing to the subsequential limit obtained in (a).

(d) We employ (b), (c) and condition (I.4) to generalise the classical methods of Neuzert [Ne] and Spohn [Sp] so as to establish the global existence and uniqueness of the solution of the Vlasov-Maxwell equations (2.32)-(2.35), and to derive these equations from the Vlasov hierarchy. The uniqueness of this solution then serves to render the subsequential convergence of (a) fully sequential.

2.5. Hydrodynamical Phase Transitions.

In a previous work [Se2], we showed that the electrostatic model supports transitions from Eulerian deterministic to stochastic flow when the initial conditions are sufficiently nonuniform. The same is evidently true of the present model, since it may readily be seen that the Vlasov-Maxwell equations (2.32)-(2.35) have solutions for which a and f vanish and m_t satisfies precisely the same conditions that gave rise to those transitions in the electrostatic case. The interesting question, of course, concerns what other hydrodynamical phase transitions the present model may support.

3. The Laser Model

3.1. The Model.

This is a version [AS] of the Dicke-Hepp-Lieb model of matter interacting with radiation. Specifically, it consists of N identical two-level atoms, coupled to n radiative modes by dipolar interactions. Furthermore, each element of the model, whether atom or mode, is an *open dissipative system*, the atoms being coupled to pumps and sinks and the radiation to sinks only.

We formulate the model as a quantum dynamical system $\Sigma = (\mathcal{A}, T, \phi)$, where $\mathcal{A}$ is a $W^\star-$algebra of observables, $\{T(t) | t \in \mathbf{R}_+\}$ is a one-parameter semigroup of normal, completely positive (CP) contractions of $\mathcal{A}$, and ϕ is a normal state on this algebra. Here, the action of the reservoirs on the system is incorporated into the structure of this semigroup. We build the model from its elements, consisting of the atoms and modes.

The Single Atom. We take the single two-level atom to be a quantum dynamical system $\Sigma_{at} = (\mathcal{A}_{at}, T_{at}, \phi_{at})$, with the following specifications.

$\mathcal{A}_{at}$, the $W^{\star}-$algebra of observables of the atom, consists of the 2-by-2 matrices with complex entries, and is therefore the linear span of the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ and the identity I. Its structure is thus given by the relations

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I;\ \sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z,\ etc. \tag{3.1}$$

We define the spin raising and lowering operators

$$\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y) \tag{3.2}$$

$\{T_{at}(t)|t\in\mathbf{R}_+\}$ is a strongly continuous one-parameter semigroup of normal CP contractions of $\mathcal{A}_{at}$, representing the dynamics of the atom. We assume that its generator, L_{at}, is given by the formula

$$L_{at}\sigma_{\pm} = -(\gamma_{\perp}\mp i\epsilon)\sigma_{\pm};\ L_{at}\sigma_z = -\gamma_{\|}(\sigma_z - \eta I);\ L_{at}I = 0 \tag{3.3}$$

where the $\epsilon(>0)$ is the energy difference between the two eigenstates of the atom, the γ's are positive damping constants, and η is a further constant representing the terminal value of σ_z for the isolated atom. In fact, the values of the γ's and η depend on the outcome of the couplings of the atom to its pump and sink, the latter constant being positive when the pumping dominates. Further, the demands of complete positivity impose the constraints [AS]

$$\gamma_{\|}\leq 2\gamma_{\perp};\ -1 < \eta < 1 \tag{3.4}$$

on these parameters.

We take ϕ_{at} to be the unique $T_{at}-$invariant state on $\mathcal{A}_{at}$, and this is given by

$$\phi_{at}(\sigma_z) = \eta;\ \phi_{at}(\sigma_{\pm}) = 0 \tag{3.5}$$

This state is faithful, and the condition that it carries an inverted population is simply that $\eta > 0$.

The Matter. We assume that the matter consists of N non-interacting copies of Σ_{at}, located on the sites $r = 1, .., N$ of a one-dimensional lattice. Thus, to each site r, we assign a copy, $\Sigma_r = (\mathcal{A}_r, T_r, \phi_r)$, of Σ_{at}, and then represent the matter as the $W^{\star}-$dynamical system $\Sigma_{mat} = (\mathcal{A}_{mat}, T_{mat}, \phi_{mat})$, where the elements of this triple are the tensor products of the $\mathcal{A}_r$'s, T_r's and ϕ_r's, respectively. $\mathcal{A}_{mat}$ is therefore faithfully represented as the linear transformations of the Hilbert space $\mathcal{H}_{mat} = \mathbf{C}^{2N}$.

We identify the spin component $\sigma_{u,r}$ $(u = x, y, z, \pm)$, of the atom at r with the element of $\mathcal{A}_{mat}$, given by the tensor product of N elements of $\mathcal{A}_{at}$, of which the r'th is σ_u and the others are I. It follows from these specifications and (3.3) that the action of the generator, L_{mat}, of T_{mat} on the observables of single spins is given by

$$L_{mat}\sigma_{\pm,r} = -(\gamma_{\perp}\mp i\epsilon)\sigma_{\pm,r};\ L_{mat}\sigma_{z,r} = -\gamma_{\|}(\sigma_{z,r} - \eta I);\ L_{mat}I = 0 \tag{3.6}$$

The Radiation. We assume that the radiation model, Σ_{rad}, corresponds to n modes, with frequencies $\omega_0, . ., \omega_{n-1}$, each mode being coupled to its own sink.

We formulate Σ_{rad} as a $W^\star-$ dynamical system $(\mathcal{A}_{rad}, T_{rad}, \phi_{rad})$, in the following way. First, we represent the radiation modes by creation and destruction operators $\{a_l^\star, a_l | l = 0, 1. ., n-1\}$ in a Hilbert space $\mathcal{H}_{rad}$, as defined by the standard conditions that

(1) there is a unit vector, Φ_{rad} in $\mathcal{H}_{rad}$, such that $a_l \Phi_{rad} = 0$ for $l = 0, .. ., n-1$;

(2) $\mathcal{H}_{rad}$ is generated by the application to Φ_{rad} of the polynomials in the $a^\star$'s; and

(3) the a's and $a^\star$'s satisfy the canonical commutation relations

$$[a_l, a_m^\star]_- = \delta_{lm} I; \ [a_l, a_m]_- = 0 \tag{3.7}$$

We then define $\mathcal{A}_{rad}$, the algebra of observables of Σ_{rad}, to be $\mathcal{L}(\mathcal{H}_{rad})$, the set of bounded operators in $\mathcal{H}_{rad}$, and we take ϕ_{rad} to be the vacuum state $(\Phi_{rad}, .\Phi_{rad})$.

We define the Weyl map $z = (z_0, .. ., z_{n-1}) \rightarrow W(z)$ of $\mathbf{C}^n$ into $\mathcal{A}_{rad}$ by the standard prescription

$$W(z) = \exp i(z.a + (z.a)^\star), \ with \ z.a = \sum_{l=0}^{n-1} z_l a_l \tag{3.8}$$

Thus, by (3.7), W satisfies the Weyl algebraic relation

$$W(z)W(z') = W(z+z')\exp(iIm(z,z')_n) \tag{3.9}$$

where $(.,.)_n$ is the $\mathbf{C}^n$ inner product. The algebra of polynomials in $\{W(z)|z \in \mathbf{C}^n\}$ is therefore just their linear span, and is ultraweakly dense in $\mathcal{A}_{rad}$.

We assume that the dynamical semigroup T_{rad} is completely positive, normal and quasi-free, and consequently is given by Vanheuverszwijn's prescription [Vh], i.e.,

$$T_{rad}(t)[W(z)] = W(\xi(t)z)\exp(-\theta(t)) \tag{3.10}$$

where $\xi(t) : \mathbf{C}^n \rightarrow \mathbf{C}^n$ and $\theta : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ are defined in terms of the frequencies ω_l and damping constants κ_l of the modes by the formulae

$$(\xi(t)z)_l = z_l \exp(-(i\omega_l + \kappa_l)t) \ for \ l = 0, .. ., n-1 \tag{3.11}$$

and

$$\theta(t) = \frac{1}{2}(\|z\|_n^2 - \|\xi(t)z\|_n^2) \tag{3.12}$$

where $\|.\|_n$ is the $\mathbf{C}^n$ norm. The generator of the semigroup T_{rad} is

$$L_{rad} = \sum_{l=0}^{n-1} (i[\omega_l a_l^\star a_l, .]_- + 2\kappa_l a_l^\star(.)a_l - \kappa_l [a_l^\star a_l, .]_+) \tag{3.13}$$

The Composite System, Σ, is obtained by coupling Σ_{mat} to Σ_{rad} by dipolar interactions, represented by the partial Hamiltonian

$$H_{int} = iN^{-1/2} \sum_{r=1}^{N} \sum_{l=0}^{n-1} \lambda_l (\sigma_{-,r} a_l^\star \exp(-2\pi i k_l r) - h.c.), \tag{3.14}$$

where k_l is the wave-number of the l'th mode and the λ's are real-valued, $N-$independent coupling constants.

We now formulate Σ as a $W^\star-$dynamical system $(\mathcal{A}, T, \phi)$, as follows.

The algebra of observables $\mathcal{A}$ is the tensor product, $\mathcal{A}_{mat}\otimes\mathcal{A}_{rad}$, of those of the matter and radiation. Thus, $\mathcal{A}=\mathcal{L}(\mathcal{H})$, where $\mathcal{H}=\mathcal{H}_{mat}\otimes\mathcal{H}_{rad}$. We identify $\mathcal{A}_{mat}\otimes I_{rad}$ and $I_{mat}\otimes\mathcal{A}_{rad}$ with $\mathcal{A}_{mat}$ and $\mathcal{A}_{rad}$, respectively, thus rendering them intercommuting subalgebras of $\mathcal{A}$. Correspondingly, if $A_{mat}\in\mathcal{A}_{mat}$ and $B_{rad}\in\mathcal{A}_{rad}$, we denote the tensor product $A_{mat}\otimes B_{rad}$ by $A_{mat}B_{rad}$.

We take the state ϕ to be the tensor product

$$\phi_{mat}\otimes\phi_{rad} \tag{3.15}$$

Finally, we construct the dynamical semigroup semigroup, T, as the perturbation of $T_{mat}\otimes T_{rad}$ corresponding to the interaction Hamiltonian H_{int}. In fact, the construction of this semigroup has been achieved [AS] by a method based on quantum stochastic processes [AFL, HP, Ku], which negotiates the problems caused by the unboundedness of the putative generator of T, namely

$$L=L_{mat}+L_{rad}+i[H_{int},.] \tag{3.16}$$

The net result of this construction is to obtain a dynamical semigroup T, such that

(1) there is a norm-dense set, $\mathcal{S}_0$, of normal states on $\mathcal{A}$, that is stable under the mapping $\rho\rightarrow\rho_t:=\rho\circ T(t)$;

(2) these states are well defined on the algebra $\mathcal{F}(\mathcal{A})$, of the affiliates of $\mathcal{A}$ given by the polynomials in the operators $a_l^\star, a_l$ and $W(z)$;* and

(3)
$$\frac{d}{dt}\rho_t(Q)=\rho_t(LQ)\ \forall\rho\in\mathcal{S}_0,\ Q\in\mathcal{F}(\mathcal{A}) \tag{3.17}$$

where L is given by equn. (3.16).

Note. This result serves to generalise the standard theory of dynamical semigroups [GK, Li] to a class of unbounded generators.

3.2. The Macroscopic Observables.

We take these to be the global intensive observables

$$s_l^{(N)}=N^{-1}\sum\nolimits_{r=1}^{N}\sigma_{-,r}\exp(-2\pi ik_lr);\ l=0,..,n-1 \tag{3.18}$$

* Thus, if ρ is the density matrix representing an $\mathcal{S}_0-$class state and $Q\in\mathcal{F}(\mathcal{A})$, then the operator ρQ is of trace class.

and

$$p_l^{(N)} = N^{-1}\sum\nolimits_{r=1}^{N} \sigma_{z,r}\exp(-2\pi i k_l r);\ l = 0, .. ., n-1 \tag{3.19}$$

together with the operators

$$\alpha_l^{(N)} = N^{-1/2} a_l;\ l = 0, .. ., n-1 \tag{3.20}$$

corresponding to a scaling of the number operators $a_l^\star a_l$ in units of N. We denote the set $\{s_l^{(N)}, s_l^{(N)\star}, p_l^{(N)}, p_l^{(N)\star}, \alpha_l^{(N)}, \alpha_l^{(N)\star} | l = 0, 1, . ., n-1\}$ by $\mathcal{M}^{(N)}$. To simplify the model, we choose

$$k_l = \frac{l}{n} \tag{3.21}$$

so that $\mathcal{M}^{(N)}$ is a Lie algebra w.r.t. commutation. Specifically, by (3.18)-(3.21), its non-zero Lie brackets are the following ones, and their adjoints.

$$[s_l^{(N)}, s_m^{(N)\star}] = -N^{-1} p_{[l-m]}^{(N)};\ [s_l^{(N)}, p_m^{(N)}] = 2N^{-1} s_{[l-m]}^{(N)};$$

$$[s_l^{(N)\star}, p_m^{(N)}] = -2N^{-1} s_{[l+m]}^{(N)\star};\ [a_l^{(N)}, a_m^{(N)\star}] = N^{-1} I \delta_{lm} \tag{3.22}$$

where $[l \pm m] = l \pm m\ (mod\ n)$. Thus, the observables $\mathcal{M}^{(N)}$ become classical in the limit $N \to \infty$. Further, by (3.2), (3.3), (3.18) and (3.19),

$$\|s_l^{(N)}\| = 1;\ \|p_l^{(N)}\| = 1\ for\ l = 0, . ., n-1 \tag{3.23}$$

and

$$p_0^{(N)\star} = p_0^{(N)};\ and\ p_l^{(N)\star} = p_{n-l}^{(N)}\ for\ l = 1, . ., n-1 \tag{3.24}$$

By (3.14) and (3.18)-(3.20), the interaction Hamiltonian H_{int} is a function of the macro-observables only, i.e.,

$$H_{int}^{(N)} = iN\sum\nolimits_{l=0}^{n-1} \lambda_l (\alpha_l^{(N)\star} s_l^{(N)} - \alpha_l^{(N)} s_l^{(N)\star}) \tag{3.25}$$

3.3. The Macroscopic Dynamics.

Our objective will be to extract the dynamics of $\mathcal{M}^{(N)}$ from the microscopic equation of motion (3.17), in a limit where $N \to \infty$ and n remains fixed and finite. Since N is not fixed here, we shall indicate the dependence of Σ, $\mathcal{A}$, T, L, H_{int} and $\mathcal{S}_0$ on this number by the superscript (N).

We shall assume that the initial state, $\rho^{(N)}$, of $\Sigma^{(N)}$ lies in $\mathcal{S}_0^{(N)}$ and that the number of photons it carries does not increase faster than N, i.e. that, for some finite constant B,

$$\rho^{(N)}(\alpha_l^{(N)\star} \alpha_l^{(N)}) < B\ for\ l = 0, .., n-1\ and\ \forall N \in \mathbf{N} \tag{3.26}$$

The following definitions provide a framework for the anticipated classical limit of the dynamics of the macro-observables.

Definition 3.1. (1) We define the space $X = \mathbf{C}^n \times \mathbf{C}^n \times \tilde{\mathbf{C}}^n$, the third factor being the subset of $\mathbf{C}^n$, whose elements $(c_0, . ., c_{n-1})$ satisfy the conditions that $c_0 = \overline{c}_0$ and $\overline{c}_l = c_{n-l}$ for $l = 1, . ., n-1$. Thus, by (3.18)-(3.20) and (3.24), X is the 'phase space' for the expectation values of the macroscopic observables, the means of $\alpha^{(N)}$, $s^{(N)}$ and $p^{(N)}$ being harboured by the factors $\mathbf{C}^n$, $\mathbf{C}^n$ and $\tilde{\mathbf{C}}^n$, respectively.

(2) We denote the points (α, s, p) (or (z, w, v)) of X by x (or y), and correspondingly, we define the measure dx (resp. dy) to be the product of those of the mutually independent real and imaginary components of α, s, p (resp. z, w, v). With these identifications, we equip X with the scalar product

$$x.y = \sum_{l=0}^{n-1} [(\alpha_l z_l + s_l w_l + c.c.) + p_l v_l] \tag{3.27}$$

and define the inverse Fourier transformation $\hat{f} \to f$ of $L^2(X, dx)$ by the formula

$$f(x) = \int_X dy \hat{f}(y) \exp(ix.y) \tag{3.28}$$

(3) We denote the Dirac measure on X with support at an arbitrary point (α, s, p) by $\delta_{\alpha,s,p}$.

(4) We define the quantum characteristic function (QCF) of the macro-observables to be the map $C^{(N)} : X \to \mathbf{C}$, given by the formulae

$$C^{(N)}(z, w, v) = \langle \rho_t^{(N)}; \exp i((z.\alpha^{(N)} + w.s^{(N)} + h.c) + v.p^{(N)}) \rangle \ \forall (z, w, v) \in X \tag{3.29}$$

where, for $z = (z_0, . ., z_{n-1})$, $w = (w_0, . ., w_{n-1})$ and $v = (v_0, . ., v_{n-1})$,

$$z.\alpha^{(N)} = \sum_{l=0}^{n-1} z_l \alpha_l^{(N)}; \ w.s^{(N)} = \sum_{l=0}^{n-1} w_l s_l^{(N)}; \ v.p^{(N)} = \sum_{l=0}^{n-1} v_l p_l^{(N)}$$

The following two theorems were proved in [AS].

Theorem 3.2. *(1) Under the above assumptions, $C_t^{(N)}$ tends pointwise, as $N \to \infty$, to the characteristic function, C_t, of a classical probability measure, m_t, on X, i.e.*

$$\lim_{N \to \infty} C_t^{(N)}(y) = \int_X \exp i(x.y) dm_t(x) \tag{3.30}$$

convergence being uniform on the $X-$compacts.

(2) The evolution of this measure is governed by the following equation of motion, which has a unique global solution.

$$\frac{d}{dt} \int_X dm_t f = \int_X dm_t \mathcal{L} f \ \forall f \in \mathcal{C}_0^{(1)}(X), \ t \in \mathbf{R}_+ \tag{3.31}$$

where

$$\mathcal{L} = A.\frac{\partial}{\partial\alpha} + S.\frac{\partial}{\partial s} + \overline{A}.\frac{\partial}{\partial\overline{\alpha}} + \overline{S}.\frac{\partial}{\partial\overline{s}} + P.\frac{\partial}{\partial p} \tag{3.32}$$

and $A = (A_0, . ., A_{n-1})$, $S = (S_0, . ., S_{n-1})$ *and* $P = (P_0, . ., P_{n-1})$ *are the functions from* X *into* $\mathbf{C}^n$, $\mathbf{C}^n$ *and* $\tilde{\mathbf{C}}^n$, *respectively, given by*

$$A_l(\alpha, s, p) = -(i\omega_l + \kappa_l)\alpha_l + \lambda_l s_l \tag{3.32a}$$

$$S_l(\alpha, s, p) = -(i\epsilon + \gamma_\perp)s_l + \sum\nolimits_{m=0}^{n-1} \lambda_m p_{[l-m]}\alpha_m \tag{3.32b}$$

and

$$P_l(\alpha, s, p) = -\gamma_\parallel(p_l - \eta\delta_{l,0}) - 2\sum\nolimits_{m=0}^{n-1} \lambda_m(\overline{\alpha}_m s_{[l+m]} + \alpha_m \overline{s}_{[m-l]}) \tag{3.32c}$$

with $[l \pm m] = l \pm m \ (mod\ n)$.

Theorem 3.3. *If* m_0 *has support at a single point* (α, s, p), *and thus is the Dirac measure* $\delta_{\alpha,s,p}$, *then* $m_t = \delta_{\alpha_t, s_t, p_t}$, *where* (α_t, s_t, p_t) *is the unique global solution of the equations of motion*

$$\frac{d\alpha_t}{dt} = A(\alpha_t, s_t, p_t); \ \frac{ds_t}{dt} = S(\alpha_t, s_t, p_t); \ \frac{dp_t}{dt} = P(\alpha_t, s_t, p_t) \tag{3.33}$$

for which

$$(\alpha_t, s_t, p_t) = (\alpha, s, p) \ at \ t = 0$$

Thus, if the macro-observables are initially dispersion-free, then they subsequently remain so and evolve according to the deterministic law (3.33).

3.4. Dynamical Phase Transitions.

We now note that the condition that m_0 has point support signifies that the initial state of the system corresponds to a pure phase, in the thermodynamic limit. Assuming this to be the case, we see from Theorem 3.3 that the evolution of the macro-state of the system is deterministic, and governed by the equations of motion (3.33). An analysis of these equations yield the following results.

(1) [HL, AS] The dynamical law (3.33) yields a stable fixed point, at which $p_0 = \eta$ and all other components of (α, s, p) vanish, provided that η is less than a certain positive critical value, η_1, that depends on the other parameters of the model. Thus, for $\eta < \eta_1$, the stable state of the system is optically quiescent.

(2) [HL, AS] As η increases through the value η_1, there is a Hopf bifurcation leading to a stable periodic orbit, in which just one mode, say the k'th, is activated: its selection depends on the parameters of the model. Specifically, the stable solution of equn. (3.33) then takes the form

$$\alpha_{l,t} = a\delta_{kl}\exp(i\nu t); \ s_{l,t} = b\delta_{kl}\exp(i\nu t); \ p_{l,t} = \eta_1\delta_{l0} \tag{3.34}$$

Thus, for η larger than, but sufficiently close to, η_1, the stable state carries coherent radiation [Gl], corresponding to a monochromatic laser beam.

(3) [AS] In the case of single mode, i.e. where $n = 1$, there is a further bifurcation when η increases through a second critical value, η_2 ($> \eta_1$); and then the radiation becomes chaotic by a mechanism precisely analogous to that governing chaos in the famous Lorenz attractor.

(4) [AS] More generally, in the multi-mode case, chaos can arise either by the Ruelle-Takens [RT] or by the Landau mechanism [LL, Section 27]. In the former case, it stems from a bifurcation of the periodic orbit of (2) into a strange attractor: in the latter one, it arises from the successive activation of a very large number of modes.

Thus, the model exhibits optically quiescent, coherent and chaotic phases.

4. Concluding Remarks

We have shown how the passage from the microscopic to the macroscopic picture of both conservative and open quantum systems can lead to rich electrodynamical structures, which support both ordered and chaotic phases.

Our treatment of both the plasma and the laser models was centred on quantum characteristic functions (QCF's) of macroscopic observables, first introduced in [NS]. This provides a natural procedure for extracting classical phenomenological dynamical laws from an underlying quantum dynamics, since, under rather general conditions, these QCF's reduce to those of a classical probability measure in a large-scale limit.

We note that both of the models treated here reduce to mean field theories, with the crucially simplifying feature that their macroscopic variables evolve according to self-contained dynamical laws. In the case of the plasma model, the mean field theoretic character stems from the macroscopic cut-off introduced in equn. (2.16). However, we conjecture (cf. [Se2]) that a proper exploitation of the Heisenberg and Pauli principles could effectively blunt the Coulomb singularity and thus remove the need for the cut-off.

In the case of the laser model, the self-contained character of the macroscopic evolution arises from (a) the fixing of the number n of radiation modes, even in the limit $N \to \infty$, and (b) the stipulation that the wave-numbers satisfy the condition (3.21), which ensures that the macro-observables $\mathcal{M}^{(N)}$ form a Lie algebra. The generalisation of the model to a more realistic one, with $n = N$, would evidently demand radically new insights, since such a model does not appear to manifest any clear separation between its microscopic and macroscopic dynamics.

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