Resonant Decay Near an Accumulation Point

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Abstract: We consider the quantum mechanics of a model system in which meta-stable states arise through perturbation of a sequence of embedded eigenvalues with an embedded accumulation point. It is shown that the embedded eigenvalues become resonances in the perturbed system. These resonances also accumulate, and the position of the accumulation point is unchanged. The positions of the resonances are estimated uniformly up to the accumulation point. The meta-stable states associated with these resonances have the usual approximately exponential decay with time.

§1: Introduction

Hamiltonians of the form

$$H(\lambda) = \begin{pmatrix} A & \lambda V \\ \lambda V^* & B \end{pmatrix}$$

where λ is a real number, and the operators A, B are self-adjoint, have been extensively studied ([Da], [F], [Ho1], [Ho2], [K1], [K2], [W1], [W2]) beginning with the work of Friedrichs on what has become known as the Friedrichs model. In the Friedrichs model B is a rank one operator with a positive eigenvalue E while A has only absolutely continuous spectrum $[0, \infty)$.

These Hamiltonians describe simple quantum mechanical models with resonant behavior. In particular, if the discrete spectrum of the operator B has non-empty intersection with the continuous spectrum of A then the Hamiltonian $H(0)$ has eigenvalues embedded in its continuous spectrum. Typically, embedded eigenvalues are unstable under perturbation and disappear for $\lambda \neq 0$. Under reasonable conditions on V, $H(\lambda)$ is expected to have only absolutely continuous spectrum in a neighborhood of an embedded eigenvalue of $H(0)$. Further, if Ψ is an eigenvector of $H(0)$ corresponding to an embedded eigenvalue then, if $\lambda \neq 0$ is small enough, Ψ is expected to behave like a quasi-stable state under the dynamics generated by $H(\lambda)$.

To discuss the perturbation of an embedded eigenvalue E, the notion of a resonance associated with E is often introduced ([WW], see [S1] [S2] for reviews). There are various ways to define a resonance. For our purposes it suffices to use the following formulation, which seems the simplest. Let Ψ be an eigenvector of $H(0)$ corresponding to E. The matrix element $\langle \Psi, \big(H(\lambda) - z\big)^{-1} \Psi \rangle$ is an analytic function of z in the upper half plane. For $\lambda = 0$ the matrix element $\langle \Psi, \big(H(0) - z\big)^{-1} \Psi \rangle$ is meromorphic in an open disk centered at E with a simple pole at $z = E$. Suppose that, for $\lambda \neq 0$, $\langle \Psi, \big(H(\lambda) - z\big)^{-1} \Psi \rangle$ has a meromorphic continuation to an open disk centered at E, whose only singularity is a simple pole below the real axis. Then this pole is called a *resonance*. Typically, the position of the pole depends on λ and approaches E as $\lambda \to 0$. For well-behaved perturbations V, this position is accurately estimated by lowest order perturbation theory.

Our interest here is in the behavior of resonances near accumulation points, an issue which does not arise in the Friedrichs model. Indeed to the best of our knowledge, ours is the first model of this type in which results have been obtained. The Hamiltonian for an atom in the quantised radiation field ([BFS], [D], [PS]) has this structure, namely an accumulation point embedded in continuous spectrum, as does the Hamiltonian for a hydrogen atom in a sufficiently strong magnetic field ([FW1], [FW2]). Although our model is considerably simpler than these models, we hope that it can point the way toward an analysis of such problems.

For the model that we consider in this paper, the operator A has purely continuous spectrum $[0, \infty)$ while the operator B has discrete spectrum with an accumulation point at the number 1 and continuous spectrum $[1, \infty)$. Note that when $\lambda = 0$ the operator $H(0)$ has embedded eigenvalues with an accumulation point at 1.

Our analysis depends on having sufficient control over $V^*(A-z)^{-1}V$ and over the resolvent of B. In order to allow a complete and straightforward analysis we make simple choices for A, B and V. We choose $A = -\frac{d^2}{dx^2}$ so that we have an explicit expression for the resolvent kernel of A. Further, we choose V to be a multiplication operator with compact support so that $V^*(A-z)^{-1}V$ has an analytic continuation across the positive real line. Finally, we choose $B = -\frac{d^2}{dx^2} - \frac{1}{x} + 1$ with Dirichlet boundary conditions at $x = 0$. The operator B is a simple example of a Schrödinger operator with an accumulation point, and we have detailed estimates for its eigenvalues, discrete eigenfunctions and continuum eigenfunctions [FW1].

Briefly, our results are as follows. For $|\lambda| > 0$ sufficiently small, each embedded eigenvalue of $H(0)$ turns into a resonance of $H(\lambda)$ in the sense discussed above. The position of the resonance agrees to lowest order with the predictions of perturbation theory. Furthermore, the resonances approach the accumulation point of the embedded spectrum in a regular way. Namely, in our model the embedded eigenvalues $E_J = 1 - \frac{1}{4J^2}$ of $H(0)$ approach 1 as $J \to \infty$, in such a way that $J^3\big(E_{J+1} - E_J\big)$ approaches a nonzero limit. The resonances z_J also approach 1 as $J \to \infty$, and in such a way that $J^3\big(z_J - E_J\big)$ approaches a nonzero limit as J$\to \infty$.

Our techniques rely heavily on the explicit form of the operators A, B and V, and they do not generalise easily to other Hamiltonians. Nevertheless we feel that it is important to have a good supply of "exactly solvable" models in quantum mechanics to illustrate the basic phenomena that can occur.

The paper is organised as follows. In section 2 we define the model, explain our goals more precisely and state our results. In section 3 we explore the difficulties presented by the embedded accumulation point, and show how to overcome them in our model. In section 4 we consider resonances near the embedded accumulation point, and prove that they behave in the regular way described above.

§2: Definition of the Model and Statement of Results

The model we consider is defined by the Hamiltonian $H(\lambda)$, acting in the state space $\mathrm{L}^2\big([0,\infty)\big) \oplus \mathrm{L}^2\big([0,\infty)\big)$. $H(\lambda)$ is given by

$$H(\lambda) = \begin{pmatrix} -\frac{d^2}{dx^2} & \lambda u \\ \lambda u & h_0 + 1 \end{pmatrix} \qquad ()$$

where h_0 is the differential operator

$$h_0 = -\frac{d^2}{dx^2} - \frac{1}{x}, \qquad ()$$

λ is a real number and u is a self-adjoint multiplication operator. $H(\lambda)$ is defined on the domain appropriate to Dirichlet boundary conditions at 0, namely

$$D\big(H(\lambda)\big) = \Big\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \,\Big|\, \phi(0) = \psi(0) = 0,\ \phi \in D(-\frac{d^2}{dx^2}) \text{ and } \psi \in D(h_0) \Big\}.$$

Here $D(A)$ is the domain of the operator A.

The function $u(x)$ is real valued and is assumed to be continuous with compact support in $[0,\infty)$. That $H(\lambda)$ is self-adjoint follows immediately from the self-adjointness of the Dirichlet problems for $-\frac{d^2}{dx^2}$ and h_0 on the half-line, from which it follows that $H(0)$ is self-adjoint, and from the fact that u is bounded, from which it then follows that $H(\lambda) - H(0)$ is bounded.

The Hamiltonian h_0 is related to the 0 angular momentum sector of the Hydrogen atom Hamiltonian, see [LL]; its spectrum is

$$\sigma(h_0) = \Big\{ -\frac{1}{4j^2} \,\Big|\, j \in \{1,2,3,\ldots\} \Big\} \cup [0,\infty).$$

The eigenvalues $-\frac{1}{4j^2}$ are simple. For each $j \in \{1,2,3,\ldots\}$ we define ψ_j to be the normalised eigenfunction of h_0 with eigenvalue $-\frac{1}{4j^2}$. In section 3 we will write down the explicit expressions for the eigenfunctions ψ_j, as well as for the continuum eigenfunctions. Similarly, the Hamiltonian $-\frac{d^2}{dx^2}$ has spectrum $\sigma\big((-\frac{d^2}{dx^2})_D\big) = [0,\infty)$ (here the subscript D indicates Dirichlet boundary conditions). It follows that the operator $H(0)$ has spectrum $[0,\infty)$ and has eigenvalues

$$E_j = 1 - \frac{1}{4j^2} \qquad ()$$

embedded in the continuous spectrum. Let

$$\Psi_j = \begin{pmatrix} 0 \\ \psi_j \end{pmatrix}$$

be the corresponding eigenvectors. Note that the eigenvalues of $H(0)$ have an embedded accumulation point at 1.

As a final comment we note that it follows from the assumption that u has compact support that $H(\lambda) - H(0)$ is not only bounded but is also a relatively compact perturbation of $H(0)$ for all λ. It follows that the essential spectrum of $H(\lambda)$ is equal to that of $H(0)$, namely $\sigma_{ess}\big(H(\lambda)\big) = [0, \infty)$.

Our main goal is to analyse resonances of $H(\lambda)$ associated to the embedded eigenvalues of $H(0)$. Define, for each $J \in \{1, 2, 3, \ldots\}$ and for $\operatorname{Im} z > 0$

$$R_J(\lambda; z) = \langle \Psi_J, \big(H(\lambda) - z\big)^{-1} \Psi_J \rangle. \qquad ()$$

Let

$$D_J = \{z \,\big|\, |z - E_J| \le \frac{1}{16J^3}\}. \qquad ()$$

Note that $R_J(0; z)$ has a meromorphic extension to D_J whose only singularity is a simple pole at $z = E_J$. We will find a meromorphic continuation of $R_J(\lambda; z)$ across the positive real half-line for $\lambda \neq 0$. The only singularity of this extension is a simple pole whose position we estimate. Explicitly, there is a $\lambda_0 > 0$, such that for all λ with $|\lambda| < \lambda_0$ and for each J, the function $R_J(\lambda; z)$ has a meromorphic extension to D_J whose only singularity is a simple pole. Let z_J be the position of this pole in D_J. Then $\operatorname{Im} z_J < 0$ and $z_J \to E_J$ as $\lambda \to 0$. We refer to z_J as the perturbative resonance of $H(\lambda)$ associated to E_J.

Further, by Stone's formula (recall that the ψ_j are real),

$$\langle \Psi_J, e^{-itH(\lambda)} \Psi_J \rangle = \frac{1}{\pi} \lim_{\eta \downarrow 0} \int e^{-it\epsilon} \operatorname{Im} R_J(\lambda; \epsilon + i\eta)\, d\epsilon. \qquad ()$$

Our results are enough to easily establish approximate exponential decay in time for $\langle \Psi_J, e^{-itH(\lambda)} \Psi_J \rangle$ and to identify the lifetime of the meta-stable state Ψ_J with the inverse of the imaginary part of the resonance z_J ([H], [S1], [S2], [WW]). We emphasize that our results are uniform in J.

Before stating our results we will need some definitions. We write $\sqrt{-z}$ for the branch of the function which is analytic off the positive real axis, and satisfies $\sqrt{-1} = +i$. Let $V(z)$ be the operator on $\mathrm{L}^2\big([0,\infty)\big)$ with integral kernel $V(z;x,y)$, given by

$$V(z;x,y) = u(x)\frac{1}{2\sqrt{-z}}\big(e^{-\sqrt{-z}\,|x-y|} - e^{-\sqrt{-z}\,(x+y)}\big)u(y). \qquad ()$$

Then a straightforward calculation shows that for $\operatorname{Im} z > 0$

$$V(z) = u(-\frac{d^2}{dx^2} - z)^{-1}u;$$

recall that we have Dirichlet boundary conditions at $x = 0$. Further, it is easy to check, using () and the fact that u has compact support, that $V(z)$ is an analytic bounded operator valued function of z in the strip $\operatorname{Re} z > \frac{1}{2}$, $|\operatorname{Im} z| < 1$.

Finally, the function

$$\varphi(x) = \sqrt{x}\, J_1(2\sqrt{x}) \qquad ()$$

for $x \geq 0$ plays an important role in our analysis. Here J_1 is the Bessel function [AS]. Note that φ is the zero energy continuum eigenfunction for h_0. In particular $h_0\varphi = 0$ as a differential equation and $\varphi(0) = 0$. We also define (with abuse of the notation for the inner product in L^2), for $\operatorname{Re} z > \frac{1}{2}$ and $|\operatorname{Im} z| < 1$,

$$\langle \varphi, V(z)\varphi\rangle = \int_0^\infty \int_0^\infty \varphi(x)V(z;x,y)\varphi(y)\,dx\,dy. \qquad ()$$

Although φ is not square integrable, () exists since u has compact support.

Results: There is a $\lambda_0 > 0$ such that for all $\lambda \in (-\lambda_0, \lambda_0)$ and for all $J \in \{1, 2, 3, \ldots\}$ the function $R_J(\lambda; z)$ has a meromorphic extension to D_J whose only singularity is a simple pole at $z = z_J$. Define $\gamma_J \in \mathbf{C}$ by

$$z_J = E_J + \frac{\gamma_J}{J^3}. \qquad ()$$

It is immediate that $\operatorname{Im}\gamma_J \leq 0$, since $H(\lambda)$ is self adjoint, and that $|\gamma_J| \leq \frac{1}{16}$, since $z_J \in D_J$.

For all J, γ_J satisfies the perturbative estimate

$$|\gamma_J + \lambda^2 J^3 \langle \psi_J, V(E_J)\psi_J\rangle| \leq \text{const}\ \lambda^4.$$

Further,

$$J^3\langle\psi_J, V(E_J)\psi_J\rangle = \frac{1}{2}\langle\varphi, V(1)\varphi\rangle + \mathcal{O}(\frac{1}{J^2}).$$

If we then make the additional assumption

$$\begin{aligned}\operatorname{Im}\langle\varphi, V(1)\varphi\rangle &= \Big(\int_0^\infty u(x)\varphi(x)\,\sin x\,dx\Big)^2 \\ &\neq 0\end{aligned} \qquad ()$$

then $\langle\varphi, V(1)\varphi\rangle \neq 0$, and there is a $J_0 > 0$ such that for $J > J_0$, $\operatorname{Im}\gamma_J < 0$. In addition $\{\gamma_J\}$ converges as $J \to \infty$.

We end this section by pointing out that, despite the eigenvalue accumulation, for all $\lambda \in (-\lambda_0, \lambda_0)$ and $J > J_0$ we get the expected approximate exponential decay away from the initial condition Ψ_J:

$$|\langle\Psi_J, e^{-itH(\lambda)}\Psi_J\rangle - e^{-itz_J}| \leq \text{const}\ \lambda^2.$$

This follows from Stone's formula () and the existence of the resonant pole, z_J, precisely as in [H].

§3: Perturbation Theory

In this section we recall the standard method for studying resonances using analytic perturbation theory. We point out why the standard method has a difficulty in this model, due to the eigenvalue accumulation, and we show how this difficulty is overcome. We will have several occasions to use the following observation from linear algebra. Let V_1 and V_2 be vector spaces and let

$$A = \begin{pmatrix} M & f \\ e & a \end{pmatrix}$$

be a linear operator on $V_1 \oplus V_2$. Here $M : V_1 \to V_1$, $f : V_2 \to V_1$, $e : V_1 \to V_2$ and $a : V_2 \to V_2$. Let M be invertible. Then A is invertible iff

$$\alpha = a - eM^{-1}f$$

is invertible. If so then

$$A^{-1} = \begin{pmatrix} M^{-1} + M^{-1}f\alpha^{-1}eM^{-1} & -M^{-1}f\alpha^{-1} \\ -\alpha^{-1}eM^{-1} & \alpha^{-1} \end{pmatrix}. \qquad ()$$

Now let $z \in D_J$. If $\mathrm{Im}\, z > 0$ then $-\frac{d^2}{dx^2} - z$ is invertible and, setting $M = -\frac{d^2}{dx^2} - z$, we can apply () to () and (). Thus $R_J(\lambda; z)$ can be rewritten

$$R_J(\lambda; z) = \langle \psi_J, \big(h_0 + 1 - z - \lambda^2 V(z)\big)^{-1} \psi_J \rangle. \qquad ()$$

Define

$$h(z) = h_0 + 1 - z - \lambda^2 V(z) \qquad ()$$

and introduce the orthogonal projection Π_J onto the eigenspace of h_0 with eigenvalue $-\frac{1}{4J^2}$. Writing $\Pi_J^\perp = 1 - \Pi_J$ we have $\mathrm{L}^2\big([0,\infty)\big) = \mathrm{Ran}\Pi_J^\perp \oplus \mathrm{Ran}\Pi_J$. In general, given an operator A in $\mathrm{L}^2\big([0,\infty)\big)$ for which the restriction $\Pi_J^\perp A \Pi_J^\perp$ is invertible on $\mathrm{Ran}\Pi_J^\perp$, we will denote by $A_{\perp,J}^{-n}$ (for $n = 1, 2$) the operator on $\mathrm{L}^2\big([0,\infty)\big)$ which equals $(\Pi_J^\perp A \Pi_J^\perp)^{-n}$ on $\mathrm{Ran}\Pi_J^\perp$, equals zero on $\mathrm{Ran}\Pi_J$, and leaves invariant the subspaces $\mathrm{Ran}\Pi_J^\perp$ and $\mathrm{Ran}\Pi_J$.

We will show that $\Pi_J^\perp h(z) \Pi_J^\perp$ is invertible on $\mathrm{Ran}\Pi_J^\perp$ and that $h(z)_{\perp,J}^{-1}$ is an analytic bounded operator valued function of z, for $z \in D_J$. Thus, another application of (), now with $M^{-1} = h(z)_{\perp,J}^{-1}$, will show that $R_J(\lambda; z)$ is analytic in z unless

$$E_J - z - \lambda^2 \langle \psi_J, V(z)\psi_J \rangle - \lambda^4 \langle \psi_J, V(z) h(z)_{\perp,J}^{-1} V(z) \psi_J \rangle = 0. \qquad ()$$

If () is not satisfied then

$$R_J(\lambda; z) = \frac{1}{\langle \psi_J, h(z)\psi_J\rangle - \lambda^4 \langle \psi_J, V(z) h(z)^{-1}_{\perp,J} V(z)\psi_J\rangle}. \qquad ()$$

If () is satisfied for some $z = z_J$ then it is immediate that

$$|z - E_J + \lambda^2 \langle \psi_J, V(z)\psi_J\rangle| \leq \text{const } \lambda^4. \qquad ()$$

where const may depend on J. It follows from the self-adjointness of H that () has no solutions if Im$z > 0$. Solutions with Im$z < 0$ are called resonances for H while real solutions are embedded eigenvalues.

In studying $h(z)^{-1}_{\perp,J}$ and () the chief difficulty to be overcome is that of obtaining estimates which are uniform in J. As J increases the unperturbed eigenvalues E_J accumulate. For large J we have

$$E_{J+1} - E_J = \frac{1}{2J^3} + \mathcal{O}(\frac{1}{J^4}).$$

As a consequence, for $z \in D_J$,

$$\|(h_0 + 1 - z)^{-1}_{\perp,J}\| \sim 2J^3 \qquad ()$$

and for a generic bounded perturbation U, $\Pi_J^{\perp}(h_0 + 1 - z + \lambda^2 U)\Pi_J^{\perp}$ is not invertible for all $z \in D_J$ unless $\lambda^2 \|U\| J^3$ is small enough. For any $\lambda \neq 0$ this fails for J sufficiently large.

It was noticed in [FW1] that this difficulty can be overcome if the perturbation U has an integral kernel $U(z; x, y)$ which approaches zero rapidly enough as either x or y goes to infinity. In the case at hand, because of our assumption that the function $u(x)$ has compact support, $V(z; x, y)$ has compact support in x and y. Let χ be the multiplication operator corresponding to the function

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \text{supp}(u) \\ 0 & \text{if } x \notin \text{supp}(u). \end{cases} \qquad ()$$

Note that, for each $J \in \{1, 2, 3, \ldots\}$, the operator $\chi(h_0 + 1 - z)^{-1}_{\perp,J}\chi$ is an analytic bounded operator valued function of z, for $z \in D_J$. We will show that $\chi(h_0 + 1 - z)^{-1}_{\perp,J}\chi$ is also bounded uniformly in J, thus overcoming the difficulty expressed by ().

As in [FW1] we introduce the eigenfunction expansion for h_0. The discrete eigenfunctions of h_0 are [FW1 Appendix B]

$$\psi_j(x) = 2^{-\frac{1}{2}} j^{-\frac{3}{2}} x e^{-\frac{1}{2j}x} M(1-j, 2, \frac{x}{j})$$

while the continuum eigenfunctions of h_0 are

$$\psi_p(x) = (1 - e^{-\frac{\pi}{|p|}})^{-\frac{1}{2}} x e^{-ipx} M(1 + \frac{i}{2p}, 2, 2ipx).$$

Here $M(a, b, z)$ is the confluent hypergeometric function [AS], $j \in \{1, 2, 3, \ldots\}$ and the functions $\psi_p(x)$ are normalized with respect to the measure $2pdp$ on $[0, \infty)$. It follows from the identity $M(a, b, z) = e^z M(b-a, b, -z)$ [AS equ. 13.1.27] that the functions $\psi_p(x)$ are real valued.

Both ψ_j and ψ_p are solutions to the differential equation

$$(\frac{d^2}{dx^2} + \frac{1}{x} + \delta)\psi(x) = 0;$$

for ψ_j we set $\delta = -\frac{1}{4j^2}$, for ψ_p we set $\delta = p^2$. It is easily checked that the following limits hold pointwise:

$$\lim_{j\to\infty} 2^{\frac{1}{2}} j^{\frac{3}{2}} \psi_j(x) = \lim_{p\to 0} \psi_p(x) = \varphi(x).$$

Here $\varphi(x)$ is the zero energy eigenfunction of h_0 given by (). We will need some information about the rates of convergence of these limits. The following estimates can be read off from the results of Appendix B of [FW1]. For x in any bounded subset of $[0, \infty)$

$$|\psi_j(x) - 2^{-\frac{1}{2}} j^{-\frac{3}{2}} \varphi(x)| \le \text{const } j^{-\frac{7}{2}} \qquad ()$$

and

$$|\psi_p(x) - \varphi(x)| \le \text{const } p^2. \qquad ()$$

Lemma : *There is a real number $K < \infty$ such that for all $J \in \{1, 2, 3, \ldots\}$ and for all $z \in D_J$*

$$\|\chi(h_0 + 1 - z)^{-1}_{\perp,J}\chi\| \le K \qquad ()$$

and

$$\|\chi(h_0 + 1 - z)^{-2}_{\perp,J}\chi\| \le K\, J^3. \qquad ()$$

proof: Let $\theta_1, \theta_2 \in \mathrm{L}^2\big([0,\infty)\big)$, fix $J \in \{1,2,3,\ldots\}$ and let P_1 be the orthogonal projection onto the spectral subspace of h_0 contained in $[-1,1]$. Then, for all $z \in D_J$,

$$\begin{aligned}\langle \theta_1, \chi(h_0+1-z)^{-1}_{\perp,J}\chi\theta_2\rangle =& \langle \theta_1, \chi(h_0+1-z)^{-1}_{\perp,J}P_1\chi\theta_2\rangle \\ &+ \langle \theta_1, \chi(h_0+1-z)^{-1}_{\perp,J}(1-P_1)\chi\theta_2\rangle.\end{aligned} \qquad ()$$

Since

$$\|(h_0+1-z)^{-1}_{\perp,J}(1-P_1)\| \le 1$$

the second term in the expression () is analytic in z and

$$|\langle \theta_1, \chi(h_0+1-z)^{-1}_{\perp,J}(1-P_1)\chi\theta_2\rangle| \le \|\theta_1\|\|\theta_2\|$$

uniformly for $z \in D_J$.

Using the eigenfunction expansion for h_0 the first term in () can be rewritten as follows:

$$\begin{aligned}\langle \theta_1, \chi(h_0+1-z)^{-1}_{\perp,J}P_1\chi\theta_2\rangle =& \sum_{j\neq J}\frac{1}{E_j-z}\langle\theta_1,\chi\psi_j\rangle\langle\chi\psi_j,\theta_2\rangle \\ &+ \int_0^1 \frac{1}{p^2+1-z}\langle\theta_1,\chi\psi_p\rangle\langle\chi\psi_p,\theta_2\rangle\, 2p dp.\end{aligned} \qquad ()$$

We now apply the estimates () and (). We have, for $\sigma \in \{1,2\}$,

$$|\langle\theta_\sigma,\chi\psi_j\rangle - 2^{-\frac{1}{2}}j^{-\frac{3}{2}}\langle\theta_\sigma,\chi\varphi\rangle| \le \text{const } j^{-\frac{7}{2}}\|\theta_\sigma\|$$

and

$$|\langle\theta_\sigma,\chi\psi_p\rangle - \langle\theta_\sigma,\chi\varphi\rangle| \le \text{const } p^2\|\theta_\sigma\|.$$

Noting that, for all $z \in D_J$,

$$\frac{1}{2}\sum_{j\neq J}\frac{j^{-r}}{E_j-z} = \begin{cases} -2\ln J + \mathcal{O}(1) & \text{if } r=3 \\ \mathcal{O}(1) & \text{if } r>3\end{cases} \qquad ()$$

and

$$\int_0^1 \frac{p^r}{p^2+1-z}\,2pdp = \begin{cases} 2\ln J + \mathcal{O}(1) & \text{if } r=0 \\ \mathcal{O}(1) & \text{if } r>0\end{cases} \qquad ()$$

we find that

$$|\langle \theta_1, \chi(h_0+1-z)^{-1}_{\perp,J}P_1\chi\theta_2\rangle| \le \text{const } \|\theta_1\|\|\theta_2\|.$$

The bound () follows.

Proving () is similar. The only difference is that we use the estimates

$$\frac{1}{2}\sum_{j\neq J}\frac{j^{-3-r}}{|E_j-z|^2}\leq \text{const}\begin{cases}J^{3-r} & \text{if } r<3\\ 1 & \text{if } r\geq 3\end{cases} \tag{}$$

and

$$\int_0^1 \frac{p^r}{|p^2+1-z|^2}\,2pdp\leq \text{const}\begin{cases}J^{2-r} & \text{if } r<2\\ \ln J & \text{if } r=2\\ 1 & \text{if } r>2\end{cases} \tag{}$$

for all $z\in D_J$. ■

An immediate consequence of () is the invertibility of $\Pi_J^\perp h(z)\Pi_J^\perp$ on $\text{Ran}\Pi_J^\perp$ for all $z\in D_J$. If λ is small enough the expansion

$$h(z)^{-1}_{\perp,J}=(h_0+1-z)^{-1}_{\perp,J}\sum_{N=0}^{\infty}\left(\lambda^2V(z)(h_0+1-z)^{-1}_{\perp,J}\right)^N \tag{}$$

converges uniformly on D_J, since it can be written as an expansion in powers of $V(z)$ times the operator (). It follows that $h(z)^{-1}_{\perp,J}$ is an analytic bounded operator valued function of z in D_J, that

$$\|\chi h(z)^{-1}_{\perp,J}\chi\|\leq \text{const} \tag{}$$

and, using (), that

$$\|\chi h(z)^{-2}_{\perp,J}\chi\|\leq \text{const }J^3. \tag{}$$

Thus () provides an analytic continuation of $R_J(\lambda;z)$ across the real line to the set $\{z\in D_J\,|\,z \text{ is not a solution of ()}\}$. To see that () has a unique solution and that this solution is a simple zero note that, using (),

$$\begin{aligned}|\frac{d}{dz}\langle\psi_J,V(z)\psi_J\rangle| &= |\langle\psi_J,V'(z)\psi_J\rangle|\\ &\leq \text{const }J^{-3}\end{aligned}$$

and, using (), () and () again, that

$$\begin{aligned}|\frac{d}{dz}\langle\psi_J,V(z)h(z)^{-1}_{\perp,J}V(z)\psi_J\rangle| \leq & |\langle\psi_J,V'(z)h(z)^{-1}_{\perp,J}V(z)\psi_J\rangle|\\ &+|\langle\psi_J,V(z)h(z)^{-1}_{\perp,J}h'(z)h(z)^{-1}_{\perp,J}V(z)\psi_J\rangle|\\ &+|\langle\psi_J,V(z)h(z)^{-1}_{\perp,J}V'(z)\psi_J\rangle|\\ \leq & \text{const }.\end{aligned}$$

Now apply the contraction mapping theorem.

Let z_J be the unique solution of () in D_J. It follows from the comments above that $R_J(\lambda; z)$ is meromorphic in z for $z \in D_J$ with a simple pole at $z = z_J$. Further, the estimate () is uniform in J. To estimate the large J behavior of z_J we can use () and () in (). Recalling the definitions () of E_J and () of γ_J we have

$$\begin{aligned}\langle \psi_J, V(E_J + \frac{\gamma_J}{J^3})\psi_J\rangle &= \frac{1}{2J^3}\langle \varphi, V(E_J + \frac{\gamma_J}{J^3})\varphi\rangle + \mathcal{O}(\frac{1}{J^5}) \\ &= \frac{1}{2J^3}\langle \varphi, V(1)\varphi\rangle + \mathcal{O}(\frac{1}{J^5}).\end{aligned}$$

Further, using (),

$$|\langle \psi_J, V(E_J + \frac{\gamma_J}{J^3})h(E_J + \frac{\gamma_J}{J^3})^{-1}_{\perp,J}V(E_J + \frac{\gamma_J}{J^3})\psi_J\rangle| \leq \text{const } \frac{1}{J^3}.$$

Thus

$$\gamma_J = -\frac{1}{2}\lambda^2\Big(\langle \varphi, V(1)\varphi\rangle + \mathcal{O}(\frac{1}{J^2})\Big) + \mathcal{O}(\lambda^4). \qquad ()$$

§4: The Large J Limit

In this section we examine the position of the resonant pole z_J as $J \to \infty$ and show that $J^3(z_J - E_J)$ has a limit. This pole is given by the unique solution of () in the disk D_J. Recall the notation introduced in (), namely $z_J = E_J + \gamma_J J^{-3}$. Let us write

$$z_J(\gamma) = E_J + \frac{\gamma}{J^3}$$

for $|\gamma| \leq \frac{1}{16}$ and define the following function of γ and J:

$$G_J(\gamma) = -\lambda^2 J^3 \langle \psi_J, V\big(z_J(\gamma)\big)\psi_J\rangle - \lambda^4 J^3 \langle \psi_J, V\big(z_J(\gamma)\big) h\big(z_J(\gamma)\big)_{\perp,J}^{-1} V\big(z_J(\gamma)\big)\psi_J\rangle. \qquad ()$$

It follows from () that γ_J is the unique fixed point of the function G_J in the disk

$$D_0 = \{\gamma \,|\, |\gamma| \leq \frac{1}{16}\}.$$

Proposition : *The sequence γ_J converges in the disk D_0 as $J \to \infty$.*

proof: The number γ_J is the solution of the equation

$$\gamma_J = G_J(\gamma_J).$$

Recall that G_J is a contraction on D_0 for $|\lambda|$ sufficiently small. Some elementary estimates now show that to prove convergence of the sequence $\{\gamma_J\}$, it is sufficient to prove that the sequence $\{G_J(\gamma)\}$ converges uniformly for $\gamma \in D_0$. Using the estimates () and (), and , this reduces to showing convergence of $\langle \varphi, V(1) h\big(z_J(\gamma)\big)_{\perp,J}^{-1} V(1)\varphi\rangle$ uniformly in γ. Using the fact that u has compact support, and recalling (), this is equivalent to the uniform convergence of the sequence of bounded operators $\chi h\big(z_J(\gamma)\big)_{\perp,J}^{-1}\chi$. By the results of , and using the convergent expansion (), it is sufficient to prove uniform convergence of $\chi(h_0 + \frac{1}{4J^2} - \frac{\gamma}{J^3})_{\perp,J}^{-1}\chi$.

As in the proof of we may restrict to the spectral subspace of h_0 with spectrum less than 1. Again let P_1 be the orthogonal projection on this spectral subspace of h_0. On the orthogonal complement to this subspace the operator $(h_0 + \frac{1}{4J^2} - \frac{\gamma}{J^3})_{\perp,J}^{-1}$ is analytic in γ and uniformly bounded by 1 on D_0. In fact we have the following equality of norm limits:

$$\lim_{J\to\infty} (1 - P_1)(h_0 + \frac{1}{4J^2} - \frac{\gamma}{J^3})_{\perp,J}^{-1} = \lim_{\eta\to 0}(1 - P_1)(h_0 + i\eta)^{-1}. \qquad ()$$

Again using the eigenfunction expansion for h_0 we have

$$\chi P_1(h_0+\frac{1}{4J^2}-\frac{\gamma}{J^3})^{-1}_{\perp,J}\chi=\sum_{j\neq J}\frac{\chi\psi_j\otimes\psi_j\chi}{\frac{1}{4J^2}-\frac{1}{4j^2}-\frac{\gamma}{J^3}}+\int_0^1\frac{\chi\psi_p\otimes\psi_p\chi}{p^2+\frac{1}{4J^2}-\frac{\gamma}{J^3}}\,2pdp.$$

Using the estimates () and () leads us back to (). Explicitly, define

$$\psi_j^{(1)}=\psi_j-2^{-\frac12}j^{-\frac32}\varphi,$$

$$\psi_p^{(1)}=\psi_p-\varphi$$

and introduce the sum and integral

$$G_J^{(0)}(\gamma)=\frac12\sum_{j\neq J}j^{-3}\frac{1}{\frac{1}{4J^2}-\frac{1}{4j^2}-\frac{\gamma}{J^3}}+\int_0^1\frac{1}{p^2+\frac{1}{4J^2}-\frac{\gamma}{J^3}}\,2pdp. \qquad ()$$

Applying (), (), () and () we have the existence of the limit (in norm)

$$\begin{aligned}\lim_{J\to\infty}&\Big(\chi P_1(h_0+\frac{1}{4J^2}-\frac{\gamma}{J^3})^{-1}_{\perp,J}\chi-G_J^{(0)}(\gamma)\,\chi\varphi\otimes\varphi\chi\Big)\\&=-4\sum_{j=1}^{\infty}j^2\big(\frac{1}{\sqrt{2j^3}}\,\chi\psi_j^{(1)}\otimes\varphi\chi+\frac{1}{\sqrt{2j^3}}\,\chi\varphi\otimes\psi_j^{(1)}\chi+\chi\psi_j^{(1)}\otimes\psi_j^{(1)}\chi\big) \qquad ()\\&\quad+2\int_0^1\frac1p\big(\chi\psi_p^{(1)}\otimes\varphi\chi+\chi\varphi\otimes\psi_p^{(1)}\chi+\chi\psi_p^{(1)}\otimes\psi_p^{(1)}\chi\big)\,dp.\end{aligned}$$

Note that convergence is uniform in γ, and the limit is independent of γ. The only possible γ dependence in the large J limit is from the term $G_J^{(0)}(\gamma)$.

We now show that $G_J^{(0)}(\gamma)$ has a limit as $J\to\infty$. It is easy to show that the integral in () satisfies the estimate

$$\int_0^1\frac{1}{p^2+\frac{1}{4J^2}-\frac{\gamma}{J^3}}\,2pdp=2\ln J+\ln 4+O(J^{-1}).$$

To estimate the sum in () we introduce the number

$$\alpha=\big(1-\frac{\gamma}{J}\big)^{-\frac12}$$

and rewrite the sum as follows:

$$\begin{aligned}\frac12\sum_{j\neq J}j^{-3}\big(\frac{1}{4J^2}-\frac{1}{4j^2}-\frac{\gamma}{J^3}\big)^{-1}&=2\alpha^2J^2\sum_{j\neq J}\frac{1}{j(j^2-\alpha^2J^2)}\\&=\sum_{j\neq J}\Big(-\frac2j+\frac{1}{j+\alpha J}+\frac{1}{j-\alpha J}\Big)\\&=\frac2J-\frac{1}{(1+\alpha)J}+\lim_{K\to\infty}\Big(-2\sum_{j=1}^{K}\frac1j+\sum_{j=1}^{K}\frac{1}{j+\alpha J}+\sum_{\substack{1\le j\le K\\ j\neq J}}\frac{1}{j-\alpha J}\Big).\end{aligned}$$

For $|z| \le \frac{1}{2}$, and $N \ge 1$ define

$$\phi_z(N) = \sum_{n=1}^{N} \frac{1}{n+z}.$$

Note that

$$\lim_{N\to\infty} \bigl(\phi_z(N) - \ln N\bigr) = C - z \sum_{n=1}^{\infty} \frac{1}{n(n+z)},$$

where C is Euler's constant, and that convergence is uniform in z. Further, let

$$\epsilon = (\alpha - 1)J;$$

it is easy to see that $|\epsilon| \le |\gamma|$ for all $J \ge 1$, and that $\epsilon \to \frac{\gamma}{2}$ as $J \to \infty$. We then have

$$\sum_{j=1}^{K} \frac{1}{j} = \phi_0(K),$$

$$\begin{aligned}\sum_{j=1}^{K} \frac{1}{j+\alpha J} &= \sum_{j=J+1}^{K+J+1} \frac{1}{j+\epsilon}\\ &= \phi_\epsilon(K+J+1) - \phi_\epsilon(J)\end{aligned}$$

and (for $K > J$)

$$\begin{aligned}\sum_{\substack{1\le j\le K\\ j\ne J}} \frac{1}{j-\alpha J} &= \sum_{\substack{1\le j\le K\\ j\ne J}} \frac{1}{(j-J)-\epsilon}\\ &= \frac{1}{J+\epsilon} - \phi_\epsilon(J) + \phi_{-\epsilon}(K-J).\end{aligned}$$

We thus arrive at the following identity:

$$\begin{aligned}\frac{1}{2}\sum_{j\ne J} j^{-3}\Bigl(\frac{1}{4J^2} - \frac{1}{4j^2} - \frac{\gamma}{J^3}\Bigr)^{-1} &= \frac{2}{J} + \frac{1}{J+\epsilon} - \frac{1}{2J+\epsilon} - 2\phi_\epsilon(J)\\ &\quad + \lim_{K\to\infty}\Bigl(-2\phi_0(K) + \phi_\epsilon(J+K+1) + \phi_{-\epsilon}(K-J)\Bigr)\\ &= \frac{2}{J} + \frac{1}{J+\epsilon} - \frac{1}{2J+\epsilon} - 2\phi_\epsilon(J) + 2\epsilon^2\sum_{n=1}^{\infty}\frac{1}{n(n^2-\epsilon^2)}.\end{aligned}$$

Substituting these expresions for the sum and integral in (), we get the following result:

$$G_J^{(0)}(\gamma) = 2\epsilon^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2-\epsilon^2)} - 2\bigl(\phi_\epsilon(J) - \ln J\bigr) + \ln 4 + O(J^{-1})$$

uniformly in γ. Letting $J \to \infty$ gives

$$\begin{aligned} G_\infty^{(0)}(\gamma) &= \lim_{J\to\infty} G_J^{(0)}(\gamma) \\ &= \frac{\gamma^2}{2}\sum_{n=1}^{\infty}\frac{1}{n(n^2-\frac{\gamma^2}{4})} + \gamma\sum_{n=1}^{\infty}\frac{1}{n(n+\frac{\gamma}{2})} - 2C + \ln 4. \end{aligned} \quad ()$$

Therefore we conclude that $\lim_{J\to\infty} G_J(\gamma)$ exists for every $\gamma \in D_0$, and that $\{G_J\}$ converges uniformly in D_0. ■

To approximate

$$\gamma_\infty = \lim_{J\to\infty} \gamma_J$$

we let

$$A(\gamma) = \lim_{J\to\infty} \chi(h_0 + \frac{1}{4J^2} - \frac{\gamma}{J^3})^{-1}_{\perp,J}\chi.$$

¿From (), () and () we find that $A(\gamma)$ exists and is given by the sum of these three terms. Note that

$$\begin{aligned} A(\gamma) - A(0) &= \Big(\frac{\gamma^2}{2}\sum_{n=1}^{\infty}\frac{1}{n(n^2-\frac{\gamma^2}{4})} + \gamma\sum_{n=1}^{\infty}\frac{1}{n(n+\frac{\gamma}{2})}\Big)\,\chi\varphi\otimes\varphi\chi \\ &= \frac{\pi^2\gamma}{6}\;\chi\varphi\otimes\varphi\chi + \mathcal{O}(\gamma^3). \end{aligned}$$

Recalling () we have

$$\chi h\big(z_J(\gamma)\big)^{-1}_{\perp,J}\chi = A(\gamma)\sum_{N=0}^{\infty}\lambda^{2N}\Big(V(1)A(\gamma)\Big)^N.$$

Sustituting in () it is straightforward to generate an expansion for γ_∞ in powers of λ^2. The first few terms are

$$\gamma_\infty = -\frac{\lambda^2}{2}\langle\varphi, V(1)\varphi\rangle - \lambda^4\langle\varphi, V(1)A(0)V(1)\varphi\rangle + \mathcal{O}(\lambda^6).$$

<u>Bibliography</u>

[BFS] V. Bach, J. Fröhlich, and I. M. Sigal, *Mathematical Theory of Nonrelativistic Matter and Radiation*, Lett. Math. Phys. **34**, 183-201 (1995).

[D] P.A.M. Dirac, *Quantum Mechanics*, Oxford University Press (1978).

[Da] E.B. Davies, *Dynamics of a Multilevel Wigner-Weisskopf Atom*, Journal of Mathematical Physics **15**, 2036-2041 (1974).

[F] K.O. Friedrichs, *On the Perturbation of Continuous Spectra*, Comm. Pure Appl. Math **1**, 361-406 (1948).

[FW1] R. Froese and R. Waxler, *The Spectrum of a Hydrogen Atom in an Intense Magnetic Field*, Reviews in Math. Physics **6**, 699-832 (1994).

[FW2] R. Froese and R. Waxler, *Ground State Resonances of a Hydrogen Atom in an Intense Magnetic Field*, Reviews in Math. Physics **7**, 311-361 (1995).

[Ho1] J. Howland: *Spectral Concentration and Virtual Poles, II*, Transactions of the A.M.S. **162**, 141-156 (1971);

[Ho2] J. Howland: *The Livsic Matrix in Perturbation Theory*, Journal of Mathematical Analysis and Applications **50**, 415-437 (1975).

[H] W. Hunziker, *Resonances, Metastable States and Exponential Decay Laws in Perturbation Theory*, Commun. Math. Phys. **132**, 177-188 (1990).

[K1] C. King: *Exponential Decay Near Resonance, without Analyticity*, Letters in Math. Phys. **23**, 215-222 (1991);

[K2] C. King: *Scattering Theory for a Model of an Atom in a Quantized Field*, Letters in Math. Phys. **25**, 17-28 (1992).

[LL] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (3rd edition)*, Pergamon (1977).

[PS] J. Parker and C.R. Stroud, Jr., *Coherence and Decay of Rydberg Wave Packets*, Phys. Rev. Letters **56** 716-719 (1986)

[RS] M. Reed and B. Simon: *Methods of Mathematical Physics vol. 2: Fourier Analysis, Self Adjointness*, Academic Press (1975);
Methods of Mathematical Physics vol. 4: Analysis of Operators, Academic Press (1978).

[S1] B. Simon: *Resonances in N-Body Quantum Systems with Dilation Analytic Potentials and the Foundations of Time-Dependent Perturbation Theory*, Annals of Math. **97**, 247-274 (1973);

[S2] B. Simon: *Resonances and Complex Scaling: a Rigorous Overview*, Int. Journal of Quant. Chem. **14**, 529-542 (1978).

[W1] R. Waxler, *The Time Evolution of a Class of Meta-stable States*, Commun. Math. Phys. **172**, 535-549 (1995).

[W2] R. Waxler, *Generalized Eigenfunction Expansion Near Resonances*, preprint

[WW] V. Weisskopf and E. Wigner, *Berechnung der Natürlichen Linienbreite auf Grund der Diracshen Lichttheorie*, Zeitschrift für Physik **63**, 54-73 (1930).