AN ALTERNATIVE APPROACH TO ALGEBRO-GEOMETRIC SOLUTIONS OF THE AKNS HIERARCHY

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Abstract. We develop an alternative systematic approach to the AKNS hierarchy based on elementary algebraic methods. In particular, we recursively construct Lax pairs for the entire AKNS hierarchy by introducing a fundamental polynomial formalism and establish the basic algebro-geometric setting including associated Burchnall-Chaundy curves, Baker-Akhiezer functions, trace formulas, Dubrovin-type equations for analogs of Dirichlet and Neumann divisors, and theta function representations for algebro-geometric solutions.

1. Introduction

The principal aim of this paper is an alternative elementary algebraic approach to the entire AKNS hierarchy in the spirit of previous treatments of the Korteweg-de Vries (KdV), Boussinesq, and Toda hierarchies. More precisely, we advocate a fundamental polynomial formalism to recursively construct Lax pairs for the AKNS hierarchy, that is, pairs (D, E_{n+1}) of matrix-valued differential expressions of order one (i.e., D) and $n+1$ (i.e., E_{n+1}) with D of the Dirac-type. In addition, we establish the basic algebro-geometric set up for special classes of solutions of the AKNS hierarchy including solitons, rational solutions, algebro-geometric quasi-periodic solutions, and limiting cases thereof. Our treatment includes a systematic approach to Burchnall-Chaundy curves, Baker-Akhiezer functions, trace formulas, Dubrovin-type equations describing the dynamics of Dirichlet and Neumann divisors, and theta function representations for algebro-geometric solutions.

Before we enter a description of the contents of each section, it seems appropriate to comment on existing treatments of this subject and to justify the addition of yet another detailed account on this topic. The theory of commuting matrix-valued differential expressions and, more generally, the algebro-geometric approach to matrix hierarchies of soliton equations has been developed in great generality by Dubrovin and Krichever. Corresponding authoritative accounts can be found, for instance, in [5], Chs. 3, 4, [14], [15], [16], [29], [30], [34], [35], [43], and the references therein. In contrast to these references, our own approach relies on two basic ingredients, an elementary polynomial approach to Lax pairs (or zero-curvature pairs) of the AKNS hierarchy and its explicit connection with a fundamental meromorphic quantity ϕ (cf. (3.10), (4.23)) which allows for a unified algebro-geometric treatment of the entire AKNS hierarchy.

In section 2 we describe a recursive approach to Lax pairs (and zero-curvature pairs) of the AKNS hierarchy following Al'ber's treatment of the KdV and nonlinear Schrödinger hierarchies [1], [2], [3] and establish its connection with the Burchnall-Chaundy theory [7], [8], [9] and hence with hyperelliptic curves. Combining the recursive formalism of

Section 2 with a polynomial approach to represent positive divisors of degree n of a hyperelliptic curve of genus n originally developed by Jacobi [32] and applied to the KdV case by Mumford [40], Section III.a.1 and McKean [37], a detailed analysis of the stationary KdV hierarchy is provided in Section 3. This includes, in particular, the theta function representation of algebro-geometric solutions of the stationary AKNS hierarchy. The corresponding time-dependent formalism is then developed in detail in Section 4. Appendix A collects the relevant material for hyperelliptic Riemann surfaces and their theta functions. Appendix B contains a simple illustration of the Riemann-Roch theorem (cf. Theorem B.1).

We emphasize that our treatment comprises, in particular, the important special case of the nonlinear Schrödinger (NLS) equation (cf. (3.77)), whose algebro-geometric solutions have been studied, for instance, in [5], Ch. 4, [15], [29], [30], [36], [38], [43]. Similarly, the case of the modified Korteweg-de Vries (mKdV) equation (cf. (3.79), whose algebro-geometric solutions have been studied, for instance, in [20], [21], are included as a special case of our formalism. Moreover, the present elementary approach is not at all restricted to the AKNS hierarchy but applies quite generally to 1+1-dimensional hierarchies of soliton equations. In fact, the KdV case has been treated in [26], the case of the Toda and Kac-van Moerbeke hierarchies in [6], and the case of the Boussinesq hierarchy in [13].

Finally, we mention that a combination of the AKNS formalism developed in this paper and the Picard-type techniques introduced in a recent explicit characterization of all elliptic solutions of the KdV hierarchy [25] (see also [24]) are expected to yield a similar characterization of all elliptic solutions of the AKNS hierarchy, a topic that continues to attract considerable interest (see, e.g., [5], Ch. 7, [10], [44]).

2. The AKNS Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we briefly review the construction of the AKNS hierarchy using a recursive approach advocated by Al'ber [1], [2], [3] (see also [12], Ch. 12, [19], [22], [23], [26]) and outline its connection with the analog of the Burchnall-Chaundy polynomial [7], [8], [9], and associated hyperelliptic curves.

Suppose $p, q \in C^\infty(\mathbb{R})$ (or meromorphic on $\mathbb{C}$) and introduce the Dirac-type matrix-valued differential expression

$$D = i\begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix}, \quad x \in \mathbb{R} \ \ (\text{or } \ \mathbb{C}). \tag{2.1}$$

In order to explicitly construct higher-order matrix-valued differential expressions E_{n+1}, $n \in \mathbb{N}_0 \ \ (= \mathbb{N} \cup \{0\})$ commuting with D, which will be used to define the stationary AKNS hierarchy later, one can proceed as follows.

Pick $n \in \mathbb{N}_0$ and define $\{f_\ell(x)\}_{0\leq \ell\leq n}$, $\{g_\ell(x)\}_{0\leq \ell\leq n+1}$, $\{h_\ell(x)\}_{0\leq \ell\leq n}$ recursively by

$$f_0(x) = -iq(x), \quad g_0(x) = 1, \quad h_0(x) = ip(x),$$
$$f_{\ell+1}(x) = \frac{i}{2} f_{\ell,x}(x) - iq(x) g_{\ell+1}(x), \quad 0 \leq \ell \leq n,$$

$$g_{\ell+1,x}(x) = p(x)f_\ell(x) + q(x)h_\ell(x), \quad 0 \le \ell \le n, \tag{2.2}$$
$$h_{\ell+1}(x) = -\frac{i}{2}h_{\ell,x}(x) + ip(x)g_{\ell+1}(x), \quad 0 \le \ell \le n.$$

Explicitly, one computes

$$\begin{aligned}
f_0 &= -iq,\\
f_1 &= \frac{1}{2}q_x + c_1(-iq),\\
f_2 &= \frac{i}{4}q_{xx} - \frac{i}{2}pq^2 + c_1(\frac{1}{2}q_x) + c_2(-iq),\\
g_0 &= 1,\\
g_1 &= c_1,\\
g_2 &= \frac{1}{2}pq + c_2,\\
g_3 &= -\frac{i}{4}(p_x q - pq_x) + c_1(\frac{1}{2}pq) + c_3,\\
h_0 &= ip,\\
h_1 &= \frac{1}{2}p_x + c_1(ip),\\
h_2 &= -\frac{i}{4}p_{xx} + \frac{i}{2}p^2 q + c_1(\frac{1}{2}p_x) + c_2(ip),
\end{aligned} \tag{2.3}$$

etc.,

where $\{c_\ell\}_{1\le\ell\le n+1}$ are integration constants. Given (2.2), one defines the matrix-valued differential expression E_{n+1} by

$$E_{n+1} = i\sum_{\ell=0}^{n+1}\begin{pmatrix} -g_{n+1-\ell} & f_{n-\ell} \\ -h_{n-\ell} & g_{n+1-\ell}\end{pmatrix} D^\ell, \quad n \in \mathbb{N}, \quad f_{-1} = h_{-1} = 0, \tag{2.4}$$

and verifies

$$[E_{n+1}, D] = \begin{pmatrix} 0 & -2if_{n+1} \\ 2ih_{n+1} & 0 \end{pmatrix}, \quad n \in \mathbb{N}_0 \tag{2.5}$$

($[\ .\ ,\ .\ ]$ the commutator symbol). The pair (E_{n+1}, D) represents the celebrated Lax pair for the AKNS hierarchy. Varying $n \in \mathbb{N}_0$, the stationary AKNS hierarchy is then defined by the vanishing of the commutator of E_{n+1} and D in (2.5), that is, by

$$[E_{n+1}, D] = 0, \quad n \in \mathbb{N}_0, \tag{2.6}$$

or equivalently, by

$$f_{n+1} = h_{n+1} = 0, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Explicitly, one obtains for the first few equations of the stationary AKNS hierarchy,

$$\begin{cases} -p_x + c_1(-2ip) = 0 \\ -q_x + c_1(2iq) = 0 \end{cases},$$

$$\begin{cases} \frac{i}{2}p_{xx} - ip^2q + c_1(-p_x) + c_2(-2ip) = 0 \\ -\frac{i}{2}q_{xx} + ipq^2 + c_1(-q_x) + c_2(2iq) = 0 \end{cases}, \tag{2.8}$$

$$\begin{cases} \frac{1}{4}p_{xxx} - \frac{3}{2}pp_xq + c_1(\frac{i}{2}p_{xx} - ip^2q) + c_2(-p_x) + c_3(-2ip) = 0 \\ \frac{1}{4}q_{xxx} - \frac{3}{2}pqq_x + c_1(-\frac{i}{2}q_{xx} + ipq^2) + c_2(-q_x) + c_3(2iq) = 0 \end{cases},$$

etc.

By definition, solutions $(p(x), q(x))$ of any of the stationary AKNS equations (2.8) are called **algebro-geometric stationary finite-gap solutions** associated with the AKNS hierarchy. If (p, q) satisfies the n^{th} equation ($n \in \mathbb{N}_0$) of (2.8) one also calls (p, q) a stationary n-gap solution.

Next, we introduce polynomials F_n, G_{n+1}, H_n with respect to $z \in \mathbb{C}$,

$$\begin{aligned} F_n(z,x) &= \sum_{\ell=0}^{n} f_{n-\ell}(x)z^\ell, \quad f_0(x) = -iq(x), \\ G_{n+1}(z,x) &= \sum_{\ell=0}^{n+1} g_{n+1-\ell}(x)z^\ell, \quad g_0(x) = 1, \\ H_n(z,x) &= \sum_{\ell=0}^{n} h_{n-\ell}(x)z^\ell, \quad h_0(x) = ip(x). \end{aligned} \tag{2.9}$$

and note that (2.6) respectively, (2.7) become

$$F_{n,x}(z,x) = -2izF_n(z,x) + 2q(x)G_{n+1}(z,x), \tag{2.10}$$

$$G_{n+1,x}(z,x) = p(x)F_n(z,x) + q(x)H_n(z,x), \tag{2.11}$$

$$H_{n,x}(z,x) = 2izH_n(z,x) + 2p(x)G_{n+1}(z,x). \tag{2.12}$$

(2.10)–(2.12) yield that

$$\left(G_{n+1}^2 - F_nH_n\right)_x = 0 \tag{2.13}$$

and hence

$$G_{n+1}(z,x)^2 - F_n(z,x)H_n(z,x) = R_{2n+2}(z), \tag{2.14}$$

where the integration constant $R_{2n+2}(z)$ is a monic polynomial in z of degree $2n+2$. Thus one may write

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{0\le m\le 2n+1} \subset \mathbb{C}. \tag{2.15}$$

Explicitly, one obtains for the first few polynomials in (2.9),

$$\begin{aligned} F_0 &= -iq, \\ F_1 &= -iqz + \frac{1}{2}q_x + c_1(-iq), \\ F_2 &= -iqz^2 + \frac{1}{2}q_xz + \frac{i}{4}q_{xx} - \frac{i}{2}pq^2 + c_1(-iqz + \frac{1}{2}q_x) + c_2(-iq), \end{aligned}$$

$$
\begin{aligned}
&G_1 = z + c_1,\\
&G_2 = z^2 + \frac{1}{2}pq + c_1 z + c_2,\\
&G_3 = z^3 + \frac{1}{2}pqz - \frac{i}{4}(p_x q - pq_x) + c_1(z^2 + \frac{1}{2}pq) + c_2 z + c_3,\\
&H_0 = ip,\\
&H_1 = ipz + \frac{1}{2}p_x + c_1(ip),\\
&H_2 = ipz^2 + \frac{1}{2}p_x z - \frac{i}{4}p_{xx} + \frac{i}{2}p^2 q + c_1(ipz + \frac{1}{2}p_x) + c_2(ip),
\end{aligned} \tag{2.16}
$$

etc.

One can use (2.10)–(2.12) and (2.14) to derive differential equations for F_n and H_n separately by eliminating G_{n+1}. One obtains for F_n,

$$
\begin{aligned}
&F_n F_{n,xx} - \frac{q_x}{q} F_n F_{n,x} - \frac{1}{2} F_{n,x}^2 + (2z^2 - 2iz\frac{q_x}{q} - 2pq)F_n^2\\
&= -2q^2 R_{2n+2}(z)
\end{aligned} \tag{2.17}
$$

and upon dividing (2.17) by q^2 and differentiating the result with respect to x,

$$
\begin{aligned}
F_{n,xxx} - 3\frac{q_x}{q}F_{n,xx} + (4z^2 - 4iz\frac{q_x}{q} - 4pq - \frac{q_{xx}}{q} + 3\frac{q_x^2}{q^2})F_{n,x}&\\
+(-4z^2\frac{q_x}{q} + 6iz\frac{q_x^2}{q^2} - 2iz\frac{q_{xx}}{q} + 2pq_x - 2p_x q)F_n = 0.&
\end{aligned} \tag{2.18}
$$

Similarly one obtains for H_n,

$$
\begin{aligned}
&H_n H_{n,xx} - \frac{p_x}{p} H_n H_{n,x} - \frac{1}{2} H_{n,x}^2 + (2z^2 + 2iz\frac{p_x}{p} - 2pq)H_n^2\\
&= -2p^2 R_{2n+2}(z),
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
H_{n,xxx} - 3\frac{p_x}{p}H_{n,xx} + (4z^2 + 4iz\frac{p_x}{p} - 4pq - \frac{p_{xx}}{p} + 3\frac{p_x^2}{p^2})H_{n,x}&\\
+(-4z^2\frac{p_x}{p} - 6iz\frac{p_x^2}{p^2} + 2iz\frac{p_{xx}}{p} + 2p_x q - 2pq_x)H_n = 0.&
\end{aligned} \tag{2.20}
$$

(2.17) and (2.19) can be used to derive recursion relations for f_ℓ and h_ℓ in the homogeneous case where all $c_\ell = 0, \quad \ell \in \mathbb{N}$ (cf. Lemma 4.5). This has interesting applications to the high-energy expansion for the Green's matrix of D as briefly discussed in Remark 4.6.

Next, we consider the kernel (i.e., the formal null space in a purely algebraic sense) of $(D - z), \quad z \in \mathbb{C}$,

$$
(D - z)\Psi = 0, \quad \Psi = \begin{pmatrix} \psi_1(z,x) \\ \psi_2(z,x) \end{pmatrix}, \quad z \in \mathbb{C} \tag{2.21}
$$

and, taking into account (2.6), that is, $[E_{n+1}, D] = 0$, compute the restriction of E_{n+1} to $\mathrm{Ker}(D - z)$. Using

$$\psi_{1,x} = -iz\psi_1 + q\psi_2,\ \psi_{2,x} = iz\psi_2 + p\psi_1, \ \text{ etc.}, \tag{2.22}$$

in order to eliminate higher-order derivatives of ψ_j, $j = 1, 2$, one obtains from (2.2), (2.4), (2.7), (2.9), and (2.10)–(2.12),

$$E_{n+1}\Big|_{\mathrm{Ker}(D-z)} = i\begin{pmatrix} -G_{n+1}(z,x) & F_n(z,x) \\ -H_n(z,x) & G_{n+1}(z,x) \end{pmatrix}\Big|_{\mathrm{Ker}(D-z)}. \tag{2.23}$$

Still assuming $f_{n+1} = h_{n+1} = 0$ as in (2.7), $[E_{n+1}, D] = 0$ in (2.6) yields an algebraic relationship between E_{n+1} and D by a celebrated result of Burchnall and Chaundy [7], [8], [9] (see also [45]). The following theorem details this relationship.

Theorem 2.1. *Assume $f_{n+1} = h_{n+1} = 0$, that is, $[E_{n+1}, D] = 0$ for some $n \in \mathbb{N}_0$. Then the Burchnall-Chaundy polynomial $\mathcal{F}_n(D, E_{n+1})$ of the pair (D, E_{n+1}) explicitly reads (cf. (2.15))*

$$\mathcal{F}_n(D, E_{n+1}) = E_{n+1}^2 + R_{2n+2}(D) = 0, \ \ R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \ \ z \in \mathbb{C}. \tag{2.24}$$

Proof. $[E_{n+1}, D] = 0$, (2.14), and (2.23) imply

$$\begin{aligned} E_{n+1}^2\Big|_{\mathrm{Ker}(D-z)} &= \Big[E_{n+1}\Big|_{\mathrm{Ker}(D-z)}\Big]^2\Big|_{\mathrm{Ker}(D-z)} \\ &= -\begin{pmatrix} G_{n+1}^2 - F_nH_n & 0 \\ 0 & G_{n+1}^2 - F_nH_n \end{pmatrix}\Big|_{\mathrm{Ker}(D-z)} = -R_{2n+2}(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big|_{\mathrm{Ker}(D-z)} \\ &= -R_{2n+2}(D)\Big|_{\mathrm{Ker}(D-z)}. \end{aligned} \tag{2.25}$$

Since $z \in \mathbb{C}$ is arbitrary one infers (2.24). ☐

Remark 2.2. Equation (2.24) naturally leads to the (possibly singular) hyperelliptic curve $\mathcal{K}_n$,

$$\mathcal{K}_n : \ \ \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0, \ \ R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \ \ n \in \mathbb{N}_0 \tag{2.26}$$

of (arithmetic) genus n.

Next, introducing a deformation parameter $t_n \in \mathbb{R}$ in (p, q) (*i.e.*, $(p(x), q(x)) \to (p(x, t_n), q(x, t_n))$), the time-dependent AKNS hierarchy (cf., e.g., [41], Chs. 3, 5 and the references therein) is defined as the collection of evolution equations (varying $n \in \mathbb{N}_0$),

$$\frac{d}{dt_n}D(t_n) - [E_{n+1}(t_n), D(t_n)] = 0, \ \ (x, t_n) \in \mathbb{R}^2, \ \ n \in \mathbb{N}_0, \tag{2.27}$$

or equivalently, by

$$\mathrm{AKNS}_n(p, q) \ = \ \begin{cases} p_{t_n}(x, t_n) - 2h_{n+1}(x, t_n) = 0 \\ \\ q_{t_n}(x, t_n) - 2f_{n+1}(x, t_n) = 0, \end{cases} , \ \ (x, t_n) \in \mathbb{R}^2, \ \ n \in \mathbb{N}_0, \tag{2.28}$$

that is, by

$$\operatorname{AKNS}_n(p,q) = \begin{cases} p_{t_n} + iH_{n,x} + 2zH_n - 2ipG_{n+1} = 0 \\ \\ q_{t_n} - iF_{n,x} + 2zF_n + 2iqG_{n+1} = 0 \end{cases}, \quad (x,t_n) \in \mathbb{R}^2, \quad n \in \mathbb{N}_0. \tag{2.29}$$

Explicitly, one obtains for the first few equations in (2.28) or (2.29),

$$\begin{aligned}
&\operatorname{AKNS}_0(p,q) = \begin{cases} p_{t_0} - p_x + c_1(-2ip) = 0 \\ \\ q_{t_0} - q_x + c_1(2iq) = 0 \end{cases}, \\
&\operatorname{AKNS}_1(p,q) = \begin{cases} p_{t_1} + \frac{i}{2}p_{xx} - ip^2q + c_1\,(-p_x) + c_2(-2ip) = 0 \\ \\ q_{t_1} - \frac{i}{2}q_{xx} + ipq^2 + c_1\,(-q_x) + c_2(2iq) \end{cases}, \\
&\operatorname{AKNS}_2(p,q) = \begin{cases} p_{t_2} + \frac{1}{4}p_{xxx} - \frac{3}{2}pp_xq + c_1(\frac{i}{2}p_{xx} - ip^2q) + c_2(-p_x) + c_3(-2ip) = 0 \\ \\ q_{t_2} + \frac{1}{4}q_{xxx} - \frac{3}{2}pqq_x + c_1(-\frac{i}{2}q_{xx} + ipq^2) + c_2(-q_x) + c_3(2iq) = 0 \end{cases},
\end{aligned} \tag{2.30}$$

etc.

Remark 2.3. We chose to start by postulating the recursion relation (2.2) and then developed the whole formalism based on (2.2), (2.4)–(2.6). Alternatively one could have started from

$$(D-z)\Psi(P) = 0, \quad (E_{n+1} - iy(p))\Psi(P) = 0, \quad P = (z, y(p)) \in \mathcal{K}_n \backslash \{\infty_\pm\} \tag{2.31}$$

and obtained the recursion relation (2.2) and the remaining stationary results of this section as a consequence of (2.9) and (2.23). Similarly, starting with

$$(D-z)\Psi(P,t_n) = 0, \quad \left(\frac{\partial}{\partial_{t_n}} - E_{n+1}\right)\Psi(P,t_n) = 0, \quad t_n \in \mathbb{R}, \tag{2.32}$$

one infers the time-dependent results (2.27)–(2.30).

Remark 2.4. Define

$$U(z,x) = \begin{pmatrix} -iz & q(x) \\ p(x) & iz \end{pmatrix}, \quad V_{n+1}(z,x) = i\begin{pmatrix} -G_{n+1}(z,x) & F_n(z,x) \\ -H_n(z,x) & G_{n+1}(z,x) \end{pmatrix}. \tag{2.33}$$

Then (2.23) implies

$$-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[E_{n+1}, D]\Big|_{\operatorname{Ker}(D-z)} = \{-V_{n+1,x}(z) + [U(z), V_{n+1}(z)]\}\Big|_{\operatorname{Ker}(D-z)} \tag{2.34}$$

and the stationary part of this section, being a consequence of $[E_{n+1}, D] = 0$, can equivalently be based on the equation

$$-V_{n+1,x} + [U, V_{n+1}] = 0. \tag{2.35}$$

In particular, the hyperelliptic curve $\mathcal{K}_n$ in (2.26) is then obtained from the characteristic equation for $V_{n+1}(z,x)$,

$$\begin{aligned} \det[yI - V_{n+1}(z,x)] &= y^2 - \det[V_{n+1}(z,x)] \\ &= y^2 - G_{n+1}(z,x)^2 + F_n(z,x)H_n(z,x) = y^2 - R_{2n+2}(z) = 0. \end{aligned} \tag{2.36}$$

Similarly, the time-dependent part (2.28)–(2.30), being based on the Lax equation (2.27), can equivalently be developed from the zero-curvature equation

$$U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0. \tag{2.37}$$

In fact, since the latter approach (2.37) is almost universally adopted in the contemporary literature on the AKNS hierarchy, we decided to recall its proper origin in connection with the Lax pair $[E_{n+1}, D]$ and based our treatment on matrix-valued differential expressions instead.

3. The Stationary AKNS Formalism

In this section we continue our discussion of the AKNS hierarchy and concentrate on the stationary case. Following [26], where the analogous treatment of the stationary KdV hierarchy can be found, we outline the connections between the polynomial approach described in Section 2 and a fundamental meromorphic function $\phi(P, x)$ defined on the hyperelliptic curve $\mathcal{K}_n$. Moreover, we discuss in detail the associated stationary Baker-Akhiezer function $\Psi(P, x, x_0)$, the common eigenfunction of D and E_{n+1} (we recall that $[E_{n+1}, D] = 0$), and associated positive divisors of degree n on $\mathcal{K}_n$ (which should be considered as the analogs of Dirichlet and Neumann divisors in the KdV context).

We recall the hyperelliptic curve (2.26),

$$\mathcal{K}_n: \ \ \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0, \qquad R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \tag{3.1}$$

where $n \in \mathbb{N}_0$ will be fixed throughout this section and denote its compactification (adding the points $\infty_\pm$) by the same symbol. Thus $\mathcal{K}_n$ becomes a (possibly singular) two-sheeted hyperelliptic Riemann surface of arithmetic genus n in a standard manner. We shall introduce a bit more notation in this context (see Appendix A for more details). Points P on $\mathcal{K}_n$ are represented as pairs $P = (z, y(P))$ satisfying (3.1) together with $\infty_\pm = (\infty, \pm\infty)$, the points at infinity. The complex structure on $\mathcal{K}_n$ is defined in the usual way by introducing local coordinates $\zeta_{P_0}: P \to (z - z_0)$ near points $P_0 \in \mathcal{K}_n$ which are neither branch nor singular points of $\mathcal{K}_n$, $\zeta_{\infty_\pm}: P \to \frac{1}{z}$ near $\infty_\pm$, and similarly at branch and/or singular points of $\mathcal{K}_n$. The holomorphic sheet exchange map (involution) $*$ is defined by

$$* : \begin{cases} \mathcal{K}_n \to \mathcal{K}_n \\ P = (z, y_j(P)) \mapsto P^* = (z, y_{j+1 \pmod 2}(P)) \ \ j = 1, 2 \end{cases}, \tag{3.2}$$

where $y_j(P)$, $j = 1, 2$ denote the two branches of $y(P)$ satisfying $\mathcal{F}_n(z, y) = 0$. A detailed description of $\mathcal{K}_n$ and its complex structure in the two most frequently discussed cases in applications where either $E_m \in \mathbb{R}$, $0 \le m \le 2n+1$ or $\{E_m\}_{0 \le m \le 2n+1} = \{E_{2m'}, \overline{E_{2m'}}\}_{0 \le m' \le n}$ is provided at the end of Appendix A.

Finally, positive divisors on $\mathcal{K}_n$ of degree are denoted by

$$\mathcal{D}_{\underline{Q}}: \begin{cases} \mathcal{K}_n \to \mathbb{N}_0 \\ P \mapsto \mathcal{D}_{\underline{Q}}(P) \begin{cases} m & \text{if } P \text{ occurs } m \text{ times in } \{Q_1, \dots, Q_n\} \\ 0 & \text{if } P \notin \{Q_1, \dots, Q_n\} \end{cases} \end{cases}, \quad \underline{Q} = (Q_1, \dots, Q_n).$$

(3.3)

Given these preliminaries, let $\Psi(P,x,x_0)$ denote the common normalized eigenfunction of D and E_{n+1}, whose existence follows from the commutativity of D and E_{n+1} (cf., eg., [7], [8] in the case of scalar differential expressions), that is, due to

$$[E_{n+1},D]=0, \tag{3.4}$$

for a given $n\in\mathbb{N}_0$, or equivalently, due to the requirement,

$$f_{n+1}=h_{n+1}=0. \tag{3.5}$$

Explicitly, this yields

$$D\Psi(P,x,x_0)=z\Psi(P,x,x_0),\quad E_{n+1}\Psi(P,x,x_0)=iy(p)\Psi(P,x,x_0),$$
$$\Psi(P,x,x_0)=\begin{pmatrix}\psi_1(P,x,x_0)\\ \psi_2(P,x,x_0)\end{pmatrix},\quad P=(z,y(P))\in\mathcal{K}_n\backslash\{\infty_\pm\},\quad x\in\mathbb{R} \tag{3.6}$$

for some fixed $x_0\in\mathbb{R}$ with the assumed normalization,

$$\psi_1(P,x_0,x_0)=1,\quad P\in\mathcal{K}_n\backslash\{\infty_\pm\}. \tag{3.7}$$

$\Psi(P,x,x_0)$ is called the Baker-Akhiezer (BA) function. Closely related to $\Psi(P,x,x_0)$ is the following meromorphic function $\phi(P,x)$ on $\mathcal{K}_n$, defined by

$$\phi(P,x)=\frac{\psi_2(P,x,x_0)}{\psi_1(P,x,x_0)},\quad P\in\mathcal{K}_n,\quad x\in\mathbb{R}. \tag{3.8}$$

Since $\phi(P,x)$ will be the fundamental object for the stationary AKNS hierarchy, we next seek its connection with the recursion formalism of Section 2. Recalling (2.23), one infers

$$E_{n+1}\Psi=i\begin{pmatrix}F_n\psi_2-G_{n+1}\psi_1\\ G_{n+1}\psi_2-H_n\psi_1\end{pmatrix}=iy\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{3.9}$$

and hence by (3.8),

$$\phi(P,x)=\frac{y(P)+G_{n+1}(z,x)}{F_n(z,x)}=\frac{-H_n(z,x)}{y(P)-G_{n+1}(z,x)},\quad P=(z,y(P))\in\mathcal{K}_n. \tag{3.10}$$

By (2.9) we may write,

$$F_n(z,x)=-iq(x)\prod_{j=1}^{n}(z-\mu_j(x)), \tag{3.11}$$

$$H_n(z,x)=ip(x)\prod_{j=1}^{n}(z-\nu_j(x)). \tag{3.12}$$

Defining

$$\hat{\mu}_j(x)=(\mu_j(x),y(\hat{\mu}_j(x)))=(\mu_j(x),G_{n+1}(\mu_j(x),x))\in\mathcal{K}_n,\quad 1\le j\le n,\quad x\in\mathbb{R}, \tag{3.13}$$

$$\hat{\nu}_j(x)=(\nu_j(x),y(\hat{\nu}_j(x)))=(\nu_j(x),-G_{n+1}(\nu_j(x),x))\in\mathcal{K}_n,\quad 1\le j\le n,\quad x\in\mathbb{R}, \tag{3.14}$$

one infers from (3.10) that the divisor $(\phi(P,x))$ of $\phi(P,x)$ is given by

$$(\phi(P,x)) = \mathcal{D}_{\underline{\hat{\nu}}(x)}(P) - \mathcal{D}_{\underline{\hat{\mu}}(x)}(P) + \mathcal{D}_{\infty_+}(P) - \mathcal{D}_{\infty_-}(P), \tag{3.15}$$
$$\underline{\hat{\nu}}(x) = (\hat{\nu}_1(x), \dots, \hat{\nu}_n(x)),\ \underline{\hat{\mu}}(x) = (\hat{\mu}_1(x), \dots, \hat{\mu}_n(x)).$$

Here we used our convention (3.3) and additive notation for divisors. Equivalently, $\infty_+, \hat{\nu}_1(x), \dots, \hat{\nu}_n(x)$, are the $n+1$ zeros of $\phi(P,x)$ and $\infty_-, \hat{\mu}_1(x), \dots, \hat{\mu}_n(x)$, its $n+1$ poles. Clearly $\mu_j(x)$ and $\nu_j(x)$ play the analogous role of Dirichlet and Neumann eigen values when comparing to the KdV case. In particular, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ represent the corresponding analogs of Dirichlet and Neumann divisors.

Next we summarize a variety of properties of $\phi(P,x)$ and $\Psi(P,x,x_0)$.

Lemma 3.1. *Assume (3.4)–(3.8), $P = (z, y(P)) \in \mathcal{K}_n \backslash \{\infty_\pm\}$, and let $(z,x,x_0) \in \mathbb{C} \times \mathbb{R}^2$. Then*

(i). $\Psi(P,x,x_0)$ *satisfies the first-oder system (cf. (2.33))*
$$\Psi_x(P,x,x_0) = U(z,x)\Psi(P,x,x_0). \tag{3.16}$$

(ii). $\phi(P,x)$ *satisfies the Ricati-type equation*
$$\phi_x(P,x) + q(x)\phi(P,x)^2 - 2iz\phi(P,x) = p(x). \tag{3.17}$$

(iii).
$$\phi(P,x)\phi(P^*,x) = \frac{H_n(z,x)}{F_n(z,x)}. \tag{3.18}$$

(iv).
$$\phi(P,x) + \phi(P^*,x) = \frac{2G_{n+1}(z,x)}{F_n(z,x)}. \tag{3.19}$$

(v).
$$\phi(P,x) - \phi(P^*,x) = \frac{2y(P)}{F_n(z,x)}. \tag{3.20}$$

(vi).
$$\psi_1(P,x,x_0) = \exp\left\{\int_{x_0}^{x} dx'[-iz + q(x')\phi(P,x')]\right\} \tag{3.21}$$
$$= \left[\frac{F_n(z,x)}{F_n(z,x_0)}\right]^{\frac{1}{2}} \exp\left\{y(P)) \int_{x_0}^{x} dx' q(x') F_n(z,x')^{-1}\right\}. \tag{3.22}$$

(vii).
$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0) = \frac{F_n(z,x)}{F_n(z,x_0)}. \tag{3.23}$$

(viii).
$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0) = \frac{H_n(z,x)}{F_n(z,x_0)}. \tag{3.24}$$

(ix).
$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0) + \psi_1(P^*,x,x_0)\psi_2(P,x,x_0) = \frac{2G_{n+1}(z,x)}{F_n(z,x_0)}. \tag{3.25}$$

Proof. (i) is an immediate consequence of (2.33) and (3.6). (ii) follows from (i), (2.22) and (3.8). (iii)–(v) are clear from (3.10). (3.21) follows from (2.22) and (3.8). (3.22) is a consequence of (iv), (v), (2.10), (3.21), and

$$\phi(P) = \frac{1}{2}[\phi(P) + \phi(P^*)] + \frac{1}{2}[\phi(P) - \phi(P^*)]$$

$$= \frac{G_{n+1}}{F_n} + \frac{y}{F_n} = \frac{1}{q}\left(\frac{F_{n,x}}{F_n} + iz\right) + \frac{y}{F_n}. \tag{3.26}$$

(vii) is clear from (3.22) and $(viii)$ is a consequence of (3.8), (iii), and (vii). Finally, (ix) is a consequence of (3.8), (3.19), and (3.23). □

In order to motivate our introduction of the basic quantity $\phi(P,x)$ we started with the common eigenfunction $\psi(P,x,x_0)$ of D and E_{n+1}. However, given (2.14) we could have defined $\phi(P,x)$ as in (3.10) and then verified that $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ defined by (3.8) and (3.21) satisfies (3.6) and (3.7).

Concerning the dynamics of the zeros $\mu_j(x)$ and $\nu_j(x)$ of $F_n(z,x)$ and $H_n(z,x)$ one obtains the following Dubrovin-type equations.

Lemma 3.2. *Assume (3.4)–(3.8), (3.11), (3.12) and let $x \in \mathbb{R}$. Then*

$$(i). \quad \mu_{j,x}(x) = \frac{-2iy(\hat{\mu}_j(x))}{\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j(x) - \mu_k(x))}, \quad 1 \le j \le n. \tag{3.27}$$

$$(ii). \quad \nu_{j,x}(x) = \frac{-2iy(\hat{\nu}_j(x))}{\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j(x) - \nu_k(x))}, \quad 1 \le j \le n. \tag{3.28}$$

Proof. Combine (2.10), (3.11), and (3.13) and (2.12), (3.12), and (3.14) in order to arrive at (3.27) and (3.28), respectively. □

Combining the polynomial approach of Section 2 with (3.11) and (3.12) readily yields trace formulas for the AKNS invariants. We indicate the first few of these below.

Lemma 3.3. *Assume (3.4)–(3.8) and let $x \in \mathbb{R}$. Then*

$$(i). \quad i\frac{p_x(x)}{p(x)} - 2c_1 = 2\sum_{j_1=1}^{n} \nu_{j_1}(x),$$

$$\frac{1}{4}\frac{p_{xx}(x)}{p(x)} - \frac{1}{2}p(x)q(x) + c_1\left(\frac{i}{2}\frac{p_x(x)}{p(x)}\right) - c_2 = -\sum_{\substack{j_1,j_2=1\\j_1<j_2}}^{n} \nu_{j_1}(x)\,\nu_{j_2}(x), \tag{3.29}$$

etc.

$$(ii). \quad i\frac{q_x(x)}{q(x)} + 2c_1 = -2\sum_{j_1=1}^{n} \mu_{j_1}(x),$$

$$\frac{1}{4}\frac{q_{xx}(x)}{q(x)} - \frac{1}{2}p(x)q(x) + c_1\left(-\frac{i}{2}\frac{q_x(x)}{q(x)}\right) - c_2 = -\sum_{\substack{j_1,j_2=1\\j_1<j_2}}^{n} \mu_{j_1}(x)\,\mu_{j_2}(x), \tag{3.30}$$

etc.

Here

$$c_1 = -\frac{1}{2}\sum_{m_1=0}^{2n+1} E_{m_1},$$

$$c_2 = \frac{1}{2} \sum_{\substack{m_1,m_2=0 \\ m_1<m_2}}^{2n+1} E_{m_1} E_{m_2} - \frac{1}{8}\Big(\sum_{m_1=0}^{2n+1} E_{m_1}\Big)^2, \tag{3.31}$$

etc.

Proof. (3.29) and (3.30) follow by comparison of powers of z substituting (3.11) into (2.9) taking into account (2.3). (3.31) follows in exactly the same way from (2.3), (2.9), and (3.12). ◻

Finally, we shall provide an explicit representation of Ψ, ϕ, p, and q in terms of the Riemann theta function associated with $\mathcal{K}_n$. We freely employ the notation established in Appendix A. In order to avoid the trivial case $n = 0$ (considered in Example 3.7) we assume $n \in \mathbb{N}$ for the remainder of this argument.

Assuming $\mathcal{K}_n$ to be nonsingular, that is,

$$E_m \neq E_{m'} \text{ for } 0 \leq m, m' \leq 2n+1, \tag{3.32}$$

we choose, without loss of generality, the base point $P_0 = (E_0, 0)$ and denote by $\underline{A}_{P_0}(.)$, $\underline{\alpha}_{P_0}(.)$ the Abel maps as defined in (A.30)–(A.32), and define $\underline{\hat{\Xi}}_{P_0}$, the vector of Riemann constants, by

$$\begin{aligned} &\underline{\Xi}_{P_0} = \underline{\hat{\Xi}}_{P_0} \pmod{L_n}, \\ &\underline{\hat{\Xi}}_{P_0} = (\hat{\Xi}_{P_0,1}, \dots, \hat{\Xi}_{P_0,n}), \quad \hat{\Xi}_{P_0,j} = \left[\frac{1+\tau_{j,j}}{2} - \sum_{\substack{k=1 \\ k\neq j}}^{n} \int_{a_k} \hat{A}_{P_0,j}\omega_k\right]. \end{aligned} \tag{3.33}$$

Next, consider the normal differential of the third kind $\omega^{(3)}_{\infty_+,\infty_-}$, which has simple poles at ∞_+ and ∞_-, corresponding residues $+1$ and -1, vanishing a-periods, and is holomorphic otherwise on K_g. Hence we have (cf. (A.36), (A.37))

$$\omega^{(3)}_{\infty_+,\infty_-} = \frac{\prod_{j=1}^n (\tilde{\pi} - \lambda_j) d\tilde{\pi}}{y}, \quad \omega^{(3)}_{\infty_-,\infty_+} = -\omega^{(3)}_{\infty_+,\infty_-}, \tag{3.34}$$

$$\int_{a_j} \omega^{(3)}_{\infty_+,\infty_-} = 0, \quad 1 \leq j \leq n, \tag{3.35}$$

$$\begin{aligned} &\underline{U}^{(3)} = (U_1^{(3)}, \dots, U_n^{(3)}), \\ &U_j^{(3)} = \frac{1}{2\pi i} \int_{b_j} \omega^{(3)}_{\infty_+,\infty_-} = \hat{A}_{\infty_-,j}(\infty_+) = 2\hat{A}_{P_0,j}(\infty_+), \quad 1 \leq j \leq n, \end{aligned} \tag{3.36}$$

$$\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-} \underset{\zeta\to 0}{=} \pm[\ln(\zeta) - \ln(\omega_0) + O(\zeta)], \quad P = (\zeta^{-1}, y(P)) \text{ near } \infty_\pm, \tag{3.37}$$

where the numbers $\{\lambda_j\}_{1\leq j\leq n}$ are determined by the normalization (3.35). The Abelian differential of the second kind $\omega^{(2)}_{\infty_\pm,0}$ (cf. (A.34), (A.35)) are chosen such that

$$\omega^{(2)}_{\infty_\pm,0} \underset{\zeta\to 0}{=} [\zeta^{-2} + O(1)]\, d\zeta \text{ near } \infty_\pm, \tag{3.38}$$

$$\int_{a_j} \omega^{(2)}_{\infty_\pm,0} = 0, \quad 1 \le j \le n, \tag{3.39}$$

$$\underline{U}^{(2)}_0 = (U^{(2)}_{0,1}, \dots, U^{(2)}_{0,n}), \;\; U^{(2)}_{0,j} = \frac{1}{2\pi i}\int_{b_j} \Omega^{(2)}_0, \quad \Omega^{(2)}_0 = \omega^{(2)}_{\infty_+,0} - \omega^{(2)}_{\infty_-,0}, \tag{3.40}$$

$$\int_{P_0}^{P} \Omega^{(2)}_0 \underset{\zeta\to 0}{=} \mp[\zeta^{-1} + e_0 + O(\zeta)], \;\; P = (\zeta^{-1}, y(P)) \text{ near } \infty_\pm. \tag{3.41}$$

Next, we formulate the following auxiliary result.

Lemma 3.4. *Let $\psi(.,x)$, $x \in \mathbb{R}$ be meromorphic on $\mathcal{K}_n \backslash \{\infty_+, \infty_-\}$ with essential singularities at $\infty_\pm$ such that $\tilde{\psi}(.,x)$ defined by*

$$\tilde{\psi}(.,x) = \psi(.,x) \exp\Big[-i(x-x_0)\int_{P_0}^{P} \Omega^{(2)}_0 \Big] \tag{3.42}$$

is multi-valued meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\tilde{\psi}(.,x)) \ge -\mathcal{D}_{\hat{\mu}(x)}. \tag{3.43}$$

Define a divisor $\mathcal{D}_0(x)$ by

$$(\tilde{\psi}(.,x)) = \mathcal{D}_0(x) - \mathcal{D}_{\hat{\mu}(x)}. \tag{3.44}$$

Then

$$\mathcal{D}_0(x) \in \sigma^n \mathcal{K}_n, \; \mathcal{D}_0(x) > 0, \; \deg(\mathcal{D}_0(x)) = n. \tag{3.45}$$

Moreover, if $\mathcal{D}_0(x)$ is nonspecial for all $x \in \mathbb{R}$, that is, if

$$i(\mathcal{D}_0(x)) = 0, \;\; x \in \mathbb{R}, \tag{3.46}$$

then $\psi(.,x)$ is unique up to a constant multiple (which may depend on x).

Proof. By the Riemann-Roch theorem (see (A.42)) there exists at least one such function $\psi(.,x)$. Suppose $\psi_j(.,x)$, $j = 1,2$ are two such functions satisfying (3.44) with corresponding divisors $\mathcal{D}_{0,j}(x)$, $j = 1,2$. Then

$$(\psi_1(.,x)/\psi_2(.,x)) = \mathcal{D}_{0,1}(x) - \mathcal{D}_{0,2}(x). \tag{3.47}$$

Since $i(\mathcal{D}_{0,2}(x)) = 0$, $\deg(\mathcal{D}_{0,2}(x)) = n$ by hypothesis, the multi-valued version of (A.42) yields $r(-\mathcal{D}_{0,2}(x)) = 1, \;\; x \in \mathbb{R}$ and hence $\psi_1(.,x)/\psi_2(.,x)$ is a constant on $\mathcal{K}_n$. (For simplicity of notation, we did not prescribe the path of integration in (3.42). This in turn forced us to use the multi-valued form of the Riemann-Roch theorem, see, e.g., [17], Sect. III.9.) $\square$

Assuming $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is , $i(\mathcal{D}_{\underline{Q}}) = 0, \;\; \underline{Q} = (Q_1, \dots, Q_n)$, a special case of Riemann's vanishing theorem yields that

$$\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_n\}. \tag{3.48}$$

Hence the divisor (3.15) of $\phi(P,x)$ suggests considering expressions of the type

$$C(x)\frac{\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}))}{\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}))} \exp\Big[\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-}\Big], \tag{3.49}$$

where $C(x)$ is independent of $P \in \mathcal{K}_n$. In fact, abbreviating

$$\underline{z}(P, \underline{Q}) = \underline{A}_{P_0}(P) - \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}}) - \underline{\Xi}_{P_0}, \tag{3.50}$$

$$\underline{z}_{\pm}(\underline{Q}) = \underline{z}(\infty_{\pm}, \underline{Q}), \quad \underline{Q} = (Q_1, \dots, Q_n), \tag{3.51}$$

one obtains the following theta function representation for ϕ, Ψ and (p, q) (the analog of the celebrated Its-Matveev formula [31] in the KdV context).

Theorem 3.5. *Let $P \in \mathcal{K}_n \backslash \{\infty_+, \infty_-\}$, $(x, x_0) \in \mathbb{R}^2$, and assume $\mathcal{K}_n$ to be nonsingular, that is, $E_m \neq E_{m'}$ for $m \neq m'$, $0 \leq m, m' \leq 2n+1$. Moreover, suppose $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = i(\mathcal{D}_{\underline{\hat{\nu}}(x)}) = 0$. Then*

$$\begin{aligned} \phi(P, x) = {} & \frac{2i}{q(x_0)\omega_0} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0)))} \frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x)))} \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(x)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))} \times \\ & \times \exp\left[\int_{P_0}^{P} \omega^{(3)}_{\infty_+, \infty_-} - 2i(x - x_0)e_0\right], \end{aligned} \tag{3.52}$$

$$\psi_1(P, x, x_0) = \frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))} \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x)))} \exp\left[i(x - x_0)\left(e_0 + \int_{P_0}^{P} \Omega_0^{(2)}\right)\right], \tag{3.53}$$

$$\begin{aligned} \psi_2(P, x, x_0) = {} & \frac{2i}{q(x_0)\omega_0} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))} \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(x)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x)))} \times \\ & \times \exp\left[\int_{P_0}^{P} \omega^{(3)}_{\infty_+, \infty_-} + i(x - x_0)\left(-e_0 + \int_{P_0}^{P} \Omega_0^{(2)}\right)\right]. \end{aligned} \tag{3.54}$$

Moreover, one derives

$$p(x) = p(x_0) \frac{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0)))}{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0)))} \frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x)))} e^{-2i(x - x_0)e_0}, \tag{3.55}$$

$$q(x) = q(x_0) \frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0)))} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x)))} e^{2i(x - x_0)e_0}, \tag{3.56}$$

$$p(x_0)q(x_0) = \frac{4}{\omega_0^2} \frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0)))} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0)))}, \tag{3.57}$$

and

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) - i(x - x_0)\underline{U}_0^{(2)}, \tag{3.58}$$

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0)}) - i(x - x_0)\underline{U}_0^{(2)}. \tag{3.59}$$

Proof. Since $\phi(P, x)e^{-\int_{P_0}^{P} \omega^{(3)}_{\infty_+, \infty_-}}$ is meromorphic on $\mathcal{K}_n$ with divisor $\mathcal{D}_{\underline{\hat{\nu}}(x)} - \mathcal{D}_{\underline{\hat{\mu}}(x)}$, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ is nonspecial (cf. the comment following (A.40)). Combining (3.7), (3.8), (3.10), (3.11), (3.12), (3.18), and (3.23) yields the asymptotic behavior

$$\phi(P, x) \underset{\zeta \to 0}{=} \begin{cases} \frac{i}{2} p(x)\zeta + O(\zeta^2), & P \text{ near } \infty_+ \\ \frac{2i}{q(x)}\zeta^{-1} + O(1), & P \text{ near } \infty_- \end{cases}, \tag{3.60}$$

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \begin{cases} e^{-i(x-x_0)\zeta^{-1}+O(\zeta)}, & P \text{ near } \infty_+ \\ \\ \left[\frac{q(x)}{q(x_0)} + O(\zeta)\right] e^{i(x-x_0)\zeta^{-1}+O(\zeta)}, & P \text{ near } \infty_- \end{cases}, \tag{3.61}$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} \begin{cases} \left[\frac{i}{2}p(x)\zeta + O(\zeta^2\right] e^{-i(x-x_0)\zeta^{-1}+O(\zeta)}, & P \text{ near } \infty_+ \\ \\ \left[\frac{2i}{q(x_0)}\zeta^{-1} + O(1)\right] e^{i(x-x_0)\zeta^{-1}+O(\zeta)}, & P \text{ near } \infty_- \end{cases}. \tag{3.62}$$

The asymptotic behavior (3.61) (near ∞_+), (3.23) and (3.48) together with Lemma 3.4 then directly yield the theta function representation (3.53) for ψ_1. (3.48) also immediately yields that $\phi(P,x)$ equals (3.49) which, together with (3.60), implies

$$p(x) = \frac{2C(x)}{i\omega_0}\frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x)))}, \tag{3.63}$$

$$q(x) = \frac{2i}{C(x)\omega_0}\frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x)))}. \tag{3.64}$$

On the other hand (3.61) (near ∞_-) and (3.53) yield (3.56). A comparison of (3.56) and (3.64) determines $C(x)$ and $p(x)$ as in (3.55), (3.57). Given $C(x)$, one determines ϕ in (3.52) from (3.49) and hence ψ_2 as in (3.54) from $\psi_2 = \phi\psi_1$. By (3.27) and a special case of Lagrange's interpolation formula,

$$\sum_{j=1}^{n} \mu_j^{k-1} \prod_{\substack{\ell=1 \\ \ell\neq j}}^{n} (\mu_j - \mu_\ell)^{-1} = \delta_{k,n}, \quad \mu_j \in \mathbb{C}, \quad 1 \le j,k \le n, \tag{3.65}$$

one infers

$$\frac{d}{dx}\underline{\alpha}_{P_0}(\underline{\hat{\mu}}(x))) = -2i\underline{c}_n, \quad \underline{c}_n = (c_{1,n}, \dots, c_{n,n}), \tag{3.66}$$

where

$$\omega_j = \sum_{k=1}^{n} c_{j,k}\frac{\tilde{\pi}\, d\tilde{\pi}}{y}, \quad 1 \le j \le n \tag{3.67}$$

abbreviates the basis of holomorphic differentials on $\mathcal{K}_n$. By (A.35) this yields

$$\frac{d}{dx}\underline{\alpha}_{P_0}(\underline{\hat{\mu}}(x))) = -i\underline{U}_0^{(2)} \tag{3.68}$$

and hence (3.58) (respectively, (3.59)). $\square$

We note that the free constant $q(x_0)$ in (3.55) (and in (3.56) using (3.57)) cannot be determined by this formalism since the AKNS equations (2.28) are invariant with respect to scale transformations. More precisely, one has

Lemma 3.6. *Suppose (p,q) satisfies one of the AKNS equations (2.28) for some $n \in \mathbb{N}_0$,*

$$AKNS_n(p,q) = 0. \tag{3.69}$$

Consider the scale transformation

$$(p(x,t_n), q(x,t_n)) \to (\check{p}(x,t_n), \check{q}(x,t_n)) = (Ap(x,t_n), A^{-1}q(x,t_n)), \quad A \in \mathbb{C}\backslash\{0\}. \tag{3.70}$$

Then

$$AKNS_n(\breve{p}, \breve{q}) = 0. \tag{3.71}$$

Proof. Let (D, E_{n+1}) and $(\breve{D}, \breve{E}_{n+1})$ be associated with (p, q) and $(\breve{p}, \breve{q})$, respectively and defined according to (2.1) and (2.4). Defining the matrix T in $\mathbb{C}^2$ by

$$T = \begin{pmatrix} (A^{\frac{1}{2}})^{-1} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \tag{3.72}$$

(fixing a particular square root branch $A^{\frac{1}{2}}$) one computes

$$TDT^{-1} = \breve{D}, \tag{3.73}$$

$$TE_{n+1}T^{-1} = i \sum_{\ell=0}^{n+1} \begin{pmatrix} -g_{n+1-\ell} & A^{-1} f_{n-\ell} \\ -Ah_{n-\ell} & g_{n+1-\ell} \end{pmatrix} \breve{D}^{\ell} = \breve{E}_{n+1}. \tag{3.74}$$

A comparison of (3.74) with

$$\breve{E}_{n+1} = i \sum_{\ell=0}^{n+1} \begin{pmatrix} -\breve{g}_{n+1-\ell} & \breve{f}_{n-\ell} \\ -\breve{h}_{n-\ell} & \breve{g}_{n+1-\ell} \end{pmatrix} \breve{D}^{\ell} \tag{3.75}$$

yields

$$\breve{f}_{n-\ell} = A^{-1} f_{n-\ell}, \quad \breve{g}_{n+1-\ell} = g_{n+1-\ell}, \quad \breve{h}_{n-\ell} = A^{-1} h_{n-\ell}, \quad 0 \le \ell \le n+1 \tag{3.76}$$

and hence (3.71), taking into account (2.28) and (3.70). $\square$

In the particular case of the nonlinear Schrödinger (NLS) hierarchy, where

$$p(x, t_n) = \pm \overline{q(x, t_n)}, \quad n \in \mathbb{N}_0, \tag{3.77}$$

(3.70) further restricts A to be unimodular, that is,

$$|A| = 1. \tag{3.78}$$

In the special case of the modified Korteweg-de Vries (mKdV) hierarchy, where

$$p(x, t_n) = \pm q(x, t_n), \quad n \in 2\mathbb{N}_0, \tag{3.79}$$

(3.70) implies the additional restriction

$$A = \pm 1. \tag{3.80}$$

Next, we briefly consider the trivial case $n = 0$ excluded in Theorem 3.5.

Example 3.7. Assume $n = 0$. Then

$\mathcal{F}_0(z, y) = y^2 - R_2(z) = y^2 - \prod_{m=0}^{1}(z - E_m)$,

$c_1 = -(E_0 + E_1)/2$,

$p(x) = p(x_0) \exp[-2ic_1(x - x_0)], \quad q(x) = q(x_0) \exp[2ic_1(x - x_0)]$,

$p(x)q(x) = -(E_0 - E_1)^2/4$,

$\phi(P, x) = \frac{y(P)+z+c_1}{-iq(x)} = \frac{ip(x)}{y(P)-z-c_1}$,

$\psi_1(P, x, x_0) = \exp[i(x - x_0)(y(P) + c_1)]$,

$\psi_2(P,x,x_0) = \frac{y(P)+z+c_1}{-iq(x_0)} \exp[i(x-x_0)(y(P)-c_1)]$.

Finally, we mention an interesting characterization of all algebro-geometric AKNS potentials due to De Concini and Johnson [11] in the special case where D generates a self-adjoint operator in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. In this case the algebro-geometric potentials are characterized by the fact that the corresponding spectrum consists of finitely many intervals and the Lyapunov exponent vanishes a.e. on the spectrum. In this context we might also point out that a detailed study of Floquet theory for periodic and self-adjoint AKNS operators (not necessarily of algebro-geometric type) generated by D can be found in [27].

4. The Time-Dependent AKNS Formalism

In our final section we indicate how to generalize the polynomial approach of Sections 2 and 3 to the time-dependent AKNS hierarchy.

Our starting point is a stationary n-gap solution $(p^{(0)}(x), q^{(0)}(x))$, associated with $\mathcal{K}_n$,

$$\begin{aligned} i\frac{p_x^{(0)}(x)}{p^{(0)}(x)} &= 2c_1 + 2\sum_{j_1=1}^{n} \nu_{j_1}^{(0)}(x), \\ i\frac{q_x^{(0)}(x)}{q^{(0)}(x)} &= -2c_1 - 2\sum_{j_1=1}^{n} \mu_{j_1}^{(0)}(x), \end{aligned} \tag{4.1}$$

satisfying

$$\mathrm{AKNS}_n(p^{(0)}, q^{(0)}) = \left\{ \begin{array}{c} -2h_{n+1} = 0 \\ -2f_{n+1} = 0 \end{array} \right. , \tag{4.2}$$

for some fixed $n \in \mathbb{N}_0$ and a given set of integration constants $\{c_\ell\}_{1\le \ell \le n+1}$. Our principal aim is to construct the r^{th} AKNS flow

$$\begin{aligned} \mathrm{AKNS}_r(p,q) &= \left\{ \begin{array}{l} p_{t_r} - 2\tilde{h}_{r+1} = 0 \\ q_{t_r} - 2\tilde{f}_{r+1} = 0 \end{array} \right. \\ &= \left\{ \begin{array}{l} p_{t_r} + i\tilde{H}_{r,x} + 2z\tilde{H}_r - 2ip\tilde{G}_{r+1} = 0 \\ q_{t_r} - i\tilde{F}_{r,x} + 2z\tilde{F}_r + 2iq\tilde{G}_{r+1} = 0 \end{array} \right. , \end{aligned} \tag{4.3}$$

$$(p(x,t_{0,r}), q(x,t_{0,r})) = (p^{(0)}(x), q^{(0)}(x)), \quad x \in \mathbb{R}$$

for $t_{0,r} \in \mathbb{R}$ and some fixed $r \in \mathbb{N}_0$. In terms of Lax pairs this amounts to solving

$$\frac{d}{dt_r} D(t_r) - [\tilde{E}_{r+1}(t_r), D(t_r)] = 0, \quad t_r \in \mathbb{R}, \tag{4.4}$$

$$[E_{n+1}(t_{0,r}), D(t_{0,r})] = 0. \tag{4.5}$$

As a consequence one obtains that

$$[E_{n+1}(t_r), D(t_r)] = 0, \quad t_r \in \mathbb{R}, \tag{4.6}$$

$$E_{n+1}(t_r)^2 = -R_{2n+2}(D(t_r)) = -\prod_{m=0}^{2n+1}(D(t_r)-E_m), \quad t_r \in \mathbb{R} \tag{4.7}$$

since the AKNS flows are isospectral deformations of $D(t_{0,r})$.

We emphasize that the integration constants $\{\tilde{c}_\ell\}$ in $\tilde{E}_{r+1}$ and $\{c_\ell\}$ in E_{n+1}, in general, are independent of each other (even if $r = n$). Hence we shall employ the notation $\tilde{E}_{r+1}$, $\tilde{V}_{r+1}$, $\tilde{F}_r$, $\tilde{G}_{r+1}$ $\tilde{H}_r$, $\tilde{f}_l$, $\tilde{g}_l$, $\tilde{h}_l$, $\tilde{c}_l$, etc., in order to distinguish it from E_{n+1}, V_{n+1}, F_n, G_{n+1}, H_n, f_l, g_l, h_l, c_l, etc. In addition, we followed a more elaborate notation inspired by Hirota's τ- function approach and indicated the individual r^{th} AKNS flow by a separate time variable $t_r \in \mathbb{R}$. (The latter notation suggests considering all AKNS flows simultaneously by introducing $\underline{t} = (t_0, t_1, t_2, \ldots\ldots..)$.)

Instead of working directly with (4.4), (4.5) and (4.6), it is more convenient to take the zero-curvature equations (2.37) as our point of departure, that is, we start from

$$U_{t_r} - \tilde{V}_{r+1,x} + [U, \tilde{V}_{r+1}] = 0, \quad (x, t_r) \in \mathbb{R}^2, \tag{4.8}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0, \quad (x, t_r) \in \mathbb{R}^2, \tag{4.9}$$

where (cf. (2.9))

$$\begin{aligned} U(z,x,t_r) &= \begin{pmatrix} -iz & q(x,t_r) \\ p(x,t_r) & iz \end{pmatrix}, \\ \tilde{V}_{r+1}(z,x,t_r) &= i\begin{pmatrix} -\tilde{G}_{r+1}(z,x,t_r) & \tilde{F}_r(z,x,t_r) \\ -\tilde{H}_r(z,x,t_r) & \tilde{G}_{r+1}(z,x,t_r) \end{pmatrix}, \\ V_{n+1}(z,x,t_r) &= i\begin{pmatrix} -G_{n+1}(z,x,t_r) & F_n(z,x,t_r) \\ -H_n(z,x,t_r) & G_{n+1}(z,x,t_r) \end{pmatrix}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} F_n(z,x,t_r) &= \sum_{\ell=0}^{n} f_{n-\ell}(x,t_r)z^\ell, \quad f_0(x,t_r) = -iq(x,t_r), \\ F_n(z,x,t_{0,r}) &= F_n^{(0)}(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}^{(0)}(x)z^\ell, \quad f_0^{(0)}(x) = -iq^{(0)}(x), \end{aligned} \tag{4.11}$$

$$\begin{aligned} G_{n+1}(z,x,t_r) &= \sum_{\ell=0}^{n+1} g_{n+1-\ell}(x,t_r)z^\ell, \quad g_0(x,t_r) = 1, \\ G_{n+1}(z,x,t_{0,r}) &= G_{n+1}^{(0)}(z,x) = \sum_{\ell=0}^{n+1} g_{n+1-\ell}^{(0)}(x)z^j, \quad g_0^{(0)}(x) = 1, \end{aligned} \tag{4.12}$$

$$H_n(z,x,t_r) = \sum_{\ell=0}^{n} h_{n-\ell}(x,t_r)z^\ell, \quad h_0(x,t_r) = ip(x,t_r), \tag{4.13}$$

$$H_n(z,x,t_{0,r}) = H_n^{(0)}(z,x) = \sum_{\ell=0}^{n} h_{n-\ell}^{(0)}(x) z^\ell, \quad h_0^{(0)}(x) = ip^{(0)}(x)$$

for fixed $t_{0,r} \in \mathbb{R}$, $n \in \mathbb{N}_0$, $r \in \mathbb{N}_0$. Here $f_\ell(x,t_r)$ and $f_\ell^{(0)}(x)$ are defined as in (2.2) with $(p(x), q(x))$ replaced by $(p(x,t_r), q(x,t_r))$ and $(p^{(0)}(x), q^{(0)}(x))$, respectively. Explicitly, (4.8) and (4.9) are equivalent to

$$p_{t_r} = -i\tilde{H}_{r,x} - 2z\tilde{H}_r + 2ip\tilde{G}_{r+1}, \tag{4.14}$$
$$q_{t_r} = i\tilde{F}_{r,x} - 2z\tilde{F}_r - 2iq\tilde{G}_{r+1}, \tag{4.15}$$
$$\tilde{G}_{r+1,x} = p\tilde{F}_r + q\tilde{H}_r \tag{4.16}$$

and (cf. (2.10)–(2.12))

$$F_{n,x} = -2izF_n + 2qG_{g+1}, \tag{4.17}$$
$$G_{n+1,x} = pF_n + qH_n, \tag{4.18}$$
$$H_{n,x} = 2izH_n + 2pG_{g+1}, \tag{4.19}$$

respectively. In particular, (2.14) holds in the present t_r-dependent setting, that is,

$$G_{n+1}^2 - F_n H_n = R_{2n+2}. \tag{4.20}$$

In analogy to (3.11) and (3.12) we write

$$F_n(z,x,t_r) = -iq(x,t_r) \prod_{j=1}^{n} (z - \mu_j(x,t_r)), \tag{4.21}$$

$$H_n(z,x,t_r) = ip(x,t_r) \prod_{j=1}^{n} (z - \nu_j(x,t_r)) \tag{4.22}$$

and define in analogy to (3.10), the following meromorphic function $\phi(P,x,t_r)$ on $\mathcal{K}_n$, the fundamental ingredient for constructing algebro-geometric solutions of the time-dependent AKNS hierarchy,

$$\phi(P,x,t_r) = \frac{y(P) + G_{n+1}(z,x,t_r)}{F_n(z,x,t_r)} = \frac{-H_n(z,x,t_r)}{y(P) - G_{n+1}(z,x,t_r)}, \quad P = (z, y(P)) \in \mathcal{K}_n. \tag{4.23}$$

As in (3.13) and (3.14) one introduces

$$\hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), y(\hat{\mu}_j(x,t_r))) = (\mu_j(x,t_r), G_{n+1}(\mu_j(x,t_r), x, t_r)) \in \mathcal{K}_n, \quad 1 \le j \le n, \ (x,t_r) \in \mathbb{R}^2, \tag{4.24}$$

$$\hat{\nu}_j(x,t_r) = (\nu_j(x,t_r), y(\hat{\nu}_j(x,t_r))) = (\nu_j(x,t_r), -G_{n+1}(\nu_j(x,t_r), x, t_r)) \in \mathcal{K}_n, \quad 1 \le j \le n, \ (x,t_r) \in \mathbb{R}^2, \tag{4.25}$$

and infers that the divisor $(\phi(P,x,t_r))$ of $\phi(P,x,t_r)$ is given by

$$(\phi(P,x,t_r)) = \mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}(P) - \mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}(P) + \mathcal{D}_{\infty_+}(P) - \mathcal{D}_{\infty_-}(P). \tag{4.26}$$

Next we define the time-dependent BA-function $\Psi(P,x,x_0,t_r,t_{0,r})$ by

$$\Psi(P,x,x_0,t_r,t_{0,r}) = \begin{pmatrix} \psi_1(P,x,x_0,t_r,t_{0,r}) \\ \psi_2(P,x,x_0,t_r,t_{0,r}) \end{pmatrix}, \tag{4.27}$$

$$\begin{aligned} \psi_1(P,x,x_0,t_r,t_{0,r}) &= \exp\bigg\{ \int_{x_0}^{x} dx' [-iz + q(x',t_r)\phi(P,x',t_r)] \\ &\quad + i \int_{t_{0,r}}^{t_r} ds [\tilde F_r(z,x_0,s)\phi(P,x_0,s) - \tilde G_{r+1}(z,x_0,s)] \bigg\}, \end{aligned} \tag{4.28}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r}) = \phi(P,x,t_r)\psi_1(P,x,x_0,t,t_{0,r}), \tag{4.29}$$

$$P \in \mathcal{K}_n \backslash \{\infty_\pm\}, \quad (x,t_r) \in \mathbb{R}^2,$$

with fixed $(x_0,t_{0,r}) \in \mathbb{R}^2$. The following Lemma records properties of $\phi(P,x,t_r)$ and $\Psi(P,x,x_0,t_r,t_{0,r})$ in analogy to the stationary case discussed in Lemma 3.1.

Lemma 4.1. *Assume (4.14)–(4.20), $P=(z,y(P)) \in \mathcal{K}_n\backslash\{\infty_\pm\}$, and let $(z,x,x_0,t_r,t_{0,r}) \in \mathbb{C}\times\mathbb{R}^4$. Then*

(i). $\phi(P,x,t_r)$ satisfies

$$\phi_x(P,x,t_r) + q(x,t_r)\phi(P,x,t_r)^2 - 2iz\phi(P,x,t_r) = p(x,t_r), \tag{4.30}$$

$$[q(x,t_r)\phi(P,x,t_r)]_{t_r} = i\partial_x[\tilde F_r(z,x,t_r)\phi(P,x,t_r) - \tilde G_{r+1}(z,x,t_r)], \tag{4.31}$$

$$\begin{aligned} \phi_{t_r}(P,x,t_r) &= 2i\tilde G_{r+1}(z,x,t_r)\phi(P,x,t_r) + \frac{1}{q(x,t_r)}[-i\tilde G_{r+1,x}(z,x,t_r) \\ &\quad + i\tilde F_r(z,x,t_r)\phi_x(P,x,t_r) + 2z\tilde F_r(z,x,t_r)\phi(P,x,t_r)]. \end{aligned} \tag{4.32}$$

(ii). $\psi_j(P,x,x_0,t_r,t_{0,r})$, $j=1,2$ satisfy

$$\psi_{1,x}(P,x,x_0,t_r,t_{0,r}) = [q(x,t_r)\phi(P,x,t_r) - iz]\psi_1(P,x,x_0,t_r,t_{0,r}), \tag{4.33}$$

$$\begin{aligned} \psi_{1,t_r}(P,x,x_0,t_r,t_{0,r}) &= i[\tilde F_r(z,x,t_r)\phi(P,x,t_r) - \tilde G_{r+1}(z,x,t_r)] \times \\ &\quad \times \psi_1(P,x,x_0,t_r,t_{0,r}), \end{aligned} \tag{4.34}$$

$$\psi_{2,x}(P,x,x_0,t_r,t_{0,r}) = [p(x,t_r)\phi(P,x,t_r)^{-1} + iz]\psi_1(P,x,x_0,t_r,t_{0,r}), \tag{4.35}$$

$$\begin{aligned} \psi_{2,t_r}(P,x,x_0,t_r,t_{0,r}) &= -i[\tilde H_r(z,x,t_r)\phi(P,x,t_r)^{-1} - \tilde G_{r+1}(z,x,t_r)] \times \\ &\quad \times \psi_2(P,x,x_0,t_r,t_{0,r}), \end{aligned} \tag{4.36}$$

or equivalently,

$$\begin{aligned} \Psi_x(P,x,x_0,t_r,t_{0,r}) &= U(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \\ \Psi_{t_r}(P,x,x_0,t_r,t_{0,r}) &= V(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}) \end{aligned} \tag{4.37}$$

(i.e., $(D-z)\Psi = 0$, $(E_{n+1} - iy)\Psi = 0$, $\Psi_{t_r} = \tilde E_{r+1}\Psi$).

(iii). $$\phi(P,x,t_r)\phi(P^*,x,t_r) = \frac{H_n(z,x,t_r)}{F_n(z,x,t_r)}. \tag{4.38}$$

$$(iv).\ \ \phi(P,x,t_r)+\phi(P^*,x,t_r)=\frac{2G_{n+1}(z,x,t_r)}{F_n(z,x,t_r)}. \tag{4.39}$$

$$(v).\ \ \phi(P,x,t_r)-\phi(P^*,x,t_r)=\frac{2y(P)}{F_n(z,x)}. \tag{4.40}$$

Proof. (4.30) follows from (4.9) and (4.23). (4.31) can be proven as follows. Using (4.3) and (4.30) one infers by a straight forward (but rather lengthy) calculation that

$$(\partial_x+2q\phi-2iz-\frac{q_x}{q})[(q\phi)_t-i(\tilde{F}_r\phi-\tilde{G}_{r+1})_x]=0. \tag{4.41}$$

Thus

$$(q\phi)_t-i(\tilde{F}_r\phi-\tilde{G}_{r+1})_x=Ce^{\int^x dx'\,[2iz+\frac{q_{x'}}{q}-2q\phi]}, \tag{4.42}$$

where C is independent of x (but may depend on P and t_r). By inspection of (4.23), the left-hand side of (4.42) is meromorphic on $\mathcal{K}_n$ while the right-hand side of (4.42) is not meromorphic at ∞_+ and ∞_- unless $C=0$. Hence one infers $C=0$ and thus (4.31). (4.32) is then an immediate consequence of (4.3) (i.e., the AKNS equation for q_{t_r}) and (4.31). (4.33) is clear from (4.28) and (4.35) is obvious from (4.29), (4.30), and (4.33). (4.34) follows from (4.28) and (4.31), and (4.36) is a straightforward consequence of (4.29), and (4.34). Finally, (iii)–(v) are proved as in Lemma 3.1. □

Next we consider the t_r-dependence of $F_n(z,x,t_r)$, $G_{n+1}(z,x,t_r)$, and $H_n(z,x,t_r)$.

Lemma 4.2. *Assume (4.14)–(4.20) and let $(z,x,t_r)\in\mathbb{C}\times\mathbb{R}^2$. Then*

$$(i).\ \ F_{n,t_r}(z,x,t_r)=2i[\tilde{F}_r(z,x,t_r)G_{n+1}(z,x,t_r)-F_n(z,x,t_r)\tilde{G}_{r+1}(z,x,t_r)]. \tag{4.43}$$

$$(ii).\ \ G_{n+1,t_r}(z,x,t_r)=i[\tilde{F}_r(z,x,t_r)H_n(z,x,t_r)-F_n(z,x,t_r)\tilde{H}_r(z,x,t_r)]. \tag{4.44}$$

$$(iii).\ H_{n,t_r}(z,x,t_r)=-2i[\tilde{H}_r(z,x,t_r)G_{n+1}(z,x,t_r)-H_n(z,x,t_r)\tilde{G}_{r+1}(z,x,t_r)]. \tag{4.45}$$

Proof. By (4.23), (4.32), (4.39), and (4.40) one infers

$$\phi_{t_r}(P)-\phi_{t_r}(P^*)=-\frac{2y(P)F_{n,t_r}}{F_n^2}=\frac{4iy(P)}{F_n^2}(\tilde{G}_{r+1}F_n-\tilde{F}_rG_{n+1}), \tag{4.46}$$

which proves (4.43). Similarly, differentiating (4.39) with respect to t_r, using (4.30), (4.32), (4.38)–(4.40), and (4.16), proves (4.44). (4.45) finally follows from $(G_{n+1}^2-F_nH_n)_{t_r}=0$ (cf. (4.20)), (4.43), and (4.44). □

The remaining items (vi)–(ix) of Lemma 3.1 in the present time-dependent setting then read

Lemma 4.3. *Assume (4.14)–(4.20), $P=(z,y(P))\in\mathcal{K}_n\backslash\{\infty\}$, and let $(z,x,x_0,t_r,t_{0,r})\in\mathbb{C}\times\mathbb{R}^4$. Then*

$$(i).\ \ \psi_1(P,x,x_0,t_r,t_{0,r})\psi_1(P^*,x,x_0,t_r,t_{0,r})=\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}. \tag{4.47}$$

$$(ii).\ \ \psi_2(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r})=\frac{H_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}. \tag{4.48}$$

$$(iii).\ \ \psi_1(P,x,x_0,t_r,t_{0,r}) = \left[\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}\right]^{\frac{1}{2}} \times$$

$$\times \exp\left\{\int_{x_0}^{x} dx' \frac{q(x',t_r)}{F_n(z,x',t_r)} + \int_{t_{0,r}}^{t_r} ds \frac{\tilde{F}_r(z,x_0,s)}{F_n(z,x_0,s)}\right\}. \tag{4.49}$$

$$(iv).\ \ \psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) + \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r})$$

$$= \frac{2G_{n+1}(z,x,t_r)}{F_n(z,x_0,t_{0,r})}. \tag{4.50}$$

Proof. (4.47) follows from (4.17), (4.28), (4.39), and (4.43). (4.48) is clear from (4.29), (4.38), and (4.47). (4.49) follows from (4.17), (4.28), (4.39), (4.40), and (4.43) by splitting $\phi(P) = \frac{1}{2}[\phi(P) + \phi(P^*)] + \frac{1}{2}[\phi(P) - \phi(P^*)]$ in (4.28). Finally, (4.50) is clear from (4.29), (4.39), and (4.47). ☐

The dynamics of the zeros $\mu_j(x,t_r)$ and $\nu_j(x,t_r)$ of $F_n(z,x,t_r)$ and $H_n(z,x,t_r)$, in analogy to Lemma 3.2, is then described in

Lemma 4.4. *Assume (4.14)–(4.22) and let $(x,t_r) \in \mathbb{R}^2$. Then*

$$(i).\quad \mu_{j,x}(x,t_r) = \frac{-2iy(\hat{\mu}_j(x,t_r))}{\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j(x,t_r) - \mu_k(x,t_r))}, \quad 1 \leq j \leq n, \tag{4.51}$$

$$\mu_{j,t_r}(x,t_r) = \frac{2\tilde{F}_r(\mu_j(x,t_r),x,t_r)y(\hat{\mu}_j(x,t_r))}{q(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j(x,t_r) - \mu_k(x,t_r))}, \quad 1 \leq j \leq n. \tag{4.52}$$

$$(ii).\quad \nu_{j,x}(x,t_r) = \frac{-2iy(\hat{\nu}_j(x,t_r))}{\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j(x,t_r) - \nu_k(x,t_r))}, \quad 1 \leq j \leq n, \tag{4.53}$$

$$\nu_{j,t_r}(x,t_r) = \frac{-2\tilde{H}_r(\nu_j(x,t_r),x,t_r))y(\hat{\nu}_j(x,t_r))}{p(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j(x,t_r) - \nu_k(x,t_r))}, \quad 1 \leq j \leq n. \tag{4.54}$$

Proof. (4.51) and (4.53) are proved as in Lemma 3.2 and follow from (4.21), (4.22), (4.17), (4.24) and (4.25). Similarly, (4.52) and (4.54) follow from (4.21), (4.24), (4.43) and (4.22), (4.25), (4.45), respectively. ☐

The initial condition

$$(p(x,t_{0,r}),q(x,t_{0,r})) = (p^{(0)}(x),q^{(0)}(x)), \quad x \in \mathbb{R} \tag{4.55}$$

in (4.3) is taken care of by

$$\hat{\mu}_j(x,t_{0,r}) = \hat{\mu}_j^{(0)}(x), \quad \hat{\nu}_j(x,t_{0,r}) = \hat{\nu}_j^{(0)}(x), \quad 1 \leq j \leq n, \quad x \in \mathbb{R} \tag{4.56}$$

(cf. (4.11), (4.13) and (4.21), (4.22)).

The trace relations in Lemma 3.3 extend in a one-to-one manner to the present time-dependent setting by substituting

$$(p(x),q(x)) \to (p(x,t_r),q(x,t_r)), \tag{4.57}$$

$$(\mu_j(x), \nu_j(x)) \to (\mu_j(x,t_r), \nu_j(x,t_r)), \quad 1 \le j \le n,$$

keeping $\{c_\ell\}_{1\le\ell\le n}$ as in (3.31) since $\mathcal{K}_n$ is t_r-dependent.

It remains to provide the explicit theta function representation of Ψ, ϕ, p, and q. We rely on the notation established in Section 3 and Appendix A in the following, assuming $\mathcal{K}_n$ to be nonsingular as in (3.32). As in Section 3 we assume $n \in \mathbb{N}$ for the remainder of this argument.

In addition to (3.33)–(3.41) we need to introduce the Abelian differentials of the second kind $\omega^{(2)}_{\infty_\pm,r}$ (cf. (A.34), (A.35)) defined by

$$\omega^{(2)}_{\infty_\pm,r} \underset{\zeta\to 0}{=} [\zeta^{-2-r} + O(1)]\, d\zeta \text{ near } \infty_\pm, \ \ r \in \mathbb{N}_0, \tag{4.58}$$

$$\int_{a_j} \omega^{(2)}_{\infty_\pm,r} = 0, \quad 1 \le j \le n, \tag{4.59}$$

$$\underline{\tilde U}^{(2)}_r = (\tilde U^{(2)}_{r,1}, \dots, \tilde U^{(2)}_{r,n}), \ \ \tilde U^{(2)}_{r,j} = \frac{1}{2\pi i}\int_{b_j} \tilde\Omega^{(2)}_r, \ \ \tilde\Omega^{(2)}_r = \pm\sum_{q=0}^{r} \tilde c_{r-q}(\omega^{(2)}_{\infty_+,q} - \omega^{(2)}_{\infty_-,q}), \tag{4.60}$$

$$\int_{P_0}^{P} \tilde\Omega^{(2)}_r \underset{\zeta\to 0}{=} \mp\left[\sum_{q=0}^{r} \tilde c_{r-q}\zeta^{-1-r} + \tilde e_r + O(\zeta)\right], \ \ P = (\zeta^{-1}, y(P)) \text{ near } \infty_\pm, \tag{4.61}$$

with $\{\tilde c_\ell\}_{1\le\ell\le r}$, $\tilde c_0 = 1$ the integration constants in $\tilde F_r$. Moreover, writing

$$\omega_j = \left(\sum_{m=0}^{\infty} d_{j,m}(\infty_\pm)\zeta^m\right) d\zeta = \pm\left(\sum_{m=0}^{\infty} d_{j,m}(\infty_+)\zeta^m\right) d\zeta \text{ near } \infty_\pm, \tag{4.62}$$

relation (A.35) yields

$$\tilde U^{(2)}_{r,j} = 2\sum_{q=0}^{r} \tilde c_{r-q} d_{j,q}(\infty_+), \quad 1 \le j \le n. \tag{4.63}$$

Before we can prove the main result of this section we need the following auxiliary result which is of independent interest due to its implications for the Green's matrix of the differential expression D.

Lemma 4.5. *Let $P \in \mathcal{K}_n\backslash\{\infty_+, \infty_-\}$ and $(x,t_r) \in \mathbb{R}^2$. Then, for $P = (\zeta^{-1}, y(P))$ near $\infty_\pm$, one obtains the asymptotic expansions*

$$\frac{F_n(\zeta^{-1}, x, t_r)}{y(P)} \underset{\zeta\to 0}{=} \mp\zeta\sum_{k=0}^{\infty} \hat f_k(x,t_r)\zeta^k, \quad \hat f_0(x,t_r) = -iq(x,t_r), \tag{4.64}$$

$$\frac{H_n(\zeta^{-1}, x, t_r)}{y(P)} \underset{\zeta\to 0}{=} \mp\zeta\sum_{k=0}^{\infty} \hat h_k(x,t_r)\zeta^k, \quad \hat h_0(x,t_r) = ip(x,t_r), \tag{4.65}$$

where $\hat{f}_k(x,t_r)$ and $\hat{h}_k(x,t_r)$ denote the homogeneous coefficients $f_k(x,t_r)$ and $h_k(x,t_r)$ in (4.11) and (4.13) (i.e., the ones satisfying (2.2) with all integration constants $c_\ell = 0$, $\ell \in \mathbb{N}$). Explicitly, $\hat{f}_k(x,t_r)$ and $\hat{h}_k(x,t_r)$ can be computed from the recursion relations,

$$\begin{aligned}
&\hat{f}_0 = -iq, \quad \hat{f}_1 = \frac{1}{2}q_x, \\
&\hat{f}_k = \sum_{\ell=0}^{k-2}\bigg(-\frac{i}{4q}\hat{f}_\ell\hat{f}_{k-2-\ell,xx} + \frac{iq_x}{4q^2}\hat{f}_\ell\hat{f}_{k-2-\ell,x} + \frac{i}{8q}\hat{f}_{\ell,x}\hat{f}_{k-2-\ell,x} \\
&\quad +\frac{ip}{2}\hat{f}_\ell\hat{f}_{k-2-\ell}\bigg) - \frac{q_x}{2q^2}\sum_{\ell=0}^{k-1}\hat{f}_\ell\hat{f}_{k-1-\ell} - \frac{i}{2q}\sum_{\ell=1}^{k-1}\hat{f}_\ell\hat{f}_{k-\ell}, \quad k \geq 2,
\end{aligned} \tag{4.66}$$

and

$$\begin{aligned}
&\hat{h}_0 = ip, \quad \hat{h}_1 = \frac{1}{2}p_x, \\
&\hat{h}_k = \sum_{\ell=0}^{k-2}\bigg(\frac{i}{4p}\hat{h}_\ell\hat{h}_{k-2-\ell,xx} - \frac{ip_x}{4p^2}\hat{h}_\ell\hat{h}_{k-2-\ell,x} - \frac{i}{8p}\hat{h}_{\ell,x}\hat{h}_{k-2-\ell,x} \\
&\quad -\frac{iq}{2}\hat{h}_\ell\hat{h}_{k-2-\ell}\bigg) - \frac{p_x}{2p^2}\sum_{\ell=0}^{k-1}\hat{h}_\ell\hat{h}_{k-1-\ell} + \frac{i}{2p}\sum_{\ell=1}^{k-1}\hat{h}_\ell\hat{h}_{k-\ell}, \quad k \geq 2.
\end{aligned} \tag{4.67}$$

Proof. Define

$$\hat{F}(P,x,t_r) = \frac{F_n(\zeta^{-1},x,t_r)}{y(P)} \tag{4.68}$$

then

$$\hat{F}\hat{F}_{xx} - \frac{q_x}{q}\hat{F}\hat{F}_x - \frac{1}{2}\hat{F}_{xx}^2 + 2(\zeta^{-2} - i\zeta^{-1}\frac{q_x}{q} - pq)\hat{F}^2 = -2q^2 \tag{4.69}$$

by (2.17). Moreover,

$$\hat{F}(P,x,t_r) = \mp\zeta\frac{\sum_{\ell=0}^{n} f_\ell(x,t_r)\zeta^\ell}{[\prod_{m=0}^{2n+1}(1-E_m)\zeta]^{\frac{1}{2}}}, \quad P \in \Pi_\pm \tag{4.70}$$

$$\underset{\zeta\to 0}{=} \mp\zeta\sum_{k=0}^{\infty}\hat{f}_k(x,t_r)\zeta^k \quad \text{near} \quad \infty_\pm, \quad f_0(x,t_r) = \hat{f}_0(x,t_r) = -iq(x,t_r), \tag{4.71}$$

where we chose the branch

$$\left[\prod_{m=0}^{2n+1}(1-E_m\zeta)\right]^{\frac{1}{2}} \underset{\zeta\to 0}{=} 1 - \frac{1}{2}\left(\sum_{m=0}^{2n+1}E_m\right)\zeta + O(\zeta^2). \tag{4.72}$$

Insertion of (4.71) into (4.69) yields the recursion relation (4.66) which represents the homogeneous solutions for the f_ℓ due to the lack of any possible integration constants in (4.66). (4.65) and (4.67) are proved in the same manner using (2.19). □

Lemma 4.5 has interesting consequences for the asymptotic high-energy expansion of the Green's matrix $G(z,x,x')$ of D (i.e., the integral kernel of the resolvent $(D-z)^{-1}$) as described in the following remark.

Remark 4.6. Let D given by (2.1) (p, q not necessarily algebro-geometric coefficients) and assume that for some $z \in \mathbb{C}$, and all $x_0 \in \mathbb{R}$, $\psi_{1,\pm}(z,.)$, $\psi_{2,\pm}(z,.) \in L^2((x_0,\pm\infty))$ satisfy (2.22), that is,

$$\psi_{1,\pm,x} = -iz\psi_{1,\pm} + q\psi_{2,\pm},\ \psi_{2,\pm,x} = iz\psi_{2,\pm} + p\psi_{1,\pm}. \tag{4.73}$$

Then the Green's matrix $G(z,x,x')$, $x \neq x'$ of D is given by

$$G(z,x,x') = \frac{i}{W(z)} \begin{cases} \begin{pmatrix} \psi_{1,+}(z,x)\psi_{2,-}(z,x') & \psi_{1,+}(z,x)\psi_{1,-}(z,x') \\ \psi_{2,+}(z,x)\psi_{2,-}(z,x') & \psi_{2,+}(z,x)\psi_{1,-}(z,x') \end{pmatrix}, & x > x' \\ \\ \begin{pmatrix} \psi_{1,-}(z,x)\psi_{2,+}(z,x') & \psi_{1,-}(z,x)\psi_{1,+}(z,x') \\ \psi_{2,-}(z,x)\psi_{2,+}(z,x') & \psi_{2,-}(z,x)\psi_{1,+}(z,x') \end{pmatrix}, & x < x' \end{cases}, \tag{4.74}$$

where the Wronskian

$$W(z) := \begin{vmatrix} \psi_{1,-}(z,x) & \psi_{1,+}(z,x) \\ \psi_{2,-}(z,x) & \psi_{2,+}(z,x) \end{vmatrix} = \psi_{1,-}(z,x)\psi_{2,+}(z,x) - \psi_{2,-}(z,x)\psi_{1,+}(z,x) \tag{4.75}$$

is x-independent. Note that $G(z,x,x')$ is continuous at $x = x'$ in its off-diagonal elements but discontinuous on the diagonal.
In the special algebro-geometric context we may identify

$$\psi_{j,+}(z,x) = \psi_j(P,x,x_0),\ \psi_{j,-}(z,x) = \psi_j(P^*,x,x_0),\ j = 1,2,\ P = (z,y(P)) \tag{4.76}$$

and (cf. (3.7), (3.8), and (3.20))

$$W(z) := \begin{vmatrix} 1 & 1 \\ \phi(P^*,x_0) & \phi(P,x_0) \end{vmatrix} = \phi(P,x_0) - \phi(P^*,x_0) = \frac{2y(P)}{F_n(z,x_0)} \tag{4.77}$$

since W is x-independent. Substituting (4.76) and (4.77) into (4.74), denoting the result by $G(P,x,x')$, then yields

$$\begin{aligned} \frac{1}{2}[G(P,x,x+0) + G(P,x,x-0) &= \frac{1}{2}[G(P,x-0,x) + G(P,x+0,x) \\ &= \frac{i}{2y(P)} \begin{pmatrix} G_{n+1}(z,x) & F_n(z,x) \\ H_n(z,x) & G_{n+1}(z,x) \end{pmatrix} = \frac{i}{2} \begin{pmatrix} \hat{G}(P,x) & \hat{F}(P,x) \\ \hat{H}(P,x) & \hat{G}(P,x) \end{pmatrix},\ P = (z,y(P)), \end{aligned} \tag{4.78}$$

where (cf. (4.68))

$$\hat{F}(P,x) = \frac{F_n(z,x)}{y(P)},\ \hat{G}(P,x) = \frac{G_{n+1}(z,x)}{y(P)},\ \hat{H}(P,x) = \frac{H_n(z,x)}{y(P)} \tag{4.79}$$

denote homogeneous quantities encountered in Lemma 4.5. Since $G(P,x,x')$ is discontinuous at $x = x'$, we introduced the arithmetic mean of the corresponding one-sided limits following the usual treatment of first-order systems (see, e.g., [4], Sect. 9.4). In fact, the arithmetic mean in (4.78) leads to the charecteristic function of D (in the terminology of [4], Sect. 9.5), the fundamental object for studying spectral properties of D. The asymptotic expansions (4.64) and (4.65) for $\hat{F}(P,x)$ and $\hat{H}(P,x)$ as $P \to \infty_\pm$ then determine the off-diagonal asymptotic high-energy expansions of of the arithmetic mean of the diagonal Green's matrix in (4.78). Similarly, using (2.10) or (2.12),

$$\hat{G}(P,x) = \frac{1}{2q(x)}[\hat{F}_x(P,x) + 2iz\hat{F}_x(P,x)] \tag{4.80}$$

$$= \frac{1}{2p(x)}[\hat{H}_x(P,x) - 2iz\hat{H}_x(P,x)],$$

one obtains the asymptotic high-energy expansion of (4.78) as $P \to \infty_\pm$ for its diagonal elements. Even though (4.64), (4.65), (4.78), and (4.80) were derived in the special algebro-geometric context, we emphasize, however, that the asymptotic expansion of (4.78) as $P \to \infty_\pm$ only involves the homogeneous coefficients $\hat{f}_k(x)$, $\hat{h}_k(x)$ which are universal differential polynomials in $(p(x), q(x))$. Thus, identifying $\Psi_\pm(z,x)$ and $\Psi(P,x,x_0)$, $\Psi(P^*,x,x_0)$ as in (4.76), yields the universal high-energy expansion of the arithmetic mean of the diagonal Greens matrix $G(z,x,x')$ of D as $z \to \infty$ in the general (not necessarily algebro-geometric) case. The recursive and hence systematic approach to this high-energy expansion, based on (4.64)–(4.67) and (4.80), appears to be new.

The theta function representations for ϕ, Ψ, and (p,q) then finally read as follows.

Theorem 4.7. *Let $P \in \mathcal{K}_n \backslash \{\infty_+, \infty_-\}$, $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$, and assume $\mathcal{K}_n$ to be nonsingular, that is, $E_m \neq E_{m'}$ for $0 \le m, m' \le 2\,n+1$. Moreover, suppose $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) = i(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}) = 0$. Then*

$$\phi(P,x,t_r) = \frac{2i}{q(x_0,t_{0,r})\omega_0} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x,t_r)))} \frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))} \times$$
$$\times \exp\left[\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-} - 2i(x-x_0)e_0 - 2i(t_r - t_{0,r})\tilde{e}_r\right], \tag{4.81}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0,,t_{0,r})))} \frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x,t_r)))} \times$$
$$\times \exp\left[i(x-x_0)\left(e_0 + \int_{P_0}^{P} \Omega_0^{(2)}\right) + i(t_r - t_{0,r})\left(\tilde{e}_r + \int_{P_0}^{P} \tilde{\Omega}_r^{(2)}\right)\right], \tag{4.82}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r}) = \frac{2i}{q(x_0,t_{0,r})\omega_0} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))} \frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x,t_r)))} \times$$
$$\times \exp\left[\int_{P_0}^{P} \omega^{(3)}_{\infty_+,\infty_-} + i(x-x_0)\left(-e_0 + \int_{P_0}^{P} \Omega_0^{(2)}\right) + i(t_r - t_{0,r})\left(-\tilde{e}_r + \int_{P_0}^{P} \tilde{\Omega}_r^{(2)}\right)\right]. \tag{4.83}$$

Moreover, one derives

$$p(x,t_r) = p(x_0,t_{0,r}) \frac{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x,t_r)))} \times$$
$$\times \exp[-2i(x-x_0)e_0 - 2i(t_r - t_{0,r})\tilde{e}_r], \tag{4.84}$$

$$q(x,t_r) = q(x_0,t_{0,r}) \frac{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0,t_{0,r})))} \frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x,t_r)))} \times$$
$$\times \exp[2i(x-x_0)e_0 + 2i(t_r - t_{0,r})\tilde{e}_r], \tag{4.85}$$

$$p(x_0,t_{0,r})q(x_0,t_{0,r}) = \frac{4}{\omega_0^2}\frac{\theta(\underline{z}_+(\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}_-(\underline{\hat{\nu}}(x_0,t_{0,r})))}\frac{\theta(\underline{z}_-(\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}_+(\underline{\hat{\mu}}(x_0,t_{0,r})))}, \tag{4.86}$$

and

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) - i(x-x_0)\underline{U}_0^{(2)} - i(t_r-t_{0,r})\underline{\tilde{U}}_r^{(2)}, \tag{4.87}$$

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}) - i(x-x_0)\underline{U}_0^{(2)} - i(t_r-t_{0,r})\underline{\tilde{U}}_r^{(2)}. \tag{4.88}$$

Proof. We first prove the θ-function representation (4.82) for ψ_1. Without loss of generality it suffices to treat the homogeneous case $\hat{c}_0=1,\ \ \hat{c}_q=0,\ \ 1\le q\le r$. Define the left-hand side of (4.82) to be $\tilde{\psi}_1(P,x,x_0,t_r,t_{0,r})$; we need to prove $\psi_1=\tilde{\psi}_1$ with ψ_1 given by (4.28). For that purpose we first investigate the local zeros and poles of ψ_1. Since they can only come from zeros of $F_n(z,x_0,s),\ F_n(z,x',t_r)$ in (4.28), we note that

$$\begin{aligned}q(x',t_r)\phi(P,x',t_r) \underset{P\to\hat{\mu}_j(x',t_r)}{=} q(x',t_r)\frac{2y(\hat{\mu}_j(x',t_r))}{-iq(x',t_r)\prod_{\substack{k=1\\k\neq j}}^n(\mu_j(x',t_r)-\mu_k(x',t_r))}\times\\ \times\frac{1}{z-\mu_j(x',t_r)}+O(1) \underset{P\to\hat{\mu}_j(x',t_r)}{=} \frac{-\mu_{j,x'}(x',t_r)}{z-\mu_j(x',t_r)}+O(1),\end{aligned} \tag{4.89}$$

$$\begin{aligned}i\hat{F}_r(z,x_0,s)\phi(P,x_0,s) \underset{P\to\hat{\mu}_j(x_0,s)}{=} \frac{2i\hat{F}_r(z,x_0,s)y(\hat{\mu}_j(x_0,s))}{-iq(x_0,s)\prod_{\substack{k=1\\k\neq j}}^n(\mu_j(x_0,s)-\mu_k(x_0,s))}\times\\ \times\frac{1}{z-\mu_j(x_0,s)}+O(1) \underset{P\to\hat{\mu}_j(x_0,s)}{=} \frac{-\mu_{j,s}(x_0,s)}{z-\mu_j(x_0,s)}+O(1)\end{aligned} \tag{4.90}$$

using (4.23), (4.24), (4.51), and (4.52). Thus

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \begin{cases}(z-\mu_j(x,t_r))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r)\neq\hat{\mu}_j(x_0,t_{0,r})\\[2ex] O(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r)=\hat{\mu}_j(x_0,t_{0,r})\\[2ex] (z-\mu_j(x_0,t_r))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_{0,r})\neq\hat{\mu}_j(x,t_r)\end{cases}, \tag{4.91}$$

with $O(1)\neq 0$ and hence ψ_1 and $\tilde{\psi}_1$ have identical zeros and poles on $\mathcal{K}_n\backslash\{\infty_+,\infty_-\}$ which are all simple. It remains to study the behavior of ψ_1 near $\infty_\pm$. One infers from (4.21)–(4.23) that

$$\phi(P,x,t_r) \underset{\zeta\to 0}{=} \begin{cases}\frac{i}{2}p(x,t_r)\zeta+O(\zeta^2), & P \text{ near } \infty_+\\[2ex] \frac{2i}{q(x,t_r)}\zeta^{-1}+O(1), & P \text{ near } \infty_-\end{cases}. \tag{4.92}$$

Thus (4.23), (4.43), (4.92), and Lemma 4.5 yield

$$\begin{aligned}&\int_{x_0}^{x} dx'[-i\zeta^{-1}+q(x,t_r)\phi(P,x,t_r)]\\ &+\int_{t_{0,r}}^{t_r} ds[\tilde{F}_r(\zeta^{-1},x_0,s)\phi(P,x_0,s)-\tilde{G}_{r+1}(\zeta^{-1},x_0,s)]\end{aligned}$$

$$
\begin{aligned}
&\underset{\zeta\to 0}{=} \left[\mp i(x-x_0)\zeta^{-1} + \left\{\begin{array}{ll} O(\zeta), & P\to\infty_+ \\ O(1), & P\to\infty_- \end{array}\right\}\right] \\
&+ \int_{t_{0,r}}^{t_r} ds \left[\frac{i\tilde F_r(\zeta^{-1},x_0,s)y(P)}{F_n(\zeta^{-1},x_0,s)} + \frac{1}{2}\frac{F_{n,t_r}(\zeta^{-1},x_0,s)}{F_n(\zeta^{-1},x_0,s)}\right] \\
&\underset{\zeta\to 0}{=} \left[\mp i(x-x_0)\zeta^{-1} + \left\{\begin{array}{ll} O(\zeta), & P\to\infty_+ \\ O(1), & P\to\infty_- \end{array}\right\}\right] \\
&+ \int_{t_{0,r}}^{t_r} ds \left[\mp i\zeta^{-r-1}\frac{\sum_{\ell=0}^{r}\tilde f_\ell(x_0,s)\zeta^\ell}{\sum_{\ell=0}^{\infty}\tilde f_\ell(x_0,s)\zeta^\ell} + \frac{1}{2}\frac{q_{t_r}(x_0,s)}{q(x_0,s)} + O(\zeta)\right] \\
&\underset{\zeta\to 0}{=} \left[\mp i(x-x_0)\zeta^{-1} + \left\{\begin{array}{ll} O(\zeta), & P\to\infty_+ \\ O(1), & P\to\infty_- \end{array}\right\}\right] \\
&+ \int_{t_{0,r}}^{t_r} ds \left[\mp i\zeta^{-r-1} \pm \frac{i\tilde f_{r+1}(x_0,s)}{\tilde f_0(x_0,s)} + \frac{1}{2}\frac{q_{t_r}(x_0,s)}{q(x_0,s)} + O(\zeta)\right] \\
&\underset{\zeta\to 0}{=} \mp i(x-x_0)\zeta^{-1} \mp i(t_r-t_{0,r})\zeta^{-r-1} + \left\{\begin{array}{ll} O(\zeta), & P\to\infty_+ \\ & \\ O(1), & P\to\infty_- \end{array}\right. , \qquad (4.93)
\end{aligned}
$$

where we used $\tilde f_0 = -iq$ and

$$
q_{t_r} = 2\tilde f_{r+1} \tag{4.94}
$$

(cf. (4.3)) in the homogeneous case $\tilde c_0 = 1,\quad \tilde c_q = 0,\quad 1 \le q \le r$. (4.93) yields the correct essential singularity structure of $\tilde\psi_1$ near $\infty_\pm$. Moreover, (3.41), (4.61), and the $O(\zeta)$-term in (4.93) as $P\to\infty_+$ also prove that ψ_1 and $\tilde\psi_1$ are identically normalized (near ∞_+) and hence coincide by the t-dependent analog of Lemma 3.4 (replacing $-i(x-x_0)\int_{P_0}^P \Omega_0^{(2)}$ by $-i(x-x_0)\int_{P_0}^P \Omega_0^{(2)} - i(t_r-t_{0,r})\int_{P_0}^P \tilde\Omega_r^{(2)}$). This proves (4.82). The expression (4.26) for the divisor of ϕ then yields

$$
\phi(P,x,t_r) = C(x,t_r)\frac{\theta(\underline{z}(P,\underline{\hat\nu}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat\mu}(x,t_r)))}e^{\int_{P_0}^P \omega^{(3)}_{\infty_+,\infty_-}}, \tag{4.95}
$$

where $C(x,t_r)$ is independent of $P\in\mathcal{K}_n$. Thus (4.92) implies

$$
p(x,t_r) = \frac{2C(x,t_r)}{i\omega_0}\frac{\theta(\underline{z}_+(\underline{\hat\nu}(x,t_r)))}{\theta(\underline{z}_+(\underline{\hat\mu}(x,t_r)))}, \tag{4.96}
$$

$$
q(x,t_r) = \frac{2i}{C(x,t_r)\omega_0}\frac{\theta(\underline{z}_-(\underline{\hat\mu}(x,t_r)))}{\theta(\underline{z}_-(\underline{\hat\nu}(x,t_r)))}. \tag{4.97}
$$

Re-examining the asymptotic behavior (4.93) of ψ_1 near ∞_-, taking into account (3.61), yields

$$
\begin{aligned}
\psi_1(P,x,x_0,t_r,t_{0,r}) &\underset{\zeta\to 0}{=} \frac{q(x,t_r)}{q(x_0,t_r)}\exp[i(x-x_0)\zeta^{-1} + O(\zeta)]\times \\
&\times\frac{q(x_0,t_r)}{q(x_0,t_{0,r})}\exp[i(t_r-t_{0,r})\zeta^{-1-r} + O(\zeta)] \qquad (4.98)
\end{aligned}
$$

$$\underset{\zeta\to 0}{=} \frac{q(x,t_r)}{q(x_0,t_{0,r})} \exp[i(x-x_0)\zeta^{-1} + i(t_r - t_{0,r})\zeta^{-1-r} + O(\zeta)] \quad \text{for } P \text{ near } \infty_-.$$

A comparison of (4.82), (4.97), and (4.98) then proves (4.85). A further comparison of (4.85) and (4.97) then determines $C(x,t_r)$ and hence yields (4.84) and (4.86). Given $C(x,t_r)$ one determines ϕ in (4.81) from (4.95) and hence ψ_2 in (4.83) from $\psi_2 = \phi\psi_1$. Finally, the linearization property of the Abel map in (4.87) and (4.88) can be proved directly using Lagrange interpolation as in the proof of Theorem 3.5. However, for increasing values of r this method becomes exceedingly cumbersome and it is simpler to resort to a standard investigation (cf., e.g., [42], p. 141–144) of the differential $\Omega_1(x,x_0,t_r,t_{0,r}) = d\ln\psi_1(.,x,x_0,t_r,t_{0,r})$ (respectively, $\Omega_2(x,x_0,t_r,t_{0,r}) = d\ln\psi_2(.,x,x_0,t_r,t_{0,r})$ in order to prove (4.87) (respectively, (4.88)). $\square$

The open constant $q(x_0,t_{0,r})$ in (4.84)–(4.86) is inherent to the AKNS formalism as discussed in Lemma 3.6.

In analogy to Example 3.7, the special case $n = 0$ (excluded in Theorem 4.7) yields solutions $(p(x,t_r), q(x,t_r))$ as in (4.84), (4.85) replacing the theta quotients by 1.

We note again that the results for Ψ and (p,q) in Theorem 4.6 are known and can be found, for instance, in [5], Ch.4, [14], [15], [29], [30], and [43]. Our main new contribution to this circle of ideas is the elementary alternative derivation of Theorem 4.6 based on the fundamental meromorphic object ϕ and its connection with the polynomial recursion formalism of Section 2.

Appendix A. Hyperelliptic Curves and Theta Functions

We briefly summarize our notation and some of the basic facts on hyperelliptic curves and their theta functions as employed in Sections 3 and 4. For background information on this standard material we refer, for instance, to [17], Chs. I–III, IV, [18], [28], Ch. 2, [33], Ch. X, [39], Ch. 2.

Consider the points

$$\{E_m\}_{0\le m\le 2n+1} \subset \mathbb{C}, \quad n \in \mathbb{N}_0 \tag{A.1}$$

and introduce an appropriate set of $n+1$ (nonintersecting) cuts $\mathcal{C}_j$ joining E_j and $E_{k(j)}$, where $E_{k(j)} = E_j$ for some j is permitted in order to include singular curves. Denote

$$\mathcal{C} = \bigcup_{j\in J} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset \text{ for } j \ne k, \tag{A.2}$$

where the finite index set $J \subset \{0,1,\dots,2n+1\}$ has cardinality $n+1$ and define the cut plane Π,

$$\Pi := \mathbb{C} \setminus \mathcal{C}. \tag{A.3}$$

Next, introduce the holomorphic function

$$R_{2n+2}(.)^{1/2} : \begin{cases} \Pi \to \mathbb{C} \\ z \mapsto [\prod_{m=0}^{2n+1}(z-E_m)]^{1/2} \end{cases} \tag{A.4}$$

on Π with a definite choice of the square root branch (A.4). Given the holomorphic function (A.4) one defines the set

$$M_n = \{(z, \sigma R_{2n+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{+,-\}\} \cup \{\infty_+, \infty_-\} \tag{A.5}$$

and

$$B_s = \{(E_m, 0)\}_{0\leq m\leq 2n+1}, \tag{A.6}$$

the set of branch and/or singular points. M_n becomes a Riemann surface upon introducing appropriate charts (U_{P_0}, ζ_{P_0}) defined in a standard manner. Let

$$\begin{gathered} P_0 = (z_0, \sigma_0 R_{2n+2}(z_0)^{1/2}) \text{ or } P_0 = \infty_\pm, \\ P = (z, \sigma R_{2n+2}(z)^{1/2}) \in U_{P_0} \subset M_n,\ V_{P_0} = \zeta_{P_0}(U_{P_0}) \subset \mathbb{C}. \end{gathered} \tag{A.7}$$

$\underline{P_0 \notin \{B_s \cup \{\infty_+, \infty_-\}\}}$:

$$\begin{aligned} U_{P_0} =& \{P \in M_n \mid |z - z_0| < C_{z_0},\ \sigma R_{2n+2}(z)^{1/2} \text{ the branch obtained by straight line} \\ & \quad \text{analytic continuation starting from } z_0\}, \quad C_{z_0} = \min_m |z_0 - E_m|, \\ V_{P_0} =& \{\zeta \in \mathbb{C} \mid |\zeta| < C_{z_0}\}, \end{aligned}$$

$$\zeta_{P_0} : \begin{cases} U_{P_0} \to V_{P_0} \\ P \mapsto (z - z_0) \end{cases}, \quad \zeta_{P_0}^{-1} : \begin{cases} V_{P_0} \to U_{P_0} \\ \zeta \mapsto (z_0 + \zeta,\ \sigma R_{2n+2}(z_0 + \zeta)^{1/2}) \end{cases}. \tag{A.8}$$

$\underline{P_0 = \infty_\pm}$:

$$U_{P_0} = \{P \in M_n \mid |z| > C_\infty\},\ C_\infty = \max_m |E_m|,\ V_{P_0} = \{\zeta \in \mathbb{C} \mid |\zeta| < C_\infty^{-1}\},$$

$$\zeta_{P_0} : \begin{cases} U_{P_0} \to V_{P_0} \\ P \mapsto z^{-1} \\ \infty_\pm \mapsto 0 \end{cases}, \quad \zeta_{P_0}^{-1} : \begin{cases} V_{P_0} \to U_{P_0} \\ \zeta \mapsto (\zeta^{-1}, \mp[\Pi_m(1 - E_m\zeta)]^{1/2}\zeta^{-n-1}) \\ 0 \mapsto \infty_\pm \end{cases}, \tag{A.9}$$

$[\Pi_m(1-E_m\zeta)]^{1/2} = 1 - \frac{1}{2}(\sum_m E_m)\zeta + O(\zeta^2)$.

Similarly, local coordinates for branch and/or singular points $P_0 \in B_s$ are defined as $\zeta_{P_0}(P) = (z - z_0)^{r/2}$ for appropriate $r \in \mathbb{N}$. For the reader's convenience we provide a detailed treatment of branch points in the nonsingular case (where $E_m \neq E_{m'}$ for $m \neq m'$) for the two most frequently occurring situations, the self-adjoint case where $\{E_m\}_{0\leq m\leq 2n+1} \subset \mathbb{R}$ and the case where $\{E_m\}_{0\leq m\leq 2n+1} = \{\epsilon_\ell, \overline{\epsilon_\ell}\}_{0\leq \ell\leq n}$ consists of complex conjugate pairs at the end of this appendix.

In addition, it is useful to consider the subsets $\Pi_\pm \subset M_n$ (i.e., upper and lower sheets)

$$\Pi_\pm = \{(z, \pm R_{2n+2}(z)^{1/2}) \in M_n \mid z \in \Pi\} \tag{A.10}$$

and the associated charts

$$\zeta_\pm : \begin{cases} \Pi_\pm \to \Pi \\ P \mapsto z \end{cases}. \tag{A.11}$$

(A.8), (A.9), and the corresponding charts for $P_0 \in B_s$ define a complex structure on M_n. We shall denote the resulting Riemann surface by $\mathcal{K}_n$. In general, $\mathcal{K}_n$ is a (possibly singular) curve of (arithmetic) genus n.

Next, consider the holomorphic sheet exchange map (involution)

$$* : \begin{cases} \mathcal{K}_n \to \mathcal{K}_n \\ (z, \sigma R_{2n+2}(z)^{1/2}) \mapsto (z, \sigma R_{2n+2}(z)^{1/2})^* = (z, -\sigma R_{2n+2}(z)^{1/2}) \\ \infty_\pm \mapsto \infty_\pm^* = \infty_\mp \end{cases} \tag{A.12}$$

and the two meromorphic projection maps

$$\tilde{\pi} : \begin{cases} \mathcal{K}_n \to \mathbb{C} \cup \{\infty\} \\ (z, \sigma R_{2n+2}(z)^{1/2}) \mapsto z \\ \infty_\pm \mapsto \infty \end{cases}, \quad R_{2n+2}^{1/2} : \begin{cases} \mathcal{K}_n \to \mathbb{C} \cup \{\infty\} \\ (z, \sigma R_{2n+2}(z)^{1/2}) \mapsto \sigma R_{2n+2}(z)^{1/2}. \\ \infty_\pm \mapsto \infty \end{cases} \tag{A.13}$$

$\tilde{\pi}$ has poles of order 1 at $\infty_\pm$ and poles of order $n+1$ at $\infty_\pm$. Moreover,

$$\tilde{\pi}(P^*) = \tilde{\pi}(P), \ R_{2n+2}^{1/2}(P^*) = -R_{2n+2}^{1/2}(P), \quad P \in \mathcal{K}_n. \tag{A.14}$$

Thus $\mathcal{K}_n$ is a two-sheeted ramified covering of the Riemann sphere $\mathbb{C}_\infty (\cong \mathbb{C} \cup \{\infty\})$, $\mathcal{K}_n$ is compact (since $\tilde{\pi}$ is open and $\mathbb{C}_\infty$ is compact), and $\mathcal{K}_n$ is hyperelliptic (since it admits the meromorphic function $\tilde{\pi}$ of degree two).

In the following we abbreviate

$$P = (z, y(P)), \quad P \in \mathcal{K}_n \backslash \{\infty_+, \infty_-\}, \tag{A.15}$$

(i.e., we define $y(P) = R_{2n+2}^{1/2}(P)$, see (A.13)).

Next we turn to nonsingular curves $\mathcal{K}_n$ where

$$E_m \neq E_{m'}, \ \text{for } m \neq m', \ 0 \leq m, m' \leq 2n+1. \tag{A.16}$$

One infers that for $n \in \mathbb{N}$, $d\tilde{\pi}/y$ is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $n-1$ at $\infty_\pm$ and hence

$$\eta_j = \frac{\tilde{\pi}^{j-1} d\tilde{\pi}}{y}, \quad 1 \leq j \leq n \tag{A.17}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$.

Next we introduce a canonical homology basis $\{a_j, b_j\}_{1 \leq j \leq n}$ for $\mathcal{K}_n$ where the cycles are chosen such that their intersection matrix reads

$$a_j \circ b_k = \delta_{j,k}, \quad 1 \leq j, k \leq n. \tag{A.18}$$

Introducing the invertible matrix C in $\mathbb{C}^n$,

$$\begin{gathered} C = (C_{j,k})_{1 \leq j,k \leq n}, \ C_{j,k} = \int_{a_k} \eta_j, \\ \underline{c}(k) = (c_1(k), \dots, c_n(k)), \ c_j(k) = (C^{-1})_{j,k}, \end{gathered} \tag{A.19}$$

the normalized differentials ω_j, $1 \leq j \leq n$,

$$\omega_j = \sum_{\ell=1}^n c_j(\ell) \eta_\ell, \ \int_{a_k} \omega_j = \delta_{j,k}, \quad 1 \leq j, k \leq n \tag{A.20}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_n$. The matrix τ in $\mathbb{C}^n$ of b-periods,

$$\tau = (\tau_{j,k})_{1\leq j,k\leq n}, \quad \tau_{j,k} = \int_{b_k} \omega_j \tag{A.21}$$

satisfies

$$\tau_{j,k} = \tau_{k,j}, \quad 1 \leq j,k \leq n, \tag{A.22}$$

$$\mathrm{Im}(\tau) = \frac{1}{2i}(\tau - \tau^*) > 0. \tag{A.23}$$

In the charts $(U_{\infty_\pm}, \zeta_{\infty_\pm} \equiv \zeta)$ induced by $1/\tilde{\pi}$ near $\infty_\pm$ one infers

$$\begin{aligned} \underline{\omega} &= \pm \sum_{j=1}^{n} \underline{c}(j) \frac{\zeta^{n-j}\, d\zeta}{[\Pi_m(1-E_m\zeta)]^{1/2}} \\ &= \pm \Big\{ \underline{c}(n) + \Big[\frac{1}{2}\underline{c}(n)\sum_{m=0}^{2n+1} E_m + \underline{c}(n-1)\Big]\zeta + O(\zeta^2)\Big\}\, d\zeta. \end{aligned} \tag{A.24}$$

Associated with the homology basis $\{a_j, b_j\}_{1\leq j\leq n}$ we also recall the canonical dissection of $\mathcal{K}_n$ along its cycles yielding the simply connected interior $\hat{\mathcal{K}}_n$ of the fundamental polygon $\partial\hat{\mathcal{K}}_n$ given by

$$\partial\hat{\mathcal{K}}_n = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n^{-1} b_n^{-1}. \tag{A.25}$$

The Riemann theta function associated with $\mathcal{K}_n$ is defined by

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^n} \exp[2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \tau\underline{n})], \quad \underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n, \tag{A.26}$$

where $(\underline{u}, \underline{v}) = \sum_{j=1}^{n} \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^n$. It has the fundamental properties

$$\begin{aligned} &\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}), \\ &\theta(\underline{z} + \underline{m} + \tau\underline{n}) = \exp[-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \tau\underline{n})]\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^n. \end{aligned} \tag{A.27}$$

A divisor $\mathcal{D}$ on $\mathcal{K}_n$ is a map $\mathcal{D} : \mathcal{K}_n \to \mathbb{Z}$, where $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_n$. The set of all divisors on $\mathcal{K}_n$ will be denoted by $\mathrm{Div}(\mathcal{K}_n)$. With L_n we denote the period lattice

$$L_n := \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{m} + \tau\underline{n},\ \underline{m}, \underline{n} \in \mathbb{Z}^n\} \tag{A.28}$$

and the Jacobi variety $J(\mathcal{K}_n)$ is defined by

$$J(\mathcal{K}_n) = \mathbb{C}^n / L_n. \tag{A.29}$$

The Abel maps $\underline{A}_{P_0}(.)$ respectively $\underline{\alpha}_{P_0}(.)$ are defined by

$$\underline{A}_{P_0} : \begin{cases} \mathcal{K}_n \to J(\mathcal{K}_n) \\ P \mapsto \underline{A}_{P_0}(P) = \int_{P_0}^{P} \underline{\omega} \mod (L_n) \end{cases}, \tag{A.30}$$

$$\underline{\alpha}_{P_0} : \begin{cases} \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n) \\ \mathcal{D} \mapsto \underline{\alpha}_{P_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_n} \mathcal{D}(P)\underline{A}_{P_0}(P) \end{cases}, \tag{A.31}$$

with $P_0 \in \mathcal{K}_n$ a fixed base point. (In the main text we agree to fix $P_0 = (E_0, 0)$ for convenience.)

In connection with (A.25) we shall also need the maps (cf. (3.33))

$$\underline{\hat{A}}_{P_0} : \begin{cases} \hat{\mathcal{K}}_n \to \mathbb{C}^n \\ P \mapsto \int_{P_0}^P \underline{\omega} \end{cases}, \quad \underline{\hat{\alpha}}_{P,0} : \begin{cases} \operatorname{Div}(\mathcal{K}_n) \to \mathbb{C}^n \\ \mathcal{D} \mapsto \sum_{P\in\hat{\mathcal{K}}_n} \mathcal{D}(P)\underline{\hat{A}}_{P_0}(P) \end{cases}, \tag{A.32}$$

with path of integration lying in $\hat{\mathcal{K}}_n$.

Let $\mathcal{M}(\mathcal{K}_n)$ and $\mathcal{M}^1(\mathcal{K}_n)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_n$. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_n)$ at a point $Q_0 \in \mathcal{K}_n$ is defined by

$$\operatorname*{res}_{Q_0}(\nu) = \frac{1}{2\pi i}\int_{\gamma_{Q_0}} \nu, \tag{A.33}$$

where γ_{Q_0} is a counterclockwise oriented smooth simple closed contour encircling Q_0 but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind (dfk). Abelian differentials of the second kind (dsk) $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_n)$ are characterized by the property that all their residues vanish. They are normalized, for instance, by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad 1 \le j \le n. \tag{A.34}$$

If $\omega^{(2)}_{P_1,n}$ is a dsk on $\mathcal{K}_n$ whose only pole is $P_1 \in \hat{\mathcal{K}}_n$ with principal part $\zeta^{-n-2}\,d\zeta$, $n \in \mathbb{N}_0$ near P_1 and $\omega_j = (\sum_{m=0}^\infty d_{j,m}(P_1)\zeta^m)\,d\zeta$ near P_1, then

$$\int_{b_j} \omega^{(2)}_{P_1,n} = \frac{2\pi i}{n+1} d_{j,n}(P_1). \tag{A.35}$$

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_n$ not of the first or second kind is said to be of the third kind (dtk). A dtk $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_n)$ is usually normalized by the vanishing of its a-periods, that is,

$$\int_{a_j} \omega^{(3)} = 0, \quad 1 \le j \le n. \tag{A.36}$$

A normal dtk $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \hat{\mathcal{K}}_n$, $P_1 \ne P_2$ by definition has simple poles at P_1 and P_2 with residues $+1$ at P_1 and -1 at P_2 and vanishing a-periods. If $\omega^{(3)}_{P,Q}$ is a normal dtk associated with P, $Q \in \hat{\mathcal{K}}_n$, holomorphic on $\mathcal{K}_n\backslash\{P, Q\}$, then

$$\int_{b_j} \omega^{(3)}_{P,Q} = 2\pi i \int_Q^P \omega_j, \quad 1 \le j \le n, \tag{A.37}$$

where the path from Q to P lies in $\hat{\mathcal{K}}_n$ (i.e., does not touch any of the cycles a_j, b_j).

We shall always assume (without loss of generality) that all poles of dsk's and dtk's on $\mathcal{K}_n$ lie on $\hat{\mathcal{K}}_n$ (i.e., not on $\partial\hat{\mathcal{K}}_n$).

For $f \in \mathcal{M}(\mathcal{K}_n)\backslash\{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_n)\backslash\{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \mathrm{Div}(\mathcal{K}_n)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_n)\backslash\{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \mathrm{Div}(\mathcal{K}_n) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(n-1),\ f \in \mathcal{M}(\mathcal{K}_n)\backslash\{0\},\ \omega \in \mathcal{M}^1(\mathcal{K}_n)\backslash\{0\}, \tag{A.38}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P\in\mathcal{K}_n} \mathcal{D}(P)$. It is custom to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid f = 0 \text{ or } (f) \geq \mathcal{D}\},\ r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.39}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_n) \mid \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\},\ i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}), \tag{A.40}$$

($i(\mathcal{D})$ the index of specialty of $\mathcal{D}$) one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_n)$, $\omega \in \mathcal{M}^1(\mathcal{K}_n)\backslash\{0\}$. Then*
(i).

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad n \in \mathbb{N}_0. \tag{A.41}$$

(ii) (Riemann-Roch theorem).

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - n + 1, \quad n \in \mathbb{N}_0. \tag{A.42}$$

(iii) (Abel's theorem). $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_n)$, $n \in \mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{P_0}(\mathcal{D}) = \underline{0}. \tag{A.43}$$

(iv) (Jacobi's inversion theorem). Assume $n \in \mathbb{N}$, then $\underline{\alpha}_{P_0} : \mathrm{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n)$ is surjective.

For notational convenience we agree to abbreviate

$$\mathcal{D}_Q : \begin{cases} \mathcal{K}_n \to \{0,1\} \\ P \mapsto \begin{cases} 1, & P = Q \\ 0, & P \neq Q \end{cases} \end{cases} \tag{A.44}$$

and, for $\underline{Q} = (Q_1, \dots, Q_n) \in \sigma^n\mathcal{K}_n$ ($\sigma^n\mathcal{K}_n$ the n-th symmetric power of $\mathcal{K}_n$),

$$\mathcal{D}_{\underline{Q}} : \begin{cases} \mathcal{K}_n \to \{0,1,\dots,n\} \\ P \mapsto \begin{cases} k & \text{if } P \text{ occurs } k \text{ times in } \{Q_1,\dots,Q_n\} \\ 0 & \text{if } P \notin \{Q_1,\dots,Q_n\} \end{cases} \end{cases}. \tag{A.45}$$

Moreover, $\sigma^n\mathcal{K}_n$ can be identified with the set of positive divisors $0 < \mathcal{D} \in \mathrm{Div}(\mathcal{K}_n)$ of degree n.

Lemma A.2. *Let $\mathcal{D}_{\underline{Q}} \in \sigma^n\mathcal{K}_n$, $\underline{Q} = (Q_1, \dots, Q_n)$. Then*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s(\leq n/2) \tag{A.46}$$

if and only if there are s pairs of the type $(P, P^*) \in \{Q_1, \dots, Q_n\}$ (this includes, of course, branch points for which $P = P^*$).

Finally, still assuming the nonsingular case (A.16) for simplicity, we consider two frequently encountered special cases, namely

Case I: The self-adjoint case, where

$$\{E_m\}_{0\leq m\leq 2n+1}\subset\mathbb{R},\quad E_0<E_1<\ldots<E_{2n+1} \tag{A.47}$$

and

Case II: Complex conjugate pairs of branch points, that is,

$$\{E_m\}_{0\leq m\leq 2n+1}=\{\epsilon_\ell,\overline{\epsilon_\ell}\}_{0\leq\ell\leq n}. \tag{A.48}$$

Without loss of generality we assume

$$\mathrm{Re}(\epsilon_\ell)\leq\mathrm{Re}(\epsilon_{\ell+1}),\ 0\leq\ell\leq n-1,\ \mathrm{Im}(\epsilon_\ell)<\mathrm{Im}(\overline{\epsilon_\ell}),\ 0\leq\ell\leq n. \tag{A.49}$$

We start with

Case I: Define

$$\mathcal{C}_j=[E_{2j},E_{2j+1}],\quad 0\leq j\leq n, \tag{A.50}$$

and extend $R_{2n+2}(.)^{1/2}$ in (A.4) to all of $\mathbb{C}$ by

$$R_{2n+2}(\lambda)^{1/2}=\lim_{\epsilon\downarrow 0}R_{2n+2}(\lambda+i\epsilon)^{1/2},\quad \lambda\in\mathcal{C}, \tag{A.51}$$

with the sign of the square root chosen according to

$$R_{2n+2}(\lambda)^{1/2}=|R_{2n+2}(\lambda)^{1/2}|\begin{cases}-1, & \lambda\in(E_{2n+1},\infty)\\ (-1)^{n+j+1}, & \lambda\in(E_{2j+1},E_{2j+2}),\ 0\leq j\leq n-1\\ (-1)^n, & \lambda\in(-\infty,E_0)\\ i(-1)^{n+j}, & \lambda\in(E_{2j},E_{2j+1}),\ 0\leq j\leq n\end{cases}. \tag{A.52}$$

In this case (A.8) and (A.9) are supplemented as follows.

$P_0=(E_{m_0},0)$:

$$U_{P_0}=\{P\in M_n\;\big|\;|z-E_{m_0}|<C_{m_0}\},\ C_{m_0}=\min_{m\neq m_0}|E_{m_0}-E_m|,$$
$$V_{P_0}=\{\zeta\in\mathbb{C}\;\big|\;|\zeta|<C_{m_0}^{1/2}\},$$

$$\zeta_{P_0}:\begin{cases}U_{P_0}\to V_{P_0}\\ P\mapsto\sigma(z-E_{m_0})^{1/2}\end{cases},\quad \begin{aligned}&(z-E_{m_0})^{1/2}=|(z-E_{m_0})^{1/2}|e^{(i/2)\arg(z-E_{m_0})},\\ &\arg(z-E_{m_0})\in\begin{cases}[0,2\pi), & m_0\text{ even}\\ (-\pi,\pi], & m_0\text{ odd}\end{cases},\end{aligned}$$

$$\zeta_{P_0}^{-1}:\begin{cases}V_{P_0}\to U_{P_0}\\ \zeta\mapsto(E_{m_0}+\zeta^2,\ [\prod_{m\neq m_0}(E_{m_0}-E_m+\zeta^2)]^{1/2}\zeta)\end{cases},$$

$$\big[\textstyle\prod_{m\neq m_0}(E_{m_0}-E_m+\zeta^2)\big]^{1/2}=(-1)^n i^{-m_0-1}\left|\big[\prod_{m\neq m_0}(E_{m_0}-E_m)\big]^{1/2}\right|\big[1+\tfrac{1}{2}\big(\sum_{m\neq m_0}(E_{m_0}-E_m)^{-1}\big)\zeta^2+O(\zeta^4)\big]. \tag{A.53}$$

Case II: Define

$$\mathcal{C}_\ell=\{z\in\mathbb{C}\mid z=\epsilon_\ell+t(\overline{\epsilon_\ell}-\epsilon_\ell),\ 0\leq t\leq 1\},\quad 0\leq\ell\leq n \tag{A.54}$$

and extend $R_{2n+2}(.)^{1/2}$ in (A.4) to all of $\mathbb{C}$ by

$$R_{2n+2}(z)^{1/2} = \lim_{\epsilon\downarrow 0} R_{2n+2}(z+(-1)^{n+\ell}\epsilon)^{1/2}, \quad z \in \mathcal{C}_\ell,\ 0 \le \ell \le n, \tag{A.55}$$

with the sign of the square root chosen according to

$$R_{2n+2}(\lambda)^{1/2} = |R_{2n+2}(\lambda)^{1/2}| \begin{cases} -1, & \mathrm{Re}(\lambda) \in (\epsilon_n, \infty) \\ (-1)^{n+\ell+1}, & \lambda \in (\mathrm{Re}(\epsilon_\ell), \mathrm{Re}(\epsilon_{\ell+1})),\ 0 \le \ell \le n-1 \\ (-1)^n, & \lambda \in (-\infty, (\mathrm{Re}(\epsilon_0)) \end{cases}. \tag{A.56}$$

In this case (A.8) and (A.9) are supplemented as follows.

$\underline{P_0 = (E_{m_0}, 0)}$:

$$U_{P_0} = \{P \in M_n \mid |z - E_{m_0}| < C_{m_0}\},\ C_{m_0} = \min_{m \ne m_0} |E_{m_0} - E_m|,$$
$$V_{P_0} = \{\zeta \in \mathbb{C} \mid |\zeta| < C_{m_0}^{1/2}\},$$

$$\zeta_{P_0} : \begin{cases} U_{P_0} \to V_{P_0} \\ P \mapsto \sigma(z - E_{m_0})^{1/2} \end{cases}, \quad (z - E_{m_0})^{1/2} = |(z - E_{m_0})^{1/2}| e^{(i/2)\arg(z - E_{m_0})}, \tag{A.57}$$

$$\arg(z - \epsilon_\ell) \in \begin{cases} (\frac{\pi}{2}, \frac{5\pi}{2}],\ \ell \text{ even} \\ \\ [\frac{\pi}{2}, \frac{5\pi}{2}),\ \ell \text{ odd} \end{cases}, \quad \arg(z - \overline{\epsilon_\ell}) \in \begin{cases} [-\frac{\pi}{2}, \frac{3\pi}{2}),\ \ell \text{ even} \\ \\ (-\frac{\pi}{2}, \frac{3\pi}{2}],\ \ell \text{ odd} \end{cases}, \quad n \text{ even},$$

$$\arg(z - \epsilon_\ell) \in \begin{cases} [\frac{\pi}{2}, \frac{5\pi}{2}),\ \ell \text{ even} \\ \\ (\frac{\pi}{2}, \frac{5\pi}{2}],\ \ell \text{ odd} \end{cases}, \quad \arg(z - \overline{\epsilon_\ell}) \in \begin{cases} (-\frac{\pi}{2}, \frac{3\pi}{2}],\ \ell \text{ even} \\ \\ [-\frac{\pi}{2}, \frac{3\pi}{2}),\ \ell \text{ odd} \end{cases}, \quad n \text{ odd},$$

$$\zeta_{P_0}^{-1} : \begin{cases} V_{P_0} \to U_{P_0} \\ \zeta \mapsto (E_{m_0} + \zeta^2,\ [\prod_{m \ne m_0}(E_{m_0} - E_m + \zeta^2)]^{1/2}\zeta) \end{cases},$$
$$\big[\textstyle\prod_{m\ne m_0}(E_{m_0}-E_m+\zeta^2)\big]^{1/2} = e^{(i/2)\sum_{m\ne m_0}\arg(E_{m_0}-E_m)}\, \Big|\big[\prod_{m\ne m_0}(E_{m_0}-E_m)\big]^{1/2}\Big| \times$$
$$\times\big[1+\tfrac{1}{2}\big(\textstyle\sum_{m\ne m_0}(E_{m_0}-E_m)^{-1}\big)\zeta^2+O(\zeta^4)\big],$$

where $\exp[(i/2)\sum_{m\ne m_0}\arg(E_{m_0} - E_m)]$ can be determined from (A.56) by analytic continuation.

Cases I and II are of course compatible with our general choice of

$$y(P) = R_{2n+2}^{1/2}(P) \underset{\zeta\to 0}{=} \mp\Big[1 - \frac{1}{2}\Big(\sum_{m=0}^{2n+1} E_m\Big)\zeta + O(\zeta^2)\Big]\zeta^{-n-1} \text{ as } P \to \infty_\pm. \tag{A.58}$$

Appendix B. An Explicit Illustration of the Riemann-Roch Theorem

We provide a brief illustration of the Riemann-Roch theorem in connection with hyperelliptic curves $\mathcal{K}_n$ of the type (2.26) and explicitly determine a basis for the vector space $\mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$, $k \in \mathbb{N}_0$. (The corresponding case of hyperelliptic curves $\mathcal{K}_n$ branched at infinity has been discussed in Appendix B of [26].)

We freely use the notation introduced in Appendix A and refer, in particular, to the definition (A.39) of $\mathcal{L}(\mathcal{D})$ and the Riemann-Roch theorem stated in Theorem A.1 (ii). In addition, we use the short-hand notation

$$k\mathcal{D}_{\infty_-} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)} = \sum_{\ell=1}^{k} \mathcal{D}_{\infty_-} + \sum_{j=1}^{n} \mathcal{D}_{\hat{\mu}_j(x_0)}, \quad k \in \mathbb{N}_0, \;\; \underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \tag{B.1}$$

and recall that

$$\mathcal{L}(-k\mathcal{D}_{\infty_-} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid f = 0 \text{ or } (f) + k\mathcal{D}_{\infty_-} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)} \geq 0\}, \quad n \in \mathbb{N}_0. \tag{B.2}$$

With $\phi(P,x)$, $\psi_j(P,x,x_0)$, $j = 1,2$ defined as in (3.8), (3.10), (3.21) we obtain the following result.

Theorem B.1. *Assume $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ to be nonspecial (i.e., $i(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = 0$) and of degree $n \in \mathbb{N}$. For $k \in \mathbb{N}_0$, a basis for the vector space $\mathcal{L}(-k\mathcal{D}_{\infty_-} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ is given by*

$$\begin{cases} \{1\}, & k = 0 \\ \{\tilde{\pi}^\ell\}_{0 \leq \ell \leq (k-1)/2} \cup \{\tilde{\pi}^\ell \phi(\,.\,, x_0)\}_{0 \leq \ell \leq (k-1)/2}, & k \text{ odd} \\ \{\tilde{\pi}^\ell\}_{0 \leq \ell \leq k/2} \cup \{\tilde{\pi}^\ell \phi(\,.\,, x_0)\}_{0 \leq \ell \leq (k-2)/2}, & k \text{ even} \end{cases}, \tag{B.3}$$

or equivalently,

$$\begin{aligned} &\mathcal{L}(-k\mathcal{D}_{\infty_-} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) \\ &= \operatorname{span}\left\{ \begin{array}{l} \big\{\frac{\partial^\ell}{\partial x^\ell}\psi_1(\,.\,,x,x_0)\big|_{x=x_0}\big\}_{0\leq\ell\leq k/2} \cup \big\{\frac{\partial^\ell}{\partial x^\ell}\psi_2(\,.\,,x,x_0)\big|_{x=x_0}\big\}_{0\leq\ell\leq(k-2)/2}, \; k \text{ even} \\ \\ \big\{\frac{\partial^\ell}{\partial x^\ell}\psi_1(\,.\,,x,x_0)\big|_{x=x_0}\big\}_{0\leq\ell\leq(k-1)/2} \cup \big\{\frac{\partial^\ell}{\partial x^\ell}\psi_2(\,.\,,x,x_0)\big|_{x=x_0}\big\}_{0\leq\ell\leq(k-1)/2}, \; k \text{ odd} \end{array} \right\}. \end{aligned} \tag{B.4}$$

Proof. The elements in (B.3) are easily seen to be linearly independent and belonging to $\mathcal{L}(-k\mathcal{D}_{\infty_-} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$. It remains to be shown that they are maximal. Since $i(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = i(k\mathcal{D}_{\infty_-} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = 0$, the Riemann-Roch theorem (A.42) implies $r(-k\mathcal{D}_{\infty_-} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = k+1$ proving (B.3). (B.4) can be proven by induction using $\psi_2 = \psi_1\phi$ and (2.26). $\square$

Replacing ϕ by ϕ^{-1} one can discuss $\mathcal{L}(-k\mathcal{D}_{\infty_+} - \mathcal{D}_{\underline{\hat{\nu}}(x_0)})$, $k \in \mathbb{N}_0$ in an analogous fashion.

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