Generalized Eigenfunction Expansion Near Resonances

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Abstract: The non-relativistic quantum mechanical description of meta-stable states which arise by perturbation of embedded eigenvalues is considered. The model given by the Hamiltonian

$$H(\lambda) = \begin{pmatrix} -\frac{d^2}{dx^2} & \lambda u \\ \lambda u & -\frac{d^2}{dx^2} + x^2 \end{pmatrix}$$

is studied for small λ. This model is a simple resonant model. At $\lambda = 0$ there is a sequence of embedded eigenvalues, $\{1, 3, 5, \ldots\}$, which become resonances for $\lambda \neq 0$. The eigenfunction expansion for $H(\lambda)$ is obtained for small λ. Systematic approximations for the continuum eigenfunctions are obtained which are uniform in the spectral parameter. The scattering theory for this model is considered and it is shown that scattering cross sections have the expected Breit-Wigner peaks near the resonances.

§1: Introduction and Statement of Results

One of the mechanisms which produces meta-stable states in Quantum Mechanics is the Feshbach resonance. Let $H(\lambda)$ be the Hamiltonian describing some physical system; λ is some real parameter. Feshbach resonances arise when $H(0)$ has eigenvalues embedded in its continuous spectrum while $H(\lambda)$, for $\lambda \neq 0$, does not. Thus there are bound states at $\lambda = 0$ which become continuum states for any $\lambda \neq 0$; these are the so called meta-stable states. In particular, the $\lambda \to 0$ limit of the time evolution operator $e^{-itH(\lambda)}$ is singular. The physics associated with this singular behavior is that of meta-stability and resonant scattering.

Formal schemes for estimating some of the physical quantities relevant to meta-stability and resonant scattering were developed early in the history of Quantum Mechanics ([GW], [WW]). Some important features of meta-stability are exponential decay and natural line width ([WW]). Important to resonant scattering are the Breit-Wigner peaks ([LL]). Let ϕ be a meta-stable state for $H(\lambda)$ corresponding to an unperturbed embedded eigenvalue of $H(0)$, ϵ_0. Exponential decay refers to the expectation that there is some number $\Gamma > 0$, depending on λ, for which

$$|\langle \phi, e^{-itH(\lambda)}\phi \rangle| \sim e^{-t\Gamma} + \mathcal{O}(\Gamma).$$

Natural line width refers to the expectation that in the $t \to \infty$ limit $e^{-itH(\lambda)}\phi$ is contained in the absolutely continuous spectral subspace of $H(0)$ and that, in a representation in which $H(0)$ is diagonal,

$$\lim_{t\to\infty} |\big(e^{-itH(\lambda)}\phi\big)(\epsilon)|^2 \sim \text{const}\ \frac{1}{(\epsilon-\epsilon_0)^2+\Gamma^2};$$

here $H(0)$ is diagonal in the sense that $\big(H(0)\psi\big)(\epsilon) = \epsilon\psi(\epsilon)$ for all ψ in the absolutely continuous spectral subspace of $H(0)$; ϵ is the spectral parameter. The Breit-Wigner peaks refer to peaks in the scattering cross sections again of the form $\frac{\text{const}}{(\epsilon-\epsilon_0)^2+\Gamma^2}$. These peaks presumably reflect the excitation and subsequent decay of a meta-stable state. In all of these expressions Γ is the same positive real number which goes to zero as $\lambda \to 0$ (if $H(\lambda)$ is a polynomial in λ then typically $\Gamma \sim \lambda^2$).

There has been only partial success in mathematically justifying these formal expectations. An early review of the problems is found in [S]. The strongest results on exponential decay are found in [H]. There are very few rigorous results on either line width ([W]) or resonant scattering for Feshbach resonances. In this paper a model is studied which is exactly solvable in the sense that a closed form expression for the continuum eigenfunctions of $H(\lambda)$ can be obtained. These continuum eigenfunctions are then used to control the scattering theory in this model and to establish the existence of Breit-Wigner peaks. In a subsequent paper it is shown how to obtain small λ asymptotics for $e^{-itH(\lambda)}$ which are uniform in t.

The model considered is described formally by the Hamiltonian

$$H(\lambda) = \begin{pmatrix} -\frac{d^2}{dx^2} & \lambda u \\ \lambda u & -\frac{d^2}{dx^2} + x^2 \end{pmatrix}$$

acting in $\mathrm{L}^2(\mathbf{R}) \oplus \mathrm{L}^2(\mathbf{R})$. Here u is the multiplication operator corresponding to the function $u(x)$. We take u to be continuous and compactly supported in $\mathbf{R}$. This model is a simple resonant model and is very similar to the model studied in [W]. For $\lambda = 0$ the spectrum of $H(0)$ is $[0, \infty)$ and consists of absolutely continuous spectrum and the embedded eigenvalues $\{2j+1 \,|\, j \in \mathbf{N}\}$. For $\lambda \neq 0$ the essential spectrum of $H(\lambda)$ is still $[0, \infty)$ since $H(\lambda) - H(0)$ is a relatively compact perturbation. There are, however, no longer any embedded eigenvalues since the equation $\big(H(\lambda) - \epsilon\big)\Psi = 0$ has no non-zero L^2 solutions if $\epsilon \geq 0$.

In this paper the spectral measure for $H(\lambda)$, $\frac{d\mu}{d\epsilon}(\lambda, \epsilon)$, is constructed for λ sufficiently small. We will see that the spectrum of $H(\lambda)$ consists of one negative eigenvalue, $E(\lambda^2)$, and absolutely continuous spectrum $[0, \infty)$. Our chief concern is with the absolutely continuous spectrum and the resonant behavior caused by the unperturbed embedded eigenvalues $2j+1$. Using $\frac{d\mu}{d\epsilon}(\lambda, \epsilon)$ a generalized eigenfunction expansion associated to $H(\lambda)$ is obtained. The continuum eigenfunctions, $\Psi(\lambda, k)$, are found to be meromorphic functions of k in an open neighborhood of the real line excluding $k = 0$. The $\Psi(\lambda, k)$ have simple poles in k when $k^2 = E(\lambda^2)$ as well as for $k^2 = \epsilon_j(\lambda^2)$, for some numbers $\epsilon_j(\lambda^2)$. Here the number $\epsilon_j(\lambda^2)$ is generally referred to as the resonance associated with the unstable eigenv! alue $2j + 1$. It is shown that the resonances are analytic functions of λ^2, have negative imaginary part and are given, for each $j \in \{0, 1, 2, \ldots\}$, by $\epsilon_j(\lambda^2) = 2j + 1 + \mathcal{O}(\lambda^2)$.

The techniques used are constructive and provide systematic approximations to $\Psi(\lambda, k)$ for small λ. These approximations are uniform in k and have a particularly simple form in leading order when k^2 is sufficiently close to $2j + 1$. It is clear that $\Psi(\lambda, k)$ cannot be analytic in λ since the nature of the spectrum of $H(\lambda)$, for $\lambda \neq 0$, is drastically different from that of $H(0)$. Rather, we will see that $\Psi(\lambda, k)$ is given by a term which is singular as $\lambda \to 0$ plus a term which is analytic in λ.

Finally the scattering theory for this model is considered. The comparison operator appropriate to scattering in this model is $-\frac{d^2}{dx^2}$ acting in $\mathrm{L}^2(\mathbf{R})$. The stationary wave operators ([RS] [Y]) for this system are easily extracted from $\Psi(\lambda, k)$. Standard methods ([RS]) then show that the stationary wave operators are in fact the wave operators

$$\Omega_\pm = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH(\lambda)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{it\frac{d^2}{dx^2}}.$$

The S matrix, $S = \Omega_-^* \Omega_+$, is then computed and is found to have the expected Breit-Wigner peaks.

The results obtained are stated below.

Results: *There is a* $\lambda_0 > 0$ *such that if* $0 < |\lambda| < \lambda_0$ *then the following is true:*

a) The spectrum of $H(\lambda)$ *consists of one negative eigenvalue and absolutely continuous spectrum* $[0, \infty)$. *Explicitly,*

$$\sigma\big(H(\lambda)\big) = \{E(\lambda^2)\} \cup [0, \infty),$$

$\sigma_{pp}\big(H(\lambda)\big) = \{E(\lambda^2)\}$, $\sigma_{ac}\big(H(\lambda)\big) = [0, \infty)$ *and* $\sigma_{sing}\big(H(\lambda)\big) = \emptyset$.

Further, for each $j \in \{0, 1, 2, \ldots\}$ *there is a resonance* $\epsilon_j(\lambda^2)$ *associated with the unperturbed embedded eigenvalue* $2j+1$. *The resonances* $\epsilon_j(\lambda^2)$ *are estimated by*

$$|\epsilon_j(\lambda^2) - \lambda^2 \delta_j + i\lambda^2 \gamma_j| \leq \text{const } \lambda^4$$

where δ_j *and* γ_j *are real numbers and*

$$\gamma_j = \frac{|\langle e^{i\sqrt{2j+1}x} u, \phi_j \rangle|^2}{2\sqrt{2j+1}}.$$

Here ϕ_j *is the eigenfunction of* $-\frac{d^2}{dx^2} + x^2$ *corresponding to the eigenvalue* $2j+1$.

b) The eigenvalue $E(\lambda^2)$, *satisfies the estimate*

$$E(\lambda^2) = -\frac{\lambda^4}{2} \Big(\langle u, \big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u \rangle \Big)^2 + \mathcal{O}(\lambda^6).$$

Further, $E(\lambda^2)$ *is simple, and its eigenfunction satisfies the* L^2 *norm estimate*

$$\Psi_0 = \frac{1}{\mathcal{N}_0} \begin{pmatrix} -\lambda\big(-\frac{d^2}{dx^2} - E(\lambda^2)\big)^{-1} u \big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u \\ \big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u \end{pmatrix} + \mathcal{O}(\lambda^2).$$

Here

$$\mathcal{N}_0 = \sqrt{\lambda^2 \|\big(-\frac{d^2}{dx^2} - E(\lambda^2)\big)^{-1} u\big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u\|^2 + \|\big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u\|^2}$$

is a normalization constant.

c) The generalized eigenfunctions are given by

$$\Psi(\lambda, k; x) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{-ikx} + i\frac{\lambda^2}{2|k|} \int e^{i|k|\,|x-r|} u(r)\psi(k;r)\,dr \\ \lambda\psi(k;x) \end{pmatrix}$$

where $\psi(k) \in \mathrm{L}^2(\mathbf{R})$ *satisfies the integro-differential equation*

$$\big(-\frac{d^2}{dx^2} + x^2 - k^2\big)\psi(k;x) - i\frac{\lambda^2}{2|k|} u(x) \int e^{i|k|\,|x-r|} u(r)\psi(k;r)\,dr = e^{-ikx} u(x)$$

and, for k *such that* $\min\{\frac{1}{2}, 2j - \frac{1}{2}\} < k^2 < 2j + \frac{5}{2}$, *satisfies the estimate*

$$\|\psi(k) - \frac{\langle \phi_j, e^{-ik\cdot} u\rangle}{k^2 - \epsilon_j(\lambda^2)}\,\phi_j\| \leq \text{const} \ .$$

As above, ϕ_j *is the eigenfunction of* $-\frac{d^2}{dx^2} + x^2$ *corresponding to the eigenvalue* $2j+1$.

d) The scattering matrix is given in momentum space by

$$S(k,q) = \alpha(k)\delta(k-q) + \beta(k)\delta(k+q);$$

here $S(k,q)$ *is defined by*

$$[S\phi](x) = \frac{1}{\sqrt{2\pi}} \int S(q,k)\hat{\phi}(k) e^{iqx}\,dk\,dq,$$

$\hat{\phi}$ *is the Fourier transform of* ϕ. *For* k *such that* $\min\{\frac{1}{2}, 2j - \frac{1}{2}\} < k^2 < 2j + \frac{5}{2}$ *the coefficients* α *and* β *have the expected Breit-Wigner form*

$$|\alpha(k)|^2 = \frac{\lambda^4 |\langle \phi_j, u e^{ik\cdot}\rangle|^4}{(k^2 - 2j - 1 - \lambda^2\delta_j)^2 + \lambda^4\gamma_j^2} + \mathcal{O}(\lambda^2)$$

and

$$|\beta(k)|^2 = \frac{\lambda^4 |\langle \phi_j, u e^{-ik\cdot}\rangle|^4}{(k^2 - 2j - 1 - \lambda^2\delta_j)^2 + \lambda^4\gamma_j^2} + \mathcal{O}(\lambda^2).$$

§2: Preliminary Results

As stated in the introduction, the model being considered is described formally by the Hamiltonian

$$H(\lambda)=\begin{pmatrix} -\frac{d^2}{dx^2} & \lambda u \\ \lambda u & -\frac{d^2}{dx^2}+x^2 \end{pmatrix}$$

acting in $\mathrm{L}^2(\mathbf{R})\oplus\mathrm{L}^2(\mathbf{R})$. Here u is the multiplication operator corresponding to the continuous, compactly supported function $u(x)$. The spectrum of $H(0)$ is very different from that of $H(\lambda)$ for $\lambda\neq 0$. The point spectrum of $H(0)$ consists of the sequence of positive embedded eigenvalues

$$\sigma_{pp}\big(H(0)\big)=\{1,3,5,\ldots\}.$$

The proposition below establishes that for any $\lambda\neq 0$ the point spectrum of $H(\lambda)$ consists of one negative eigenvalue, $E(\lambda^2)\sim-\lambda^4$.

Proposition : *The operator $H(\lambda)$ has no non-negative eigenvalues. Further, if λ is small enough, $H(\lambda)$ has a unique eigenvalue*

$$E(\lambda^2)=-\frac{\lambda^4}{2}\Big(\langle u,\big(-\frac{d^2}{dx^2}+x^2\big)^{-1}u\rangle\Big)^2+\mathcal{O}(\lambda^6).$$

This eigenvalue is simple and has eigenfunction

$$\Psi_0=\frac{1}{\mathcal{N}_0}\begin{pmatrix} -\lambda(-\frac{d^2}{dx^2}-E(\lambda^2))^{-1}u\big(-\frac{d^2}{dx^2}+x^2\big)^{-1}u \\ \big(-\frac{d^2}{dx^2}+x^2\big)^{-1}u \end{pmatrix}+\mathcal{O}(\lambda^2).$$

Here

$$\mathcal{N}_0=\sqrt{\lambda^2\|\big(-\frac{d^2}{dx^2}-E(\lambda^2)\big)^{-1}u\big(-\frac{d^2}{dx^2}+x^2\big)^{-1}u\|^2+\|\big(-\frac{d^2}{dx^2}+x^2\big)^{-1}u\|^2}$$

is a normalization constant.

proof: Consider the equation

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix}&=\begin{pmatrix} -\frac{d^2}{dx^2}-\epsilon & \lambda u \\ \lambda u & -\frac{d^2}{dx^2}+x^2-\epsilon \end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix}\\ &=\begin{pmatrix} (-\frac{d^2}{dx^2}-\epsilon)\psi+\lambda u\phi \\ (-\frac{d^2}{dx^2}+x^2-\epsilon)\phi+\lambda u\psi \end{pmatrix}.\end{aligned}$$

For $\epsilon \geq 0$ the equation $(-\frac{d^2}{dx^2} - \epsilon)\psi + \lambda u\phi = 0$ has no solutions with $\psi \in \mathrm{L}^2(\mathbf{R})$, regardless of ϕ, except for the trivial solution $\psi = \phi = 0$. It follows that $H(\lambda)$ has no non-negative eigenvalues.

For $\epsilon < 0$ we have

$$\psi = -\lambda(-\frac{d^2}{dx^2} - \epsilon)^{-1} u\phi \qquad ()$$

so that

$$\Big(-\frac{d^2}{dx^2} + x^2 - \epsilon - \lambda^2 u(-\frac{d^2}{dx^2} - \epsilon)^{-1} u\Big)\phi = 0. \qquad ()$$

Writing $\epsilon = -\kappa^2$, for $\kappa > 0$, we have the following integral kernel for $V(\epsilon) = u(-\frac{d^2}{dx^2} - \epsilon)^{-1} u$.

$$\begin{aligned} V(\kappa^2; x, y) &= u(x)\frac{1}{2\kappa}e^{-\kappa|x-y|}u(y) \\ &= \frac{1}{2\kappa}u(x)u(y) + u(x)\frac{e^{-\kappa|x-y|} - 1}{2\kappa}u(y) \\ &= \frac{1}{2\kappa}u(x)u(y) + V^{(1)}(\kappa; x, y). \end{aligned}$$

For all $\kappa > 0$ we have $\|V^{(1)}(\kappa)\| \leq \text{const}$. Thus, if λ is small enough, $-\frac{d^2}{dx^2} + x^2 + \kappa^2 - \lambda^2 V^{(1)}(\kappa)$ is invertible. It follows that () is equivalent to

$$\phi = \langle u, \phi\rangle \Big(-\frac{d^2}{dx^2} + x^2 + \kappa^2 - \lambda^2 V^{(1)}(\kappa)\Big)^{-1} u \qquad ()$$

Thus, () has a non-trivial solution in $\mathrm{L}^2(\mathbf{R})$ if and only if κ satisfies

$$1 = \frac{\lambda^2}{2\kappa}\langle u, \Big(-\frac{d^2}{dx^2} + x^2 + \kappa^2 - \lambda^2 V^{(1)}(\kappa)\Big)^{-1} u\rangle. \qquad ()$$

Note that, as a function of κ, the right side of () is continuous, non-zero and, from considering large κ, positive. Further, since

$$\|\Big(-\frac{d^2}{dx^2} + x^2 + \kappa^2 - \lambda^2 V^{(1)}(\kappa)\Big)^{-1}\| \leq \text{const}\ \frac{1}{1+\kappa^2}$$

if λ is small enough, () has at least one solution, $\kappa \sim \mathcal{O}(\lambda^2)$. For such κ it is clear that

$$\langle u, \Big(-\frac{d^2}{dx^2} + x^2 + \kappa^2 - \lambda^2 V^{(1)}(\kappa)\Big)^{-1} u\rangle = \langle u, \big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u\rangle + \mathcal{O}(\lambda^2).$$

Finally, for $\kappa \sim \mathcal{O}(\lambda^2)$, the derivative with respect to κ of the right side of () is equal to $-\frac{1}{\lambda^2} + \mathcal{O}(1)$. Thus, () has a unique solution. Along with () and () this finishes the proof. ∎

We will see that the spectrum of $H(\lambda)$ consists of the one negative eigenvalue, $E(\lambda^2)$, from , and absolutely continuous spectrum $[0,\infty)$. Our chief concern is with the absolutely continuous spectrum and the resonant behavior caused by the unperturbed embedded eigenvalues $2j+1$. To this end we construct the spectral measure for $H(\lambda)$, $\frac{d\mu}{d\epsilon}(\lambda,\epsilon)$.

¿From Stone's formula we have

$$\frac{d\mu}{d\epsilon}(\lambda,\epsilon)=\frac{1}{2\pi i}\operatorname*{s-lim}_{\zeta\downarrow 0}\Big(\big(H(\lambda)-\epsilon-i\zeta\big)^{-1}-\big(H(\lambda)-\epsilon+i\zeta\big)^{-1}\Big). \qquad ()$$

To estimate () we will use an integral kernel for $\big(H(\lambda)-z\big)^{-1}$ which will be constructed in §3 and §4. Let $\mathcal{G}(\lambda,z;x,y)$ be this integral kernel; note that it is a 2 by 2 matrix valued function of x and y. The equation $\big(H(\lambda)-z\big)\big(H(\lambda)-z\big)^{-1}=1$ implies that

$$\begin{pmatrix} -\frac{d^2}{dx^2}-z & \lambda u(x) \\ \lambda u(x) & -\frac{d^2}{dx^2}+x^2-z \end{pmatrix}\mathcal{G}(\lambda,z;x,y)=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\delta(x-y)$$

and $\big(H(\lambda)-z\big)^{-1}\big(H(\lambda)-z\big)=1$ that $\mathcal{G}(\lambda,z;x,y)=\mathcal{G}^t(\lambda,z;y,x)$. Here the superscript t denotes the 2 by 2 matrix transpose.

Given $\mathcal{G}(\lambda,z;x,y)$ we have ([A])

$$\begin{aligned}
&\mathcal{G}(\lambda,\epsilon+i0^+;x,y)-\mathcal{G}(\lambda,\epsilon-i0^+;x,y)\\
&\quad=\int_{-\infty}^{\infty}\Bigg\{\mathcal{G}(\lambda,\epsilon+i0^+;x,w)\begin{pmatrix} -\frac{d^2}{dw^2}-\epsilon & \lambda u(w) \\ \lambda u(w) & -\frac{d^2}{dw^2}+w^2-\epsilon \end{pmatrix}\mathcal{G}(\lambda,\epsilon-i0^+;w,y)\\
&\qquad\qquad-\left[\begin{pmatrix} -\frac{d^2}{dw^2}-\epsilon & \lambda u(w) \\ \lambda u(w) & -\frac{d^2}{dw^2}+w^2-\epsilon \end{pmatrix}\mathcal{G}(\lambda,\epsilon+i0^+;w,x)\right]^t\mathcal{G}(\lambda,\epsilon-i0^+;w,y)\Bigg\}dw\\
&\quad=-\int_{-\infty}^{\infty}\Bigg\{\mathcal{G}(\lambda,\epsilon+i0^+;x,w)\frac{d^2}{dw^2}\mathcal{G}(\lambda,\epsilon-i0^+;w,y)\\
&\qquad\qquad-\Big(\frac{d^2}{dw^2}\mathcal{G}(\lambda,\epsilon+i0^+;x,w)\Big)\mathcal{G}(\lambda,\epsilon-i0^+;w,y)\Bigg\}dw\\
&\quad=-\Bigg\{\mathcal{G}(\lambda,\epsilon+i0^+;x,w)\frac{d\mathcal{G}(\lambda,\epsilon-i0^+;w,y)}{dw}\\
&\qquad\qquad-\frac{d\mathcal{G}(\lambda,\epsilon+i0^+;x,w)}{dw}\mathcal{G}(\lambda,\epsilon-i0^+;w,y)\Bigg\}\Bigg|_{w=-\infty}^{\infty}
\end{aligned}$$

so that the integral kernel for $\frac{d\mu}{d\epsilon}(\lambda,\epsilon)$ is

$$\begin{aligned}
\frac{d\mu}{d\epsilon}(\lambda,\epsilon;x,y)=-\frac{1}{2\pi i}\Bigg\{&\mathcal{G}(\lambda,\epsilon+i0^+;x,w)\frac{d\mathcal{G}(\lambda,\epsilon-i0^+;w,y)}{dw}\\
&-\frac{d\mathcal{G}(\lambda,\epsilon+i0^+;x,w)}{dw}\mathcal{G}(\lambda,\epsilon-i0^+;w,y)\Bigg\}\Bigg|_{w=-\infty}^{\infty}.
\end{aligned} \qquad ()$$

Using () and the constructions of the following two sections it will be shown that, for $\epsilon \geq 0$ and λ sufficiently small, $\frac{d\mu}{d\epsilon}$ is absolutely continuous with respect to Lebesgue measure. In addition, the continuum eigenfunctions for $H(\lambda)$ will be extracted. The techniques used are constructive and provide systematic small λ approximations to the continuum eigenfunctions as well as to $\frac{d\mu}{d\epsilon}$. These approximations cannot take the form of power series in λ, since the nature of the spectrum of $H(\lambda)$, for $\lambda \neq 0$, is drastically different from that of $H(0)$. In particular, $\frac{d\mu}{d\epsilon}$ cannot be analytic in λ.

§3: The Resonant Poles

For $z \notin \mathbf{R}$ the operator $H - z$ is invertible. We have

$$\big(H - z\big)^{-1} = \begin{pmatrix} A(z) & -\lambda\ B(z) \\ -\lambda\ C(z) & \big(h(z) - z\big)^{-1} \end{pmatrix} \qquad ()$$

where

$$h(z) = -\frac{d^2}{dx^2} + x^2 - \lambda^2 u(-\frac{d^2}{dx^2} - z)^{-1} u \qquad ()$$

and

$$\begin{aligned} A(z) &= (-\frac{d^2}{dx^2} - z)^{-1} + \lambda^2(-\frac{d^2}{dx^2} - z)^{-1} u\big(h(z) - z\big)^{-1} u(-\frac{d^2}{dx^2} - z)^{-1} \\ B(z) &= (-\frac{d^2}{dx^2} - z)^{-1} u\big(h(z) - z\big)^{-1} \\ C(z) &= \big(h(z) - z\big)^{-1} u(-\frac{d^2}{dx^2} - z)^{-1}. \end{aligned} \qquad ()$$

Define, for $z \notin [0, \infty)$,

$$V(z) = u\big(-\frac{d^2}{dx^2} - z\big)^{-1} u. \qquad ()$$

Recall that $V(z)$ has an integral kernel, $V(z; x, y)$, given by

$$V(z; x, y) = \frac{1}{2\sqrt{-z}} u(x)\, e^{-\sqrt{-z}\,|x-y|}\, u(y).$$

This integral kernel allows us to construct two analytic (as bounded operator valued functions of z) extensions of $V(z)$, $V_{\pm}(z)$, given by their integral kernels,

$$V_{\pm}(z; x, y) = \mp\frac{1}{2i\sqrt{z}} u(x)\, e^{\pm i\sqrt{z}\,|x-y|}\, u(y) \qquad ()$$

for $z \notin (-\infty, 0]$. In this paper square roots are always chosen to have their branch cut on the negative real axis. In particular, for all $\epsilon \in (0, \infty)$,

$$\lim_{\eta \downarrow 0} \sqrt{-\epsilon \mp i\eta} = \mp i\sqrt{\epsilon}. \qquad ()$$

Lemma : *Let $z \notin (-\infty, 0]$ and define*

$$h_{\pm}(z) = -\frac{d^2}{dx^2} + x^2 - \lambda^2 V_{\pm}(z).$$

Let

$$\mathcal{N} = \{z \in \mathbf{C} \,\big|\, \mathrm{Re}\, z > 0 \text{ and } |\mathrm{Im}\, z| < 2\}.$$

If λ is small enough there is a discrete set, $\mathcal{R}_\pm = \{\epsilon_j^{(\pm)}(\lambda^2) \,|\, j \in \{0,1,2,\ldots\}\} \subset \mathcal{N}$ such that $h_\pm(z) - z$ is invertible for $z \in \mathcal{N} \setminus \mathcal{R}_\pm$. The inverses $\big(h_\pm(z) - z\big)^{-1}$ are meromorphic (as bounded operator valued functions of z) in $\mathcal{N}$ with simple poles at $\epsilon_j^{(\pm)}(\lambda^2)$. Further,

$$|\operatorname{Re} \epsilon_j^{(\pm)}(\lambda^2) - 2j - 1 - \lambda^2 \delta_j| \leq \text{const } \lambda^4$$

and

$$|\operatorname{Im} \epsilon_j^{(\pm)}(\lambda^2) \pm \lambda^2 \gamma_j| \leq \text{const } \lambda^4$$

where δ_j and γ_j are real constants with

$$\gamma_j = \frac{|\langle e^{i\sqrt{2j+1}x} u, \phi_j\rangle|^2}{2\sqrt{2j+1}},$$

and ϕ_j is the eigenfunction of $-\frac{d^2}{dx^2} + x^2$ corresponding to the eigenvalue $2j+1$.

proof: For such $z \in \mathcal{N}$ that $|z| > \lambda^2$ the lemma is a simple consequence of the bound

$$\begin{aligned} \|V_\pm(z)\| &\leq \sqrt{\int |V_\pm(z;x,y)|^2 \, dx \, dy} \\ &\leq \frac{1}{2|z|^{\frac{1}{2}}} \sup_{x,y \in supp(u)} \left| e^{\pm i\sqrt{z}\,|x-y|} \right| \int |u(x')|^2 \, dx' \\ &\leq \text{const } \frac{e^{K|\operatorname{Im}\sqrt{z}\,|}}{|z|^{\frac{1}{2}}}, \end{aligned} \qquad ()$$

where

$$K = 2 \sup_{x \in supp(u)} |x|,$$

and analytic perturbation theory.

For $z \in \mathcal{N}$ with $|z| \leq \lambda^2$ we use

$$\begin{aligned} V_\pm(z;x,y) &= \mp \frac{1}{2i\sqrt{z}} u(x)\, u(y) \mp u(x) \, \frac{e^{\pm i\sqrt{z}\,|x-y|} - 1}{2i\sqrt{z}} \, u(y) \\ &= \mp \frac{1}{2i\sqrt{z}} u(x)\, u(y) + V_\pm^{(1)}(z;x,y). \end{aligned}$$

Then $V_\pm^{(1)}(z)$ is bounded analytic in z for $z \notin (-\infty, 0]$ and satisfies the bound $\|V_\pm^{(1)}(z)\| \leq$ const $e^{K|\operatorname{Im}\sqrt{z}\,|}$. Thus, for λ small enough,

$$h_\pm^{(1)}(z) - z = -\frac{d^2}{dx^2} + x^2 - z - \lambda^2 V_\pm^{(1)}(z)$$

is invertible for all $z \in \mathcal{N}$. To see that

$$h_\pm(z) - z = h_\pm^{(1)}(z) - z \pm \frac{\lambda^2}{2i\sqrt{z}} u \otimes u$$

is invertible we note that $h_\pm(z) - h_\pm^{(1)}(z)$ is rank one so that $h_\pm(z) - z$ is invertible if and only if $\lambda^2 \langle u, \big(h_\pm^{(1)}(z) - z\big)^{-1} u\rangle \neq \mp 2i\sqrt{z}$. If so then

$$\begin{aligned}\big(h_\pm(z) - z\big)^{-1} &= \big(h_\pm^{(1)}(z) - z\big)^{-1} \\ &\mp \frac{\lambda^2}{2i\sqrt{z} \pm \lambda^2 \langle u, \big(h_\pm^{(1)}(z) - z\big)^{-1} u\rangle} \big(h_\pm^{(1)}(z) - z\big)^{-1} u \otimes u \big(h_\pm^{(1)}(z) - z\big)^{-1}.\end{aligned} \tag{}$$

To finish the proof of the lemma it must be shown that $\lambda^2 \langle u, \big(h_\pm^{(1)}(z) - z\big)^{-1} u\rangle \neq \mp 2i\sqrt{z}$ for $z \in \mathcal{N}$ with $|z| \leq \lambda^2$. For such z

$$\begin{aligned}\|\big(h_\pm^{(1)}(z) - z\big)^{-1} - \big(-\frac{d^2}{dx^2} + x^2\big)^{-1}\| &\leq \|\big(h_\pm^{(1)}(z) - z\big)^{-1} - \big(-\frac{d^2}{dx^2} + x^2 - z\big)^{-1}\| \\ &\quad + \|\big(-\frac{d^2}{dx^2} + x^2 - z\big)^{-1} - \big(-\frac{d^2}{dx^2} + x^2\big)^{-1}\| \\ &\leq \text{const } \lambda^2.\end{aligned}$$

If ϕ_j is the eigenfunction of $-\frac{d^2}{dx^2} + x^2$ corresponding to the eigenvalue $2j+1$ then it follows that

$$\begin{aligned}\langle u, \big(h_\pm^{(1)}(z) - z\big)^{-1} u\rangle &= \langle u, \big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u\rangle + \mathcal{O}(\lambda^2) \\ &= \sum_{j=0}^{\infty} \frac{|\langle \phi_j, u\rangle|^2}{2j+1} + \mathcal{O}(\lambda^2).\end{aligned}$$

Thus, if $\lambda^2 \langle u, \big(h_\pm^{(1)}(z) - z\big)^{-1} u\rangle = \mp 2i\sqrt{z}$ then $z = -\lambda^4 \big(\langle u, \big(-\frac{d^2}{dx^2} + x^2\big)^{-1} u\rangle\big)^2 + \mathcal{O}(\lambda^6)$. It follows that, for λ small enough, $\operatorname{Re} z < 0$ and thus $z \notin \mathcal{N}$. ■

Using () to estimate $\big(H - z\big)^{-1}$ it is sufficient to estimate $\big(h(z) - z\big)^{-1}$. Let $h_\pm(z)$ be the operators obtained by replacing $u(-\frac{d^2}{dx^2} - z)^{-1} u$ with $V_\pm(z)$ in (). Let $\operatorname{Re} z > 0$ and $|z - 2j - 1| < \frac{3}{2}$. Using we can extract the singular parts from $\big(h_\pm(z) - z\big)^{-1}$. We have, for $j > 0$ (the generalization to $j = 0$ is straightforward),

$$\big(h_\pm(z) - z\big)^{-1} = \frac{1}{z - \epsilon_j^{(\pm)}(\lambda^2)} E_\pm^{(j)}(\lambda^2) + R_\pm^{(j)}(\lambda^2, z) \tag{}$$

where

$$E_{\pm}^{(j)}(\lambda^2) = \frac{1}{2\pi i}\int_\Gamma \big(h_\pm(w) - w\big)^{-1}\,dw, \tag{}$$

$$R_{\pm}^{(j)}(\lambda^2, z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{w-z}\big(h_\pm(w) - w\big)^{-1}\,dw \tag{}$$

and $\Gamma = \{z \in \mathbf{C}\,\big|\,|z-2j-1| = \frac{3}{2}\}$. Note that $R_{\pm}^{(j)}(\lambda^2, z)$ is analytic in z for $|z-2j-1| < \frac{3}{2}$. Since $V_\pm(z)$ is uniformly bounded for $|z-2j-1| \le \frac{3}{2}$ we have, for λ sufficiently small,

$$E_{\pm}^{(j)}(\lambda^2) = \sum_{N=0}^{\infty} \lambda^{2N} E_{\pm;N}^{(j)}, \tag{}$$

where

$$E_{\pm;N}^{(j)} = \frac{1}{2\pi i}\int_\Gamma \big(-\frac{d^2}{dx^2} + x^2 - w\big)^{-1}\Big(V_\pm(w)\big(-\frac{d^2}{dx^2} + x^2 - w\big)^{-1}\Big)^N\,dw,$$

and

$$R_{\pm}^{(j)}(\lambda^2, z) = \sum_{N=0}^{\infty} \lambda^{2N} R_{\pm;N}^{(j)}(z), \tag{}$$

where

$$R_{\pm;N}^{(j)}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{w-z}\big(-\frac{d^2}{dx^2} + x^2 - w\big)^{-1}\Big(V_\pm(w)\big(-\frac{d^2}{dx^2} + x^2 - w\big)^{-1}\Big)^N\,dw.$$

Note that both $E_{\pm;N}^{(j)}$ and $R_{\pm;N}^{(j)}(z)$ are uniformly bounded for $|z-2j-1| \le \frac{3}{2}$.

To estimate $E_{\pm}^{(j)}(\lambda^2)$ and $R_{\pm}^{(j)}(\lambda^2, z)$ one must evaluate () and (). For example, if ϕ_j is the eigenfunction of $-\frac{d^2}{dx^2} + x^2$ corresponding to the eigenvalue $2j+1$ and $P_{j\perp}$ is the orthogonal projection onto the orthogonal complement to ϕ_j in $\mathrm{L}^2(\mathbf{R})$ then

$$E_{\pm;0}^{(j)} = \phi_j \otimes \phi_j.$$

This is enough to show that if $|z-2j-1| \le \frac{3}{2}$ then

$$\|\big(h_\pm(z) - z\big)^{-1} - \frac{1}{z - \epsilon_j^{(\pm)}(\lambda^2)}\phi_j \otimes \phi_j\| \le \text{const}\ . \tag{}$$

To get an estimate uniformly correct to $\mathcal{O}(\lambda^2)$ we need both

$$\begin{aligned} E_{\pm;1}^{(j)} = {}& \langle \phi_j, V_\pm'(2j+1)\phi_j\rangle \phi_j \otimes \phi_j \\ & - P_{j\perp}\big(-\frac{d^2}{dx^2} + x^2 - k^2\big)^{-1} P_{j\perp} V_\pm(2j+1)\phi_j \otimes \phi_j \\ & - \phi_j \otimes \phi_j V_\pm(2j+1) P_{j\perp}\big(-\frac{d^2}{dx^2} + x^2 - k^2\big)^{-1} P_{j\perp} \end{aligned}$$

and, if $|z-2j-1|\le \frac{3}{2}$,

$$R^{(j)}_{\pm;0}(z)=P_{j\perp}\big(-\frac{d^2}{dx^2}+x^2-z\big)^{-1}P_{j\perp}.$$

Then, for such z,

$$\|\big(h_\pm(z)-z\big)^{-1}-\frac{1}{z-\epsilon_j^{(\pm)}(\lambda^2)}\Big(\phi_j\otimes\phi_j+\lambda^2E^{(j)}_{\pm;1}\Big)-P_{j\perp}\big(-\frac{d^2}{dx^2}+x^2-z\big)^{-1}P_{j\perp}\|\le \text{const }\lambda^2.$$

§4: The Spectral Density and the Continuum Eigenfunctions

In this section we find an integral kernel for $\big(H(\lambda)-z\big)^{-1}$ and apply () to obtain an integral kernel for the spectral density $\frac{d\mu}{d\epsilon}$. We will then be able to extract expressions for the continuum eigenfunctions for $H(\lambda)$.

Recall that the integral kernel for $\big(-\frac{d^2}{dx^2}+x^2-z\big)^{-1}$, $g^{(0)}(z;x,y)$, is given by

$$g^{(0)}(z;x,y)=\frac{2^{-\frac{z}{2}-1}}{\pi}\Gamma(\frac{3-z}{4})\Gamma(\frac{1-z}{4})\,U(-\frac{z}{2},-\sqrt{2}\,x_<)U(-\frac{z}{2},\sqrt{2}\,x_>).$$

Here $U(a,x)$ is the parabolic cylinder function, $x_<=\min\{x,y\}$ and $x_>=\max\{x,y\}$. $U(a,x)$ is analytic in a and satisfies

$$U(a,x)=x^{-a-\frac{1}{2}}e^{-\frac{1}{4}x^2}\big(1+\mathcal{O}(\frac{1}{|x|})\big)$$

as $x\to\infty$ and

$$U(a,x)=\frac{\sqrt{\pi}}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}a)}+\frac{\sqrt{\pi}}{2^{\frac{1}{2}a-\frac{1}{4}}\Gamma(\frac{1}{4}+\frac{1}{2}a)}\,x+\mathcal{O}(x^2)$$

for small x. Using the fact that $V_\pm(z)$ is an integral operator with compact support convergent expansions for the integral kernels for $E_\pm^{(j)}(\lambda^2)$ and $R_\pm^{(j)}(\lambda^2,z)$ follow. Then for z satisfying $|z-2j-1|\le\frac{3}{2}$ if $j>0$ and $|z-1|\le\frac{3}{2}$, $\operatorname{Re}z\ge\frac{1}{2}$ if $j=0$

$$g_\pm(z;x,y)=\frac{1}{z-\epsilon_j^{(\pm)}(\lambda^2)}E_\pm^{(j)}(\lambda^2;x,y)+R_\pm^{(j)}(\lambda^2,z;x,y)$$

is the resulting integral kernel for $\big(h_\pm(z)-z\big)^{-1}$. For $0<\operatorname{Re}z<\frac{1}{2}$ we obtain an integral kernel from (). These integral kernels are symmetric in x and y and all approach zero as any one of their arguments, say y, approaches $\pm\infty$. In general, as $y\to\infty$ these integral kernels and their derivatives are all bounded by $f(x)\,e^{-\frac{1}{4}y^2}$. Here f is continuous and positive.

We can now use () to estimate $\frac{d\mu}{d\epsilon}(\lambda,\epsilon)$. Let the subscript $\pm$ denote the operator obtained on replacing $V(z)$ with $V_\pm(z)$. If $\pm\operatorname{Im}z>0$ then

$$\mathcal{G}(\lambda,z;x,y)=\begin{pmatrix}A_\pm(z;x,y) & -\lambda\ B_\pm(z;x,y)\\ -\lambda\ B_\pm(z;y,x) & g_\pm(z;x,y)\end{pmatrix}$$

where

$$\begin{aligned}A_\pm(z;x,y)=&\mp\frac{1}{2i\sqrt{z}}e^{\pm i\sqrt{z}\,|x-y|}\\ &-\lambda^2\frac{1}{4z}\int e^{\pm i\sqrt{z}\,|x-r_1|}u(r_1)g_\pm(z;r_1,r_2)u(r_2)e^{\pm i\sqrt{z}\,|r_2-y|}\,dr_1\,dr_2,\end{aligned}$$

and

$$B_\pm(z;x,y) = \mp\frac{1}{2i\sqrt{z}}\int e^{\pm i\sqrt{z}\,|x-r|}u(r)g_\pm(z;r,y)\,dr.$$

Let $\sigma,\delta\in\{\pm\}$ and define

$$\psi_\delta^{(\sigma)}(z;x) = \int e^{-\sigma\delta i\sqrt{z}\,r}u(r)g_\sigma(z;r,x)\,dr. \qquad ()$$

Note that $\overline{\psi_\delta^{(\sigma)}(z;x)} = \psi_\delta^{(-\sigma)}(\bar{z};x)$. With this notation we find that, as $x\to\delta\infty$,

$$A_\sigma(z;x,y) = \frac{-\sigma}{2i\sqrt{z}}e^{\sigma\delta i\sqrt{z}\,(x-y)} - \lambda^2\frac{1}{4z}e^{\sigma\delta i\sqrt{z}\,x}\int\psi_\delta^{(\sigma)}(z;r)u(r)e^{\sigma i\sqrt{z}\,|r-y|}\,dr$$

and

$$B_\sigma(z;x,y) = \frac{-\sigma}{2i\sqrt{z}}e^{\sigma\delta i\sqrt{z}\,x}\psi_\delta^{(\sigma)}(z;y).$$

Both $g_\sigma(z;x,y)$ and $B_\sigma(z;y,x)$, as well as their derivatives with respect to x, are bounded, for fixed y, by const $\cdot\, e^{-\frac14 x^2}$.

Substituting these last expressions in () we find that

$$\frac{d\mu}{d\epsilon}(\lambda,\epsilon;x,y) = \begin{pmatrix} \frac{d\mu_{1,1}}{d\epsilon} & \frac{d\mu_{1,2}}{d\epsilon} \\ \frac{d\mu_{2,1}}{d\epsilon} & \frac{d\mu_{2,2}}{d\epsilon}\end{pmatrix}$$

where

$$\frac{d\mu_{1,1}}{d\epsilon} = \frac{1}{4\pi\sqrt{\epsilon}}\sum_{\delta=\pm}\Big(e^{-\delta i\sqrt{\epsilon}\,x} + i\frac{\lambda^2}{2\sqrt{\epsilon}}\int e^{i\sqrt{\epsilon}\,|x-r|}u(r)\psi_\delta^{(+)}(\epsilon;r)\,dr\Big)$$
$$\cdot\Big(e^{\delta i\sqrt{\epsilon}\,y} - i\frac{\lambda^2}{2\sqrt{\epsilon}}\int e^{-i\sqrt{\epsilon}\,|y-r|}u(r)\psi_\delta^{(-)}(\epsilon;r)\,dr\Big),$$

$$\frac{d\mu_{1,2}}{d\epsilon} = \frac{-\lambda}{4\pi\sqrt{\epsilon}}\sum_{\delta=\pm}\Big(e^{-\delta i\sqrt{\epsilon}\,x} + i\frac{\lambda^2}{2\sqrt{\epsilon}}\int e^{i\sqrt{\epsilon}\,|x-r|}u(r)\psi_\delta^{(+)}(\epsilon;r)\,dr\Big)\psi_\delta^{(-)}(\epsilon;y),$$

$$\frac{d\mu_{2,1}}{d\epsilon} = \frac{-\lambda}{4\pi\sqrt{\epsilon}}\sum_{\delta=\pm}\psi_\delta^{(+)}(\epsilon;x)\Big(e^{\delta i\sqrt{\epsilon}\,y} - i\frac{\lambda^2}{2\sqrt{\epsilon}}\int e^{-i\sqrt{\epsilon}\,|y-r|}u(r)\psi_\delta^{(-)}(\epsilon;r)\,dr\Big)$$

and

$$\frac{d\mu_{2,2}}{d\epsilon} = \frac{\lambda^2}{4\pi\sqrt{\epsilon}}\sum_{\delta=\pm}\psi_\delta^{(+)}(\epsilon;x)\psi_\delta^{(-)}(\epsilon;y).$$

It follows that if we let

$$\Psi_\delta^{(\pm)}(\lambda,\epsilon;x) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} e^{\mp\delta i\sqrt{\epsilon}\,x} \pm i\frac{\lambda^2}{2\sqrt{\epsilon}}\int e^{\pm i\sqrt{\epsilon}\,|x-r|}u(r)\psi_\delta^{(\pm)}(\epsilon;r)\,dr \\ -\lambda\psi_\delta^{(\pm)}(\epsilon;x)\end{pmatrix} \qquad ()$$

then

$$\frac{d\mu}{d\epsilon}(\lambda,\epsilon;x,y) = \frac{1}{2\sqrt{\epsilon}}\sum_{\delta=\pm}\Psi_\delta^{(+)}(\lambda,\epsilon;x)\otimes_{\mathbf{C}^2}\Psi_\delta^{(-)\,t}(\lambda,\epsilon;y). \qquad ()$$

The tensor product $\otimes_{\mathbf{C}^2}$ in () is the tensor product in $\mathbf{C}^2$.

The functions $\Psi_\delta^{(\pm)}(\lambda,\epsilon;x)$ are the continuum eigenfunctions for $H(\lambda)$. They satisfy $\overline{\Psi_\delta^{(+)}(\lambda,\epsilon;x)} = \Psi_\delta^{(-)}(\lambda,\epsilon;x)$. By Stone's formula they are complete in the sense that

$$\frac{1}{2}\Big(P_{[a,b]}(x,y)+P_{(a,b)}(x,y)\Big) = \int_a^b \frac{1}{2\sqrt{\epsilon}}\sum_{\delta=\pm}\Psi_\delta^{(+)}(\lambda,\epsilon;x)\otimes_{\mathbf{C}^2}\Psi_\delta^{(-)\,t}(\lambda,\epsilon;y)\,d\epsilon$$

where $P_A(x,y)$ is the integral kernel for the spectral projection of $H(\lambda)$ on $A\subset\mathbf{R}$. It follows immediately that $H(\lambda)$ has purely absolutely continuous spectrum in $(0,\infty)$. Note that $P_{(0,\infty)} = P_{ac}$ is the spectral projection on the absolutely continuous spectrum of $H(\lambda)$.

As usual, it is convenient to parameterize the spectral density by the "wave number", $k=\sqrt{\epsilon}$, rather than by the energy, ϵ. Define

$$\Psi^{(\pm)}(\lambda,k;x) = \Psi^{(\pm)}_{\pm\operatorname{sgn}k}(\lambda,k^2;x)$$

and

$$\psi^{(\pm)}(k;x) = \psi^{(\pm)}_{\pm\operatorname{sgn}k}(k^2;x)$$

where $\operatorname{sgn}k = \frac{k}{|k|}$. Explicitly

$$\Psi^{(\pm)}(\lambda,k;x) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} e^{-ikx}\pm i\frac{\lambda^2}{2|k|}\int e^{\pm i|k|\,|x-r|}u(r)\psi^{(\pm)}(k;r)\,dr \\ -\lambda\psi^{(\pm)}(k;x)\end{pmatrix} \qquad ()$$

and

$$\psi^{(\pm)}(k) = \big(h_\pm(k^2)-k^2\big)^{-1}e^{-ik\cdot}u(\cdot). \qquad ()$$

Here $e^{-ik\cdot}u(\cdot)\in \mathrm{L}^2(\mathbf{R})$ is the function $e^{\mp ikx}u(x)$. Note that the functions $\Psi^{(\pm)}(\lambda,k)$ are defined in such a way that they are solutions of the Lippmann-Schwinger equations

$$\begin{pmatrix} 1 & \lambda(-\frac{d^2}{dx^2}-k^2\mp i0^+)^{-1}u \\ \lambda u & -\frac{d^2}{dx^2}+x^2-k^2\end{pmatrix}\Psi^{(\pm)}(\lambda,k) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} e^{-ikx} \\ 0\end{pmatrix}.$$

With these definitions we have

$$\frac{d\mu}{dk}(\lambda,k) = \Psi^{(+)}(\lambda,k)\otimes\Psi^{(+)}(\lambda,k). \qquad ()$$

The notation $\Psi_1 \otimes \Psi_2$ is that of a rank one operator in $\mathrm{L}^2 \oplus \mathrm{L}^2$ and is defined by

$$(\Psi_1 \otimes \Psi_2)\Phi = \Big(\int \overline{\Psi_2^t(x)} \Phi(x)\, dx \Big) \Psi_1.$$

The parameter k ranges over $(-\infty, \infty)$. Note that

$$\overline{\Psi^{(+)}(\lambda, k; x)} = \Psi^{(-)}(\lambda, -k; x) \qquad ()$$

and that

$$\int \overline{\Psi^{(\pm)}(\lambda, q; x)} \Psi^{(\pm)}(\lambda, k; x)\, dx = \delta(q-k). \qquad ()$$

We get approximations for $\Psi^{(\pm)}(\lambda, k)$ by approximating the function $\psi^{(\pm)}(k)$. If we fix j and let k satisfy $\min\{\frac{1}{2}, 2j - \frac{1}{2}\} < k^2 < 2j + \frac{5}{2}$ then, recalling (), we have

$$\psi^{(\pm)}(k) = \Big(\frac{1}{k^2 - \epsilon_j^{(\pm)}(\lambda^2)} E_\pm^{(j)}(\lambda^2) + R_\pm^{(j)}(\lambda^2, k^2) \Big) e^{-ik\cdot} u.$$

Recall that both $\|E_\pm^{(j)}(\lambda^2)\| \le \text{const}$ and $\|R_\pm^{(j)}(\lambda^2, k^2)\| \le \text{const}$ uniformly in k for the range of k considered here. Using () we have

$$\|\psi^{(\pm)}(k) - \frac{\langle \phi_j, e^{-ik\cdot} u \rangle}{k^2 - \epsilon_j^{(\pm)}(\lambda^2)}\, \phi_j \| \le \text{const}\ .$$

Thus, to leading order in λ, we have

$$\Psi^{(\pm)}(\lambda, k; x) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{-ikx} \pm \lambda^2 \frac{\langle \phi_j, e^{-ik\cdot} u \rangle}{k^2 - \epsilon_j^{(\pm)}(\lambda^2)} \int \frac{i}{2|k|}\, e^{\pm i|k|\,|x-r|} u(r) \phi_j(r)\, dr \\ \lambda \frac{\langle \phi_j, e^{-ik\cdot} u \rangle}{k^2 - \epsilon_j^{(\pm)}(\lambda^2)}\, \phi_j \end{pmatrix} + \mathcal{O}(\lambda)$$

pointwise in x uniformly in k.

§5: The Scattering Matrix

In this section we use the results of §3 to obtain the scattering matrix for $H(\lambda)$, acting in $\mathrm{L}^2(\mathbf{R}) \otimes \mathrm{L}^2(\mathbf{R})$, with respect to $-\frac{d^2}{dx^2}$, acting in $\mathrm{L}^2(\mathbf{R})$. We first define the wave operators $\Omega_\pm : \mathrm{L}^2(\mathbf{R}) \to \mathrm{L}^2(\mathbf{R}) \otimes \mathrm{L}^2(\mathbf{R})$. For $\phi \in \mathrm{L}^2(\mathbf{R})$ let $\hat{\phi}$ be the Fourier transform of ϕ,

$$\hat{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int \phi(x) e^{ikx}\, dx.$$

Then $\Omega_\pm$ is given by

$$\big[\Omega_\pm \phi\big](x) = \int \hat{\phi}(k) \Psi^{(\pm)}(\lambda, k; x)\, dk \tag{}$$

and its adjoint, $\Omega_\pm^* : \mathrm{L}^2(\mathbf{R}) \otimes \mathrm{L}^2(\mathbf{R}) \to \mathrm{L}^2(\mathbf{R})$, is given by

$$\begin{aligned} \big[\Omega_\pm^* \Phi\big](x) &= \frac{1}{\sqrt{2\pi}} \int \Big(\int \overline{\Psi^{(\pm)\,t}(\lambda, k; y)}\, \Phi(y)\, dy \Big) e^{-ikx}\, dk \\ &= \frac{1}{\sqrt{2\pi}} \int \Big(\int \Psi^{(\mp)\,t}(\lambda, k; y)\, \Phi(y)\, dy \Big) e^{-ikx}\, dk. \end{aligned}$$

We now show, using standard arguments [RS], that $\Omega_\pm$ are the wave operators

$$\Omega_\pm = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH(\lambda)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{it\frac{d^2}{dx^2}}.$$

Proposition : *For all* $\phi \in \mathrm{L}^2(\mathbf{R})$

$$\lim_{t \to \mp\infty} \Big\| \Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{it\frac{d^2}{dx^2}} - e^{-itH(\lambda)} \Omega_\pm \Big] \phi \Big\| = 0.$$

proof: We have

$$\begin{aligned} \eta^{(\sigma)}(\phi; t, x) &= \Big(\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{it\frac{d^2}{dx^2}} - e^{-itH(\lambda)} \Omega_\sigma \Big] \phi \Big)(x) \\ &= \int e^{-itk^2} \Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2\pi}} e^{-ikx} - \Psi^{(\sigma)}(\lambda, k; x) \Big]\, \hat{\phi}(k)\, dk. \end{aligned} \tag{}$$

It suffices to consider ϕ for which $\hat{\phi} \in C_0^\infty\big(\mathbf{R} \setminus \{0\}\big)$ since such ϕ form a dense subset of $\mathrm{L}^2(\mathbf{R})$ and $\eta^{(\sigma)}(\phi; t)$ is a bounded linear function of ϕ.

Let $M > 0$ be large enough so that $\mathrm{supp}(u) \subset (-M, M)$ and let χ be the characteristic function of the interval $(-M, M)$,

$$\chi(x) = \begin{cases} 1 & \text{if } x \in (-M, M) \\ 0 & \text{if } x \notin (-M, M). \end{cases} \tag{}$$

Changing integration variable in () from k to $\kappa = t^{\frac{1}{2}}k$ we see that $\eta^{(\sigma)}(\phi;t,x) \to 0$ as $t \to \pm\infty$ uniformly in x. It follows that $\|\eta^{(\sigma)}(\phi;t)\chi\| \to 0$ as $t \to \pm\infty$.

If $x \notin (-M, M)$ then

$$\eta^{(\sigma)}(\phi;t,x) = \int e^{-itk^2} \begin{pmatrix} \lambda^2 f^{(\sigma)}(\lambda, \operatorname{sgn} x; k) e^{\sigma i|k|\,|x|} \\ \lambda\psi^{(\sigma)}(k;x) \end{pmatrix} \hat{\phi}(k)\, dk \qquad ()$$

where $\operatorname{sgn} x = \frac{x}{|x|}$ and

$$\begin{aligned} f^{(\sigma)}(\lambda, \pm; k) &= \frac{\sigma i}{2|k|} \int e^{\mp\sigma i|k|r} u(r) \psi^{(\sigma)}(k;r)\, dr \\ &= \frac{\sigma i}{2|k|} \langle e^{\mp\sigma i|k|\cdot} u, \left(h_\sigma(k^2) - k^2\right)^{-1} e^{-ik\cdot} u \rangle. \end{aligned} \qquad ()$$

Recalling that $\psi^{(\sigma)}(k;x)$ is real analytic in k for $k \neq 0$ and that $|\psi^{(\sigma)}(k;x)| \leq \text{const}\ e^{-\frac{1}{4}x^2}$ it follows that

$$\lim_{t\to\pm\infty} \| \int e^{-itk^2} \psi^{(\sigma)}(k)\, \hat{\phi}(k)\, dk \| = 0.$$

Noting that $-tk^2 \pm |k||x| > 0$ when $\mp t > 0$ it follows from

$$\begin{aligned} |\int e^{-itk^2} e^{\pm i|k|\,|x|}\, \hat{\phi}(k)\, dk| &= |\int e^{-itk^2 \pm i|k|\,|x|} \frac{d}{dk}\Big(\frac{i}{2tk \mp \operatorname{sgn}(k)\,|x|} \hat{\phi}(k) \Big)\, dk| \\ &\leq K(\phi) \frac{1}{|t \mp |x||}; \end{aligned}$$

here $K(\phi)$ is a constant, recall that $\hat{\phi} \in C_0^\infty\big(\mathbf{R} \setminus \{0\}\big)$; that

$$\lim_{t\to\mp\infty} \|\eta^{(\pm)}(\phi;t)(1-\chi)\| = 0.$$

This establishes the proposition. ■

Note that $\Omega_\pm^* \Omega_\pm = 1$ and that $\Omega_\pm(-\frac{d^2}{dx^2}) = H(\lambda)\Omega_\pm$. Further, asymptotic completeness, $\operatorname{Ran}\Omega_+ = \operatorname{Ran}\Omega_-$, follows from the relation $\overline{\Psi^{(+)}(\lambda,k;x)} = \Psi^{(-)}(\lambda,k;x)$ so that the scattering matrix $S = \Omega_-^*\Omega_+ : \mathrm{L}^2(\mathbf{R}) \to \mathrm{L}^2(\mathbf{R})$ exists. We have

$$[S\phi](x) = \frac{1}{\sqrt{2\pi}} \int S(q,k) \hat{\phi}(k) e^{iqx}\, dk\, dq$$

where

$$S(q,k) = \int \overline{\Psi^{(-)\,t}(\lambda,q;x)}\, \Psi^{(+)}(\lambda,k;x)\, dx.$$

The relation $\Omega_\pm(-\frac{d^2}{dx^2}) = H(\lambda)\Omega_\pm$ implies that S commutes with $-\frac{d^2}{dx^2}$. It follows that $S(q,k) = S(k,k)\delta(q-k) + S(-k,k)\delta(q+k)$.

The most straightforward way to compute $S(\pm k, k)$ is to note that, as differential equations, $\big(H(\lambda) - k^2\big)\Psi^{(-)}(\lambda, \pm k; x) = \big(H(\lambda) - k^2\big)\Psi^{(+)}(\lambda, k; x) = 0$ so that

$$\Psi^{(+)}(\lambda, k; x) = \alpha(k)\,\Psi^{(-)}(\lambda, k; x) + \beta(k)\,\Psi^{(-)}(\lambda, -k; x). \qquad ()$$

By the orthogonality relation () we then have $S(k,k) = \alpha(k)$ and $S(-k,k) = \beta(k)$. For $x \notin \operatorname{supp} u$ we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}e^{-ikx} + \lambda^2 f^{(+)}(\lambda, \operatorname{sgn} x; k)e^{i|k|\,|x|} =& \alpha(k)\Big(\frac{1}{\sqrt{2\pi}}e^{-ikx} + \lambda^2 f^{(-)}(\lambda, \operatorname{sgn} x; k)e^{-i|k|\,|x|}\Big) \\ &+ \beta(k)\Big(\frac{1}{\sqrt{2\pi}}e^{ikx} + \lambda^2 f^{(-)}(\lambda, \operatorname{sgn} x; -k)e^{-i|k|\,|x|}\Big). \end{aligned}$$

¿From the case $kx < 0$ we find

$$\alpha(k) = 1 + \lambda^2\sqrt{2\pi} f^{(+)}(\lambda, -\operatorname{sgn} k; k),$$

and from $kx > 0$

$$\beta(k) = \lambda^2\sqrt{2\pi} f^{(+)}(\lambda, \operatorname{sgn} k; k).$$

In this one dimensional setting $\alpha(k) - 1$ is called the reflection coefficient and $\beta(k)$ the transmission coefficient.

Define the operator $T(z)$ by

$$T(z) = u\big(h_+(z) - z\big)^{-1}u; \qquad ()$$

$T(z)$ has the integral kernel $T(z; x, y) = u(x)g_+(z; x, y)u(y)$. The operator $-\lambda^2 T(z)$ is the so called T matrix ([RS]) for this model. If

$$T(z; q, k) = \frac{1}{2\pi}\int\int e^{iqx}T(z; x, y)e^{-iky}\,dx\,dy$$

then it follows from () that

$$S(q,k) = \delta(q-k) + \frac{i\lambda^2}{\sqrt{2\pi}}T(k^2; q, k)\delta(q^2 - k^2). \qquad ()$$

Using () in () we see that there is anamolously large scattering when k^2 is close to one of the resonant energies $2j+1$. We find from () that for, say, $|k^2-2j-1|<\frac{1}{2}$

$$T(k^2;q,k)=\frac{1}{k^2-\epsilon_j^{(+)}(\lambda^2)}\langle e^{iq\cdot}u,\phi_j\rangle\langle\phi_j,ue^{ik\cdot}\rangle+\mathcal{O}(1).$$

In particular, recalling the estimate for $\epsilon_j^{(+)}$ from , near a resonance the scattering cross sections $|\alpha(k)|^2$ and $|\beta(k)|^2$ have the expected Breit-Wigner form,

$$|\alpha(k)|^2=\frac{\lambda^4|\langle\phi_j,ue^{ik\cdot}\rangle|^4}{(k^2-2j-1-\lambda^2\delta_j)^2+\lambda^4\gamma_j^2}+\mathcal{O}(\lambda^2)$$

and

$$|\beta(k)|^2=\frac{\lambda^4|\langle\phi_j,ue^{-ik\cdot}\rangle|^4}{(k^2-2j-1-\lambda^2\delta_j)^2+\lambda^4\gamma_j^2}+\mathcal{O}(\lambda^2).$$

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