Hydrodynamic Scaling, Convex Duality,

and

Asymptotic Shapes of Growth Models

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Abstract. We present a technique for simultaneously deriving two related results: Hydrodynamic scaling limits for one-dimensional asymmetric particle systems and asymptotic shapes for growth models. The idea is to specify the particle dynamics in terms of a microscopic Lax-Oleinik formula which leads directly to the macroscopic description in terms of a nonlinear conservation law. The law of large numbers required for this link comes from the growth model that is embedded in the particle system. In the limit, the asymptotic shape of the growth model becomes the convex conjugate of the flux of the conservation law, and the latter is computable from the particle system in equilibrium. The asymptotic shape is then obtained from the duality relation. The method is illustrated with four applications.

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Introduction

We present a procedure that combines two tasks to mutual benefit: The method finds asymptotic shapes of growth models while simultaneously proving a hydrodynamic scaling limit for an accompanying particle system. No general theorem is attempted, but we describe four examples where the technique works. The idea is to embed the growth model in a particle system and to carry out a hydrodynamic scaling limit of the particle system in such a way that the embedded growth model converges to its asymptotic shape. In our examples the particle systems scale to solutions of nonlinear conservation laws. The function describing the asymptotic shape of the growth model becomes the convex conjugate of the flux function of the conservation law, which in turn is computable from the particle system in equilibrium. The key insight that makes these examples work is that the particle dynamics can be described by a microscopic analogue of the Lax-Oleinik formula for Hamilton-Jacobi equations. Additionally, a crucial technical point is that a one-parameter family of invariant measures of the particle systems can be identified.

Here is the Lax-Oleinik formula: Suppose H is convex, and consider the Hamilton-Jacobi equation

$$\begin{aligned} &v_t + H(v_x) = 0 \text{ on } \mathbf{R} \times (0, \infty), \\ &v(x,0) = v_0(x). \end{aligned} \tag{0.1}$$

A weak solution to this equation can be defined as follows: Let H^* be the convex conjugate of H and set

$$v(x,t) = \inf_{q \in \mathbf{R}} \left\{ v_0(q) + t\, H^*\left(\frac{x-q}{t}\right)\right\}, \quad x \in \mathbf{R},\, t > 0. \tag{0.2}$$

Under some regularity assumptions on v_0, it can be proved that $v(x,t)$ is the unique viscosity solution of (0.1) (see [BE, Ev, LSV]).

The ideas of this paper have their origin in techniques developed to solve Ulam's problem. This is the evaluation of the limit $c = \lim_{n\to\infty} n^{-1/2} L_n$, where L_n is the length of the longest increasing subsequence of a random permutation on n letters. The definition of Hammersley's particle process by Aldous and Diaconis may contain the first occurrence of a microscopic Lax-Oleinik formula (equation (11) in [AD]). Aldous and Diaconis used hydrodynamic scaling of Hammersley's process to give a new solution to Ulam's problem. Motivated by their work, this author utilized the link between the microscopic and macroscopic Lax-Oleinik formulas to connect the solution of Ulam's problem with a hydrodynamic limit of the so-called stick model [Se]. The present paper arose from a realization that behind the arguments of [Se] lies a method of some general applicability.

Here is a quick overview of the paper. We begin with a familiar example: The growth model of Example 1 is a random Young diagram in continuous time, and the associated particle model is the totally asymmetric zero range process. The shape result was proved by Rost [Ro] in 1981 in one of the seminal papers of hydrodynamic scaling. The hydrodynamic limit of the zero range process has been proved by many authors for successively more general initial profiles. We give a fairly thorough description of Example 1, and omit most of the details for the subsequent examples.

Example 2 is a variant of the first one. The particle system is the totally asymmetric simple exclusion process, whose hydrodynamic limit is also well-known. The growth model associated with it is new, composed for this paper.

The growth model of Example 3 comes from work on domino tilings of the Aztec diamond [CEP, JPS]. We present the key step of the proof of the so-called arctic ellipse theorem: Finding the asymptotic shape of a random Young diagram growing in discrete time. As in Example 1, the associated particle system is a zero range process, but now evolving in discrete time.

Example 4 is composed for illustration of the technique. It is a partly discretized version of Ulam's problem. It differs from the other three in that the flux function is convex, while the three first examples lead to concave flux functions, and this changes some formulas in the development. The particle system is a coupled random walk. In a curious coincidence, the asymptotic shape of the growth model is a hyperbola for Example 4, while it was a parabola for Examples 1 and 2, and an ellipse for Example 3.

The strength of the approach is that it gives both the asymptotic shape for the growth model and the hydrodynamic limit for the particle system by playing them against each other. Once the existence of the asymptotic shape is established, the need for delicate moment or entropy estimates is virtually eliminated. Thus the proofs we give for known results are much easier than their original proofs. Another consequence is that the hydrodynamic limit is obtained for general initial distributions of the particle process, including deterministic initial configurations, and for a broad class of initial macroscopic profiles, such as Radon measures without densities.

1. First Example

1.1. Results. Consider this growth process on the plane: Divide the plane into left-open right-closed unit squares with integer corners. Initially the positive quadrant is white, and the rest of the plane is black. A white square turns black with rate one, independently of everything else, if the squares below it and to its left are black. A black square stays forever black. Let B_t denote the union of the black squares in the positive quadrant at time $t \geq 0$. (B_t) is an example of a randomly growing Young diagram. Label the unit squares of $(0,\infty)^2$ by their upper right corners, so that $\lambda = (\lambda_1, \lambda_2) \in \mathbf{N} \times \mathbf{N}$ represents the square $(\lambda_1 - 1, \lambda_1] \times (\lambda_2 - 1, \lambda_2]$. Let $T_{0,\lambda}$ denote the time when the square λ joins the shape B_t. We shall rigorously construct the process $\{T_{0,\lambda}\}$, and then use the subadditive ergodic theorem to deduce the existence of a limiting shape for (B_t).

Give each square β an Exp(1) random variable Y_β, independently of the other squares. By an *up-right path* from square κ to square λ we mean a sequence

$$\gamma = \{\kappa + (1,1) = \beta^{(1)}, \beta^{(2)}, \dots, \beta^{(s)} = \lambda\} \tag{1.1}$$

of squares such that if $\beta^{(\ell)} = (i,j)$ then $\beta^{(\ell+1)} = (i+1,j)$ or $(i,j+1)$. Let $\Gamma_{\kappa,\lambda}$

denote the set of such paths from κ to λ. Set

$$T_{\kappa,\lambda} = \max_{\gamma \in \Gamma_{\kappa,\lambda}} \sum_{\beta \in \gamma} Y_\beta. \tag{1.2}$$

Even though $(0,0)$ is not among the squares considered, it makes perfect sense to take $\kappa = 0 = (0,0)$ in the above definitions. $T_{0,\lambda}$ is the quantity computed in (1.2) by maximizing over paths that start with lower left corner at $(0,0)$. For any $x = (x_1, x_2) \in (0,\infty)^2$ set $T_{0,x} = T_{0,\lambda}$ where λ is the square that contains the point x. The random Young diagram described in the opening paragraph can now be defined by

$$B_t = \{x \in (0,\infty)^2 : T_{0,x} \le t\}.$$

A nonnegative superadditive process $X_{m,n}$, $0 \le m < n$, comes from setting $X_{m,n} = T_{m\lambda,n\lambda}$ for a fixed $\lambda \in \mathbf{N} \times \mathbf{N}$, with $n\lambda = (n\lambda_1, n\lambda_2)$. Once we have the appropriate moment bounds, the subadditive ergodic theorem and straightforward approximations imply that the limit

$$\tau(x_1, x_2) = \lim_{n\to\infty} \frac{1}{n} T_{0,nx} \tag{1.3}$$

exists in probability, for each $x = (x_1, x_2) \in (0,\infty)^2$. The function $\tau(x_1, x_2)$ is finite, concave, continuous, homogeneous ($\tau(cx_1, cx_2) = c\tau(x_1, x_2)$ for $c > 0$), and strictly increasing in each coordinate. It also extends continuously to $[0,\infty)^2$.

By homogeneity, to derive $\tau(x_1, x_2)$ it suffices to derive the function

$$h(x_2) = \inf\{x_1 \ge 0 : \tau(x_1, x_2) \ge 1\}, \quad x_2 \ge 0. \tag{1.4}$$

This function determines a subset B of $(0,\infty)^2$ by

$$B = \{(x_1, x_2) : x_2 > 0, 0 < x_1 < h(x_2)\}.$$

B is the limiting shape of the random shape B_t in the sense that, for any $\varepsilon > 0$,

$$\lim_{t\to\infty} P\{(1-\varepsilon)B \subset t^{-1}B_t \subset (1+\varepsilon)B\} = 1.$$

The result about the limiting functions is:

Theorem 1A. *We have* $\tau(x_1, x_2) = (\sqrt{x_1} + \sqrt{x_2})^2$ *or, equivalently,* $h(z) = (1 - \sqrt{z})^2$ *for* $0 \le z < 1$, $h(z) = 0$ *for* $z \ge 1$.

To compute $h(z)$ and thereby $\tau(x_1, x_2)$ we embed the growing shape B_t in a particle system. Let $\eta(t) = (\eta_i(t))_{i\in\mathbf{Z}}$, $t \ge 0$, $\eta_i(t) \in \{0, 1, 2, \dots\}$, be a totally asymmetric zero range process on the one-dimensional integer lattice $\mathbf{Z}$, with generator

$$Lf(\eta) = \sum_{i\in\mathbf{Z}} \mathrm{I}_{\eta_i \ge 1}[f(\eta^{i,i+1}) - f(\eta)], \tag{1.5}$$

where

$$\eta_j^{i,i+1} = \begin{cases} \eta_i - 1, & j = i \\ \eta_{i+1} + 1, & j = i+1 \\ \eta_j, & j \neq i, i+1. \end{cases} \tag{1.6}$$

Let us imagine that the sites $i \in \mathbf{Z}$ represent a doubly infinite sequence of servers that we label as z_i. Each server serves customers in the order in which they arrive, one customer at a time with exponential service rate equal to one. Customers move from left to right, joining the queue at server z_{i+1} once service with server z_i is completed. Then η_i is the number of customers at server z_i, both those waiting in queue and the one in service (if any). We refer to $\eta(t)$ as the customer process. It is well-known that geometrically distributed i.i.d. queues are invariant for this process.

An equivalent way to represent this situation is to keep the customers fixed in space and let the servers jump from one customer to the next upon completion of service. Let $z = (z_k)_{k\in\mathbf{Z}}$ denote a configuration of server particles, an element of the space

$$\mathcal{Z} = \{z \in \mathbf{Z}^{\mathbf{Z}} : z_{k-1} \leq z_k \text{ for all } k \in \mathbf{Z}\ \}. \tag{1.7}$$

The sites of the underlying space $\mathbf{Z}$ now represent the customers. We let the servers jump from right to left, so the queue at server z_k is

$$\eta_k = z_k - z_{k-1}. \tag{1.8}$$

The server process $z(t) = (z_k(t))_{k\in\mathbf{Z}}$ is constructed, simultaneously for all initial configurations (z_i), by the usual graphical representation: Give each site $\ell \in \mathbf{Z}$ an independent rate one Poisson process of possible jump times, and declare that server z_k jumps to $\ell - 1$ at time t if $z_k(t-) = \ell$, t is a possible jump time at site ℓ, and $z_{k-1}(t-) \leq \ell - 1$. (See [Gr] for details of this construction.) Through (1.8) we also obtain a construction of the customer process for any initial queues.

Fix a site ℓ, and consider starting the server process from the configuration $x_i = \ell$ for $i \geq 0$, $x_i = -\infty$ for $i < 0$. (Think of an infinitely long queue of customers at server x_0, while the queues for x_i, $i > 0$, start off empty. The servers x_i for $i < 0$ do not enter into the present discussion, so we place them at $-\infty$ where they never move and do not influence the evolution of $(x_i)_{i\geq 0}$. The notation $x(\cdot)$ is reserved for the server process started from this particular type of initial configuration.) Let $\xi^\ell(i,t)$ be the number of customers served by server x_i by time t so that

$$x_i(t) = \ell - \xi^\ell(i,t).$$

The random variables $\xi^\ell(i,t)$ provide yet another equivalent way of defining the random shape B_t, and we can recover the shape function $h(z)$ from

$$h(z) = \lim_{t\to\infty} \frac{1}{t}\xi^\ell([tz],t), \quad z \geq 0. \tag{1.9}$$

This limit holds in probability, even for varying $\ell = \ell(t)$ because the distribution of $\xi^\ell(i,t)$ is independent of ℓ.

The key observation is that the server process $z(\cdot)$ with an arbitrary initial configuration (z_i) can be realized, sample path by sample path, by the microscopic Lax-Oleinik formula

$$z_k(t) = \sup_{i \le k}\{z_i - \xi^{z_i}(k-i,t)\}. \tag{1.10}$$

In other words: Using one realization of the Poisson jump times and a given initial configuration (z_i), the quantities $\xi^{z_i}(k-i,t)$ for $i \le k$, $t \ge 0$, are constructed as above. Then $z_k(t)$, $k \in \mathbf{Z}$, $t \ge 0$, computed from the graphical representation coincides with that given by equation (1.10).

We have linked the random shape B_t to the evolution of the server process $z(t)$ through the random variables $\xi^\ell(i,t)$. To make use of this connection we study a scaling limit of the server process that simultaneously produces the limit in (1.9). Let $N = 1,2,3,\dots$ denote the scaling parameter. Assume given a right-continuous nondecreasing function $U_0(x)$ on $\mathbf{R}$, and define $U(x,t)$ by

$$U(x,t) = \sup_{q \le x}\left\{U_0(q) - t\,h\left(\frac{x-q}{t}\right)\right\}, \quad x \in \mathbf{R}, t > 0. \tag{1.11}$$

$U(\cdot,t)$ is a well-defined, right-continuous nondecreasing function for each $t > 0$. Pick a sequence of (possibly random) initial server configurations $(z_i^N)_{i\in\mathbf{Z}}$, $N = 1,2,3,\dots$, so that

$$\lim_{N\to\infty} \frac{1}{N} z^N_{[Nq]} = U_0(q) \tag{1.12}$$

holds in probability for all $q \in \mathbf{R}$. Then, as comparison of (1.9)–(1.12) suggests,

$$\lim_{N\to\infty} \frac{1}{N} z^N_{[Nx]}(Nt) = U(x,t) \tag{1.13}$$

holds in probability for $t > 0$ and $x \in \mathbf{R}$.

It is from the connection (1.10)–(1.13) that all our results flow. Firstly, Theorem 1A follows by studying a particular explicitly computable solution $U(x,t)$. Secondly, we can derive hydrodynamic scaling limits for the particle models. To fully explain this, some preliminaries need to be developed.

Consider a sequence $\eta^N(\cdot)$ of customer processes indexed by N, with probability measures P^N on the sample space. The dynamics is the same for each N. The initial distribution varies with N so that the following condition is satisfied, for some Radon measure m_0 on $\mathbf{R}$:

For any $-\infty < a < b < \infty$ and $\varepsilon > 0$,

$$\lim_{N\to\infty} P^N\left\{\left|\frac{1}{N}\sum_{i=[Na]+1}^{[Nb]} \eta_i^N(0) - m_0(a,b]\right| \ge \varepsilon\right\} = 0. \tag{1.14}$$

This says that m_0 is the macroscopic initial profile for the customer queues. The customer configuration is tracked through its empirical measure

$$\alpha_t^N = N^{-1}\sum_{i\in\mathbf{Z}} \eta_i^N(t)\delta_{i/N}, \tag{1.15}$$

a random Radon measure on $\mathbf{R}$. Pick $U_0(x)$ so that $U_0(b) - U_0(a) = m_0(a,b]$ for all $a < b$, and define $(z_i^N)_{i\in\mathbf{Z}}$ so that (1.8) and (1.12) hold. Then (1.13) implies that α_{Nt}^N converges to the Lebesgue-Stieltjes measure m_t of $U(x,t)$, defined by $m_t(a,b] = U(b,t) - U(a,t)$.

This result tells us that, in the scaling limit, the customer profile obeys a nonlinear conservation law: Assume that

$$m_0(dx) = u_0(x)dx \text{ for a function } u_0 \in L^\infty(\mathbf{R}). \tag{1.16}$$

Then $U(x,t)$ has a space derivative

$$u(x,t) = U_x(x,t) \tag{1.17}$$

which is a weak solution of the equation

$$\begin{aligned} &u_t + f(u)_x = 0 \text{ on } \mathbf{R}\times(0,\infty), \\ &\lim_{t\searrow 0} u(x,t)dx = m_0(dx). \end{aligned} \tag{1.18}$$

The weak interpretation of (1.18) means that, for all $\phi \in C_0^1(\mathbf{R})$ (compactly supported, continuously differentiable) and $t > 0$,

$$\int_{\mathbf{R}} \phi(x)\, u(x,t)\, dx - \int_{\mathbf{R}} \phi(x)\, m_0(dx) = \int_0^t \int_{\mathbf{R}} \phi'(x) f(u(x,s))\, dx\, ds. \tag{1.19}$$

Equation (1.18) has multiple solutions, among which $u(x,t)$ is the unique entropy solution. (See [La1, La2, Le, Sm] for the precise definition of an entropy solution.)

The shape function $h(z)$ and the flux function $f(u)$ are related by convex duality: Let $g(z) = -h(z)$. This $g(z)$ is a nondecreasing, continuous concave function on $[0,\infty)$, and has a well-defined monotone conjugate

$$g^-(u) = \inf_{z\geq 0}\{uz - g(z)\}, \qquad u \geq 0, \tag{1.20}$$

that satisfies $g(z) = g^{--}(z)$. (See p. 111 in [Rf].) It turns out that $f(u) = g^-(u)$, and consequently $g(z) = f^-(z)$. This enables us to identify $h(z)$ through $f(u)$.

Let us state the hydrodynamic limit for the customer process as a theorem:

Theorem 1B. *Let m_0 be a Radon measure on $\mathbf{R}$ satisfying* (1.16) *and assume* (1.14). *Then for all $t > 0$, α_{Nt}^N converges in probability to a deterministic measure $u(x,t)dx$ as $N \to \infty$, in the sense that*

$$\lim_{N\to\infty} P^N\left\{\left|\alpha_{Nt}^N(\phi) - \int_{\mathbf{R}} \phi(x)u(x,t)dx\right| \geq \varepsilon\right\} = 0$$

for $\varepsilon > 0$ *and* $\phi \in C_0^1(\mathbf{R})$. *The function* $u(x,t)$ *is the unique entropy solution of the conservation law* (1.18) *with flux function* $f(u) = u/(1+u)$.

There is no probabilistic reason for assumption (1.16): Theorem 1B follows from (1.8) and (1.13) which are valid for any choice of U_0. To remove assumption (1.16) we need the right existence and uniqueness theory for solutions of (1.18) started from an arbitrary Radon measure. To keep the focus of this paper compact we shall not embark on such a development. Let us only remark that the required theory must admit measures as solutions. For example, if we start equation (1.18) with flux $f(u) = u/(1+u)$ from a mass point of size $r > 0$ at the origin, $m_0 = r \cdot \delta_0$, the solution is

$$m_t(dx) = u(x,t)dx + (r-t)_+ \cdot \delta_0(dx)$$

for a function $u(x,t)$, so the singular part will be smoothed away only after time r. The reason is that the flux at infinity is finite: $f(\infty) = 1$. In Sect. 4 the flux is convex and satisfies $f(\infty) = \infty$. Then a singular initial profile is immediately regularized, and the only condition needed on m_0 is a bound on growth at $-\infty$ to prevent blow-up in finite time.

Instead of the conservation law, one can work with the corresponding Hamilton-Jacobi equation

$$\begin{aligned} &U_t + f(U_x) = 0 \text{ on } \mathbf{R} \times (0,\infty) \\ &U(x,0) = U_0(x), \end{aligned} \tag{1.21}$$

and study discontinuous viscosity solutions. Assumption (1.16) implies that $U_0(x)$ is uniformly Lipschitz, and under this assumption formula (1.11) gives the unique viscosity solution to (1.21) as proved in [BE].

In addition to Theorem 1B we can write down a similar theorem for the empirical measure of the server particles. Also, it is possible to deduce from (1.13) an ordinary differential equation that governs the path of an individual server in the scaling limit. Such tagged particle results have been proved by Rezakhanlou [Re2] for the zero range and exclusion processes. It may be worthwile to note that the logic of our approach is opposite to that of Rezakhanlou's: In [Re2] tagged particle results are consequences of hydrodynamic scaling limits, while for us the tagged particle result comes first.

Some further bibliographical remarks before starting on the proofs:

Young diagrams are a counting tool in the combinatorics of partitions. The reader may consult Ch. 15 in [LW] for background material. Theorem 1A was first proved by Rost [Ro] in 1981. He did this by proving a hydrodynamic scaling limit for the totally asymmetric simple exclusion process, with all sites to the left of the origin initially occupied and to the right initially empty. This corresponds to the $x(\cdot)$ server process we introduced above. A generalization appears at the end of Sect. VIII.5 of [Lg2].

The zero range process was intoduced by Spitzer [Sp] in 1970, and its properties have since been studied in a number of papers, see for example those by Holley [Ho], Liggett [Lg1] and Andjel [An]. Andjel and Kipnis [AK] and Benassi and Fouque [BF2] were the first to study hydrodynamic scaling of the asymmetric zero range process. The most recent version of Theorem 1B is due to Rezakhanlou

[Re1]. Compared to his result, ours admits more general initial profiles and initial distributions for the process, while his is not restricted to the special case (1.5) of the zero range process and is valid in higher space dimensions. Ferrari's [Fe2] review paper is a good source for further references on hydrodynamics of asymmetric processes. General references on hydrodynamic scaling are the lectures of DeMasi and Presutti [DP] and Spohn's monograph [Sp].

Use of a correspondence such as that between our customer and server processes goes back at least to Harris's [Hr] work on 'motions of point processes'. Recently it has been used by several authors to link the exclusion and zero range processes, see for example the papers by Kipnis [Ki] and Ferrari [Fe1]. Obviously we also could have defined the server process $z(\cdot)$ as an exclusion process, after replacing (1.8) with $\eta_k = z_k - z_{k-1} - 1$ and performing corresponding changes in the proofs.

1.2. Proofs. The first thing to note is the precise connection between the random variables $\xi^\ell(i,t)$ and the random shape B_t discussed in the beginning of the section. The location ℓ will not be relevant until later, so take $\ell = 0$ and write $\xi(i,t) = \xi^0(i,t)$. For $(n,k) \in \mathbf{N} \times \mathbf{N}$, let

$$\widetilde{T}_{n,k} = \inf\{t > 0 : \xi(k-1,t) \geq n\}$$

and

$$\widetilde{B}_t = \bigcup\{(n-1,n] \times (k-1,k] \subset (0,\infty)^2 : \widetilde{T}_{n,k} \leq t\}.$$

Then the processes $(B_t : t \geq 0)$ and $(\widetilde{B}_t : t \geq 0)$ are identical in distribution, because both are jump processes on the countable state space of Young diagrams in $(0,\infty)^2$, both start from the empty set, and both add a new square with the same rule. ¿From this follows the distributional equality

$$\{T_{(0,0),(n,k)}\}_{(n,k)\in\mathbf{N}\times\mathbf{N}} \overset{\mathcal{D}}{=} \{\widetilde{T}_{n,k}\}_{(n,k)\in\mathbf{N}\times\mathbf{N}}, \tag{1.22}$$

since these processes are functions of (B_t) and $(\widetilde{B}_t)$, respectively.

Next we derive moment bounds on $\widetilde{T}_{n,k}$. Return to the interpretation of $\xi(i,t)$ as the number of customers served by server x_i by time t, when the system starts with infinitely many customers in queue at server x_0 and with empty queues for servers x_i, $i > 0$. $\widetilde{T}_{n,k}$ is the time when the nth customer leaves server x_{k-1}. To bound $\widetilde{T}_{n,k}$ we compare this setting with a system in equilibrium (see Sect. 2.1–2.2 in [Ke] for the missing details). Fix a parameter $\nu \in (0,1)$. Assume the servers x_i, $i \geq 0$, start with queues in equilibrium so that their queues are i.i.d. with common distribution Geom$(1-\nu)$. Let the new customers arrive as a Poisson stream with intensity ν. Then equilibrium is preserved, and the waiting times of a customer passing through the queues are i.i.d. Exp$(1-\nu)$ variables. Compared with the original setting, the nth customer is slowed down, so $\widetilde{T}_{n,k}$ is stochastically dominated by $S_n^\nu + S_k^{1-\nu}$. Here S_n^ν denotes a sum of n i.i.d. Exp(ν) variables, similarly for $S_k^{1-\nu}$. The first sum S_n^ν is the time when the nth customer arrives in the system, and the second sum $S_k^{1-\nu}$ is the time it takes her to get through k servers.

Fix a square $\lambda = (\lambda_1, \lambda_2)$. Taking expectations,

$$ET_{0,n\lambda} = E\widetilde{T}_{n\lambda_1,n\lambda_2} \le ES^{\nu}_{n\lambda_1} + ES^{1-\nu}_{n\lambda_2} = \frac{n\lambda_1}{\nu} + \frac{n\lambda_2}{1-\nu}.$$

Picking ν to minimize the right-hand side gives

$$ET_{0,n\lambda} \le n\big(\sqrt{\lambda_1} + \sqrt{\lambda_2}\,\big)^2. \tag{1.23}$$

Inequality (1.23) gives the moment bound required by the subadditive ergodic theorem (see for example Thm. 6.1 in [Du]) so that the limit

$$\tau(\lambda_1, \lambda_2) = \lim_{n\to\infty} \frac{1}{n} T_{0,n\lambda} \tag{1.24}$$

exists and is finite for all $\lambda = (\lambda_1, \lambda_2) \in \mathbf{N} \times \mathbf{N}$. Extension of the limit from $(\lambda_1, \lambda_2) \in \mathbf{N} \times \mathbf{N}$ to all $(x_1, x_2) \in (0, \infty)^2$ and the properties of the function $\tau(x_1, x_2)$ stated immediately after formula (1.3) are routine and left to the reader. Note that continuity follows automatically from finiteness and concavity (Thm. 10.1 in [Rf]), concavity in turn from superadditivity and homogeneity. Inequality (1.23) together with an obvious lower bound give

$$x_1 + x_2 \le \tau(x_1, x_2) \le (\sqrt{x_1} + \sqrt{x_2})^2 \tag{1.25}$$

for $(x_1, x_2) \in (0, \infty)^2$. From this it is clear that $\tau(x_1, x_2)$ extends continuously to $[0, \infty)^2$ and

$$\tau(z, 0) = \tau(0, z) = z, \quad z \ge 0. \tag{1.26}$$

It is interesting to note that our upper bound in (1.25) already exactly matches $\tau(x_1, x_2)$, as seen from the statement of Thm. 1A.

¿From the properties of $\tau(x_1, x_2)$ it follows that $h(z)$ of (1.4) is a decreasing continuous convex function for $z \ge 0$. The limit (1.9) is a consequence of (1.22). Continuity of $h(z)$ at $z = 0$ would have been a problem without the continuous extension of $\tau(x_1, x_2)$ to the boundary of $(0, \infty)^2$. Without it, the limit in (1.9) would be valid only for $z > 0$. This continuity of $h(z)$ is also required for the final arguments of convex duality that identify it in terms of $f(u)$.

We now turn to study the evolution $z(\cdot)$ of the server particles. Assume given an initial configuration $(z_i)_{i\in\mathbf{Z}}$, an element of the space $\mathcal{Z}$. For each $i \in \mathbf{Z}$, let $(x^i_j(t))_{j\ge 0}$ be a configuration of server particles such that $x^i_j(0) = z_i$ for all $j \ge 0$, and the evolution $x^i_j(\cdot)$ is defined by the same graphical construction as $z(\cdot)$. The point is that the equality

$$x^i_j(t) = z_i - \xi^{z_i}(j, t) \tag{1.27}$$

holds by definition, so to prove (1.10) one checks that

$$z_k(t) = \sup_{i\le k} x^i_{k-i}(t) \tag{1.28}$$

holds for all $k \in \mathbf{Z}$ and $t \geq 0$. In one space dimension this is easy: At time 0, z_k is contained in a finite block of sites $\{\ell_0, \ldots, \ell_1\}$ such that, up to time t_0, there are no possible jump times at sites ℓ_0 and $\ell_1 + 1$. Thus the position $z_k(t)$, $0 \leq t \leq t_0$, can be computed from the (finitely many) Poisson jump times in $\{\ell_0, \ldots, \ell_1\} \times [0, t_0]$. Equation (1.28) holds at time 0 (take $i = k$), and can be proved for $0 < t \leq t_0$ by induction on the successive jumps.

Assume given a nondecreasing right-continuous function $U_0(q)$ on $\mathbf{R}$, and define $U(x,t)$ by (1.11). From (1.9) it follows that $h(z) = 0$ for $z \geq 1$, so by the continuity of $h(z)$, the supremum in (1.11) is in fact realized at some $q \in [x-t, x]$. Thus $U(x,t)$ is a well-defined finite number, and monotonicity and right-continuity of $U(\cdot, t)$ are easy to prove.

Fix $(x,t) \in \mathbf{R} \times (0, \infty)$. Our next goal is the scaling limit (1.13). Thus assume that the initial server locations $(z_i^N)_{i \in \mathbf{Z}}$, $N = 1, 2, 3, \ldots$, satisfy (1.12). Begin by rewriting (1.11) as

$$\frac{1}{N} z_{[Nx]}^N(Nt) = \sup_{i \leq [Nx]} \left\{ \frac{1}{N} z_i^N - \frac{1}{N} \xi^{z_i^N}\big([Nx] - i, Nt\big) \right\}. \tag{1.29}$$

The limit (1.9) implies that

$$\lim_{N \to \infty} \frac{1}{N} \xi^{z_i^N}\big([Nx] - [Nq], Nt\big) = t\, h\!\left(\frac{x-q}{t}\right) \tag{1.30}$$

in probability, for all $q \leq x$.

To prove (1.13), we first check the easy half, namely that

$$\lim_{N \to \infty} P^N\big\{N^{-1} z_{[Nx]}^N(Nt) \geq U(x,t) - \varepsilon\big\} = 1. \tag{1.31}$$

This comes immediately from (1.12), (1.29) and (1.30) by taking $i = [Nq]$ for a number q that realizes the supremum in (1.11).

For the other half we need to restrict the range of i in (1.29). For $r < x$, let

$$\zeta^{N,r} = \max_{[Nr] < i \leq [Nx]} \big\{ z_i^N - \xi^{z_i^N}\big([Nx] - i, Nt\big)\big\}.$$

Lemma 1.1. *If $r < x - t$, then*

$$\lim_{N \to \infty} P^N\big\{z_{[Nx]}^N(Nt) \neq \zeta^{N,r}\big\} = 0.$$

Proof. Forgetting about the scaling momentarily, suppose $\xi^{z_\ell}(k - \ell, t) = 0$ for some $\ell < k$. Then for $j < \ell$, by the monotonicity of the initial servers (z_i),

$$z_j - \xi^{z_j}(k - j, t) \leq z_j \leq z_\ell = z_\ell - \xi^{z_\ell}(k - \ell, t),$$

so

$$z_k(t) = \max_{\ell \leq i \leq k} \{z_i - \xi^{z_i}(k - i, t)\}.$$

Thus, taking $p \in (r, x-t)$,

$$P^N\{z^N_{[Nx]}(Nt) \neq \zeta^{N,r}\} \leq P^N\{\xi^{z^N_{[Np]}}\big([Nx]-[Np], Nt\big) > 0\}$$
$$\to 0 \qquad \text{as } N \to \infty,$$

because $\xi^{z^N_{[Np]}}([Nx]-[Np], Nt) > 0$ only if the first customer has completed service with at least $[Nx]-[Np]$ servers by time Nt, yet each service takes an independent Exp(1) distributed random amount of time and $x - p > t$. □

Pick and fix $r < x - t$. To complete the proof of (1.13), it suffices to show that

$$\lim_{N\to\infty} P^N\{N^{-1}\zeta^{N,r} \geq U(x,t) + \varepsilon\} = 0. \tag{1.32}$$

Pick a partition

$$r = r_0 < r_1 < \cdots < r_m = x$$

so that

$$t\,h\left(\frac{x - r_{\ell+1}}{t}\right) \leq t\,h\left(\frac{x-r_\ell}{t}\right) + \frac{\varepsilon}{4} \tag{1.33}$$

for $\ell = 0, \dots, m-1$. By the monotonicity built into the server dynamics,

$$x^i_{[Nx]-i}(Nt) \leq x^{[Nr_{\ell+1}]}_{[Nx]-[Nr_\ell]}(Nt)$$

whenever $[Nr_\ell] < i \leq [Nr_{\ell+1}]$, so remembering (1.27),

$$\begin{aligned}\zeta^{N,r} &= \max_{0\leq \ell < m} \max_{[Nr_\ell] < i \leq [Nr_{\ell+1}]} x^i_{[Nx]-i}(Nt) \\ &\leq \max_{0 \leq \ell < m}\{z^N_{[Nr_{\ell+1}]} - \xi^{z^N_{[Nr_{\ell+1}]}}\big([Nx]-[Nr_\ell], Nt\big)\}.\end{aligned}$$

Thus the probability in (1.32) is at most

$$\begin{aligned}\sum_{\ell=0}^{m-1} P^N\bigg\{&N^{-1} z^N_{[Nr_{\ell+1}]} - N^{-1}\xi^{z^N_{[Nr_{\ell+1}]}}\big([Nx]-[Nr_\ell], Nt\big) \\ &\geq U_0(r_{\ell+1}) - t\,h\left(\frac{x-r_{\ell+1}}{t}\right) + \varepsilon\bigg\},\end{aligned}$$

which vanishes as $N \to \infty$, by (1.12), (1.30), and (1.33). This completes the proof of the scaling limit (1.13) for the server dynamics.

Proof of Theorem 1A. Let $g(z) = -h(z)$ for $z \geq 0$, and $f(u) = g^-(u)$ as defined by (1.20). Fix $u \geq 0$ and set $U_0(q) = uq$. Then (1.11) gives, for $x = 0$,

$$\begin{aligned}U(0,t) &= \inf_{q\leq 0}\{uq - t\,h(-q/t)\} \\ &= -t \cdot \inf_{z\geq 0}\{uz - g(z)\} \\ &= -t\,f(u).\end{aligned} \tag{1.34}$$

We shall compute $U(0,t)$ through a microscopic evolution that approximates it. For the initial configuration, set $z_0 = 0$ and let $\{z_i - z_{i-1} : i \in \mathbf{Z}\}$ be i.i.d. random variables with common distribution $\mathrm{Geom}(1/(1+u))$. Then

$$\lim_{N\to\infty} N^{-1} z_{[Nq]} = U_0(q)$$

for all $q \in \mathbf{R}$, by the law of large numbers. (Since the initial configuration does not vary with N, the superscript in z_i^N is not needed.) The queues $\eta_i(t) = z_i(t) - z_{i-1}(t)$, $i \in \mathbf{Z}$, remain in equilibrium for all time. So the server $z_0(t)$ always sees an equilibrium queue $\eta_0(t) \sim \mathrm{Geom}(1/(1+u))$ to its left and jumps with rate one whenever this queue is nonempty. Thus

$$E[N^{-1} z_0(Nt)] = -N^{-1} \int_0^{Nt} P[\eta_0(s) > 0]\, ds = -t \cdot \frac{u}{1+u}, \tag{1.35}$$

while

$$N^{-1} z_0(Nt) \to U(0,t) \tag{1.36}$$

in probability by (1.13). Uniform integrability of $\{N^{-1} z_0(Nt)\}$ comes from noting that $|z_0(Nt)|$ is stochastically dominated by a Poisson(Nt) variable. Thus we have convergence of expectations in (1.36), and from (1.34) and (1.35) we get

$$f(u) = \frac{u}{1+u}. \tag{1.37}$$

Working backwards, $g(z)$ comes from the duality $g(z) = f^-(z)$, the shape function $h(z)$ from $h(z) = -g(z)$, and $\tau(x_1, x_2)$ from $\tau(h(x_2), x_2) = 1$ $(0 \le x_2 \le 1)$ and homogeneity. $\square$

The proof of Theorem 1B is contained in (1.37) and in the remarks preceding the statement of the theorem.

2. Second Example

2.1. Results. The second example is a variant of the first. While the associated customer process of Example 1 was the zero range process, this time it is the other most thoroughly studied particle model for nonlinear conservation laws, namely the totally asymmetric simple exclusion process. The queues have maximum length one, or more accurately, no queues are allowed to build up and a customer may jump from server z_i to server z_{i+1} only if server z_{i+1} is empty.

The general setting is the same as in Example 1, except that the admissible paths γ may now take steps up, to the right, and rightward diagonally down: In (1.1), if $\beta^{(\ell)} = (i,j)$ then $\beta^{(\ell+1)}$ is any one of $(i+1,j)$, $(i,j+1)$, or $(i-1,j+1)$. The squares that can be reached from the origin along such paths are labeled by coordinates from the set

$$E_0 = \{(\lambda_1, \lambda_2) : \lambda_1 \in \mathbf{N}, \lambda_2 \in \mathbf{Z}, \lambda_2 \ge -\lambda_1 + 2\}.$$

As before

$$T_{0,\lambda} = \max_{\gamma \in \Gamma_{0,\lambda}} \sum_{\beta \in \gamma} Y_\beta,$$

where the $\{Y_\beta : \beta \in E_0\}$ are i.i.d. Exp(1) variables.

The rule for the growth of the shape B_t is such that a white square turns black with rate one, provided the squares to its south, west, and northwest are black. Initially the complement of E_0 is black. We let B_t be the union of the black squares on the right half-plane. Then $T_{0,\lambda}$ denotes the time when square λ joins B_t.

The limits $\tau(x_1, x_2)$ of (1.3) exist in probability for

$$(x_1, x_2) \in E = \{(x_1, x_2) : x_1 > 0, x_2 > -x_1\}.$$

The function

$$h(x_2) = \inf\{x_1 : x_1 > 0 \vee (-x_2),\ \tau(x_1, x_2) \geq 1\}, \quad x_2 \in \mathbf{R}, \tag{2.1}$$

is again well-defined, finite, convex, and consequently continuous. The asymptotic shape for B_t is given by

$$B = \{(x_1, x_2) : x_2 \in \mathbf{R}, 0 < x_1 < h(x_2)\}.$$

The result is as follows:

Theorem 2A. *For $(x_1, x_2) \in E$, $\tau(x_1, x_2) = x_2^2(\sqrt{x_1} - \sqrt{x_1 + x_2})^{-2}$ if $x_2 \neq 0$, and $\tau(x_1, 0) = 4x_1$. Equivalently,*

$$h(z) = \begin{cases} -z & z \leq -1 \\ \frac{1}{4}(1-z)^2 & -1 < z < 1 \\ 0 & z \geq 1. \end{cases}$$

Recall that, in the server/customer interpretation, square (n, k) joins the shape B_t when customer n leaves server x_{k-1} to join the queue at x_k. Now this will not happen unless square $(n-1, k+1)$ has already joined B_t, in other words, unless customer $n-1$ has left server x_k and the queue at x_k is empty. Thus the customer process $\eta(\cdot)$ evolves as a totally asymmetric simple exclusion process. It has state space $\mathcal{Y} = \{0,1\}^{\mathbf{Z}}$ and generator

$$Lf(\eta) = \sum_{i \in \mathbf{Z}} \eta_i(1 - \eta_{i+1})[f(\eta^{i,i+1} - f(\eta)],$$

with $\eta^{i,i+1}$ defined as in (1.6). The empirical measure α_t^N is as before in (1.15). Let the initial profile be a measurable function $0 \leq u_0(x) \leq 1$, and set $m_0(dx) = u_0(x)dx$.

Theorem 2B. *Assume* (1.14). *For all* $t > 0$, α^N_{Nt} *converges in probability to a deterministic measure* $u(x,t)dx$ *as* $N \to \infty$. *The function* $u(x,t)$ *is the unique entropy solution of* (1.18) *with* $f(u) = u(1-u)$, *and the flux function* $f(u)$ *is the concave conjugate of* $-h(z)$.

As already mentioned, the first proof of Theorem 2B for a special initial profile is due to Rost [Ro] in 1981. His results were subsequently generalized by Liggett [Lg2] and Benassi and Fouque [BF1]. Progress in the hydrodynamic scaling of the exclusion process has typically come in conjunction with progress on the zero range process, so for further references the reader may consult the literature cited in Sect. 1.1.

2.2. Proofs. In this example we do not need moment estimates, because $h(z)$ is defined for all $z \in \mathbf{R}$ so the delicate boundary question is not present, and because convergence in probability always holds for nonnegative superadditive processes. This follows from the standard theorem by truncation, and is easy to prove from scratch. Thus the limit (1.24) holds for $(\lambda_1, \lambda_2) \in E_0$ and a routine approximation procedure extends it to E. A priori the limit function satisfies $0 \le \tau(x_1, x_2) \le \infty$. But concavity, superadditivity and homogeneity hold, so $h(z)$ has the properties stated below equation (2.1).

The argument proceeds as in Example 1. The main difference comes from the exclusion rule: A server particle has to look both left *and right* to decide whether to jump. Consequently the variables $\xi^\ell(j,t)$ must be defined for all $j \in \mathbf{Z}$.

The server process $z(t) = (z_k(t))_{k\in\mathbf{Z}}$ has state space

$$\mathcal{Z} = \{z \in \mathbf{Z}^{\mathbf{Z}} : z_{k-1} \le z_k \le z_{k-1} + 1 \text{ for all } k \in \mathbf{Z}\},$$

and z_k jumps left at rate one, except that a jump violating any of the inequalities $z_{k-1} \le z_k \le z_{k-1}+1$ is suppressed. The process $z(\cdot)$ is constructed by a graphical representation. Consider the initial configuration of server particles defined by $x_j = \ell$ for $j \ge 0$, $x_j = \ell + j$ for $j < 0$. Then let $\xi^\ell(j,t)$ be the displacement of x_j from ℓ at time t: $x_j(t) = \ell - \xi^\ell(j,t)$. As in Example 1, for an arbitrary initial configuration $(z_i)_{i\in\mathbf{Z}} \in \mathcal{Z}$, the process $z(\cdot)$ obeys the formula

$$z_k(t) = \sup_{i\in\mathbf{Z}}\{z_i - \xi^{z_i}(k-i,t)\}, \tag{2.2}$$

as can be proved by induction.

On the way to Theorem 2A, the first task again is to prove that

$$\lim_{N\to\infty} \frac{1}{N} z^N_{[Nx]}(Nt) = U(x,t), \tag{2.3}$$

with

$$U(x,t) = \sup_{q\in\mathbf{R}}\left\{U_0(q) - t\,h\left(\frac{x-q}{t}\right)\right\} \tag{2.4}$$

for $x \in \mathbf{R}$, $t > 0$. This time $U_0(q)$ is a continuous, nondecreasing function with $0 \le U_0'(q) \le 1$ whenever the derivative exists, and (1.12) is again assumed. Easy estimates show that

$$h(z) = 0 \text{ for } z \ge 1 \text{ and } h(z) = -z \text{ for } z \le -1, \tag{2.5}$$

so the supremum in the definition (2.4) of $U(x,t)$ is attained at some $q \in [x-t, x+t]$. The limit

$$\lim_{N\to\infty} \frac{1}{N} \xi^{z_i^N}\big([Nx]-[Nq], Nt\big) = t\, h\Big(\frac{x-q}{t}\Big)$$

is again valid, in probability, for all $q \in \mathbf{R}$.

The steps in the proof of (2.3) are the same as those for Example 1: One half is immediate. For the other half, estimates of sums of exponential random variables show that, if $y_0 < x - t$ and $y_1 > x + t$, and

$$\zeta^N = \max_{[Ny_0] < i \le [Ny_1]} \{z_i^N - \xi^{z_i^N}\big([Nx]-i, Nt\big)\},$$

then

$$\lim_{N\to\infty} P^N\{z_{[Nx]}^N(Nt) \ne \zeta^N\} = 0.$$

The proof of (2.3) is completed with the partition argument as was done after the proof of Lemma 1.1 in Sect. 1.2.

Proof of Theorem 2A. Let $f(u)$ be the concave conjugate of $-h(z)$:

$$f(u) = \inf_{z\in\mathbf{R}} \{uz + h(z)\}.$$

Property (2.5) implies that $f(u) \equiv -\infty$ for $u < 0$ and $u > 1$. For $0 \le u \le 1$ equation (2.4) with $U_0(q) = uq$ gives $U(0,t) = -tf(u)$. To compute $U(0,t)$ from a microscopic evolution, set initially $z_0 = 0$ and let $\{\eta_i = z_i - z_{i-1} : i \in \mathbf{Z}\}$ be i.i.d. Bernoulli random variables with expectation $E[\eta_i] = u$. Reasoning as in the proof of Thm. 1A,

$$\begin{aligned} U(0,t) &= \lim_{N\to\infty} E[N^{-1} z_0(Nt)] \\ &= -\lim_{N\to\infty} N^{-1} \int_0^{Nt} P[\eta_0(s) = 1, \eta_1(s) = 0]\, ds \\ &= -tu(1-u), \end{aligned}$$

so $f(u) = u(1-u)$. From this $h(z)$ and $\tau(x_1, x_2)$ follow. $\square$

With this, the limit (2.3), and the definition (1.17) of $u(x,t)$ we also consider Theorem 2B proved.

3. Third Example

3.1. Results. In the third example everything is as in Example 1, except that time marches in discrete units ($\tau = 0, 1, 2, 3, \ldots$). Fix a parameter $p \in (0,1)$. The Young diagram B_τ evolves according to the following rule: Initially $(0,\infty)^2$ is white and its complement is black. At each time step, each white square turns black with probability p, independently of everything else, if the squares below it and to its left are black. B_τ denotes the union of the black squares in $(0,\infty)^2$ at time τ.

Equivalently, give the squares a collection of i.i.d. random variables $\{Y_\beta \colon \beta \in \mathbf{N}\times\mathbf{N}\}$ with common distribution $P(Y_\beta = k) = (1-p)p^{k-1}$, $k \in \mathbf{N}$ (in other words, $Y_\beta - 1 \sim \mathrm{Geom}(p)$). Then define $T_{0,\lambda}$, the time for square λ to join the shape, in terms of up-right paths as in (1.2). The superadditive limit comes about and the functions $\tau(x_1, x_2)$, $(x_1, x_2) \in (0,\infty)^2$, and $h(z)$, $z \geq 0$, are defined exactly as for Example 1. This time the result is:

Theorem 3A. *We have*

$$\tau(x_1, x_2) = p^{-1}\left\{x_1 + x_2 + 2\big[(1-p)x_1x_2\big]^{1/2}\right\}$$

or, equivalently,

$$h(z) = \left[\sqrt{p(1-z)} - \sqrt{z(1-p)}\,\right]^2 \text{ for } 0 \leq z < p,\ h(z) = 0 \text{ for } z \geq p.$$

The customer process is a discrete time zero range process: The state space is $\mathcal{Y} = \{0, 1, 2, \ldots\}^{\mathbf{Z}}$. At each time step, independently at each site i, if $\eta_i \geq 1$ one customer jumps from site i to $i+1$ with probability p. Denote the process by $\eta(\tau) = (\eta_i(\tau))_{i\in\mathbf{Z}}$, $\tau = 0, 1, 2, \ldots$. Here is a one-parameter family of invariant distributions for this process: The parameter is $r \in (0, p)$. Set $w = r(1-p)/p(1-r) \in (0,1)$. The η_i are i.i.d. random variables with common distribution

$$P[\eta_i = 0] = \frac{p-r}{p} \quad \text{and} \quad P[\eta_i = k] = \frac{p-r}{p(1-p)} w^k \text{ for } k = 1, 2, 3, \ldots \tag{3.1}$$

Define the empirical measure by

$$\alpha_t^N = N^{-1} \sum_{i\in\mathbf{Z}} \eta_i([t])\delta_{i/N}.$$

The hydrodynamic limit now reads:

Theorem 3B. *Let m_0 be a Radon measure on $\mathbf{R}$ satisfying* (1.16) *and assume* (1.14)*. Then for all $t > 0$, α_{Nt}^N converges in probability to a deterministic measure $u(x,t)dx$ as $N \to \infty$. The function $u(x,t)$ is the unique entropy solution of* (1.18) *with*

$$f(u) = \frac{1}{2}\left\{1 + u - \big[(1+u)^2 - 4pu\big]^{1/2}\right\},$$

and the concave flux function $f(u)$ is the monotone conjugate of $-h(z)$.

A coordinate change transforms the curve $w = h(z)$ into an arc of the ellipse

$$\frac{x^2}{p} + \frac{y^2}{1-p} = 1.$$

Theorem 3A was first proved as the "arctic ellipse theorem" for domino tilings of the Aztec diamond by Jockusch, Propp, and Shor [JPS]. They employed a Rost-like analysis of the discrete time exclusion process, and covered the case $p \le 1/2$. The case $p > 1/2$ was done by Cohn, Elkies, and Propp [CEP] with complex analytic tools.

3.2. Proofs. We start by outlining the proof of some facts about a discrete time queueing system.

Lemma 3.1. *The i.i.d. queues* $(\eta_i)_{i \in \mathbf{Z}}$ *with common distribution defined by* (3.1) *are invariant for the customer process* $\eta(\cdot)$ *with jump probability* p. *In equilibrium, a customer passing through the system experiences i.i.d. waiting times* $\{\sigma_i\}$ *with common distribution specified by* $\sigma_i - 1 \sim Geom((p-r)/(1-r))$.

Proof. The argument follows the development of [Ke] for continuous time queues. Consider first one single server queue operating as follows: During the time interval $(\tau - 1, \tau)$, $\tau = 1, 2, 3, \ldots$, one service is completed with probability p and one new customer arrives with probability r. Easy calculation shows that P of (3.1) is invariant for this Markov chain, and in equilibrium the departure process is Bernoulli(r). Next one verifies that, in equilibrium, the size of the queue at time τ is independent of the departure process up to time τ. ¿From this it follows that the i.i.d. queues are invariant for the process. To describe the waiting times, one first checks that a customer arriving in an equilibrium system experiences a waiting time distributed as σ_0 at the first server. That waiting times at subsequent servers are independent of σ_0 follows from proving that the departure process up to the random time $\sigma_0 - 1$ is independent of σ_0. $\square$

The server process $z(\cdot)$ has the same state space $\mathcal{Z}$ as in (1.7). It evolves according to this rule: At each time step τ, each server z_k jumps one step to the left with probability p, provided $z_{k-1}(\tau - 1) \le z_k(\tau - 1) - 1$. Its construction via the graphical representation is the same as in Example 1 except that now the Poisson point processes of potential jump times are replaced by Bernoulli(p) variables that signify possible jumps at integer times τ.

The construction of the variables $\xi^\ell(i, \tau)$ in terms of the special server process $x(\cdot)$ is done as for Example 1, except for the discrete time: Initially $x_i = \ell$ for all $i \ge 0$, and $\xi^\ell(i, \tau)$ is the number of customers served by (equivalently, the displacement of) x_i by time τ. Then we can think of $T_{0,(n,k)}$ as the time when the kth server x_{k-1} has served n customers. Comparing this with the equilibrium system gives moment bounds: If instead of queueing up immediately at server x_0 with all other queues empty, customers arrive in a Bernoulli(r) stream to a system in equilibrium with parameter r, we find that

$$ET_{0,(n,k)} \le n \cdot \frac{1}{r} + k \cdot \frac{1-r}{p-r}.$$

The first term on the right accounts for being the nth success in a sequence of Bernoulli(r) trials, and the second term for the waiting times $\sigma_0 + \cdots + \sigma_{k-1}$ from Lemma 3.1. For $\lambda = (\lambda_1, \lambda_2) \in \mathbf{N} \times \mathbf{N}$ and $n \in \mathbf{N}$ we get, by optimizing over

$r \in (0, p)$,

$$\frac{1}{n} ET_{0,n\lambda} \leq p^{-1}\big[\lambda_1 + \lambda_2 + 2\sqrt{(1-p)\lambda_1\lambda_2}\,\big]. \tag{3.2}$$

After the limit $n \to \infty$ and the approximations that extend it to all $(x_1, x_2) \in (0, \infty)^2$ we combine (3.2) with a trivial lower bound to write

$$p^{-1}(x_1 + x_2) \leq \tau(x_1, x_2) \leq p^{-1}\big[x_1 + x_2 + 2\sqrt{(1-p)x_1 x_2}\,\big]. \tag{3.3}$$

¿From this follow the continuity properties again, and in particular $h(z)$ is well-defined by (1.4) as a continuous convex function for $z \geq 0$.

The microscopic Lax-Oleinik formula

$$z_k(\tau) = \inf_{i \leq k}\{z_i - \xi^{z_i}(k - i, \tau)\}$$

is valid as before. The details of the proof of the scaling limit for the server process are the same as for Example 1, so we skip them and consider this verified:

$$\lim_{N\to\infty} \frac{1}{N} z^N_{[Nx]}([Nt]) = U(x, t)$$

holds in probability for $t > 0$ and $x \in \mathbf{R}$, whenever (1.12) is assumed and $U(x, t)$ is defined by (1.11) for a nondecreasing right-continuous function $U_0(q)$.

Proof of Theorem 3A. Let

$$f(u) = \inf_{z \geq 0}\{uz + h(z)\}, \quad u \geq 0,$$

denote the monotone conjugate of $-h(z)$. For $u \geq 0$, pick the parameter $r = r(u)$ in the invariant distribution (3.1) of $\eta(\cdot)$ so that $E[\eta_i] = u$. Let (η_i) be i.i.d. queues with this common distribution, and define the initial server configuration by $z_0 = 0$, $z_i - z_{i-1} = \eta_i$ for $i \in \mathbf{Z}$. Because the queues remain in equilibrium,

$$E[z_0(\tau)] = -p \sum_{m=0}^{\tau-1} P[\eta_0(m) > 0] = -p\tau\left(1 - \frac{p - r}{p}\right) = -\tau r.$$

By the reasoning of the proof of Thm. 1A, and after some algebra,

$$f(u) = r = \frac{1}{2}\big\{1 + u - \big[(1 + u)^2 - 4pu\big]^{1/2}\big\}.$$

¿From this the function $h(z)$ is recovered by

$$h(z) = -\inf_{u \geq 0}\{uz - f(u)\}, \quad z \geq 0,$$

and $\tau(x_1, x_2)$ by homogeneity. $\square$

Theorem 3B follows exactly as Theorem 1B did.

4. Fourth Example

4.1. Results. Our last growth model does not involve squares, but it too takes place on the positive quadrant of the plane. The parameters n and N of the superadditive process and the scaling limit, respectively, are now identical, so we omit N in favor of n. Thus the reader should take $N = n$ when reference is made to formulas from past sections. The vertical axis will serve as the time axis for the particle model, so to anticipate this we denote points in $\mathbf{R} \times [0, \infty)$ by (x, t). Let $\gamma > 0$ be a parameter. Give each vertical half-line $\{r\} \times (0, \infty)$, $r \in \gamma\mathbf{Z} = \{\dots, -2\gamma, -\gamma, 0, \gamma, 2\gamma, \dots\}$, a rate γ Poisson point process. We shall refer to this point process as Poisson points on the γ-grid. For $-\infty < a < b < \infty$ and $0 \le s < t$, let $L_\gamma((a, s), (b, t))$ denote the length of a longest up-right path of Poisson points contained in the rectangle $[a, b) \times (s, t]$. In other words, $L_\gamma((a, s), (b, t))$ is the maximal integer $m \ge 0$ such that there is a sequence $(x_1, t_1), \dots, (x_m, t_m)$ of Poisson points that satisfy

$$a \le x_1 \le x_2 \le \cdots \le x_m < b \qquad \text{and} \qquad s < t_1 \le t_2 \le \cdots \le t_m \le t.$$

For $n \in \mathbf{N}$, define the random shape by

$$B_n(\gamma) = \{(x, t) \in (0, \infty)^2 : L_\gamma((0, 0), (x, t)) \le n - 1\}.$$

We shall characterize the asymptotic shape B_γ of $n^{-1}B_n(\gamma_n)$, assuming that $\gamma_n \to \gamma$ as $n \to \infty$. The varying parameter γ_n is introduced to control the effect of restricting the Poisson points to lie on the γ-grid. In the $\gamma \to 0$ limit the Poisson points on the γ-grid become a planar rate one Poisson point process.

So assume throughout that the limit $\gamma_n \to \gamma$ exists. The interesting limiting case is $\gamma = 0$, so we might as well assume that $0 < \gamma_n \le 1$ and consequently $0 \le \gamma \le 1$. Superadditive ergodicity and moment bounds give the existence of a finite limit in probability, for $(x, t) \in (0, \infty)^2$:

$$\psi_\gamma(x, t) = \lim_{n \to \infty} \frac{1}{n} L_{\gamma_n}((0, 0), (nx, nt)). \tag{4.1}$$

The values of the limits are as follows:

Theorem 4A. *For $0 \le \gamma \le 1$, $\psi_\gamma(x, t) = \gamma t + 2\sqrt{xt}$. For $\gamma > 0$, the asymptotic shape is given by*

$$B_\gamma = \{(x, t) \in (0, \infty)^2 : 0 < t < \gamma^{-1}, 0 < x < (1 - \gamma t)^2/4t\},$$

and for $\gamma = 0$ by

$$B_0 = \{(x, t) \in (0, \infty)^2 : 4xt < 1\}.$$

To compute these limits we embed the random variables $L_{\gamma_n}((a, s), (b, t))$ in a customer process utilizing the following idea: Let $z^n = (z_i^n)_{i \in \mathbf{Z}}$ be an initial configuration on the lattice $\gamma_n\mathbf{Z}$, and select a realization of the Poisson points on the γ_n-grid. Fix k and a time $t > 0$. For each z_i^n to the left of z_k^n, a potential location for z_k^n at time t is given by the endpoint of an up-right path of length $k - i$ starting at $(z_i^n, 0)$ and staying below the time-t line. Of these potential locations,

z_k^n chooses the leftmost. Some restrictions have to be placed on z^n to prevent z_k^n from reaching $-\infty$ in finite time. The right choice for a state space is

$$(4.2)\qquad \mathcal{Z}_n = \left\{ z \in (\gamma_n \mathbf{Z})^{\mathbf{Z}} : z_{k-1} \le z_k \text{ for all } k \in \mathbf{Z},\ \lim_{k\to-\infty} k^{-2} z_k = 0 \right\}.$$

(The superscript n appears in z^n whenever we refer to the process, as opposed to a generic element of a state space.) To rigorously define the above evolution, set

$$(4.3)\qquad \Gamma_{\gamma_n}(r,\ell,t) = \inf\{b > 0 : L_{\gamma_n}((r,0),(r+b,t)) \ge \ell\}$$

for $r \in \mathbf{R}$, $\ell \in \{0,1,2,\dots\}$, and $t > 0$. And then

$$(4.4)\qquad z_k^n(t) = \inf_{i \le k}\{ z_i^n + \Gamma_{\gamma_n}(z_i^n, k-i, t)\}.$$

Formula (4.4) defines the server process as a $\mathcal{Z}_n$-valued Markov process. The rule that is realized is this: A Poisson point appears at rate γ_n at each site of $\gamma_n\mathbf{Z}$, and whenever that happens, the nearest server particle to the right is moved to the location of the Poisson point. This particle evolution was originally derived by Aldous and Diaconis [AD] from Hammersley's ideas in [Ha], so the process $z^n(\cdot)$ could be called Hammersley's process on a lattice.

By homogeneity, to deduce Theorem 4A it suffices to identify the function $h_\gamma(x) = \psi_\gamma(x,1)$, $x > 0$. This function $h_\gamma(x)$ is nonnegative, concave and strictly increasing, so it has a convex inverse $g_\gamma(x)$. To have $g_\gamma(x)$ defined and continuous for all $x \ge 0$, set $g_\gamma(x) = 0$ for $0 \le x \le h_\gamma(0+)$. The limit (4.1) implies that

$$(4.5)\qquad \lim_{n\to\infty} \frac{1}{n} \Gamma_{\gamma_n}(r_n, [nx], nt) = t\, g_\gamma\left(\frac{x}{t}\right)$$

holds in probability, for $x \ge 0$, $t > 0$, and any sequence $\{r_n\}$ of base points.

¿From the server process we construct the customer process by

$$(4.6)\qquad \eta_i^n(t) = z_{i+1}^n(t) - z_i^n(t).$$

The process $\eta^n(\cdot)$ is a coupled random walk where each customer particle has 'weight' γ_n. At each site $i \in \mathbf{Z}$ the size η_i of the queue is a multiple of γ_n. Particles jump from i to $i+1$ according to the rule: At rate η_i, pick a number $k\gamma_n$ uniformly at random from $\{\gamma_n, 2\gamma_n, \dots, \eta_i\}$, and move amount $k\gamma_n$ from site i to $i+1$. The generator is

$$(4.7)\qquad \mathcal{L}_n f(\eta) = \sum_{i\in\mathbf{Z}} \gamma_n \sum_{k=1}^{\eta_i/\gamma_n} [f(\eta^{k\gamma_n,i,i+1}) - f(\eta)],$$

where

$$\eta_j^{r,i,i+1} = \begin{cases} \eta_i - r, & j = i \\ \eta_{i+1} + r, & j = i+1 \\ \eta_j, & j \ne i, i+1. \end{cases}$$

We shall not enter into a discussion about the proper existence of the generator, and offer it only as an intuition-enhancing tool for readers who prefer to read the dynamics off the generator. Equation (4.6) is the rigorous definition of the process $\eta^n(\cdot)$. Paralleling (4.2), the state space for $\eta^n(\cdot)$ becomes

$$\mathcal{Y}_n = \left\{ \eta \in \{0, \gamma_n, 2\gamma_n, 3\gamma_n \dots\}^{\mathbf{Z}} : \lim_{k\to-\infty} k^{-2} \sum_{i=-1}^{k} \eta_i = 0 \right\}.$$

The distributions on $\mathcal{Y}_n$ under which the $(\gamma_n^{-1}\eta_i)_{i\in\mathbf{Z}}$ are geometrically distributed i.i.d. random variables are invariant for the process.

The initial macroscopic profile for the customer process is a Radon measure m_0 on $\mathbf{R}$, assumed to satisfy

$$\lim_{x\to-\infty} x^{-2} m_0[x,0) = 0. \tag{4.8}$$

Pick a left-continuous function $U_0(x)$ such that

$$m_0[x,y) = U_0(y) - U_0(x) \tag{4.9}$$

for all $-\infty < x < y < \infty$, and define

$$U(x,t) = \inf_{q\le x} \left\{ U_0(q) + t\, g_\gamma\left(\frac{x-q}{t}\right) \right\} \tag{4.10}$$

by the Lax-Oleinik formula. Then it is proved in the Appendix of [Se] that the derivative

$$u(x,t) = U_x(x,t) \tag{4.11}$$

exists almost everywhere, is locally bounded in $\mathbf{R}\times(0,\infty)$, and is a weak solution of the conservation law (1.18) where $f(u) = f_\gamma(u)$ is the monotone conjugate of $g_\gamma(x)$:

$$f_\gamma(u) = \sup_{x\ge 0}\{ux - g_\gamma(x)\}, \quad u \ge 0. \tag{4.12}$$

Furthermore, if m_0 has an $L^\infty(\mathbf{R})$ derivative then this $u(x,t)$ is the unique entropy solution of (1.18). For general m_0 an alternative uniqueness criterion is developed in [Se].

The hydrodynamic scaling limit for the empirical measure

$$\alpha_t^n = \frac{1}{n}\sum_{i\in\mathbf{Z}} \eta_i^n(t)\delta_{i/n}$$

is valid as in the previous sections. Write P^n for the probabilities of the process $\eta^n(\cdot)$, and assume that the distributions of $\eta^n(0)$ are chosen so that

$$\lim_{n\to\infty} P^n\left\{ \left| \frac{1}{n}\sum_{i=[na]}^{[nb]-1} \eta_i^n(0) - m_0[a,b) \right| \ge \varepsilon \right\} = 0 \tag{4.13}$$

for any $a < b$ and $\varepsilon > 0$. This time we need an additional assumption due to the combination of unbounded rates with unbounded queues:

(4.14)
$$\begin{gathered}\text{There is a number } b \in \mathbf{R} \text{ such that}\\ \text{for all } \varepsilon > 0 \text{ we can find } q \text{ and } n_0 \text{ that satisfy}\\ \sup_{n \geq n_0} P^n \left\{ \sup_{k \leq nq} \frac{k}{n^2} \sum_{i=k}^{[nb]-1} \eta_i^n(0) \geq \varepsilon \right\} \leq \varepsilon.\end{gathered}$$

Among the admissible initial distributions are the deterministic queues $\eta_i^n(0) = n\, m_0[i/n, (i+1)/n)$. Under more stringent assumptions on m_0 also independent geometrically distributed queues with the above expectations satisfy (4.13)–(4.14). See Sect. 10 in [Se] for further discussion on this point.

Theorem 4B. *Let m_0 be a Radon measure on $\mathbf{R}$ and assume* (4.8), (4.13), *and* (4.14). *Then for all $t > 0$, α_{nt}^n converges in probability to the deterministic measure $u(x,t)dx$ as $n \to \infty$, where $u(x,t)$ is the function defined in* (4.11). *The flux function is given by $f_\gamma(u) = u^2 + \gamma u$ for $0 \leq \gamma \leq 1$, $u \geq 0$.*

Ulam's problem concerns the evaluation of the limits in Theorem 4A in the case $\gamma_n \equiv 0$, in other words, when the random variables $L_0((a,s),(b,t))$ are defined in terms of a planar Poisson process. The theorem is that

$$\lim_{n\to\infty} \frac{1}{n} L_0((0,0),(n,n)) = 2. \tag{4.15}$$

The first proof of (4.15) appeared in 1977 and is due to Vershik and Kerov [VK]. Independently, at the same time, Logan and Shepp [LS] proved that the limit is ≥ 2. Both proofs are combinatorial in nature and use Young diagrams. Aldous and Diaconis [AD] gave a hydrodynamic argument, similar in spirit to Rost's [Ro] early paper. An argument utilizing the Lax-Oleinik formula appeared in [Se]. See Hammersley's paper [Ha] for the original link between increasing subsequences of random permutations and the random variables $L_0((0,0),(n,n))$.

The particle process $\eta^n(\cdot)$ appeared in its reversible guise in [ES], where it was studied under diffusion scaling. The case $\gamma = 0$ is the interesting one in both settings: Here Theorem 4B gives the Burgers equation, while in the diffusive case the limit is governed by the porous medium equation.

The corresponding Hamilton-Jacobi equation (1.21) with $f(u) = u^2$ was studied in the appendix of [LSV], where it is proved that the Lax-Oleinik formula defines a continuous viscosity solution for very general initial data $U_0(x)$.

4.2. Proofs.

We begin with a moment bound.

Lemma 4.1. *For any $x, t > 0$, $0 < \gamma \leq 1$, and whenever $k \geq 2e^4 t^{1/2}[(x+1)\vee t]^{1/2}$,*

$$P\big\{L_\gamma((0,0),(x,t)) \geq k\big\} \leq e^{-2k}. \tag{4.16}$$

Proof. Take $\gamma = 1$ first. Let $n \leq x+1$ be the number of lattice points in $[0,x)$. Suppose $L_\gamma((0,0),(x,t)) \geq k$. Then among the ℓ $(\geq k)$ Poisson points in $[0,x) \times$

$(0,t]$ there are k that form an up-right path. Suppose k_i of them fall on the half-line $\{i-1\} \times (0,t]$, $i = 1, \dots, n$, $k_1 + \cdots + k_n = k$. Of the $k!$ possible orderings of the t-coordinates of these k points, $k_1!k_2! \cdots k_n!$ orderings render the set into an up-right path. Thus, letting $\ell = \ell_1 + \cdots + \ell_n$ be the decomposition of the ℓ points among the n half-lines, the probability in (4.16) is at most

$$\sum_{\substack{\ell_1, \dots, \ell_n \\ \ell_1 + \cdots + \ell_n \geq k}} \frac{e^{-t} t^{\ell_1}}{\ell_1!} \cdots \frac{e^{-t} t^{\ell_n}}{\ell_n!} \sum_{\substack{k_1, \dots, k_n \\ k_1 + \cdots + k_n = k \\ 0 \leq k_i \leq \ell_i,\, i=1, \dots, n}} \binom{\ell_1}{k_1} \cdots \binom{\ell_n}{k_n} \frac{k_1! k_2! \cdots k_n!}{k!}.$$

Fairly straightforward calculations and estimates show that this is at most

$$\left(\frac{e^2 t}{k} \left(1 + \frac{n}{k} \right) \right)^k.$$

The statement follows by considering separately the cases $k < n$ and $k \geq n$. The result for $\gamma < 1$ follows from the scaling relation

$$L_\gamma((0,0),(x,t)) \stackrel{\mathcal{D}}{=} L_1((0,0),(\gamma^{-1}x, \gamma t)). \quad \square \tag{4.17}$$

The first item is the rigorous construction of the server process by formula (4.4), in terms of Poisson points on the γ_n-grid and an initial configuration $(z_i^n)_{i \in \mathbf{Z}} \in \mathcal{Z}_n$. A similar construction was performed in Sect. 4 of [Se], so we shall simply record the outcome: For a.e. realization of the Poisson points, (4.4) defines a Markov process $z^n(\cdot)$ with paths in $D([0,\infty), \mathcal{Z}_n)$. The exponential bound (4.16) is exactly what is needed for this task: Equation (4.5) of [Se] follows again, and the same arguments can be repeated.

Once the server process has been constructed, the customer process $\eta^n(\cdot)$ starting from a configuration $(\eta_i^n)_{i \in \mathbf{Z}} \in \mathcal{Y}_n$ is defined by

(i) first choosing a point $r_0 \in \gamma_n \mathbf{Z}$ and setting $z_0^n = r_0$, $z_i^n = r_0 + \sum_{j=0}^{i-1} \eta_j^n$ for $i > 0$, and $z_i^n = r_0 - \sum_{j=i}^{-1} \eta_j^n$ for $i < 0$;
(ii) then constructing the process $z^n(\cdot)$ from this initial configuration $(z_i^n)_{i \in \mathbf{Z}}$ by formula (4.4);
(iii) and lastly by defining $\eta^n(\cdot)$ by formula (4.6).

¿From this definition one can derive the form of the generator (4.7) and the invariance of i.i.d. geometric queues. See Sect. 5–8 in [Se] for an example of such a development.

Now assume $\gamma_n \to \gamma$ and $\gamma > 0$. We shall deal with the case $\gamma = 0$ separately. Superadditivity and the bound (4.16) guarantee a finite limit $\psi_1(x,t)$ in (4.1) for the case $\gamma_n \equiv 1$. Superadditivity, homogeneity, concavity, and continuity of $\psi_1(x,t)$ are easy to prove. For the general case $\gamma_n \to \gamma > 0$ the limit (4.1) follows from the scaling relation (4.17).

We pause to check a monotonicity property. As mentioned above, for $\gamma = 0$ the random variable $L_0((0,0),(x,t))$ is the maximal length of an up-right path of points from a planar Poisson point process.

Lemma 4.2. *For any* $0 \le \gamma < \kappa$, $L_{\kappa+\gamma}\big((0,0),(x(1+\gamma/\kappa),t)\big)$ *stochastically dominates* $L_\gamma((0,0),(x,t))$.

Proof. Apply the map $(y,s) \mapsto ([y/\kappa]\kappa, s)$ to a realization of Poisson points on the γ-grid. The image points are distinct with probability one, so the lengths of all upright paths are preserved. After applying the map, each vertical line of the κ-grid has a Poisson point process of rate at most $(\kappa/\gamma+1)\gamma = \kappa+\gamma$. Add more points to those lines with rates below this and dilate the x-axis by the factor $1+\gamma/\kappa$ to transform this setting into Poisson points on the $(\kappa+\gamma)$-grid. $\square$

¿From this follows that $\lim_{x\to\infty}\psi_\gamma(x,t) = \infty$, and then concavity implies that $h_\gamma(x) \equiv \psi_\gamma(x,1)$ is strictly increasing for all $x > 0$. The existence of the inverse $g_\gamma(x) = h_\gamma^{-1}(x)$ and the limit (4.5) have thus been verified.

Moving on to the scaling limits, assume the Radon measure m_0 satisfies (4.8), and the (possibly random) initial customers $\eta^n(0)$ satisfy (4.13) and (4.14). Fix $U_0(x)$ so that (4.9) holds, and define $U(x,t)$ by (4.10). Choose the initial servers $z^n(0)$ so that (4.6) and

$$\lim_{n\to\infty}\frac{1}{n}z^n_{[nq]}(0) = U_0(q) \ \text{ in probability, } q \in \mathbf{R}, \tag{4.18}$$

hold. The scaling limit

$$\lim_{n\to\infty}\frac{1}{n}z^n_{[nx]}(nt) = U(x,t), \quad x \in \mathbf{R},\ t > 0, \tag{4.19}$$

follows as it did in the previous sections: For the nontrivial half, set

$$\zeta^{n,r} = \min_{[nr]<i\le[nx]}\big\{z_i^n + \Gamma_{\gamma_n}\big(z_i^n, [nx]-i, nt\big)\big\}.$$

As in Lemma 8.11 of [Se], assumption (4.14) and the bound (4.16) allow us to deduce

$$\lim_{n\to\infty} P^n\big\{z^n_{[nx]}(nt) \ne \zeta^{n,r}\big\} = 0$$

for a suitably chosen r. The partition argument completes the proof of (4.19).

Proof of Theorem 4A for the case $\gamma > 0$. Setting $U_0(q) = uq$ for $u \ge 0$ in (4.10) gives, via (4.12), $U(0,t) = -tf_\gamma(u)$. For the microscopic side, set initially $z_0^n = 0$, and let $(\eta_i^n)_{i\in\mathbf{Z}}$ be i.i.d. so that $\gamma_n^{-1}\eta_i^n$ has Geom$(\gamma_n/(u+\gamma_n))$ distribution. Then $E^n[\eta_i^n] = u$, and (4.14) and (4.18) hold. The rate of advance for server z_0^n is

$$\begin{aligned}
&\underset{\text{(rate of jumping)}}{\eta_0^n} \cdot \sum_{k=1}^{\eta_0^n/\gamma_n}\Big[\underset{\text{(size of jump)}}{-k\gamma_n} \cdot \underset{\text{(uniform probability)}}{\gamma_n/\eta_0^n}\Big] \\
&= -\frac{1}{2}\eta_0^n(\eta_0^n+\gamma_n).
\end{aligned}$$

Since z_0^n sees an equilibrium queue to its left at all times,

$$\begin{aligned}
E^n[n^{-1}z_0^n(nt)] &= -\frac{1}{2}n^{-1}\int_0^{nt} E^n\big[\eta_0^n(s)(\eta_0^n(s)+\gamma_n)\big]\,ds \\
&= -t(u^2+\gamma_n u).
\end{aligned}$$

A similar argument gives a uniform moment bound

$$\sup_n E^n[(n^{-1}z_0^n(nt))^2] < \infty,$$

so by uniform integrability,

$$f_\gamma(u) = -t^{-1}U(0,t) = -t^{-1} \cdot \lim_{n\to\infty} E^n[n^{-1}z_0^n(nt)] = u^2 + \gamma u.$$

By duality,

$$g_\gamma(x) = \begin{cases} 0 & 0 \le x \le \gamma \\ (x-\gamma)^2/4 & x > \gamma. \end{cases}$$

¿From this $\psi_\gamma(x,t) = \gamma t + 2\sqrt{xt}$, and B_γ follows from $B_\gamma = \{\psi_\gamma < 1\}$. $\square$

Let us now turn to the case $\gamma_n \to \gamma = 0$. We borrow the result (4.15) in the form

$$\lim_{n\to\infty} \frac{1}{n} L_0((0,0),(nx,nt)) = 2\sqrt{xt}, \quad x,t > 0. \tag{4.20}$$

For arbitrary $\varepsilon, \kappa > 0$, $\gamma_n < \varepsilon\kappa$ for large enough n. Writing $\lesssim$ for stochastic dominance, Lemma 4.2 gives the two-sided bounds

$$\begin{aligned} L_0((0,0),(nx,nt)) &\lesssim L_{\gamma_n}((0,0),(nx,nt)) \\ &\lesssim L_{\kappa+\gamma_n}((0,0),(nx(1+\varepsilon),nt)). \end{aligned} \tag{4.21}$$

The limit (4.1) with $\psi_0(x,t) = 2\sqrt{xt}$ follows from (4.20), (4.21), and the already proved, upon letting $\kappa, \varepsilon \searrow 0$. The limit (4.5) and the scaling limit (4.19) then follow as above, as does Theorem 4A.

Finally, Theorem 4B is a consequence of (4.19) as in the earlier examples.

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