Virial Relations for Nonlinear Wave Equations and Nonexistence of Almost-Periodic Solutions

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Abstract

We present a formalism for constructing integral identities involving time almost-periodic solutions of nonlinear wave equations. As an application we prove a nonexistence theorem that is applicable to a large class of nonlinearities.

1 Introduction

A virial relation is an integral identity involving the solution of a differential equation. The identity can be derived from the differential equation itself by multiplying the equation by a combination of derivatives of the solution and then integrating over the range of the independent variables or, if the equation can be formulated as a variational problem, from infinitesimal variations of the action functional associated to the equation. A well known example from physics relates the time-average kinetic and potential energies of an n-particle system under the influence of central forces (usually referred to as the virial theorem; see for example [LL]). In mathematics virial relations have been used extensively, for example, in deriving necessary conditions for the existence of solutions of differential equations beginning with the work of Pohozaev [Po].

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In this article we present a systematic approach to deriving virial relations for almost periodic solutions of nonlinear wave equations (NLW's) of the form

$$\partial_t^2\varphi - \Delta\varphi + f(\varphi) = 0. \tag{1.1}$$

Here $\varphi : {}_x^N \times {}_t \to \ ,$ $\partial_t^2\varphi = \partial^2\varphi/\partial t^2$, $\Delta\varphi = \sum_{i=1}^N \partial^2\varphi/\partial x_i^2$, and $f : \ \to \$ with $f(0) = 0$. By almost periodic solution we understand solutions that are almost periodic in time t and $L^2({}^N)$ in space x. Almost periodic solutions have special significance in field theories in that they represent bound states, i.e., solutions that are localized in space uniformly in time (cf. below). Our objective is twofold. First, we want to illuminate a method that has been known and used in various guises for many years in mathematics and physics and is well-suited for problems arising in nonlinear differential equations. Second, we apply this method to derive necessary conditions for the existence of almost periodic solutions of NLW.

We remark about the notion of a periodic solution. Due to the invariance of NLW under the Poincaré group, an almost periodic solution may not appear to be almost periodic in another Lorentz frame, although it still satisfies NLW in this frame. For example, if the second frame is one which is moving at a constant (nonzero) velocity with respect to the first frame (in which the solution is almost periodic), then the solution is not almost periodic in this moving frame. This means that when we are considering an almost periodic solution we are tacitly referring to a particular Lorentz frame (it is not unique). Thus, for us an almost periodic solution of NLW is a solution that is almost periodic in *some* Lorentz frame.

In this paper we are concerned solely with almost periodic solutions and do not address the question of the existence of general solutions. In particular, our results are that NLW's with certain nonlinearities do not possess almost periodic solutions. Never the less, global existence theory does apply to some of these equations (for a survey see [GV] or [Str2]). Thus, together these two results imply that there are NLW's that possess global (in time) solutions but that none of these solutions are almost periodic - that the set of solutions of these NLW's do not contain functions with a certain temporal behavior (almost periodicity). An example of such a NLW is given below (cf. Example 1 in Section 3).

When NLW is viewed as an evolution equation in time it is natural to attempt to characterize solutions by their temporal behavior. The work of this article is a start in that direction. In deciding what sorts of behavior are appropriate for such a characterization, we are motivated by two observations. The first is that $L^2({}^N)$-valued almost periodic functions have the property of being uniformly bounded with respect to time. That is, for any $\varepsilon > 0$ there exists a ball $B(\rho) \subset {}^N$ of radius $\rho = \rho(\varepsilon)$ such that

$$\int_{B^c(\rho)} |\varphi(x,t)|^2 \, dx \ < \ \varepsilon \tag{1.2}$$

for all t (cf. Definition 4.5 and Lemma 4.7 below). We refer to functions satisfying this condition as being *bound states*. Secondly, an important theorem in quantum mechanics due to Ruelle [R] states that bound state solutions of the Schrödinger equation are almost periodic in time. The converse is true as we have just seen. Thus, for the Schrödinger equation bound states are characterized by their temporal behavior and by this alone. We believe that a similar statement can be made for some nonlinear equations such as NLW and the nonlinear Schrödinger equation [Py]. That is, solutions of these equations that are bound states, as defined above, are almost periodic in time (we may have to first transform to an appropriate coordinate system via a symmetry of the equation). Therefore, from this perspective the work presented in this article aims to contribute towards a classification of nonlinear wave equations according to whether they possess bound states or not.

One of the few NLW's known to possess periodic solutions that are localized in x (often referred to as "breathers") is the sine-Gordon equation $\partial_t^2\varphi - \partial_x^2\varphi + \sin(\varphi) = 0$. Here the solutions are given by the formula

$$\varphi(x,t;\omega) \ = \ 4\tan^{-1}\left(\frac{\varepsilon \sin \omega t}{\omega \cosh \varepsilon x}\right), \quad \varepsilon^2 + \omega^2 = 1. \tag{1.3}$$

Being an integrable system, these solutions can be found using techniques related to the inverse scattering transform [MNPZ]. However, many NLW's of interest are not integrable, for example, the "φ^4" equation; $\partial_t^2\varphi - \partial_x^2\varphi + \varphi - \varphi^3 = 0$, which is an important model in particle physics. It is not known whether this latter equation possesses periodic solutions. The method we describe in this article does not depend on the integrability properties of the equation.

We briefly review previous studies concerning periodic solutions of nonlinear wave equations. The structural stability of the sine-Gordon breather has been an object of study for a number of years (for recent work see [BMW] and [D], for an earlier study see [MS]). Here one looks for periodic solutions of the perturbed sine-Gordon equation that are close to the sine-Gordon breather. Because the sine-Gordon breather is known explicitly, a detailed analysis can be carried out. These references reveal a complex interaction between the breather and nonlinearity $\sin(\varphi)$ that is easily upset by a perturbation. The conclusion is that typically the perturbed equation does not have a periodic solution close to the sine-Gordon breather. This result is corroborated by the work [Ki] in which the sine-Gordon equation is singled-out as essentially the only NLW on $^{1+1}$ having (analytic) breathers. In the same vein one can look at how periodic solutions of *linear* equations behave under nonlinear perturbations. This was investigated in [Si] where it was shown that these solutions, like the sine-Gordon breather, are generically unstable under nonlinear structural perturbations. A necessary condition for the existence of periodic solutions on ^{1} follows from a result of Coron's [Co] which states that if $\varphi \in C^2(,)$ is a $2\pi/\omega$ -periodic solution of NLW, then $\omega^2 \leq f'(0)$. Recently, we have been able to extend this result to multi-spatial dimensions [PS].

The situation is very different on a bounded or semi-bounded spatial domain; here periodic and almost periodic solutions are abundant. In [SV],[V2],[V3], and [We], periodic, quasiperiodic, or almost periodic solutions of NLW on the half-line $_+$ were constructed using methods from centre manifold theory. Here the one dimensionality of the spatial variable is essential as NLW is treated as a dynamical system in a phase space of periodic functions with x playing the role of the dynamical variable. The (one-sided) exponentially decaying periodic solutions are points in the stable or unstable manifolds of the zero solution. The failure of these manifolds to intersect prohibits extending these existence results to the entire real line [V3]. [Sc] and [Sm] apply a similar analysis for radially symmetric periodic solutions on $^{N+1}$. For NLW on an interval, [CW] and [Wa] used infinite dimensional KAM theory to prove the existence of families of periodic solutions, while [BCN] applied variational methods and [H] used bifurcation theory to prove the existence of periodic solutions.

Although the above results indicate that almost periodic solutions of nonlinear wave equations on $^{N+1}$ are rare, the question of existence is still very much open. With this in mind we set out to investigate what conditions the nonlinearity must satisfy in order for NLW on the infinite spatial domain ^{N} to support solutions that are time periodic or, more generally, time almost periodic. Although in this paper we will be considering the unbounded spatial domain, we could also carry out our analysis on bounded or semi-bounded spatial domains.

Our approach to this problem was motivated by the study [V1] where the periodic NLW on $^{1+1}$ was formulated as a variational problem. Since periodic solutions are critical points of the action functional, the Fréchet derivative of the functional is zero at these points. Evaluating the Fréchet derivative on certain functions derived from the solution itself results in integral identities involving the solution and nonlinearity. ¿From these identities several nonexistence theorems for classical (i.e., $C^2(^2)$) periodic solutions was proven. Subsequently, the development of our ideas in the direction of describing virial relations as infinitesimal variations of the action functional (cf. the appendix to the present paper) was aided by a recent article [M] which surveys the role of virial relations in physical theories that are based on an action principle.

The work presented in this paper extends the results of [V1] in several directions. First, we work with more general quasiperiodic solutions (multiple, but finitely many, frequencies instead of one) and in several spatial dimensions. Our admissible set of solutions is also larger, being of class H^1 in space and time (weak solutions). After deriving a class of virial relations for quasiperiodic solutions and using these to state necessary conditions for the existence of such solutions, we will see that by using ideas from ergodic theory we can express our results in a way that makes sense for the larger class of almost periodic solutions (i.e., allowing for infinitely many frequencies). In the last section of this paper we show that these formulae are in fact valid for almost periodic solutions.

Historically, the idea of formulating a nonlinear equation as a minimization problem to obtain necessary conditions for the existence of solutions has its origins in a work of Pohozaev [Po] who used the method to study solutions of a nonlinear elliptic boundary problem. Subsequent applications of this technique can be found, for example, in [Str1] and [B] where it was used to study stationary states of some nonlinear wave equations, and in [BL] and [BrLi] where it was used in a study of a class of nonlinear elliptic variational problems.

The main results of this article are the following. We first derive a class of integral identities that must be satisfied by almost periodic solutions of NLW (Theorems 2.7 and 4.10). Then, by choosing a particular subclass of these we are able to prove the following theorem, which we state here without all the technical details (cf. Theorems 3.5 and 5.1).

Theorem *Suppose* φ *is an almost periodic solution of NLW such that* $\varphi(\cdot,t) \in H^1(^N)$. *Let* $F(z) = \int_0^z f(w)\,dw$. *If* $F(z) - czf(z) \leq 0$ *for some* $c \in [\frac{N-2}{2N}, \frac{1}{2}]$ *and for all* z *such that* $|z| \leq \|\varphi\|_{L^\infty(^{N+1})}$, *then* φ *is independent of time.*

Application of this theorem can be widened by considering small amplitude solutions since then only properties of the nonlinearity near zero enters. For such solutions we present two nonexistence results (Corollaries 3.6 and 3.7). The first has relevance, in particular, to nonlinear Klein-Gordon equations (i.e., $f'(0) > 0$) with odd nonlinearity. For the second we combine Wirtinger's inequality with virial relations to obtain a result similar to Coron's [Co] in multi-spatial dimensions. By taking advantage of the decay of solutions as $|x| \to \infty$ we can, by "localizing" the virial relation in a neighborhood of spatial infinity, extend some of the results about small amplitude solutions to solutions of arbitrary amplitude (Corollary 3.8).

Before concluding this introduction we present some heuristics to help motivate and describe our approach. We begin with the observation that a function $\varphi(x,t) \in H^1(^{N+1})$ can be viewed as a function from to $H^1(^N)$ via the map $t \mapsto \varphi(\cdot,t)$. Furthermore, if φ is $2\pi/\omega$ -periodic then we can write $\varphi(x,t) = \gamma(x,\omega t)$ where $\gamma :\ S^1 \to H^1(^N)$ is a function defined on the unit circle S^1. ¿From this it is clear how to define quasiperiodic functions: φ is quasiperiodic if $\varphi(x,t) = \gamma(x,\omega t)$ for some $\gamma :\ ^l \to H^1(^N)$ where ^{l} is the l-torus $S^1 \times \cdots \times S^1$, and for some $\omega \in {}^l$. We call γ the *generating function* of φ and ω the *frequency* of φ. Given a quasiperiodic solution φ of NLW with frequency $\omega \in {}^l$, by the chain rule we derive an equation satisfied by its generating function γ;

$$\mathcal{D}_\omega^2\gamma - \Delta\gamma + f(\gamma) = 0.$$

Here $\mathcal{D}_\omega^2 = \sum_{i,j=1}^l \omega_i\omega_j\mathcal{D}_i\mathcal{D}_j$, $\mathcal{D}_i = \partial/\partial\theta_i$, and $\theta_1, \ldots, \theta_l$ are coordinates on ^{l}. We call this equation the *nonlinear wave equation on* ${}^N \times {}^l \equiv \Omega_{N,l}$ with frequency ω. To derive virial relations for quasiperiodic solutions φ then, we derive virial relations for solutions γ of NLW on $\Omega_{N,l}$ and transfer these back to φ using the identification $\varphi(x,t) = \gamma(x, \omega t)$.

In general, an almost periodic function is a quasiperiodic function with infinitely many (independent) frequencies. Almost periodic functions φ can be characterized by generating functions γ that are defined on the infinite dimensional torus ${}^\infty$; $\gamma : {}^\infty \to H^1({}^N)$. Then, there is a dense embedding $\Gamma : \to {}^\infty$ such that $\varphi(t) = \gamma(\Gamma(t))$. For almost periodic solutions though, we will work directly with the function itself rather than with its generating function. This is facilitated by passing from ${}^\infty$ to via the Bohr compactification of the real line which is realized by the formula

$$\int_\infty \gamma(\theta)\, d\theta \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(\gamma \circ \Gamma\Big)(t)\, dt. \tag{1.4}$$

Here $d\theta$ is the (normalized) Haar measure on ${}^\infty$. That is, instead of integrating γ over the torus ${}^\infty$, which is its space average, we take its time average along the curve Γ.

To illustrate the general strategy as well as to describe some aspects of our approach, we present an example of a virial relation. Suppose γ is a solution of NLW on $\Omega_{N,l}$ with frequency ω and let $\beta : {}^l \to H^1({}^N)$ be some function on ^{l}. Then, multiplying NLW by β and integrating we have,

$$\int_l \int_N \beta\Big(\mathcal{D}_\omega^2\gamma - \Delta\gamma + f(\gamma)\Big) \;=\; 0.$$

Integrating by parts we obtain

$$\int_l \int_N \Big\{ -\mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta + \nabla\gamma\cdot\nabla\beta + f(\gamma)\beta \Big\} \;=\; 0,$$

which expresses that γ is a weak solution. Now take for the multiplier β some combination of γ and its derivatives, for example, $\beta = x \cdot \nabla\gamma$. Using this β in the latter formula and removing a divergence term we arrive at the virial relation

$$\int_{l}\int_{N} \Big\{\frac{1}{2}(\mathcal{D}_\omega\gamma)^2 + \Big(\frac{2-N}{2N}\Big) \mid \nabla\gamma \mid^2 - F(\gamma)\Big\} \;\; = \;\; 0, \qquad F' = f. \tag{1.5}$$

We see right away that in spatial dimensions 1 or 2, F cannot be a nonpositive function. Thus, we have a necessary condition on the nonlinearity for the existence of solutions of NLW on $\Omega_{N,l}$. The virial relation for the quasiperiodic solution φ can be recovered from this by the identification $\varphi(x,t) = \gamma(x, \omega t)$ and where the integral over ^{l} is replaced by the time-mean over (cf. (1.4)).

What is required to make this argument rigorous is that γ be sufficiently regular and integrable. How regular and integrable depends on how β is defined. One could, at the start, consider only those solutions for which the formal manipulations remain valid. However, the final formula (the virial relation) that one obtains requires less of the solution than the rigorous analysis asks for. In the above example we see that the virial relation makes sense (i.e., the integral in (1.5) is finite) if $\gamma \in H^1(\Omega_{N,l})$ and $F(\gamma) \in L^1(\Omega_{N,l})$. In our analysis we assume from a solution and nonlinearity only what is needed to make the (formally derived) virial relation a well-defined formula. This defines then the largest possible class of solutions (for a given nonlinearity) for which the virial relation is valid. To make the derivation rigorous we regularize the multiplier $\beta \mapsto \beta_\varepsilon$, $\beta_0 = \beta$, derive the virial relation corresponding to β_ε, and then regain the virial relation corresponding to β in the limit $\varepsilon \to 0$.

Because we allow for infinitely many frequencies in the almost periodic case, the construction of virial relations for almost periodic solutions differs from that for periodic and quasiperiodic solutions. But since periodic and quasiperiodic solutions are also almost periodic, the latter results include the former. However, the method of proof for quasiperiodic solutions is different than the proof for almost periodic solutions. We have included both proofs here because the method for quasiperiodic solutions could be adapted to handle other partial differential equations; those defined on open subsets Ω of ^{m} (bounded or unbounded) where the solutions are in $L^2(\Omega)$. The technical tools for this are a regularization of the solution and Lebesgue's dominated convergence theorem. The dominated convergence theorem cannot be applied to almost periodic solutions because they are not square integrable on $^{N+1}$. Instead, we utilize the uniform boundedness of almost periodic functions (cf. (1.2)) in conjunction with the uniform (in t) convergence of β_ε on compact subsets of ^{N}.

In the above example we used the function $\beta = x \cdot \nabla\gamma$ to derive a virial relation. One can view x as a vector field v on ^{N}: $v(x) = x$. Conversely, for any vector field v we may derive a virial relation by setting $\beta = v \cdot \nabla\gamma$. In our study we use this identification to characterize a class of virial relations. Furthermore, these vector fields generate flows $\Phi_\lambda : {}^N \to {}^N$ which in turn define transformation groups T_λ acting on functions via the formula $T_\lambda\varphi \equiv \varphi \circ \Phi_\lambda$. In general, every transformation group will lead to a virial relation via the infinitesimal variations of the action associated to NLW under this group. If T_λ is a symmetry of the action then there is a corresponding differential identity (a conservation law: Noether's theorem) which can be thought of as a "trivial" virial relation. This is discussed more thoroughly in the appendix.

We outline the contents of this paper. The next section deals with quasiperiodic solutions of NLW. We define quasiperiodic functions and the types of quasiperiodic solutions we will be concerned with, as well as discuss our hypothesis on the nonlinearity. Then we state and prove the main result of that section (Theorem 2.7): a class of virial relations for quasiperiodic solutions of NLW that are characterized by vector fields on ^{N}. In the following section, Section 3, we use these virial relations to derive several nonexistence results. Sections 4 and 5 deal with more general almost periodic solutions. The main results there, Theorems 4.10 and 5.1, characterize a class of virial relations for almost periodic solutions by vector fields on ^{N} and state a necessary condition for the existence of such solutions. In the appendix we show how virial relations may be derived by formulating NLW as a variational problem. This will elucidate the relationship between virial relations and conservation laws.

<u>Notation:</u> ∇ denotes the gradient operator on ^{N}: $\nabla = (\partial_1, ..., \partial_N)$, $\partial_i = \partial/\partial x_i$, and for multi-index $a \in {}^N \geq 0$, $\partial^a = \partial_1^{a_1}, ...\partial_N^{a_N}$. By the vector $\omega \in {}^l$ we will always mean an l-tuple of incommensurate numbers ("frequencies"). $\mathbf{1}$ will stand for the identity matrix. ^{l} will denote the l-torus $S^1 \times \cdots \times S^l$, and $\Omega_{N,l} \equiv {}^N \times {}^l$. Given a frequency vector $\omega \in {}^l$ we