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BEREZIN-TOEPLITZ QUANTIZATION OF COMPACT KÄHLER MANIFOLDS

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Abstract. In this lecture results are reviewed obtained by the author together with Martin Bordemann and Eckhard Meinrenken on the Berezin-Toeplitz quantization of compact Kähler manifolds. Using global Toeplitz operators, approximation results for the quantum operators are shown. From them it follows that the quantum operators have the correct classical limit. A star product deformation of the Poisson algebra is constructed.

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1. Introduction

Let me start with some mathematical aspects of quantization. As a mathematician, especially as an algebraic geometer, I find the following concepts very fascinating. Dear reader if you are a physicist or a fellow mathematician working in a different field (e.g. in measure theory) you will probably prefer other aspects of the quantization. So please excuse if these other important concepts are not covered here.

The arena of classical mechanics is as follows. One starts with a phase space M, which locally should represent position and momentum. We assume M to be a differentiable manifold. The physical observables are functions on M. One needs a symplectic form ω, a non-degenerate antisymmetric closed 2-form, which roughly speaking opens the

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possibility to introduce dynamics. This form defines a Poisson structure on M in the following way. One assign to every function f its Hamiltonian vector field X_f via

$$f \in C^\infty(M) \;\mapsto\; X_f, \quad \text{with } \; X_f \; \text{ defined by } \quad \omega(X_f, .) = df(.) \;.$$

A Lie algebra structure on $C^\infty(M)$ is now defined by the product

$$\{f, g\} := \omega(X_f, X_g) \;.$$

The Lie product fulfils the compatibility

$$\text{for all} \quad f, g, h \in C^\infty(M): \quad \{f \cdot g, h\} = f \cdot \{g, h\} + \{f, h\} \cdot g \;.$$

This says that $(C^\infty(M), \cdot, \{.., ..\})$ is a Poisson algebra. The pair (M, ω) is called a symplectic manifold. A Hamiltonian system (M, ω, H) is given by fixing a function $H \in C^\infty(M)$, the so called Hamiltonian function.

The first part of quantization (and only this step will be discussed here) consists in replacing the commutative algebra of functions by something noncommutative. But there is the fundamental requirement, that the classical situation (including the Poisson structure) should be recovered again as “limit”. There are some methods to achieve at least partially this goal. I do not want to give a review of these methods. Let me just mention a few. There is the “canonical quantization”, the deformation quantization using star product, geometric quantization, Berezin quantization using coherent states and Berezin symbols, Berezin-Toeplitz quantization, and so on. I am heading here for Berezin-Toeplitz quantization which has relations to the more known geometric quantization as introduced by Kostant and Souriau. In the following section I will recall some necessary definitions for the case I will consider later on. For a systematic treatment see [31], [35].

2. Geometric Quantization

Here I will assume (M, ω) to be a Kähler manifold, i.e. M is a complex manifold and ω a Kähler form. This says that ω is a positive, non-degenerate closed 2-form of type $(1, 1)$. If $\dim_{\mathbb{C}} M = n$ and $z_1, z_2, \ldots, z_n$ are local holomorphic coordinates then it can be written as

$$\omega = \mathrm{i} \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\overline{z}_j \;, \qquad g_{ij} \in C^\infty(M, \mathbb{C}) \;,$$

where the matrix $(g_{ij}(z))$ is for every z a positive definite hermitian matrix. Obviously (M, ω) is a symplectic manifold. A further data is (L, h, ∇) , with L a holomorphic line bundle, h a hermitian metric on L (conjugate-linear in the first argument), and ∇ a connection which is compatible with the metric and the complex structure. With respect to local holomorphic coordinates and with respect to a local holomorphic frame of the bundle it can be given as $\nabla = \partial + \partial \log h + \overline{\partial}$. The curvature of L is defined as

$$F(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \;.$$

Definition. The Kähler manifold (M,ω) is called quantizable, if there is such a triple (L,h,∇) with

$$F(X,Y) = -\,\mathrm{i}\,\omega(X,Y)\ . \tag{1}$$

The condition (1) is called the prequantum condition. The bundle (L,h,∇) is called a (pre)quantum line bundle. Usually we will drop h and ∇ in the notation.

Example 1. The flat complex space $\mathbb{C}^n$ with

$$\omega = \mathrm{i}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j\ .$$

Example 2. The Riemann sphere, the complex projective line, $\mathbb{P}(\mathbb{C}) = \mathbb{C}\cup\{\infty\} \cong S^2$. With respect to the quasi-global coordinate z the form can be given as

$$\omega = \frac{\mathrm{i}}{(1+z\overline{z})^2} dz\wedge d\overline{z}\ .$$

The quantum line bundle L is the hyperplane bundle. For the Poisson bracket one obtains

$$\{f,g\} = \mathrm{i}\,(1+z\overline{z})^2\left(\frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{z}}\right)\ .$$

Example 3. The (complex-) one dimensional torus M. Up to isomorphy it can be given as $M = \mathbb{C}/\Gamma_\tau$ where $\Gamma_\tau := \{n + m\tau \mid n,m\in\mathbb{Z}\}$ is a lattice with $\operatorname{Im}\tau > 0$. As Kähler form we take

$$\omega = \frac{\mathrm{i}\,\pi}{\operatorname{Im}\tau} dz\wedge d\overline{z}\ ,$$

with respect to the coordinate z on the covering space $\mathbb{C}$. The corresponding quantum line bundle is the theta line bundle of degree 1, i.e. the bundle whose global sections are multiples of the Riemann theta function.

Example 4. A compact Riemann surface M of genus $g\ge 2$. Such an M is the quotient of the open unit disc $\mathcal{E}$ in $\mathbb{C}$ under the fractional linear transformations of a Fuchsian subgroup of $SU(1,1)$. If $R = \begin{pmatrix} a & b\\ \overline{b} & \overline{a}\end{pmatrix}$ with $|a|^2 - |b|^2 = 1$ (as an element of $SU(1,1)$) then the action is

$$z \mapsto R(z) := \frac{az+b}{\overline{b}z+\overline{a}}\ .$$

The Kähler form
$$\omega = \frac{2\,\mathrm{i}}{(1-z\overline{z})^2} dz \wedge d\overline{z}$$
of $\mathcal{E}$ is invariant under the fractional linear transformations. Hence it defines a Kähler form on M. The quantum bundle is the canonical bundle, i.e. the bundle whose local sections are the holomorphic differentials. Its global sections can be identified with the automorphic forms of weight 2 with respect to the Fuchsian group.

Example 5. The complex projective space $\mathbb{P}^n(\mathbb{C})$. This generalizes Example 2. The points in $\mathbb{P}^n(\mathbb{C})$ are given by their homogeneous coordinates $(z_0 : z_1 : \ldots : z_n)$. In the affine chart with $z_0 \neq 0$ we take $w_j = z_j/z_0$ with $j = 1, \ldots, n$ as holomorphic coordinates. The Kähler form is the Fubini-Study fundamental form
$$\omega_{FS} := \mathrm{i}\,\frac{(1+|w|^2)\sum_{i=1}^n dw_i \wedge d\overline{w}_i - \sum_{i,j=1}^n \overline{w}_i w_j dw_i \wedge d\overline{w}_j}{(1+|w|^2)^2} \ .$$
The quantum line bundle is the hyperplane bundle H, i.e. the line bundle whose global holomorphic sections can be identified with the linear forms in the $n+1$ variables z_i.

Example 6. Projective Kähler submanifolds. Let M be a complex submanifold of $\mathbb{P}^N(\mathbb{C})$ and denote by $i : M \hookrightarrow \mathbb{P}^N(\mathbb{C})$ the embedding, then the pull-back of the Fubini-Study form $i^*(\omega_{FS}) = \omega_M$ is a Kähler form on M and the pull-back of the hyperplane bundle $i^*(H) = L$ is a quantum line bundle for the Kähler manifold (M, ω_M). Note that by general results $i(M)$ is an algebraic manifold.

There is an important observation. If M is a compact Kähler manifold which is quantizable then from the prequantum condition (1) we get for the Chern form of the line bundle the relation
$$c(L) = \frac{\mathrm{i}}{2\pi}F = \frac{\omega}{2\pi} \ .$$
This implies that L is a positive line bundle. In the terminology of algebraic geometry it is an ample line bundle. By the Kodaira embedding theorem M can be embedded (as algebraic submanifold) into projective space $\mathbb{P}^N(\mathbb{C})$ using a basis of the global holomorphic sections s_i of a suitable tensor power L^{m_0} of the bundle L
$$z \ \mapsto \ (s_0(z) : s_1(z) : \ldots : s_N(z)) \in \mathbb{P}^N(\mathbb{C}) \ .$$
These algebraic manifolds can be described as zero sets of homogeneous polynomials. Note that the dimension of the space $\Gamma_{hol}(M, L^{m_0})$ consisting of the global holomorphic sections of L^{m_0}, can be determined by the Theorem of Grothendieck-Hirzebruch-Riemann-Roch, see [19], [30]. So even if we start with an arbitrary Kähler manifold the quantization condition will force the manifold to be an algebraic manifold and we

are in the realm of algebraic geometry. This should be compared with the fact that there are "considerable more" Kähler manifolds than algebraic manifolds. This tight relation between quantization and algebraic geometry can also be found in the theory of coherent states as explained by A. Odijewicz [27] and S. Berceanu [4].

Here a warning is in order. With the help of the embedding into projective space we obtain by pull-back of the Fubini-Study form another Kähler form on M and by pull-back of the hyperplane bundle another quantum bundle on M. As holomorphic bundles the two bundles are the same, but in general the Kähler form and the metric of the bundle and hence the connection will be different. Essentially, these data will only coincide if M is a Kähler submanifold, or in other words if the embedding is an isometric Kähler embedding. The situation is very much related to Calabi's diastatic function [12], [11, 2nd ref.], see also Section 4.

Now we have to deal with the functions and how to assign operators to them. In geometric quantization such an assignment is given by

$$P : (C^\infty(M), \{.\,,.\}) \to \mathrm{End}(\Gamma_\infty(M,L), [.\,,.]), \quad f \mapsto P_f := -\nabla_{X_f} + \mathrm{i}\, f \cdot id \ .$$

Here $\Gamma_\infty(M,L)$ is the space of differentiable global sections of the bundle L. Due to the prequantum condition this is a Lie homomorphism.

Unfortunately one has too many degrees of freedom. The fields depend locally on position and momentum. Physical reasons imply that they should depend only on half of them. Such a choice of "half of the variables" is called a polarization. In general there is no unique choice of polarization. However, for Kähler manifolds there is a canonical choice of coordinates: the splitting into holomorphic and anti-holomorphic coordinates. To obtain a polarization we consider only sections which depend holomorphically on the coordinates. This is called the Kähler (or holomorphic) polarization.

If we denote by

$$\Pi : \Gamma_\infty(M,L) \to \Gamma_{hol}(M,L),$$

the projection operator from the space of differentiable sections onto the subspace consisting of holomorphic sections then the quantum operators are defined as

$$Q : C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M,L)), \qquad f \mapsto Q_f = \Pi\, P_f\, \Pi \ .$$

This map is still a linear map. But it is not a Lie homomorphism anymore.

3. Berezin-Toeplitz Quantization

Let the situation be as in the last section. We assume everywhere in the following that M is compact. We take $\Omega = \frac{1}{n!}\omega^n$ as volume form on M. On the space of section

$\Gamma_\infty(M,L)$ we have the scalar product

$$\langle\varphi,\psi\rangle := \int_M h(\varphi,\psi)\,\Omega\ , \qquad ||\varphi|| := \sqrt{\langle\varphi,\varphi\rangle}\ . \tag{2}$$

Let $\mathrm{L}^2(M,L)$ be the L^2-completion of the space of C^∞-sections of the bundle L and $\Gamma_{hol}(M,L)$ be its finite-dimensional closed subspace of holomorphic sections. Again let $\Pi : \mathrm{L}^2(M,L) \to \Gamma_{hol}(M,L)$ be the projection.

Definition. For $f \in C^\infty(M)$ the Toeplitz operator T_f is defined to be

$$T_f := \Pi\,(f\cdot) : \quad \Gamma_{hol}(M,L) \to \Gamma_{hol}(M,L)\ .$$

In words: One multiplies the holomorphic section with the differentiable function f. This yields only a differentiable section. To obtain a holomorphic section again, we have to project it back.

The linear map

$$T : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M,L)\big), \qquad f \to T_f\ ,$$

will be our Berezin-Toeplitz quantization. It is neither a Lie algebra homomorphism nor an associative algebra homomorphism, because in general

$$T_f\,T_g = \Pi\,(f\cdot)\,\Pi\,(g\cdot)\,\Pi \neq \Pi\,(fg\cdot)\,\Pi\ .$$

From the point of view of Berezin's approach [5], T_f is the operator with contravariant symbol f (see also [33]). At the end of this section I will give some more references.

Due to the compactness of M this defines a map from the commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. A lot of information will get lost. To recover this information one should consider not just the bundle L alone but all its tensor powers L^m and apply all the above constructions for every m. In this way one obtains a family of matrix algebras and maps

$$T^{(m)} : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M,L^m)\big), \qquad f \to T_f^{(m)}\ .$$

This infinite family should in some sense "approximate" the algebra $C^\infty(M)$.(See [7] for a definition of such an approximation.)

For the Riemann sphere $\mathbb{P}(\mathbb{C})$ we obtain with the help of an integral kernel the following explicit expression for the Toeplitz operator

$$(T_f^{(m)}s)(z) = \frac{m+1}{2\pi}\int_{\mathbb{C}} \frac{(1+z\overline{\zeta})^m f(\zeta)s(\zeta)}{(1+\zeta\overline{\zeta})^m}\,\frac{\mathrm{i}\,d\zeta\wedge d\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}\ .$$

Here the function s is representing a holomorphic section of L^m. The Toeplitz operator in our situation has always an integral kernel. Let $k(m) := \dim \Gamma_{hol}(M, L^m)$ and take an orthonormal basis $s_i,\ i = 1, \ldots, k(m)$ of the space $\Gamma_{hol}(M, L^m)$ then

$$(T_f^{(m)} s)(z) = \int_M \sum_{i=1}^{k(m)} h^{(m)}\big(s_i(w), f(w)s(w)\big) \cdot s_i(z)\ \Omega(w)\ . \tag{3}$$

These Toeplitz operators are still complicated but they are easier to handle than the quantum operators. For compact M we have the following relation

$$Q_f^{(m)} = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f} = \mathrm{i}\ \left(T_f^{(m)} - \frac{1}{2m} T^{(m)}_{\Delta f} \right)\ .$$

This is a result of Tuynman [32, Thm.2.1] reinterpreted in our context, see also [7]. Here the Laplacian Δ has to be calculated with respect to the metric $g(X,Y) = \omega(X, IY)$, where I is the complex structure. We see that for $m \to \infty$ the quantum operator of geometric quantization will asymptotically be equal to the quantum operator of the Berezin-Toeplitz quantization.

For the following let us assume that L is already very ample. This says that its global sections will already do the embedding. If this is not the case we would have to start with a certain m_0-tensor power of L and the form $m_0\,\omega$. The following three theorems were obtained in joint work with Martin Bordemann and Eckhard Meinrenken [8].

Theorem 1. *For every $f \in C^\infty(M)$ there is some $C > 0$ such that*

$$||f||_\infty - \frac{C}{m} \ \leq\ ||T_f^{(m)}|| \ \leq\ ||f||_\infty \quad \text{as} \quad m \to \infty\ .$$

Here $||f||_\infty$ is the sup-norm of f on M and $||T_f^{(m)}||$ is the operator norm on $\Gamma_{hol}(M, L^m)$. In particular, we have $\lim_{m\to\infty} ||T_f^{(m)}|| = ||f||_\infty$.

Theorem 2. *For every $f, g \in C^\infty(M)$ we have*

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T^{(m)}_{\{f,g\}}|| \quad = \quad O(\frac{1}{m}) \quad \text{as} \quad m \to \infty\ .$$

The proofs can be found in the above mentioned article [8]. I will give some ideas of them in the next section.

These theorems give two approximating sequences of maps

$$(C^\infty(M), ||..||_\infty) \to (\mathfrak{gl}(n, \mathbb{C}), ||..||_m := \frac{1}{m}||..||) \qquad f \mapsto \mathrm{i}\,m T_f^{(m)}, \quad f \mapsto m Q_f^{(m)}\ .$$

Restricted to real valued functions the maps take values in $\mathfrak{u}(k)$, for $k = \dim \Gamma_{hol}(M, L^m)$. These families of maps are only linear maps, not Lie homomorphism with respect to the Poisson bracket. But by Theorem 1 they are nontrivial and by Theorem 2 they are approximatively Lie homomorphisms. So every Poisson algebra of a Kähler manifold is a $\mathfrak{u}(k)$, $k \to \infty$ limit. This was a conjecture in [7] and our starting point was the aim to prove this conjecture. In [8] also a Egorov type theorem is presented.

If one puts $\hbar = \frac{1}{m}$ in Theorem 2 one can rewrite it as

$$\lim_{\hbar \to 0} || \frac{\mathrm{i}}{\hbar}[T_f^{(1/\hbar)}, T_g^{(1/\hbar)}] - T_{\{f,g\}}^{(1/\hbar)} || = 0 \ .$$

One should compare this with the definition of a star product deformation of $C^\infty(M)$ (see [3], [34]) based on the deformation theory of algebras as developed by Gerstenhaber. Because there are different variants let me recall the definition we are using.

Let $\mathcal{A} = C^\infty(M)[[\hbar]]$ be the algebra of formal power series in the variable $\hbar$ over the algebra $C^\infty(M)$. A product $*$ on $\mathcal{A}$ is called a (formal) star product if it is an associative $\mathbb{C}[[\hbar]]$-linear product such that

(1) $\mathcal{A}/\hbar\mathcal{A} \cong C^\infty(M)$, i.e. $\ f * g \bmod \hbar = f \cdot g$,

(2) $\dfrac{1}{\hbar}(f * g - g * f) \bmod \hbar = -\,\mathrm{i}\,\{f, g\}$.

Note that $f*g = \sum_{i=0}^{\infty} C_i(f,g)\hbar^i$ with $\mathbb{C}$-bilinear maps $C_i : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. With this we calculate

$$C_0(f,g) = f \cdot g, \quad \text{and} \quad C_1(f,g) - C_1(g,f) = -\,\mathrm{i}\,\{f,g\} \ . \tag{4}$$

Theorem 3. *There exists a unique (formal) star product on $C^\infty(M)$*

$$f * g := \sum_{j=0}^{\infty} \hbar^j C_j(f,g), \quad C_j(f,g) \in C^\infty(M), \tag{5}$$

in such a way that for $f, g \in C^\infty(M)$ and for every N we have

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| = K_N(f,g) \left(\frac{1}{m}\right)^N \tag{6}$$

for $m \to \infty$, with suitable constants $K_N(f,g)$.

We do not say anything about the convergence of the series (5). Hence we do not claim to obtain a "strict deformation quantization" as introduced by Rieffel [29].

We obtain a star product deformation not just by cohomological techniques as [16] but a geometrically induced one. There are other geometric constructions of a star product deformation for Poisson algebras. An important one is given by Fedosov [17].[1] (See Omori, Maeda, Yoshioka [28] and Karasev, Maslow [23] for related ones.) As pointed out by Deligne [15] it would be interesting to examine the relations between the two different approaches.

Here it is not the place and in fact I am not the expert to give a complete list of references on the Berezin-Toeplitz quantization. So let me just quote few of them. Berezin-Toeplitz quantization was mainly examined for certain complex symmetric domains. For older work see besides Berezin [5] also Berger-Coburn [6]. Similar results as stated in Theorem 1 and Theorem 2 were recently obtained in these cases. To give a few names: Klimek-Lesniewski [24], Borthwick-Lesniewski-Upmeier [9], Coburn [13], ... As I will explain in Section 4 the techniques in these cases are very different from ours. They will not work in the case of a general Kähler manifold. On the contrary, our methods are closely related to the compactness. So the results are at two different edges of the theory. Let me add that the case of compact Riemann surfaces of arbitrary genus has been proven by the "classical techniques" ([24, 2nd ref.] for $g \geq 2$ and [7] for $g = 1$). In some cases the relation to star product deformations have been studied [14].

Closely related to the Berezin-Toeplitz quantization is the quantization via Berezin's coherent states using the Berezin symbols [5] in the formulation of Cahen, Gutt and Rawnsley [11]. This technique was also used to define star products. See also the construction of star products by Moreno and Ortega-Navarro [26], [25]. For the idea of relating asymptotics to a deformation of the Poisson bracket see Karasev and Maslov [22]

Let me close this section with the remark that Berezin-Toeplitz quantization fits into the concept of "prime quantization" introduced by Ali and Doebner [2], [1].

4. Some remarks on the proofs

One way to prove the theorems is to represent the sections of L in a certain way, write down the projection operator as integral operator and calculate norms of the Toeplitz operators. This was done by Bordemann, Hoppe, Schaller and Schlichenmaier in [7] for the case of the n-dimensional complex torus using theta functions, and for the Riemann sphere (unpublished). For Riemann surfaces of genus $g \geq 2$ it was done by Klimek and Lesniewski [24] using automorphic forms. Similar techniques work for symmetric domains. In all these cases it was important that one could represent the sections as ordinary functions on some simple covering of the manifold under consideration.

[1] See also the Bourbaki exposé by Weinstein [34] and the review by Flato and Sternheimer [18].

For general Kähler manifolds this does not work. We need a different approach. The principal idea is to group all $T_f^{(m)}$ together to a global object. Take $(U,k) := (L^*, h^{-1})$ the dual of the quantum line bundle, Q the unit circle bundle inside U (with respect to the metric k) and $\tau : Q \to M$ the projection. Note that for the projective space the bundle U is just the tautological bundle whose fibre over the point $z \in \mathbb{P}^N(\mathbb{C})$ consists of the line in $\mathbb{C}^{N+1}$ which is represented by z. In particular the total space of U without the zero section can be identified with $\mathbb{C}^{N+1} \setminus \{0\}$.

Starting from the function $\hat{k}(\lambda) := k(\lambda,\lambda)$ on U we define $\tilde{a} := \dfrac{1}{2\,\mathrm{i}}(\partial - \overline{\partial})\log \hat{k}$ on $U \setminus 0$ (with respect to the complex structure on U) and restrict it to Q. Denote this restriction by α. Now $d\alpha = \tau_Q^*\omega$ (with $d = d_Q$) and $\nu = \frac{1}{2\pi}\tau^*\Omega \wedge \alpha$ is a volume form on Q. With respect to this form we take the L^2-completion $\mathrm{L}^2(Q,\nu)$ of the space of functions on Q. The generalized Hardy space $\mathcal{H}$ is the closure of the functions of $\mathrm{L}^2(Q,\nu)$ which can be extended to holomorphic functions on the whole disc bundle. The generalized Szegö projector is the projection $\Pi : \mathrm{L}^2(Q,\nu) \to \mathcal{H}$.

By the natural circle action Q is a S^1-bundle and the tensor powers of U can be viewed as associated bundles. The space $\mathcal{H}$ is preserved by this action. It is the (completed) direct sum $\mathcal{H} = \sum_{m=0}^{\infty} \mathcal{H}^{(m)}$ where $c \in S^1$ acts on $\mathcal{H}^{(m)}$ as multiplication by c^m. Sections of $L^m = U^{-m}$ can be identified with functions ϕ on Q which satisfy the equivariance condition $\phi(c\lambda) = c^m \phi(\lambda)$. This identification is an isometry. Hence, restricted to the holomorphic objects

$$\Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)} \ .$$

There is the notion of Toeplitz structure (Σ, Π) as developed by Guillemin and Boutet de Monvel in [10], [20]. Here is not the place to go into the details of the general definitions. Let me just explain what is needed here. Here Σ is the symplectic submanifold of the tangent bundle of Q with the zero section removed,

$$\Sigma = \{\ t\alpha(\lambda) \mid \lambda \in Q,\ t > 0\ \} \ \subset\ T^*Q \setminus 0\ ,$$

and Π is the above projection. A (generalized) Toeplitz operator of order k is an operator $A : \mathcal{H} \to \mathcal{H}$ of the form $A = \Pi \cdot R \cdot \Pi$ where R is a pseudodifferential operator (ΨDO) of order k on Q. The Toeplitz operators build a ring. The (principal) symbol of A is the restriction of the principal symbol of R (which lives on T^*Q) to Σ. Note that R is not fixed by A but Guillemin and Boutet de Monvel showed that the (principal) symbols are well-defined and that they obey the same rules as the symbols of ΨDOs

$$\sigma(A_1A_2) = \sigma(A_1)\sigma(A_2), \qquad \sigma([A_1,A_2]) = \mathrm{i}\,\{\sigma(A_1),\sigma(A_2)\}_\Sigma. \tag{7}$$

Here we use the 2-form $\omega_0 = \sum_i dq_i \wedge dp_i$ on T^*Q to define the Poisson bracket there. We are only dealing with two Toeplitz operators:

(1) The generator of the circle action gives the operator $D_\varphi = \dfrac{1}{\mathrm{i}}\dfrac{\partial}{\partial\varphi}$. It is an operator of order 1 with symbol t. It operates on $\mathcal{H}^{(m)}$ as multiplication by m.
(2) For $f \in C^\infty(M)$ let M_f be the multiplication operator on $\mathrm{L}^2(Q,\nu)$, i.e. $M_f(g)(\lambda) := f(\tau(\lambda))g(\lambda)$. We set $T_f = \Pi \cdot M_f \cdot \Pi : \mathcal{H} \to \mathcal{H}$. Because M_f is constant along the fibres, T_f commutes with the circle action. Hence $T_f = \bigoplus_{m=0}^{\infty} T_f^{(m)}$, where $T_f^{(m)}$ is the restriction of T_f to $\mathcal{H}^{(m)}$. After the identification of $\mathcal{H}^{(m)}$ with $\Gamma_{hol}(M,L^m)$ we see that these $T_f^{(m)}$ are exactly the Toeplitz operators $T_f^{(m)}$ introduced in Section 3. In this sense we call T_f also the global Toeplitz operator and the $T_f^{(m)}$ the local Toeplitz operators. T_f is an operator of order 0 and its symbol is just f pull-backed to Q and further to T^*Q (and restricted to Σ). Let us denote by $\tau_\Sigma^* : \Sigma \supseteq \tau^* Q \to Q \to M$ the composition then we obtain for its symbol $\sigma(T_f) = \tau_\Sigma^*(f)$.
This is the set-up more details can be found in [8].

Proof of Theorem 2. Now we are able to proof Theorem 2. The commutator $[T_f, T_g]$ is a Toeplitz operator of order -1. Using $\omega_{0|t\alpha(\lambda)} = -t\tau_\Sigma^*\omega$ for t a fixed positive number, we obtain[2] with (7) that its principal symbol is

$$\sigma([T_f,T_g])(t\alpha(\lambda)) = \mathrm{i}\,\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma(t\alpha(\lambda)) = -\,\mathrm{i}\, t^{-1}\{f,g\}_M(\tau(\lambda)) \ .$$

Now consider

$$A := D_\varphi^2\,[T_f,T_g] + \mathrm{i}\,D_\varphi\,T_{\{f,g\}} \ .$$

Formally this is an operator of order 1. Using $\sigma(T_{\{f,g\}}) = \tau_\Sigma^*\{f,g\}$ and $\sigma(D_\varphi) = t$ we see that its principal symbol vanishes. Hence it is an operator of order 0. Now M and hence Q are compact manifolds. This implies that A is a bounded operator (ΨDOs of order 0 are bounded). It is obviously S^1-invariant and we can write $A = \sum_{m=0}^\infty A^{(m)}$ where $A^{(m)}$ is the restriction of A on the space $\mathcal{H}^{(m)}$. For the norms we get $||A^{(m)}|| \le ||A||$. But

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m^2[T_f^{(m)}, T_g^{(m)}] + \mathrm{i}\,m T_{\{f,g\}}^{(m)}.$$

Taking the norm bound and dividing it by m we get the claim of Theorem 2. □

Proof of Theorem 3. This proof is a modification of the above approach. One constructs inductively $C_j(f,g) \in C^\infty(M)$ such that

$$A_N = D_\varphi^N T_f T_g - \sum_{j=0}^{N-1} D_\varphi^{N-j} T_{C_j(f,g)}$$

[2] Unfortunately, in [8] the minus sign was missing. This causes in Thm. 4.2 of that article also the wrong sign.

is a zero order Toeplitz operator. Because A_N is S^1-invariant and it is of zero order its principal symbol descends to a function on M. Take this function to be $C_N(f,g)$. Then $A_N - T_{C_N(f,g)}$ is of order -1 and $A_{N+1} = D_\varphi(A_N - T_{C_N(f,g)})$ is of order zero. The induction starts with $A_0 = T_f T_g$ which implies $\sigma(A_0) = \sigma(T_f)\sigma(T_g) = f \cdot g = C_0(f,g)$. As a zero order operator A_N is bounded, hence this is true for the component operators $A_N^{(m)}$. We obtain

$$||m^N T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{N-1} m^{N-j} T^{(m)}_{C_j(f,g)}|| \le ||A_N|| \ .$$

dividing this by m^N we obtain the asymptotics (6) of the theorem. Writing this explicitly for $N = 2$ we obtain for the pair (f,g)

$$||m^2 T_f^{(m)} T_g^{(m)} - m^2 T^{(m)}_{f\cdot g} - m T^{(m)}_{C_1(f,g)}|| \le K \ ,$$

and a similar expression for the pair (g,f). By subtracting the corresponding operators, using the triangle inequality, dividing by m and multiplying with i we obtain

$$||m\,\mathrm{i}\,(T_f^{(m)} T_g^{(m)} - T_g^{(m)} T_f^{(m)}) - T^{(m)}_{\mathrm{i}\,\left(C_1(f,g) - C_1(g,f)\right)}|| = O(\frac{1}{m}) \ .$$

With Theorem 2 this yields $||T^{(m)}_{\{f,g\}-\mathrm{i}\,\left(C_1(f,g)-C_1(g,f)\right)}|| = O(\frac{1}{m})$. But Theorem 1 says that the left hand side has as limit $|\{f,g\} - \mathrm{i}\,(C_1(f,g) - C_1(g,f))|_\infty$, hence $\{f,g\} = \mathrm{i}\,(C_1(f,g) - C_1(g,f))$. This shows equation (4). Uniqueness of the $C_N(f,g)$ follows inductively in the same way from (6), again using Theorem 1. The associativity follows from the definition by operator products. $\square$

Unfortunately, Theorem 1 has a rather complicated proof using Fourier integral operators, oscillatory integrals and Berezin's coherent states. (At least we have not been able to find a simpler one). For the special situation of projective Kähler submanifolds we have a much less involved proof, using Calabi's diastatic function.

Recall from Section 2 that a projective Kähler submanifold is a Kähler manifold M which can be embedded into projective space $\mathbb{P}^N(\mathbb{C})$ (with N suitable chosen) such that the Kähler form of M coincides with the pull-back of the Fubini-Study form. The pull-back of the tautological bundle is the dual of the quantum bundle. We denote this bundle by U. On the tautological bundle we have the standard hermitian metric $k(z,w) := \langle z, w\rangle = \bar{z}w$ in $\mathbb{C}^{N+1}$. By pull-back this defines a metric on U. Note that in this case the pull-back is essentially just the restriction of all objects to the submanifold. The Calabi (diastatic) function [12],[11, 2nd ref.] is defined as

$$D : M \times M \to \mathbb{R}_{\ge 0} \cup \{\infty\}, \qquad D(\tau(\lambda), \tau(\mu)) = -\log |k(\lambda,\mu)|^2$$

(where we have to choose λ and μ with $k(\lambda,\lambda)=k(\mu,\mu)=1$ representing the points of M). It is well-defined, vanishes only along the diagonal and is strictly positive outside the diagonal.

Proof of Theorem 1 for this case. First the easy part (which of course works in all cases). Note that $||T_f^{(m)}|| = ||\Pi^{(m)}\, M_f^{(m)}\, \Pi^{(m)}|| \le ||M_f^{(m)}||$ and for $\varphi\ne 0$

$$\frac{||M_f^{(m)}\varphi||^2}{||\varphi||^2} = \frac{\int_M h^{(m)}(f\varphi,f\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} = \frac{\int_M \overline{f(z)}f(z)h^{(m)}(\varphi,\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} \le ||f||_\infty^2\ .$$

Hence,

$$||T_f^{(m)}|| \le ||M_f^{(m)}|| = \sup_{\varphi\ne 0}\frac{||M_f^{(m)}\varphi||}{||\varphi||} \le ||f||_\infty .$$

To proof the first inequality, let $x_0\in M$ be a point where $|f|$ assumes its maximum, and fix a $\lambda_0\in\tau^{-1}(x_0)$ with $k(\lambda_0,\lambda_0)=1$. We define a sequence of holomorphic functions $\tilde{\Phi}^{(m)}$ by setting $\tilde{\Phi}^{(m)}(\lambda) := k(\lambda_0,\lambda)^m$. Because $\tilde{\Phi}^{(m)}(c\lambda) = c^m k(\lambda_0,\lambda)^m = c^m\tilde{\Phi}^{(m)}(\lambda)$ this defines an element $\Phi^{(m)}$ of $\Gamma_{hol}(M,L^m)$. Note that

$$h^m(\Phi^{(m)},\Phi^{(m)})(x) = \overline{\tilde{\Phi}^{(m)}(\lambda)}\tilde{\Phi}^{(m)}(\lambda) = \overline{k(\lambda_0,\lambda)}^m k(\lambda_0,\lambda)^m = \exp(-mD(x_0,x))$$

With Cauchy-Schwartz's inequality we obtain

$$\begin{aligned} ||T_f^{(m)}|| &\ge \frac{||T_f^{(m)}\Phi^{(m)}||}{||\Phi^{(m)}||} \ge \frac{|<\Phi^{(m)},T_f^{(m)}\Phi^{(m)}>|}{<\Phi^{(m)},\Phi^{(m)}>} \\ &= \frac{|\int_M f(x)h^m(\Phi^{(m)},\Phi^{(m)})(x)\Omega(x)|}{\int_M h^m(\Phi^{(m)},\Phi^{(m)})(x)\Omega(x)} = \frac{|\int_M f(x)e^{-mD(x_0,x)}\Omega(x)|}{\int_M e^{-mD(x_0,x)}\Omega(x)}\ . \end{aligned}$$

We want to consider the $m\to\infty$ limit. The part of the integral outside a small neighbourhood of x_0 will vanish exponentially. For the rest the stationary phase theorem [21] allows one to compute the asymptotics. The point $x=x_0$ is a zero of D and it is a non-degenerate critical point. Hence we obtain for the right hand side the asymptotic

$$\frac{|f(x_0)|+O(m^{-1})}{1+O(m^{-1})} = |f(x_0)|+O(m^{-1}),$$

and hence

$$||T_f^{(m)}|| \ge |f(x_0)|+O(m^{-1}) = ||f||_\infty + O(m^{-1})\ .\quad\square$$

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