ON LIEB-THIRRING INEQUALITIES FOR HIGHER ORDER OPERATORS WITH CRITICAL AND SUBCRITICAL POWERS.

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Abstract. Let $\varkappa_i(H_l(V))$ denote the negative eigenvalues of the operator $H_l(V)u := (-\Delta)^l u - V(x)u,\ V \geq 0,\ x \in \mathbb{R}^d$ on $L_2(\mathbb{R}^d)$. We prove the two-sided estimate

$$\tilde{\mathfrak{L}}(d,l)\int_{\mathbb{R}^d} V(x)dx \leq \sum_k |\varkappa_k(H_l(V))|^{1-\kappa} \leq \mathfrak{L}(d,l,1-\kappa)\int_{\mathbb{R}^d} V(x)dx,\quad \kappa = d/2l < 1.$$

We discuss bounds on the Riesz means $\sum_k |\varkappa_k(H_l(V))|^\mu$ if $0 < \mu < 1-\kappa$.

1. Introduction

1.1. We consider the quadratic form

$$\mathbf{h}_l(V)[u,u] := \int_{\mathbb{R}^d} |\nabla^l u|^2 dx - \int_{\mathbb{R}^d} V|u|^2 dx,\quad 0 \leq V \in L_1^{\mathrm{loc}}(\mathbb{R}^d),\quad l \in \mathbb{N}_+,$$

defined on functions $u \in C_0^\infty(\mathbb{R}^d)$. If the function V vanishes properly at infinity, this form can be closed. Its closure generates the self-adjoint operator

$$H_l(V) := (-\Delta)^l - V(x) \tag{1}$$

on $L_2(\mathbb{R}^d)$, the negative spectrum of which is discrete and bounded from below. Let $\{\varkappa_k(H_l(V))\}$ stand for the non-decreasing, finite or infinite sequence of the negative eigenvalues of the operator $H_l(V)$.

Estimates on the negative spectrum of operators $H_l(V)$ in terms of the potential V have been studied for many years, see e.g. [3], [6], [16], [15], [8], [13], [12], [9]. For given d, l define

$$\kappa = \kappa(d,l) := \frac{d}{2l},\qquad \nu = \nu(d,l) := 1 - \frac{d}{2l}. \tag{2}$$

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In [14] Lieb and Thirring proved the inequalities

$$S_{l,\mu}(V) := \sum_k |\varkappa_k(H_l(V))|^\mu \le \mathfrak{L}(d,l,\mu) \int_{\mathbb{R}^d} V^{\mu+\kappa}(x)dx, \tag{3}$$

for potentials $0 \le V \in L_{\mu+\kappa}(\mathbb{R}^d)$ with $\mu > \max\{0,\nu\}$ in the case $l = 1$. Their argument can easily be extended to arbitrary $l \in \mathbb{N}_+$. In [15], [8] and [13] the respective inequality was shown for $\mu = 0$ if $\nu < 0$. On the other hand it is known, that (3) fails for $0 \le \mu < \nu$ if $\nu > 0$ and for $\mu = 0$ if $\nu = 0$. In [19] the author verified (3) for $l = d = 1$ and $\mu = \nu(1,1) = 1/2$, where in fact the two-sided estimate

$$\frac{1}{4}\int_{\mathbb{R}} V dx \le S_{1,1/2}(V) \le 1.005 \int_{\mathbb{R}} V dx, \quad d = l = 1, \ 0 \le V \in L_1(\mathbb{R}), \tag{4}$$

holds, cf. [10]. In this note we prove (3) for the remaining case of a positive critical power $\mu = \nu(d,l) > 0$ for arbitrary $d, \ l \in \mathbb{N}_+$, such that $2l > d$. In analogy to (4) we find a two-sided estimate

$$\tilde{\mathfrak{L}}(d,l) \int_{\mathbb{R}^d} V(x)dx \le S_{l,\nu}(V) \le \mathfrak{L}(d,l,\nu) \int_{\mathbb{R}^d} V(x)dx, \tag{5}$$

which holds for all sumable, non-negative potentials $0 \le V \in L_1(\mathbb{R}^d)$ with certain constants $0 < \tilde{\mathfrak{L}}(d,l) \le \mathfrak{L}(d,l,\nu) < \infty$.

It is well-known that (3) is of sharp order in the limit of large potentials. This follows from the Weyl type asymptotical formula

$$S_{l,\mu}(\alpha V) = \alpha \mathfrak{L}^{\mathrm{cl}}(d,l,\mu) \int_{\mathbb{R}^d} V^{\mu+\kappa} dx + o(\alpha) \quad \text{as } \alpha \to \infty, \tag{6}$$

$$\mathfrak{L}^{\mathrm{cl}}(d,l,\mu) = \frac{\mu\Gamma(\mu)\Gamma(\kappa+1)}{2^d \pi^{d/2}\Gamma(\frac{d}{2}+1)\Gamma(\kappa+\mu+1)}, \quad \mu \ge 0, \tag{7}$$

which can be obtained for sufficiently regular non-negative potentials, and which can be closed to all potentials $0 \le V \in L_{\mu+\kappa}(\mathbb{R}^d)$ if (3) holds. On the other hand for $\nu > 0$ the operator $H_l(\alpha V), \ 0 \le V, 0 \not\equiv V$ has negative spectrum for arbitrary small $\alpha > 0$, and for sufficiently regular, non-negative potentials the asymptotics

$$S_{l,\mu}(\alpha V) = \big(\alpha \mathfrak{L}^0(d,l,\nu) \int_{\mathbb{R}^d} V dx\big)^{\mu/\nu} + o(\alpha^{\mu/\nu}) \quad \text{as} \quad \alpha \to 0, \ \mu > 0, \tag{8}$$

$$\mathfrak{L}^0(d,l,\nu) = \frac{\pi\kappa}{\sin \pi\kappa} \mathfrak{L}^{\mathrm{cl}}(d,l,0),$$

can be calculated. [1] In the case of a positive critical power $\mu = \nu > 0$ this asymptotics is of the same type as (5), and we can close (8) with

[1] We attach the proof of (8) in the Appendix.

$\mu/\nu = 1$ to all potentials $0 \le V \in L_1(\mathbb{R}^d)$. Comparing (8) and (6) we see, that a two-sided estimate can hold only in the critical case.

Naturally formula (6) agrees with the estimate (3) for supercritical powers $0 < \nu < \mu$. However, in the scale of subcritical powers $0 < \mu < \nu$ we find $\mu/\nu < \mu+\kappa$, and (6) disproves (3). Hence a proper substitute of (3) for positive subcritical powers should contain two terms on the right hand side: one of homogeneity order μ/ν serving for small coupling constants, and one of Weyl type order $\mu + \kappa$, serving as $\alpha \to \infty$. In the final section of this paper we shall prove such estimates.

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1.2. **Notations.** Below $\mathbb{Q}^d = \{x = (x_1, ..., x_d) \in \mathbb{R}^d : |x_j| \le 1/2, j = 1, ..., d\}$. Moreover $\mathbb{N} = \{n \in \mathbb{Z} : n \ge 0\}$, while $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$. For a multiindex $\boldsymbol{\iota} \in \mathbb{N}^d$ we use the notations $|\boldsymbol{\iota}| = \sum_{j=1}^d \iota_j$ and $\boldsymbol{\iota}! = \prod_{j=1}^d \iota_j!$. The vector ∇^k consists of the elements $\sqrt{\frac{k!}{\boldsymbol{\iota}!}}\frac{\partial^{\boldsymbol{\iota}}}{\partial x^{\boldsymbol{\iota}}}$ with $|\boldsymbol{\iota}| = k$. Further $\Omega_{d,k}$ stands for the $\binom{k+d}{d}$-dimensional lineal of all polynomials over $\mathbb{R}^d$, the order of which does not exceed k.

Throughout the paper κ and ν are defined as in (2).

Finally, if the self-adjoint operator T is semi-bounded from below and its lower portion of the spectrum is discrete, then $\{\varkappa_k(T)\}$ denotes the non-decreasing sequence of the respective eigenvalues (according to their multiplicity).

2. The Lieb-Thirring inequality for positive critical powers

2.1. **Main result.** In this section we shall prove

Theorem 1. *Assume $d, l \in \mathbb{N}_+$ and $\nu = 1 - d/2l > 0$. Then for all potentials $V(x) \ge 0$, $V \in L_1(\mathbb{R}^d)$, the inequality*

$$\tilde{\mathfrak{L}}(d,l)\int_{\mathbb{R}^d} V\,dx \le S_{l,\nu}(V) \le \mathfrak{L}(d,l,\nu)\int_{\mathbb{R}^d} V\,dx \tag{9}$$

holds.

2.2. **Two covering Lemmata.** We introduce

Definition 1. *Let $0 \le V(x) \in L_1(\mathbb{R}^d)$ have compact support. A family $\mathbf{Q} = \{\mathcal{Q}_\tau\}$ of cubes $\mathcal{Q}_\tau = x_\tau + a_\tau\mathbb{Q}^d$, $x_\tau \in \mathbb{R}^d$, $a_\tau > 0$, is called a A-proper covering of* supp V *of multiplicity $\Xi(\mathbf{Q})$, if* supp$V \subseteq \cup_\tau \mathcal{Q}_\tau$,

$$a_\tau^{2l-d}\int_{\mathcal{Q}_\tau} V\,dx = A, \quad A > 0, \quad \textit{and} \quad \Xi(\mathbf{Q}) := \sup_{x\in\mathbb{R}^d} \sum_{\tau: x \in \mathrm{int}\ \mathcal{Q}_\tau} 1 \ < \ \infty. \tag{10}$$

The following result dates back to Besikovic [5]. For the convenience of the reader we give its proof and follow the argument of de Guzman [11].

Lemma 1. *For each non-trivial potential* $0 \le V \in L_1(\mathbb{R}^d)$ *of compact support and any fixed* $A > 0$ *there exists some finite A-proper covering* $\mathbf{Q}(V)$ *of* supp V *of multiplicity* $\Xi(\mathbf{Q}(V)) \le 2^d$.

Proof. We can assume $V \not\equiv 0$. Then for each $x \in \mathbb{R}^d$ there exists a unique $a(x) > 0$, such that for $\mathcal{Q}_x = x + a(x)\mathbb{Q}^d$ the equality

$$a^{2l-d}(x) \int_{\mathcal{Q}_x} V dx = A$$

holds. The function $a : \mathbb{R}^d \to \mathbb{R}_+$ is continuous and bounded from below by

$$a(x) \ge \left(A^{-1} \int_{\mathbb{R}^d} V dx \right)^{\frac{1}{d-2l}} > 0. \tag{11}$$

Choose $\tilde{\mathbf{Q}} = \{\mathcal{Q}_x : x \in \text{supp } V\}$. We shall select the sought finite proper covering as an appropriate subset from $\tilde{\mathbf{Q}}$. Assume we have already chosen the points $x_i,\ i = 1, ..., m$ and the respective cubes $\mathcal{Q}_{x_i}$. Then let x_{m+1} be one of the points x, where the continuous function $a(x)$ achieves its maximum value on the compact set $x \in \text{supp}V \setminus \cup_{i=1}^m \text{int}\mathcal{Q}_{x_i}$. Since the interiors of the cubes $x_i + \frac{a(x_i)}{2}\mathbb{Q}^d$ do not intersect each other, by (11) this process stops after a finite number of iterations, and we put $\mathbf{Q}(V) = \{\mathcal{Q}_{x_i}\}$.

Evidently $\text{supp}V \subseteq \cup_i \mathcal{Q}_{x_i}$. Let us show, that $\Xi(\mathbf{Q}(V)) \le 2^d$. Each of the points x_i does not belong to the interior of any other cube than $\mathcal{Q}_{x_i}$. Fix some point $y \in \mathbb{R}^d,\ y \ne x_i$. Assume $y \in \cap_{k=1}^r \text{int}\mathcal{Q}_{x_{i_r}}$ with $x_{i_p} \ne x_{i_q}$ for all $1 \le p \ne q \le r$ and $r > 2^d$. Let $\vec{\iota} = (\iota_1, ..., \iota_d)$ denote vectors of the type $\iota_k \in \{0, 1\},\ k = 1, ..., d$. Then one of the 2^d sectors $\Sigma_{y,\vec{\iota}} := y + \otimes_{k=1}^d [0, (-1)^{\iota_k}\infty)$ should contain more then one of the points $x_{i_p},\ p = 1, ..., r$. On the other hand, if $x_{i_p}, x_{i_q} \in \Sigma_{y,\vec{\iota}},\ |y - x_{i_p}| \le |y - x_{i_q}|, p \ne q$, and $y \in \text{int}\mathcal{Q}_{x_{i_p}} \cap \text{int}\mathcal{Q}_{x_{i_q}}$, then $x_{i_p} \in \text{int}\mathcal{Q}_{x_{i_q}}$, what contradicts to the construction. Thus $r \le 2^d$. □

We supplement Lemma 1 by

Lemma 2. *Assume* $2l > d$. *Then there exists a positive constant* $\tilde{c}(d, l)$ *such, that from each finite A-proper covering* $\mathbf{Q}(V) = \{\mathcal{Q}_i\}_{i=1}^n,\ \mathcal{Q}_i = x_i + a_i\mathbb{Q}$ *of* supp V *of multiplicity* $\Xi(\mathbf{Q}(V)) \le 2^d$ *for a non-trivial potential* $0 \le V \in L_1(\mathbb{R}^d)$ *of compact support one can extract a subset* $\mathbf{Q}^\sharp(V) = \{\mathcal{Q}_i\}_{i\in I},\ I \subseteq \{1, ..., n\}$ *with the properties*

$$x_i + 2a_i\mathbb{Q} \cap x_j + 2a_j\mathbb{Q} = \emptyset \quad \textit{for all} \ \ i \ne j,\ i, j \in I, \tag{12}$$

$$\sum_{i\in I}\int_{\mathcal{Q}_i} V\,dx \geq \tilde{c}(d,l)\int_{\mathbb{R}^d} V\,dx. \tag{13}$$

Proof. Put $I_0 = J_0 = \emptyset,\ M_0 = \{1, ..., n\}$. Assume the sets I_k, J_k, M_k have already been constructed. If $M_k = \emptyset$ we abbreviate the process and take $I = I_k$. Otherwise choose i_{k+1} such that $a_{i_{k+1}} = \min_{j\in M_k} a_j$, and take

$$I_{k+1} = I_k \cup \{i_{k+1}\},\quad J_{k+1} = \{j \in M_k : x_j + 2a_j\mathbb{Q} \cap x_{i_{k+1}} + 2a_{i_{k+1}}\mathbb{Q} \neq \emptyset\}\setminus\{i_{k+1}\},$$

$$M_{k+1} = M_k\setminus\Big(J_{k+1}\cup\{i_{k+1}\}\Big).$$

Obviously $x_{i'} + 2a_{i'}\mathbb{Q} \cap x_{i''} + 2a_{i''}\mathbb{Q} = \emptyset$ for all $i', i'' \in I$. Moreover notice, that $a_j \geq a_{i_k}$ for $j \in J_k$. Thus we can decompose J_k as

$$J_k = \cup_{m\in\mathbb{N}} J_k^m,\quad J_k^m = \{j \in J_k : 2^m a_{i_k} \leq a_j < 2^{m+1}a_{i_k}\}.$$

If $j \in J_k^m$, then $\mathcal{Q}_j \subset x_{i_k} + (1 + 3\cdot 2^m)a_{i_k}\mathbb{Q}$. Since $\Xi(\mathbf{Q}(V)) \leq 2^d$ and $\operatorname{vol}\mathcal{Q}_j \geq 2^{md}a_{i_k}^d$ we find

$$\operatorname{card} J_k^m \leq \frac{(1 + 3\cdot 2^m)^d 2^d}{2^{md}} \leq 8^d.$$

Moreover

$$\sum_{j\in J_k}\int_{\mathcal{Q}_j} V\,dx = \sum_{m\in\mathbb{N}}\sum_{j\in J_k^m} Aa_j^{d-2l} \leq 8^d \sum_{m\in\mathbb{N}} 2^{m(d-2l)}Aa_{i_k}^{d-2l} = \frac{8^d}{1-2^{d-2l}}\int_{\mathcal{Q}_{i_k}} V\,dx.$$

Since $2l > d$ we conclude

$$\int_{\mathbb{R}^d} V\,dx \leq \sum_{i=1}^{n}\int_{\mathcal{Q}_i} V\,dx = \sum_{k}^{\operatorname{card} I}\Big(\int_{\mathcal{Q}_{i_k}} V\,dx + \sum_{j\in J_k}\int_{\mathcal{Q}_j} V\,dx\Big) \leq \frac{1}{\tilde{c}(d,l)}\sum_k\int_{\mathcal{Q}_{i_k}} V\,dx$$

with $\tilde{c}(d,l) = (1 - 2^{d-2l})/(1 - 2^{d-2l} + 8^d) > 0$. $\square$

2.3. **The negative spectrum of the "Neumann" problem on the cube.** In what follows put $\mathcal{Q} = a\mathbb{Q}^d$ for some $a > 0$. Let $H_{l,\mathcal{Q}}^N(V)$ be the self-adjoint operator on $L_2(\mathbb{Q})$, corresponding to the closure of the hermitian form

$$\mathbf{h}_{l,\mathcal{Q}}^N(V)[u,u] := \int_{\mathcal{Q}} |\nabla^l u|^2 dx - \int_{\mathcal{Q}} V|u|^2 dx,\ \ 0 \leq V \in L_1(\mathcal{Q}),\ \ u \in C^\infty(\mathcal{Q}).$$

For the negative spectrum of this operator the following standard fact holds.

Lemma 3. *Assume $2l > d,\ l, d \in \mathbb{N}_+$. Then there exists a positive finite constant $\hat{c}(d,l)$ such, that for all potentials $0 \leq V \in L_1(\mathcal{Q})$ with*

$$\hat{c}(d,l)a^{2l-d}\int_{\mathcal{Q}} V\,dx \leq 1,\quad \mathcal{Q} = a\mathbb{Q}^d,\quad a > 0, \tag{14}$$

the operator $H^N_{l,\mathcal{Q}}(V)$ has not more than $\binom{l+d-1}{d}$ negative eigenvalues.

Proof. By homogeneity we can take $\hat{c}(d,l)$ as the sharp constant in the inequality

$$|u(x)|^2 \le a^{2l-d}\hat{c}(d,l)\int_{\mathcal{Q}} |\nabla^l u(x)|^2, \quad \mathcal{Q} = a\mathbb{Q}^d, \quad a>0, \quad u \in W_2^l(\mathcal{Q}) \ominus_{L_2(\mathcal{Q})} \Omega_{d,l-1}|_{\mathcal{Q}}, \tag{15}$$

which holds in view of the Sobolev embedding for $2l > d$ and the Theorem on equivalent norms. Because of (14) and (15) the form $\mathbf{h}^N_{l,\mathcal{Q}}[u,u]$ is non-negative on $u \in C_0^\infty(\mathcal{Q}) \ominus_{L_2(\mathcal{Q})} \Omega_{d,l-1}$. This subspace is of codimension $\binom{l+d-1}{d}$ in $L_2(\mathcal{Q})$, what by Glazmanns Lemma completes the proof. □

2.4. **The Birman-Schwinger principle for $H^N_{l,\mathcal{Q}}(V)$.** If $2l > d$ the resolvent of the unperturbed operator $H^N_{l,\mathcal{Q}}(0)$

$$\big((H^N_{l,\mathcal{Q}}(0) - \varkappa)^{-1}u\big)(x) = \int_{\mathcal{Q}} G_{\mathcal{Q}}(x,z,\varkappa)u(z)dz$$

is an integral operator with a bounded continuous kernel $G_{\mathcal{Q}}(x,z,\varkappa) \in C(\mathcal{Q}\times\mathcal{Q})$ for any $\varkappa < 0$, see [1]. The Green function $G_{\mathcal{Q}}(x,z,\varkappa)$ obeys the homogeneity property

$$G_{\mathcal{Q}}(x,z,\varkappa) = a^{2l-d}G^d_{\mathbb{Q}}(a^{-1}x, a^{-1}z, a^{2l}\varkappa), \quad \mathcal{Q} = a\mathbb{Q}^d, \quad a>0, \ \varkappa<0. \tag{16}$$

¿From Hilberts resolvent identity one immediately concludes, that

$$\mathcal{G}_{\mathcal{Q}}(\varkappa) := \max_{x\in\mathcal{Q}} G_{\mathcal{Q}}(x,x,\varkappa)$$

is a continuous, strongly increasing function in $\varkappa < 0$. Moreover

$$\mathcal{G}_{\mathcal{Q}}(\varkappa) \to 0 \ \text{ as } \ \varkappa \to -\infty, \qquad \mathcal{G}_{\mathcal{Q}}(\varkappa) \to +\infty \ \text{ as } \ \varkappa \to -0,$$

while (16) implies

$$\mathcal{G}_{\mathcal{Q}}(\varkappa) = a^{2l-d}\mathcal{G}^d_{\mathbb{Q}}(a^{2l}\varkappa), \quad \mathcal{Q} = a\mathbb{Q}^d, \quad a>0. \tag{17}$$

Now let $\{\varkappa_k(H^N_{l,\mathcal{Q}}(V))\}_k$ denote the non-decreasing sequence of eigenvalues of $H^N_{l,\mathcal{Q}}(V)$. Consider the counting function

$$N(\varkappa, H^N_{l,\mathcal{Q}}(V)) := \sum 1 : \{k : \varkappa_k(H^N_{l,\mathcal{Q}}(V)) < \varkappa\}, \quad \varkappa < 0,$$

for the common multiplicity of the spectrum of $H^N_{l,\mathcal{Q}}(V)$ below $\varkappa < 0$. According to the Birman-Schwinger principle [6], [16] this quantity can be estimated by

$$N(\varkappa, H^N_{l,\mathcal{Q}}(V)) \le \mathrm{Tr}\left\{V^{1/2}(x)\int_{\mathcal{Q}} G_{\mathcal{Q}}(x,z,\varkappa)V^{1/2}(z)\cdot dz\right\}$$
$$\le \mathcal{G}_{\mathcal{Q}}(\varkappa)\int_{\mathcal{Q}} V(x)dx = a^{2l-d}\mathcal{G}^d_{\mathbb{Q}}(a^{2l}\varkappa)\int_{\mathcal{Q}} V(x)dx. \quad (18)$$

If we put $\varkappa = \varkappa_1(H^N_{l,\mathcal{Q}}(V)) + 0$, we find

$$1/\mathcal{G}_{\mathbb{Q}^d}(a^{2l}\varkappa_1(H^N_{l,\mathcal{Q}}(V))) \le a^{2l-d}\int_{\mathcal{Q}} V dx.$$

The monotone decreasing continuous function $1/\mathcal{G}^d_{\mathbb{Q}} : \mathbb{R}^0_- \to \mathbb{R}^0_+$ has the monotone decreasing inverse $\mathcal{F} : \mathbb{R}^0_+ \to \mathbb{R}^0_-$. Thus for the lowest eigenvalue the estimate

$$|\varkappa_1(H^N_{l,\mathcal{Q}}(V))|^\nu \le a^{d-2l}\left|\mathcal{F}\left(a^{2l-d}\int_{\mathcal{Q}} V dx\right)\right|^\nu, \quad \nu = 1 - \frac{d}{2l}, \quad (19)$$

holds.

2.5. **Proof of Theorem 1 - The estimate from above.** We start with potentials $0 \le V \in L_1(\mathbb{R}^d)$ with compact support. Let $\mathbf{Q}(V) = \{\mathcal{Q}_{x_1}, ..., \mathcal{Q}_{x_m}\}$ be a $A-$proper finite covering of supp V with multiplicity $\Xi(\mathbf{Q}(V)) \le 2^d$ and $A = 2^{-d}/\hat{c}(d,l)$. According to (14), (19) and (10) each of the operators $H^N_{l,\mathcal{Q}_{x_i}}(2^d V)$ has not more than $\binom{l+d-1}{d}$ negative eigenvalues $\varkappa_j(H^N_{l,\mathcal{Q}_{x_i}}(2^d V))$. Put $J(i) = \{j : \varkappa_j(H^N_{l,\mathcal{Q}_{x_i}}(2^d V)) < 0\}$. Then

$$\sum_{j\in J(i)} |\varkappa_j(H^N_{l,\mathcal{Q}_{x_i}}(2^d V))|^\nu \le 2^d\binom{l+d-1}{d}\hat{c}(d,l)|\mathcal{F}(\hat{c}^{-1}(d,l))|^\nu \int_{\mathcal{Q}_{x_i}} V dx. \quad (20)$$

Using the variational principle and the estimate on the multiplicity of the covering it is easy to verify, that

$$\varkappa_k(H_l(V)) \ge \varkappa_k(\hat{H}) \quad \text{for all } k: \ \varkappa_k(H_l(V)) < 0, \quad \hat{H} := \oplus_i H^N_{l,\mathcal{Q}_{x_i}}(2^d V), \quad (21)$$

where $\hat{H}$ acts on $\oplus_i L_2(\mathcal{Q}_{x_i})$. The negative eigenvalues $\{\varkappa_k(\hat{H})\}$ of $\hat{H}$ coincides as set and in its multiplicity with the union of the sets $\{\varkappa_j(H^N_{l,\mathcal{Q}_{x_i}}(V)) < 0\}$. For the sum of powers of negative eigenvalues of H_l this implies

$$\sum_k |\varkappa_k(H_l(V))|^\nu \le \sum_{k:\varkappa_k(\hat{H})<0} |\varkappa_k(\hat{H})|^\nu = \sum_{i,j\in J(i)} |\varkappa_j(H^N_{l,\mathcal{Q}_{x_i}}(2^d V))|^\nu \le \mathfrak{L}(d,l,\nu)\int_{\mathbb{R}^d} V dx.$$

The constant on the r.h.s. does not depend on the support of V. A standard argument allows one to close this inequality to all potentials $0 \le V \in L_1(\mathbb{R}^d)$.□

2.6. **Proof of Theorem 1 - The estimate from below.** Let $\hat{\mathcal{Q}}$ be some cube in $\mathbb{R}^d$ and let $H^D_{l,\hat{\mathcal{Q}}}(V)$ be the self-adjoint operator on $L_2(\hat{\mathcal{Q}})$, corresponding to the closure of the hermitian form

$$\mathbf{h}^D_{l,\hat{\mathcal{Q}}}(V)[u,u] := \int_{\hat{\mathcal{Q}}} |\nabla^l u|^2 dx - \int_{\hat{\mathcal{Q}}} V|u|^2 dx, \ \ 0 \le V \in L_1(\hat{\mathcal{Q}}), \ \ u \in C_0^\infty(\hat{\mathcal{Q}}).$$

Below $\{\varkappa_k(H^D_{l,\hat{\mathcal{Q}}}(V))\}_k$ denotes the non-decreasing sequence of eigenvalues of $H^D_{l,\hat{\mathcal{Q}}}(V)$. Fix a function $\psi \in C_0^\infty(2\mathbb{Q})$, such that $\psi \equiv 1$ on $\mathbb{Q}$. Put

$$\varsigma := \int_{2\mathbb{Q}} |\nabla^l \psi|^2 dx, \quad \vartheta := \int_{2\mathbb{Q}} |\psi|^2 dx.$$

For the lowest eigenvalue of $H^D_{l,\hat{\mathcal{Q}}}(V)$ with $\mathcal{Q} = a\mathbb{Q} + y, \ \hat{\mathcal{Q}} = 2a\mathbb{Q} + y, \ a > 0, \ y \in \mathbb{R}^d$ the variational estimate

$$\varkappa_1(H^D_{l,\hat{\mathcal{Q}}}(V)) \le \frac{\int_{\hat{\mathcal{Q}}} |\nabla^l \psi(a^{-1}(x-y))|dx - \int_{\hat{\mathcal{Q}}} V|\psi(a^{-1}(x-y))|^2 dx}{\int_{\hat{\mathcal{Q}}} |\psi(a^{-1}(x-y)|^2 dx} \le \frac{a^{d-2l}\varsigma - \int_{\mathcal{Q}} V dx}{a^d \vartheta} \tag{22}$$

holds.

For potentials $0 \le V \in L_1(\mathbb{R}^d)$ of compact support we choose a finite $\kappa^{-1}\varsigma$-proper covering of the support of V, and according to Lemma 2 extract the subset

$$\mathbf{Q}^\sharp(V) = \{\mathcal{Q}_i\}_{i \in I}, \quad \mathcal{Q}_i = x_i + a_i \mathbb{Q},$$

with the properties (12), (13). ¿From the variational principle we find, that

$$\varkappa_k(H_l(V)) \le \varkappa_k(\tilde{H}) \quad \text{for all} \ \ k: \ \varkappa_k(H_l(V)) < 0, \quad \tilde{H} := \oplus_{i \in I} H^D_{l,\tilde{\mathcal{Q}}_{x_i}}(V), \tag{23}$$

where $\tilde{H}$ acts on $L_2(\cup_{i \in I} \tilde{\mathcal{Q}}_{x_i})$ with $\tilde{\mathcal{Q}}_i = x_i + 2a_i\mathbb{Q}$ as $i \in I$. For $\hat{\mathcal{Q}} = \tilde{\mathcal{Q}}_i$ (22) turns into

$$\varkappa_1(H^D_{l,\tilde{\mathcal{Q}}_i}(V)) \le -\vartheta^{-1}\nu\kappa^{\kappa/\nu}\varsigma^{-\kappa/\nu}\Big(\int_{\mathcal{Q}_i} V dx\Big)^{1/\nu}.$$

The quantity on the r.h.s. is negative, thus $\varkappa_1(H^D_{l,\tilde{\mathcal{Q}}_i}(V)) < 0$ and

$$|\varkappa_1(H^D_{l,\tilde{\mathcal{Q}}_i}(V))|^\nu \ge \vartheta^{-\nu}\nu^\nu\kappa^\kappa\varsigma^{-\kappa} \int_{\mathcal{Q}_i} V dx. \tag{24}$$

Hence from (24) and (13) we conclude

$$\sum_k |\varkappa_k(H_l(V))|^\nu \geq \sum_{k:\varkappa_k(\tilde{H})<0} |\varkappa_k(\tilde{H})|^\nu \geq \sum_{i\in I} |\varkappa_1(H^D_{l,\tilde{\mathcal{Q}}_{x_i}}(V))|^\nu$$

$$\geq \vartheta^{-\nu}\nu^\nu\kappa^\kappa\varsigma^{-\kappa}\sum_{i\in I}\int_{\mathcal{Q}_i} V\,dx \geq \tilde{\mathfrak{L}}(d,l)\int_{\mathbb{R}^d} V\,dx,$$

with

$$\tilde{\mathfrak{L}}(d,l) = \tilde{c}(d,l)\vartheta^{-\nu}\nu^\nu\kappa^\kappa\varsigma^{-\kappa} > 0.$$

Closing this estimate to all $0 \leq V \in L_1(\mathbb{R}^d)$, we complete the proof of Theorem 1. $\square$

2.7. **Positive supercritical powers.** Following an argument of Lieb and Aizenmann one can easily show, that Theorem 1 implies

$$S_{l,\mu}(V) := \sum_k |\varkappa_k(H_l(V))|^\mu \leq \mathfrak{L}(d,l,\mu)\int_{\mathbb{R}^d} V^{\mu+\kappa}(x)dx$$

for all powers $\mu \geq \nu > 0$. As usual the condition $V \geq 0$ in the r.h.s. of Theorem 1 can be dropped, if we substitute V by $\max\{V(x),0\}$ in the integral in the r.h.s. of (9). Then

$$S_{\mu,l} = \frac{1}{B(\mu-\nu,\nu+1)}\sum_m \int_0^\infty \lambda^{\mu-\nu-1}(|\varkappa_m|-\lambda)^\nu_+ d\lambda \qquad (25)$$

$$\leq \mathfrak{L}(d,l,\nu)\int_0^\infty \frac{d\lambda}{\lambda}\lambda^{\mu-\nu}\int_0^\infty dx(V(x)-\lambda)_+ = \frac{\mathfrak{L}(d,l,\nu)B(\mu-\nu,2)}{B(\mu-\nu,\nu+1)}\int V^{\mu+\frac{d}{2l}}(x)dx\,.$$

Thus $\mathfrak{L}(d,l,\mu)$ is finite for all $\mu \geq \nu$.$\square$

2.8. **Asymptotics for small coupling constants.**

Theorem 2. *Assume $\kappa = d/2l < 1$. Then for the critical power $\nu = 1-\kappa$ the asymptotical formula*

$$S_{l,\nu}(\alpha V) = \alpha\mathfrak{L}^0(d,l,\nu)\int_{\mathbb{R}^d} V\,dx + o(\alpha) \quad as \quad \alpha \to 0, \qquad (26)$$

$$\mathfrak{L}^0(d,l,\nu) = \frac{\pi\kappa}{2^d\pi^{d/2}\Gamma(\frac{d}{2}+1)\sin\pi\kappa}$$

holds for all potentials $0 \leq V \in L_1(\mathbb{R}^d)$, and

$$\sum_{k\geq 2:\ \varkappa_k(H_l(\alpha V))<0} |\varkappa_k(H_l(\alpha V))|^\nu = o(\alpha) \quad as \quad \alpha \to 0. \qquad (27)$$

This Theorem is based on the following two known results. For the benefit of the reader we attach the proofs of these Lemmata in the Appendix.

Lemma 4. *Suppose $2l > d$ and assume the potential $0 \leq V \in L_1(\mathbb{R}^d)$ has compact support and is not identically zero. Then there exist exactly $\binom{l+[\frac{d}{2}]}{d}$ negative eigenvalues for the operator $H_l(\alpha V)$ for all sufficiently small coupling constants $0 < \alpha < \alpha_0(V)$.*

Lemma 5. *Suppose $2l > d$ and $0 \leq V \in L_1(\mathbb{R}^d)$. Then the bottom eigenvalue $\varkappa_1(H_l(\alpha V))$ of $H_l(\alpha V)$ obeys the asymptotical formula*

$$|\varkappa_1(H_l(\alpha V))|^\nu = \alpha \mathfrak{L}^0(d,l,\nu) \int_{\mathbb{R}^d} V\,dx + o(\alpha) \quad \text{as} \quad \alpha \to 0. \tag{28}$$

If $l + [\frac{d}{2}] > d$ and V is of compact support, for the subsequent negative eigenvalues the asymptotical estimates

$$|\varkappa_j(H_l(\alpha V))| = o(|\varkappa_1(H_l(\alpha V))|) \quad \text{as} \quad \alpha \to 0, \quad j \geq 2, \tag{29}$$

hold.

Remark 1. The asymptotical formula (28) is accompanied by the well-known estimate

$$|\varkappa_1(H_l(\alpha V))|^\nu \leq \alpha \mathfrak{L}^0(d,l,\nu) \int_{\mathbb{R}^d} V\,dx\ , \tag{30}$$

which holds for all $\alpha > 0$ and $0 \leq V \in L_1(\mathbb{R}^d)$.

Proof of Theorem 2. The formula (32) is an immediate consequence of the two previous Lemmata. In view of Theorem 1 we close (26) to all potentials $V \in L_1(\mathbb{R}^d)$. Finally comparing (26) and (28), we arrive at (27). □

Remark 2. Obviously

$$\tilde{\mathfrak{L}}(d,l) \leq \mathfrak{L}^{\mathrm{cl}}(d,l,\nu) < \mathfrak{L}^0(d,l,\nu) \leq \mathfrak{L}(d,l,\nu).$$

For the case $d = l = 1$ the equality $\tilde{\mathfrak{L}}(1,1) = \mathfrak{L}^{\mathrm{cl}}(1,1,1/2) = 1/4$ is known [19], [10], while Lieb and Thirring conjectured $\mathfrak{L}^0(1,1,1/2) = \mathfrak{L}(1,1,1/2) = 1/2$ [14]. This conjecture and the question, up to what extend

$$\tilde{\mathfrak{L}}(d,l) = \mathfrak{L}^{\mathrm{cl}}(d,l,\nu) \quad \text{and} \quad \mathfrak{L}^0(d,l,\nu) = \mathfrak{L}(d,l,\nu) \tag{31}$$

hold for general d, l with $2l > d$, remains unresolved.

Remark 3. If $2l > d$ for compactly supported potentials $0 \leq V \in L_1(\mathbb{R}^d)$ the asymptotics

$$S_{l,\mu}(\alpha V) = \big(\alpha \mathfrak{L}^0(d,l,\nu) \int_{\mathbb{R}^d} V\,dx\big)^{\mu/\nu} + o(\alpha^{\mu/\nu}) \quad \text{as} \quad \alpha \to 0,\ \mu > 0, \tag{32}$$

holds.

3. Lieb-Thirring type inequalities for subcritical powers

3.1. **Main result.** In this section we discuss substitutes for (3), if $0<\mu<\nu$. Below $\mathbf{E}$ denotes the sequence of shifted unit cubes

$$\{\mathcal{E}_{\vec{j}}\}_{\vec{j}\in\mathbb{Z}^d} := \{\mathbb{Q}^d + \vec{j}\}_{\vec{j}\in\mathbb{Z}^d}.$$

Moreover $\mathbf{F}$ stands for the sequence $\{\mathcal{F}_j\}_{j\in\mathbb{N}}$ with $\mathcal{F}_1 = \mathbb{Q}^d$ and $\mathcal{F}_j := 2^j\mathbb{Q}^d\backslash 2^{j-1}\mathbb{Q}^d,\ j=2,3,...$ For a locally sumable potential we introduce the notations $\boldsymbol{\beta}^{\mathbf{E}}(V) := \{\beta^{\mathbf{E}}_{\vec{j}}(V)\}_{\vec{j}\in\mathbb{Z}^d}$ and $\boldsymbol{\beta}^{\mathbf{F}}(V) := \{\beta^{\mathbf{F}}_{j}(V)\}_{j\in\mathbb{N}}$ with

$$\beta^{\mathbf{E}}_{\vec{j}}(V) := \int_{\mathcal{E}_{\vec{j}}} |V|dx \quad \text{and} \quad \beta^{\mathbf{F}}_{j}(V) := \int_{\mathcal{F}_j} |V|dx\ .$$

Norms of such sequences have been used by Birman and Solomyak [7] to give estimates on the number of negative bound states for the operator $H_l(V)$ if $2l>d$. We shall prove

Theorem 3. *Assume that for $0\le V\in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ the sequence $\boldsymbol{\beta}^{\mathbf{E}}(V)$ belongs to $\ell_{\mu/\nu},\ 0<\mu<\nu=1-\kappa,\ \kappa=d/2l$. Then the estimate*

$$S_{l,\mu}(V) \le C(d,l,\mu)\Big(\|\boldsymbol{\beta}^{\mathbf{E}}(V)\|^{\mu/\nu}_{\ell_{\mu/\nu}} + \|\boldsymbol{\beta}^{\mathbf{E}}(V)\|^{\mu+\kappa}_{\ell_{\mu+\kappa}}\Big) \tag{33}$$

holds.

Theorem 4. *Assume that for $0\le V\in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ the sequence $\boldsymbol{\beta}^{\mathbf{F}}((1+|x|)^\sigma V(x))$ belongs to $\ell_{\mu+\kappa},\ \sigma := d(\nu-\mu)/(\mu+\kappa),\ 0<\mu<\nu=1-\kappa,\ \kappa=d/2l$. Put $\theta(t) := t^{\mu/\nu}+t^{\mu+\kappa}$ for all $t\ge 0$. Then the estimate*

$$S_{l,\mu}(V) \le c(d,l,\mu)\theta\Big(\big\|\boldsymbol{\beta}^{\mathbf{F}}\big((1+|x|)^\sigma V(x)\big)\big\|_{\ell_{\mu+\kappa}}\Big) \tag{34}$$

holds.

3.2. **Proof of Theorem 3.** First we consider potentials $0\le V\in L_1(\mathbb{R}^d)$ of compact support. Let $\mathbf{Q}(V)=\{\mathcal{Q}_{x_i}\}_{i=1}^m$ be a finite A-proper covering of supp $V,\ A=2^{-d}/\hat{c}(d,l)$. Combining (21) and (20) as in the proof of Theorem 1 one finds

$$S_{l,\mu}(V) = \sum_k |\varkappa_k(H_l(V))|^\mu \le c_{3.1}(d,l,\mu)\sum_i\Big(\int_{\mathcal{Q}_{x_i}} V dx\Big)^{\mu/\nu}\ . \tag{35}$$

Put $\mathcal{P}_{i,\vec{j}} = \mathcal{Q}_{x_i} \cap \mathcal{E}_{\vec{j}}$ and $I(\vec{j}) := \{i : \text{int } \mathcal{P}_{i,\vec{j}} \neq \emptyset\}$, $N(\vec{j}) = \text{card } I(\vec{j})$. Then

$$\sum_i \Big(\int_{\mathcal{Q}_{x_i}} V dx \Big)^{\mu/\nu} \le \sum_{i,\vec{j}} \Big(\int_{\mathcal{P}_{i,\vec{j}}} V dx \Big)^{\mu/\nu} \le \sum_{\vec{j}} (N(\vec{j}))^{1-\frac{\mu}{\nu}} \Big(\sum_{i \in I(\vec{j})} \int_{\mathcal{P}_{i,\vec{j}}} V dx \Big)^{\mu/\nu}$$
$$\le 2^{d\mu/\nu} \sum_{\vec{j}} (N(\vec{j}))^{1-\frac{\mu}{\nu}} \Big(\int_{\mathcal{E}_{\vec{j}}} V dx \Big)^{\mu/\nu} . \tag{36}$$

Next we estimate the value of $N(\vec{j})$. Therefore we split the index set $I(\vec{j})$ into

$$I'(\vec{j}) := \{i \in I(\vec{j}) : \text{vol } \mathcal{Q}_{x_i} > 1\}, \quad I''(\vec{j}) = I(\vec{j}) \backslash I'(\vec{j}) . \tag{37}$$

If $i \in I'(\vec{j})$ then the interior of $\mathcal{Q}_{x_i}$ contains at least one of the corners of $\mathcal{E}_{\vec{j}}$. Since the proper covering $\mathbf{Q}(V)$ is of a multiplicity $\Xi(\mathbf{Q}(V)) \le 2^d$, we have card $I'(V) \le 2^{2d}$. On the other hand $i \in I''(\vec{j})$ implies $\mathcal{Q}_{x_i} \subset \vec{j} + 3\mathbb{Q}^d$. Thus from $\Xi(\mathbf{Q}(V)) \le 2^d$ we obtain

$$\sum_{i \in I''(\vec{j})} \text{vol } \mathcal{Q}_{x_i} \le 6^d, \tag{38}$$

while from (10) we deduce

$$\sum_{i \in I''(\vec{j})} \big(\text{vol } \mathcal{Q}_{x_i}\big)^{1-\kappa^{-1}} \le 4^d \hat{c}(d,l) \int_{\vec{j}+3\mathbb{Q}^d} V dx . \tag{39}$$

Together (38) and (39) imply

$$\big(\text{card } I''(\vec{j})\big)^{\kappa^{-1}} \le \Big(\sum_{i \in I''(\vec{j})} \text{vol } \mathcal{Q}_{x_i}\Big)^{\kappa^{-1}-1} \sum_{i \in I''(\vec{j})} \big(\text{vol } \mathcal{Q}_{x_i}\big)^{1-\kappa^{-1}} \le c_{3.2}(d,l) \int_{\vec{j}+3\mathbb{Q}^d} V dx , \tag{40}$$

thus

$$N(\vec{j}) = c_{3.3}(d,l) + c_{3.4}(d,l) \Big(\int_{\vec{j}+3\mathbb{Q}^d} V dx \Big)^{\kappa} . \tag{41}$$

Inserting this estimate into (35) and (36) we arrive at

$$S_{l,\mu}(V) \le c_{3.5}(d,l,\mu) \|\boldsymbol{\beta}^{\mathbf{E}}(V)\|_{\ell_{\mu/\nu}}^{\mu/\nu} + c_{3.6}(d,l,\mu) \Big(\sum_{\vec{j}} \int_{\vec{j}+3\mathbb{Q}^d} V dx \Big)^{\mu+\kappa},$$

what is equivalent to (33). Since the constant in this estimate does not depend on V, we can close the bound to all potentials $0 \le V$ with $\boldsymbol{\beta}(V) \in \ell_{\mu/\nu}$. $\square$

3.3. **Proof of Theorem 4.** We consider potentials of compact support and choose a A-finite proper covering $\mathbf{Q}(V)$ of multiplicity $\Xi(\mathbf{Q}(V)) \leq 2^d$ of the support of V with $A = 2^{-d}/\hat{c}(d,l)$. We put $\mathcal{P}_{i,j} = \mathcal{Q}_{x_i} \cap \mathcal{F}_j$, $j \in \mathbb{N}$, and $I(j) := \{i : \text{int } \mathcal{P}_{i,j} \neq \emptyset\}$ is of cardinality $N(j) = \text{card } I(j)$. In analogy to the previous proof we find

$$S_{l,\mu}(V) \leq c_{3.7}(d,l,\mu) \sum_j (N(j))^{1-\frac{\mu}{\nu}} \Big(\int_{\mathcal{F}_j} V \, dx \Big)^{\mu/\nu} . \tag{42}$$

Choose the decomposition

$$I'(j) := \{ i \in I(j) : \text{vol } \mathcal{Q}_{x_i} > \max\{2^{-d}, 2^{d(j-3)}\}\}, \quad I''(j) = I(j) \backslash I'(j) .$$

If $i \in I''(j)$ then

$$\mathcal{Q}_{x_i} \subset \mathcal{M}_j := \bigcup_{s=\max\{1,j-1\}}^{j+1} \mathcal{F}_s ,$$

and estimates similar to (38), (39), (40) give

$$\text{card } I''(j) \leq c_{3.8}(d,l) \Big(\text{vol } \mathcal{M}_j \Big)^{\nu} \Big(\int_{\mathcal{M}_j} V \, dx \Big)^{\kappa} . \tag{43}$$

A simply geometrical argument shows, that in view of $\Xi(\mathbf{Q}(V)) \leq 2^d$ the estimate

$$\text{card } I'(j) \leq c_{3.9}(d) \tag{44}$$

holds. Inserting $N(j) = \text{card } I'(j) + \text{card } I''(j)$ with (43) and (44) into (42), we claim

$$S_{l,\mu}(V) \leq c_{3.10}(d,l,\mu) \Big\{ \sum_j \Big(\int_{\mathcal{M}_j} V dx \Big)^{\mu/\nu} + \sum_j \Big(\text{vol } \mathcal{M}_j \Big)^{\nu-\mu} \Big(\int_{\mathcal{M}_j} V dx \Big)^{\mu+\kappa} \Big\} . \tag{45}$$

Notice that $\text{vol } \mathcal{M}_j \asymp (1+|x|)^d$ on $x \in \mathcal{M}_j$. Thus the second sum on the r.h.s. of (45) is bounded from above by $c_{3.11}(d,l,\mu) \big\| \boldsymbol{\beta}^{\mathbf{F}} \big((1+|x|)^{\sigma} V(x) \big) \big\|_{\ell_{\mu+\kappa}}^{\mu+\kappa}$. The first sum can be estimated by

$$\sum_j \Big(\int_{\mathcal{M}_j} V dx \Big)^{\mu/\nu} \leq c_{3.12}(d,l,\mu) \big\| \boldsymbol{\beta}^{\mathbf{F}} \big((1+|x|)^{\sigma} V(x) \big) \big\|_{\ell_{\mu+\kappa}}^{\mu/\nu} \Big(\sum_j \big(\text{vol } \mathcal{M}_j \big)^{-\frac{\mu\sigma q}{\nu d}} \Big)^{q^{-1}} ,$$

where we applied Hölders inequality with the powers $p = \nu(\mu+\kappa)/\mu > 1$, $q^{-1} = 1 - p^{-1}$. The sum of the negative powers of vol $\mathcal{M}_j$ converges, what completes the proof. □

3.4. **Remark.** ¿From the proofs of Theorems 3 and 4 we see, that in the respective bounds the term of homogeneity μ/ν corresponds to large cubes $\mathcal{Q}_{x_i} \in \mathbf{Q}(V)$, that means areas of low density of the potential, while the term of homogeneity $\mu+\kappa$ corresponds to small cubes $\mathcal{Q}_{x_i} \in \mathbf{Q}(V)$, that means areas of high density of the potential. This agrees with the fact, that under the conditions of these theorems we have $S_{l,\mu}(\alpha V) \asymp \alpha^{\mu/\nu}$ as $\alpha \to 0$, but $S_{l,\mu}(\alpha V) \asymp \alpha^{\mu+\kappa}$ as $\alpha \to \infty$.

4. Appendix

In the appendix we outline the proof of Lemmata 4 and 5.

Lemma 6. *Assume $2l > d$. Put $B_r = \{x \in \mathbb{R}^d : |x| < r\}$ and let $\pi_k = \pi_k(r)$ denote the orthogonal projection in $L_2(B_r)$ onto $\Omega_{d,k}|_{B_r}$. Then the inequality*

$$\|f - \pi_m f\|_{L_\infty(B_r)} \le c_{4.1}(d,l,r)\|\nabla^l f\|_{L_2(\mathbb{R}^d)}, \quad m = \left[l - \frac{d}{2}\right], \quad f \in C_0^\infty(\mathbb{R}^d), \tag{46}$$

holds.

Proof. We start from the inequalities

$$\|\nabla^n f\|_{L_{p(n)}(\mathbb{R}^d)} \le c_{4.2}(d,l,p,n)\|\nabla^l f\|_{L_2(\mathbb{R}^d)}, \quad f \in C_0^\infty(\mathbb{R}^d), \tag{47}$$

$$2^{-1} - p^{-1}(n) = (l-n)/d, \quad n \in \mathbb{N} : \ m+1 \le n \le l,$$

see e.g. [4] p. 153 Theorem 6.5.1. By the Theorem on equivalent norms on B_r we have

$$\|g\|_{L_2(B_r)} \le c_{4.3}(d,l,r)\|\nabla^l g\|_{L_2(B_r)} = c_{4.3}(d,l,r)\|\nabla^l f\|_{L_2(B_r)}, \ \ g := f - \pi_{l-1} f.$$

The Sobolev embedding $W_2^l(B_r) \hookrightarrow C(B_r)$ gives

$$\|g\|_{L_\infty(B_r)} \le c_{4.4}(d,l,r)\|\nabla^l f\|_{L_2((B_r)}. \tag{48}$$

On the other hand, applying (47) with $n = m+1$ to f and g, we find

$$\|\nabla^n \pi_{l-1} f\|_{L_{p(n)}(B_r)} = \|\nabla^n(\pi_{l-1} - \pi_m) f\|_{L_{p(n)}(B_r)} \le c_{4.5}(d,l,r)\|\nabla^l f\|_{L_2(\mathbb{R}^d)}.$$

On the finite-dimensional lineal $\Omega_{d,l-1}|_{B_r} \ominus_{L_2(B_r)} \Omega_{d,m}|_{B_r}$ the norms $\|\nabla^n \cdot \|_{L_{p(n)}(B_r)}$ and $\|\cdot\|_{L_\infty(B_r)}$ are equivalent. Thus

$$\|(\pi_{l-1} - \pi_m) f\|_{L_\infty(B_r)} \le c_{4.6}(d,l,r)\|\nabla^l f\|_{L_2(\mathbb{R}^d)}. \tag{49}$$

¿From (48) and (49) we conclude (46). $\square$

4.1. **Proof of Lemma 4. 1.** Take $r > 0$ such, that $\mathrm{supp} V \subset \{x \in \mathbb{R}^d : |x| < r\}$. From Lemma 6 we conclude

$$\int_{\mathbb{R}^d} V|u - \pi_m u|^2 dx \le c_{4.7}(d,l,r)\Big(\int_{\mathbb{R}^d} V\,dx\Big)\|\nabla^l u\|^2_{L_2(\mathbb{R}^d)},\ \ m = \Big[l - \frac{d}{2}\Big],\ \ u \in C_0^\infty(\mathbb{R}^d).$$

Thus the form $\mathbf{h}_l(\alpha V)[u,u]$ is non-negative on all

$$u \in C_0^\infty(\mathbb{R}^d) :\ \pi_m u|_{B_r} = 0, \tag{50}$$

if $0 < \alpha < 1/(c_{4.7}(d,l,r)\int V dx)$. The respective operator $H_l(\alpha V)$ has not more than rank $\pi_m = \binom{l+[\frac{d}{2}]}{d}$ negative eigenvalues.

2. Equip the linear space $\Omega_{d,k}$ with the norm $\|p\| := \max_{\boldsymbol{\iota}\in\mathbb{N}^d:|\boldsymbol{\iota}|\le k} |c_{\boldsymbol{\iota}}|$. Choose some function $\psi \in C^\infty(\mathbb{R})$, such that $\psi(t) \equiv 1$ for $t < 1$, $\psi(t) \equiv 0$ for $t > 2$ and $0 \le \psi \le 1$ for $1 \le t \le 2$. Define $\psi_\epsilon \in C_0^\infty(\mathbb{R}^d)$ by $\psi_\epsilon(x) := \psi(\epsilon \ln |x|)$, $\epsilon > 0$. A calculation shows (cf. [18], p. 123), that

$$\int |\nabla^l \psi_\epsilon p|^2 dx < \epsilon \|p\| M(d,l,\psi),\quad p \in \Omega_{d,[l-\frac{d}{2}]},\quad 0 < \epsilon < 1,$$

while

$$\int V|\psi_\epsilon p(x)|^2 dx = \int V|p(x)|^2 dx \ge \|p\| m(d,l,V),$$

for sufficiently small $\epsilon_0(V) > \epsilon > 0$ and suitable constants $0 < m(d,l,V), M(d,l,\psi) < \infty$. The quadratic form $\mathbf{h}_l(\alpha V)$ is negative on all functions $\psi_\epsilon p(x) \not\equiv 0$ from the $\binom{l+[\frac{d}{2}]}{d}$-dimensional subspace $\psi_\epsilon \Omega_{d,[l-\frac{d}{2}]}$ for $0 < \epsilon < \min\{1, \epsilon_0(V), \alpha m(d,l,V)/M(d,l,\psi)\}$. Thus $H_l(\alpha V)$ has exactly $\binom{l+[\frac{d}{2}]}{d}$ negative eigenvalues for all sufficiently small $\alpha > 0$. $\Box$

4.2. **Proof of Lemma 5.** Let $< \cdot,\cdot >$ denote the standard scalar product in $\mathbb{R}^d$. For $V \ge 0$ we put $W(x) = \sqrt{V(x)}$ and

$$\big(X_\varkappa(V)u\big)(x) := W(x)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{e^{i<\xi,x-y>}W(y)u(y)d\xi dy}{(2\pi)^d(|\xi|^{2l} - \varkappa)},\quad \varkappa < 0.$$

For $V \in L_1(\mathbb{R}^d)$ and $2l > d$ this positive integral operator acts as a Hilbert-Schmidt on $L_2(\mathbb{R}^d)$. Let $\{\lambda_n(X_\varkappa(V))\}$ denote the non-increasing sequence of the eigenvalues of $X_\varkappa(V)$. Moreover $\{\varkappa_n(\alpha)\} := \{\varkappa_n(H_l(\alpha V))\}$ denotes the non-decreasing sequence of negative eigenvalues of $H_l(\alpha V)$. According to the Birman-Schwinger principle the identities

$$\lambda_k(X_{\varkappa_k(\alpha)}(V)) = \alpha^{-1},\quad k \in \mathbb{N}, \tag{51}$$

hold. In particular one finds

$$\alpha^{-1} = \|X_{\varkappa_1(\alpha)}(V)\| \leq \ \mathrm{Tr} X_{\varkappa_1(\alpha)}(V) = |\varkappa_1(\alpha)|^{-\nu}\mathfrak{L}^0(d,l,\nu)\int_{\mathbb{R}^d} V(x)dx \ ,$$

what turns into (30).

Assume now, that $0 \leq V(x) \in L_1(\mathbb{R}^d)$ is of compact support. We decompose the operator $X_\varkappa(V)$ as

$$X_\varkappa(V) := \tilde{X}_\varkappa(V) + \hat{X}_\varkappa(V) + \dot{X}_\varkappa(V),$$

$$\big(\tilde{X}_\varkappa(V)u\big)(x) := W(x)\int_{|\xi|\geq 1}\int_{y\in\mathbb{R}^d} \frac{e^{i<\xi,x-y>}W(y)u(y)d\xi dy}{(2\pi)^d(|\xi|^{2l}-\varkappa)},$$

$$\big(\hat{X}_\varkappa(V)u\big)(x) := W(x)\int_{|\xi|< 1}\int_{y\in\mathbb{R}^d} \frac{W(y)u(y)d\xi dy}{(2\pi)^d(|\xi|^{2l}-\varkappa)},$$

$$\big(\dot{X}_\varkappa(V)u\big)(x) := W(x)\int_{|\xi|< 1}\int_{y\in\mathbb{R}^d} \frac{(e^{i<\xi,x-y>}-1)W(y)u(y)d\xi dy}{(2\pi)^d(|\xi|^{2l}-\varkappa)}.$$

Evaluating the respective Hilbert-Schmidt norms we find

$$\|\tilde{X}_\varkappa(V)\| \leq c_{4.8}(V),$$

$$\|\dot{X}_\varkappa(V)\| \leq \begin{cases} c_{4.9}(V)|\ln(e+|\varkappa|^{-1})| & \text{as } 2l = d+1, \\ c_{4.10}(V)|\varkappa|^{\frac{d+1}{2l}-1} & \text{as } 2l > d+1 \end{cases}, \quad |\varkappa| < 1.$$

Finally we represent $\hat{X}_\varkappa(V)$ as

$$\hat{X}_\varkappa(V) = \hat{X}^0_\varkappa(V) + \hat{X}^1_\varkappa(V),$$

$$\big(\hat{X}^0_\varkappa(V)u\big)(x) := W(x)\int_{\xi\in\mathbb{R}^d}\int_{y\in\mathbb{R}^d} \frac{W(y)u(y)d\xi dy}{(2\pi)^d(|\xi|^{2l}-\varkappa)}.$$

Obviously

$$\hat{X}^0_\varkappa(V) = |\varkappa|^{\frac{d}{2l}-1}\hat{X}^0_{-1}(V),$$

and

$$\|\hat{X}^1_\varkappa(V)\| \leq c_{4.11}(V), \quad \varkappa < 0.$$

We underline, that the constants $c_{4.8}, \cdots, c_{4.11}$ do not depend on $\varkappa < 0$.

¿From standard perturbation theory we conclude, that the operator

$$X_\varkappa(V) = |\varkappa|^{\frac{d}{2l}-1}\hat{X}^0_{-1}(V) + Y_\varkappa(V), \quad Y_\varkappa(V) := \hat{H}^1_\varkappa(V) + \tilde{H}_\varkappa(V) + \dot{H}_\varkappa(V)$$

has not more than $\mathrm{rank} X^0_\varkappa(V) = 1$ eigenvalue larger than $\|Y_\varkappa(V)\|$, or

$$\lambda_k(X_\varkappa(V)) \leq c_{4.12}(V)\max\{|\varkappa|^{\frac{d+1}{2l}-1}, \ln(e+|\varkappa|^{-1}), 1\} \ \text{as} \ |\varkappa| < 1, \ k \geq 2.$$

¿From (30) and (51) we conclude, that for compactly supported potentials $0 \leq V \in L_1(\mathbb{R}^d)$ the asymptotical estimates

$$|\varkappa_k(\alpha)| = o(\alpha^{1/\nu}) \ \text{ as } \ \alpha \to 0, \ \ k \geq 2,$$

hold. On the other hand for the leading eigenvalue we have

$$|\varkappa|^{\frac{d}{2l}-1}\lambda_1(\hat{X}^0_{-1}(V))-\|Y_\varkappa(V)\| \le \lambda_1(X_\varkappa(V)) \le |\varkappa|^{\frac{d}{2l}-1}\lambda_1(\hat{X}^0_{-1}(V))+\|Y_\varkappa(V)\|,$$

which mounts into

$$\lambda_1(X_\varkappa(V)) = |\varkappa|^{\frac{d}{2l}-1}\mathrm{Tr}\hat{X}^0_{-1}(V)+O\big(\max\{|\varkappa|^{\frac{d+2}{2l}-1}, |\ln(e+|\varkappa|^{-1})|\}\big) \text{ as } \varkappa \to -0.$$

Then (30) and (51) imply

$$|\varkappa_1(\alpha)|^\nu = \alpha\mathfrak{L}^0(d,l,\nu)\int V(x)dx + o(\alpha) \quad \text{as } \alpha \to 0. \tag{52}$$

In view of (30) we can close (52) to all potentials $0 \le V \in L_1(\mathbb{R}^d)$. □

Remark 4. The technique of the extraction of a diverging operator of finite rank is well-known. It can be applied to the case of non-signdefined potentials and the asymptotics of the subsequent eigenvalues can also be calculated. In particular one can show that (52) remains true for compactly supported non-signdefined potentials $V \in L_1(\mathbb{R}^d)$, if only $\int V dx > 0$. For the related results on the weakly coupled one- or twodimensional Schrödinger operator we refer to [17].

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