# ON THE $1/N$ CORRECTIONS TO THE GREEN FUNCTIONS OF RANDOM MATRICES WITH INDEPENDENT ENTRIES

A.Khorunzhy, B.Khoruzhenko, and L.Pastur
Mathematical Division, Institute for Low Temperature Physics,
47 Lenin Ave, 310164, Kharkiv, Ukraine

**Abstract**

We propose a general approach to the construction of $1/N$ corrections to the Green function $G_N(z)$ of the ensembles of random real symmetric and Hermitean $N \times N$ matrices with independent entries $H_{k,l}$. By this approach we study the correlation function $C_N(z_1, z_2)$ of normalized trace $N^{-1}\mathrm{Tr}\ G_N$ assuming that the average of $|H_{k,l}|^5$ is bounded. We found that to the leading order $C_N(z_1, z_2) = N^{-2}F(z_1, z_2)$, where $F(z_1, z_2)$ depends only on the second and the fourth moments of $H_{k,l}$. For the correlation function of the density of energy levels we obtain an expression which, in the scaling limit depends only on the second moment of $H_{k,l}$. This can be viewed as a support to the universality conjecture of random matrix theory

1. Random $N \times N$ matrices with independent entries were introduced by E.Wigner [1].

The majority of rigorous results for these matrices concerns the proof of convergence of the density $\rho_N(E)$ of their eigenvalues to the celebrated Wigner semicircle law and its generalization known as the deformed semicircle law [2, 3, 4]. These nonrandom limiting eigenvalue distributions are completely determined by two first moments of the probability distribution of random matrix entries.

Much less is known on the large-$N$ corrections, in particular on their dependence on the probability distribution of entries. The aim of this letter is to present a rigorous approach to the systematic construction of the large-$N$ corrections for moments of the Green functions of respective random matrices.

2. We consider an ensemble of real symmetric $N\times N$ random matrices which, in particular case of the Gaussian distributed entries, possesses the orthogonal invariance property (respective ensemble is known as the Gaussian orthogonal ensemble (GOE) and is an archetype model in the field [5]). Thus, our random matrices $H=\{H_{k,l}\}_{k,l=1}^{N}$ are specified by relations

$$H_{k,l}=H_{l,k}=\Big(\frac{1+\delta_{k,l}}{N}\Big)^{1/2}W_{k,l},\qquad k,l=1,\dots,N. \tag{1}$$

Here $W_{k,l}$, $k\le l$, are independent random variables such that: i) $\overline{W_{k,l}}=0$; ii) $\overline{W_{k,l}^2}\equiv v^2$; iii) $\overline{W_{k,l}^4}-3(\overline{W_{k,l}^2})^2\equiv\sigma$; iv) $\max_{k\le l}\overline{|W_{k,l}|^5}\le C$, where the bar denotes averaging over respective probability distribution and $v$, $\sigma$, and $C$ are $N$-independent. Thus $W_{k,l}$'s may have different probability distributions for different pairs $(k,l)$ but their second and fourth moments have to be the same.

Denote by $G_N(z)=\{G_{i,k}(z)\}_{i,k=1}^{N}$ the Green function $G_N(z)=(H_N-z)^{-1}$, Im $z\neq 0$, of our matrices and by $\langle G_N(z)\rangle$ its normalized trace $\langle G_N(z)\rangle=1/N\sum_{k=1}^{N}G_{k,k}(z)$. It is known [4, 6] that the moments of $\langle G_N(z)\rangle$ factorize. Namely, if

$$m_N^{(p)}(z_1,\dots,z_p)=\overline{\prod_{i=1}^{N}\langle G_N(z_i)\rangle}, \tag{2}$$

then

$$m_N^{(p)}(z_1,...,z_p)=\prod_{i=1}^{p}m_N^{(1)}(z_i)+O(1/N^2). \tag{3}$$

In particular, the correlation function

$$C_N(z_1,z_2)=\overline{\langle G_N(z_1)\rangle\langle G_N(z_2)\rangle}-\overline{\langle G_N(z_1)\rangle}\cdot\overline{\langle G_N(z_2)\rangle} \tag{4}$$

for Im $z_1$, Im $z_2\neq 0$ is of the order $1/N^2$ as $N\to\infty$.

Our aim is to construct the expansion of $m_N^{(p)}(z_1,...,z_p)$ in powers of $1/N$. We outline the main idea and find, as an example, the explicit form of $1/N^2$ correction to $C_N(z_1,z_2)$. We also discuss some implications of our results.

3. Our approach is an adaptation and extension of that proposed in [7] to study spectral characteristics of a certain class of random finite-difference operators of the order $R$, acting in $l^2(\mathbf{Z}^d) \otimes \mathbf{C}^n$ in the limiting cases when one of the parameters $R$, $n$, or $d$ tends to infinity. The main idea is to derive certain identities for the moments $m_N^{(p)}(z_1, ..., z_p)$ and then, treating the whole set of these relations as an equation in an appropriate linear space, to compute the moments in each order of $1/N$ by iterating this equation.

To derive the relations let us rewrite (2) as $1/N \sum_{k=1}^N \overline{G_{k,k}(z_1) \prod_{i=2}^p \langle G_N(z_i)\rangle}$ and replace $G_{k,k}(z_1)$ by the r. h. s. of the resolvent identity $G_{k,k}(z_1) = -1/z_1 + 1/z_1 \sum_{l=1}^N H_{k,l} G_{l,k}(z_1)$. We obtain

$$m_N^{(p)}(z_1, ...z_p) = -\frac{1}{z_1} m^{(p-1)}(z_2, ...z_p) + \sum_{k,l=1}^N \overline{H_{k,l} G_{l,k}(z_1) \prod_{i=2}^p \langle G_N(z_i)\rangle}, \qquad (5)$$

where for $p = 0$ we set $m_N^{(0)} \equiv 1$. Now we average in the second term of the r.h.s. of (5) over the random variable $H_{k,l}$ by using another resolvent identity

$$G_{m,n} = \hat{G}_{m,n} - \left[\hat{G}_{m,k} G_{l,n} - \hat{G}_{m,l} G_{k,n}\right] H_{k,l}(1 + \delta_{k,l})/2, \qquad (6)$$

where $\hat{G} = G \Big|_{H_{k,l}=0}$. Iterating (5) several times we represent any matrix element of $G_N(z)$ as a sum of powers of $H_{k,l}$ multiplied by matrix elements of $\hat{G}_N$ and of a power of $H_{k,l}$, say $H_{k,l}^a$, multiplied by a sum of matrix elements of the both Green functions $G_N$ and $\hat{G}_N$ ($a$ is determined by the number of iterations). The inequality

$$|G_{k,m}(z)| \leq \|G_N\| \leq |\mathrm{Im}\, z|^{-1}, \qquad (7)$$

and analogous inequality for $\hat{G}_N$ allows us to estimate the average of the latter by $\overline{|H_{k,l}|^a}$ (which is proportional to $1/N^{a/2}$ according to (1)) multiplied by a power of $|\mathrm{Im}\, z|^{-1}$ and by some absolute constant. Since $\hat{G}_N$ and $H_{k,l}$ are independent, all other terms in this representation will have the form of moments of matrix elements of $\hat{G}_N$ multiplied by moments of $H_{k,l}$ whose order is smaller than $a$. At last, we return back from $\hat{G}_N$ to $G_N$ by using the identity that differs from (6) by interchanging $G_N$ and $\hat{G}_N$ and replacing $H_{k,l}$ by $-H_{k,l}$. We obtain the identity

$$m_N^{(p)} = (Am_N)^{(p)} + f_N^{(p)}. \qquad (8)$$

Here $A$ is the linear operator defined as

$$(Am)^{(p)}(z_1,...,z_p) = -\frac{1-\delta_{1,p}}{z_1} m^{(p-1)}(z_2,...,z_p) - \frac{v^2}{z_1} m^{(p+1)}(z_1,z_1,...,z_p)$$

in the Banach space $B$ of bounded sequences $\{m^{(p)}\}_{p=1}^{\infty}$, $||m|| = \sup_{p\geq 1} v^p |m^{(p)}|$. $f_N^{(p)}$ for every $p$ is a sum of moments whose form is different from (2).

If $|\text{Im } z_i| > 2v$, then $||A|| \leq 2v/|\text{Im } z_i| < 1$, the operator $1-A$ is invertible and (8) has the unique solution. Since according to (7) the moments (2) are bounded above by $\prod_{i=1}^{p} |\text{Im } z_i|^{-1}$, this solution coincides with $\{m^{(p)}(z_1,\ldots,z_p)\}_{p=1}^{\infty}$ under the above condition $|\text{Im}\, z_i| > 2v$.

Furthermore, to the leading order $f_N^{(p)} = -\delta_{p,1}/z_1$ and therefore (cf (3))

$$m_N^{(p)}(z_1,...,z_p) = \prod_{i=1}^{p} r(z_i) + O(1/N), \tag{9}$$

where $r(z) = (-z+\sqrt{z^2-4v^2})/2v^2$ is the Stieltjes transform of the semicircle law, i.e. $\pi^{-1}\text{Im } r(E+\text{i}0) = {}_+\sqrt{4v^2-E^2}/2\pi v^2$, where ${}_+\sqrt{t} = \sqrt{\max(t,0)}$.

Next orders of $f_N^{(p)}$ in the Gaussian case are given by expressions

$$\frac{v^2}{z_1 N}\overline{\langle G_N(z_1)^2\rangle \prod_{j=2}^{p} \langle G_N(z_j)\rangle},\ \frac{2v^2}{z_1 N^2}\sum_{j=2}^{p} \overline{\langle G_N(z_1)G_N(z_j)^2\rangle \prod_{k=2,k\neq j}^{p} \langle G_N(z_k)\rangle},$$

etc. We see that moments of the form different from (2) do appear in these expressions. These new moments of can be found by an argument analogous to that for $m_N^{(p)}$, i.e. by deriving equations similar to (8). In general case there are additional terms in each order of $1/N$. They can be handled analogously (see e.g. the first term in the r.h.s. of Eq. (10) below). Therefore, solving (8) and these "higher order" equations step by step in each order of $1/N$ we obtain corrections to $m_N^{(p)}$ for any $p$.

4. For the Gaussian entries $1/N$ corrections were studied in the physical paper [8] basing on the formal perturbation theory with respect to $H_{k,l}$ and the diagrammatic technique. This approach is an adaptation of respective technique developed in [9] in order to construct the $1/N$ expansion of the Green function moments of the random operator describing a disordered system on $Z^d$ with $N$ orbitals at each site. It is not an easy problem to extend this technique

(especially in its rigorous version) to the non-Gaussian case. In comparison with that our approach is much less sensitive to the type of probability distribution of $H_{k,l}$. Moreover, the complicated and cumbersome combinatorial problem of rearranging of diagrams does not appear. In particular, the "dressing" procedure replacing "bare" Green function $-1/z$ by $\lim_{N\to\infty} \langle G_N(z)\rangle$ is automatic in our approach.

This is especially evident in the computation of the two-point correlation function (4). Here we can simplify our general procedure because in this case it is sufficient to iterate only a few first relations of the infinite system (8). Namely, if as before, $|\text{Im } z_i| > 2v$, then

$$\begin{aligned}\left(z_1 - 2v^2\overline{\langle G_N(z_1)\rangle}\right) C_N(z_1,z_2) &= \frac{2\sigma}{N^2}\sum_{k,l=1}^{N} \overline{G_{k,k}(z_1)G_{k,k}(z_2)G_{l,l}(z_1)G_{l,l}(z_2)^2} + \\ &\quad \frac{2v^2}{N^2}\overline{\langle G_N(z_1)G_N(z_2)^2\rangle} + O\left(\frac{1}{N^{5/2}}\right) \qquad (10)\end{aligned}$$

Due to the factor $1/N^2$ in front of the first and the second terms of the r.h.s. of this relation we can replace respective averages in these terms by their limiting (zero order) values given by the first term in the r.h.s. of (8). We obtain after some algebra

$$C_N(z_1,z_2) = N^{-2}F(z_1,z_2) + O(N^{-5/2}), \qquad (11)$$

$$\begin{aligned}F(z_1,z_2) &= \frac{1}{N^2}\left(\frac{2v^2(r_1-r_2)^2}{\beta(z_1-z_2)^2(1-v^2r_1^2)(1-v^2r_2^2)} + \frac{2\sigma r_1^3 r_2^3}{(1-v^2r_1^2)(1-v^2r_2^2)}\right) + \\ &\quad O\left(\frac{1}{N^{5/2}}\right), \qquad (12)\end{aligned}$$

where $\beta = 1$.

5. Notice that by an analogous argument we can also find $1/N$ corrections to the $N = \infty$ limit of the Green functions of an ensemble of the Hermitean random matrices. For instance, let

$$H_{k,l} = N^{-1/2}(X_{k,l} + \mathrm{i}Y_{k,l}) \qquad k,l = 1\ldots N, \qquad (13)$$

where $X_{k,l} = X_{l,k}$ and $Y_{k,l} = -Y_{l,k}$ are independent for $k \le l$ random variables such that i) $\overline{X_{k,l}} = \overline{Y_{k,l}} = 0$; ii) $\overline{X_{k,l}^2} = (1+\delta_{k,l})v^2/2,\ \overline{Y_{k,l}^2} = (1-\delta_{k,l})v^2/2$; iii) $\overline{X_{k,l}^4} - 3\left(\overline{X_{k,l}^2}\right)^2 = \sigma_X,\ \overline{Y_{k,l}^4} - 3\left(\overline{Y_{k,l}^2}\right)^2 = \sigma_Y,\ \sigma_X + \sigma_Y = \sigma$, if $k < l,$ and $\overline{X_{k,k}^4} - 3\left(\overline{X_{k,k}^2}\right)^2 = \sigma$; iv) $\sup_{k \le l} \overline{|X_{k,l}|^5 + |Y_{k,l}|^5} \le C < \infty$, where $v, \sigma,$ and $C$ are $N$-independent. Then the two-point correlation function of $1/N\mathrm{Tr}\, G_N(z)$ is given by (11) and (12) with $\beta = 2$.

6. We have presented above the scheme of rigorous construction of $1/N$ corrections (in fact expansions) for moments of normalized traces of the Green functions of random matrix ensembles (1) and (13). Now we use our result to draw certain nonrigorous conclusions on the form of the leading term of the correlation function $S_N(E_1, E_2)$ of the density of states (DOS) $\rho_N(E) = N^{-1}\mathrm{Tr}\ \delta(H-E) \equiv \langle\delta(H-E)\rangle$. Since $\rho_N(E) = 1/\pi \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\,\langle G_N(E + \mathrm{i}\epsilon)\rangle \equiv I_E\{G_N(z)\}$, we see that to obtain $S_N(E_1, E_2)$ we have to use (11) and (12) outside the domain $|\mathrm{Im}\, z| > 2v$ where they were rigorously proved. Nevertheless, since the function $F(z_1, z_2)$ given by (12) can obviously be continued up to the real axis with respect to the both variables $z_1$ and $z_2$ we can apply to this expression the operations $I_{E_1}$ and $I_{E_2}$, $E_1 \ne E_2$. It means that we are going to compute the leading term of the DOS correlation function by performing first the limit $N \to \infty$ and then the limits $\epsilon_1, \epsilon_2 \downarrow 0$. This order of limiting transitions is inverse with respect to that prescribed by the definition of this correlation function.

The resulting expression for the correlation function is

$$\begin{aligned} S_N(E_1, E_2) \;\; &= \;\; -\frac{1}{\beta\pi^2[N(E_1 - E_2)]^2}\frac{4v^2 - E_1E_2}{\sqrt{4v^2 - E_1^2}\sqrt{4v^2 - E_2^2}} \\ & \quad -\frac{2\sigma}{N^2\pi^2v^6}\frac{(2v^2 - E_1^2)(2v^2 - E_2^2)}{\sqrt{4v^2 - E_1^2}\sqrt{4v^2 - E_2^2}}. \end{aligned} \tag{14}$$

For the Gaussian orthogonal and unitary ensembles (GOE and GUE) $\sigma = 0$ and we recover result

$$S_N(E_1, E_2) = -\frac{1}{\beta\pi^2[N(E_1 - E_2)]^2}\frac{4v^2 - E_1E_2}{\sqrt{4v^2 - E_1^2}\sqrt{4v^2 - E_2^2}}. \tag{15}$$

obtained in [10, 11]. We see that in a general non-Gaussian case the respective expression depends not only on the second moment of random entries but also on the fourth moment via $\sigma$.

Let us consider now the so-called scaling limit of $S_N(E_1, E_2)$, when $E_1, E_2 \to E,\ N(E_1-E_2) \to s$ [5]. We obtain remarkably simple expression: $\lim_{N(E_1-E_2)\to s}\ S_N(E_1, E_2) = -1/(\beta\pi^2 s^2)$. According to Wigner and Dyson (see e. g. [5]) the exact large-$s$ asymptotics the DOS correlation functions of the Gaussian ensembles are: $-1/(\pi^2 s^2)$ (GOE) and $-\sin^2 \pi\rho(E)s/(\pi^2 s^2)$ (GUE). Comparing these expressions with our we see that our procedure of computing of the correlation function yields for the general case the expression coinciding with the large-$s$ asymptotics of the Gaussian correlation function smoothed over energy intervals $\Delta s \gg \rho^{-1}(E)$. This can be regarded as a support of the universality conjecture of random matrix theory [5].

Let us mention three more supports of this conjecture. The first one [12] concerns the so-called sparse (or diluted) random matrices whose entries are independently distributed random variables such that $\Pr\{H_{k,l} = 0\} = p/N$. The authors used the Grassman integral technique and found the Wigner-Dyson universal form of the DOS correlator if $p$ is large enough. The second was obtained recently [13] for the completely different ensemble, known as the unitary invariant ensemble whose probability density is $Z^{-1}\exp[-NV(H)]$, where $V(t)$ is an even polynomial. Basing on an approach known as the orthogonal polynomial technique, the authors established a number of interesting results concerning the eigenvalue statistics of this ensemble, in particular the relation (15) for $\beta = 2$. The third was obtained by present authors for the ensemble $H = \sum_{\mu=1}^{p} \tau_\mu(\cdot, \xi^\mu)\xi^\mu$, where $\tau_\mu$ and $\xi^\mu = \{\xi_1^\mu, \ldots, \xi_n^\mu\}$ are independent identically distributed random variables. For this ensemble which was introduced in [14] we obtained the analogue of (11) and (12) and showed that its scaling limit is the same as above. These results will be published elsewhere.

## Acknowledgements

This work was supported in part by the International Science Foundation under Grant No U2S000 and the State Committee for Science and Technology of Ukraine under Grant No 3/1/132.

# References


[1] Wigner, E., *Ann. Math.* **57**, 168 (1958)

[2] Pastur, L., *Russ. Math. Surveys* **28**, 1 (1973)

[3] Brody, T. A., Flores, J., French, J. B., Mello, P. A., Pandey, A., and Wong, S. S. M., *Rev. Mod. Physics* **53**, 385 (1981)

[4] Girko, V. L., *Spectral Theory of Random Matrices* (in Russian), Nauka, Moscow, 1988

[5] Mehta, M. L., *Random Matrices*, Second edition, Academic Press, New York, 1991

[6] Khorunzhy, A. M., Khoruzhenko, B. A., Pastur, L. A., and Shcherbina, M. V., in *Phase Transitions and Critical Phenomena*, Vol. 15, p.73 (ed. C.Domb and J. L. Lebowitz), Academic Press, London, 1992

[7] Khorunzhy, A. M., and Pastur, L. A., *Commun. Math. Phys.* **153**, 605 (1993)

[8] Verbaarschot, J. J. M., Weidenmuller, H. A., and Zirnbauer, M. R., *Ann. Phys. (N.Y.)* **153**, 367 (1984)

[9] Opermann, R., and Wegner, F. *Z. Phys.* **B34**, 327 (1979)

[10] French, J. B., Mello, P. A., and Pandey, A., *Ann. Phys. (N.Y.)* **113**, 277 (1978)

[11] Pandey, A., *Ann. Phys. (N.Y.)* **134**, 110 (1981)

[12] Mirlin, A. D., and Fyodorov, Ya. N., *Journ. Phys. A: Math. Gen.* **24**, 227 (1991)

[13] Brezin, E., and Zee, A., *Nucl. Phys.* **B402**, 613 (1993)

[14] Marchenko, V., and Pastur, L., *Mat. Sbornik* **72**, 507 (1967)