EXPONENTIAL INSTABILITY FOR A CLASS OF DISPERSING BILLIARDS

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ABSTRACT. The billiard in the exterior of a finite disjoint union K of strictly convex bodies in $\mathbf{R}^d$ with smooth boundaries is considered. The existence of global constants $0 < \delta < 1$ and $C > 0$ is established such that if two billiard trajectories have n successive reflections from the same convex components of K, then the distance between their jth reflection points is less than $C(\delta^j + \delta^{n-j})$ for a sequence of integers j with uniform density in $1, 2, \ldots, n$. Consequently, the billiard ball map (though not continuous in general) is expansive. As applications, an asymptotic of the number of prime closed billiard trajectories is proved which generalizes a result of T. Morita [Mor], and it is shown that the topological entropy of the billiard flow does not exceed $\frac{\log(s-1)}{a}$, where s is the number of convex components of K and a is the minimal distance between different convex components of K.

1. INTRODUCTION

Let K be a compact subset of $\mathbf{R}^d$, $d \geq 2$ of the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_s, \tag{1}$$

where K_i are compact strictly convex disjoint bodies in $\mathbf{R}^d$ with smooth *boundaries* $\Gamma_i = \partial K_i$. We assume that $s \geq 3$. Let

$$Q = \overline{\mathbf{R}^d \setminus K}.$$

Denote $D = \text{diam}(K)$ and $a = \min_{i \neq j} \text{dist}(K_i, K_j)$. Clearly $0 < a < D$. The *outer unit normal* to ∂K at $q \in \partial K$ will be denoted by $n(q)$.

Let M be the *unit tangent bundle* over Q. We will identify M with $Q \times \mathbf{S}^{d-1}$ and write the points of M in the form $x = (q, v)$, where $q \in Q$ and $v \in \mathbf{S}^{d-1}$. The *natural projection* $\pi : M \longrightarrow Q$ is given by $\pi(q, v) = q$. Then $\partial M = \pi^{-1}(\partial Q)$ is the boundary of the manifold M. Let S_t be the *billiard flow* on M (see [CFS]). Given $x = (q, v) \in M$, the trajectory $\gamma(x) = \{S_t(x) : t \geq 0\}$, is the usual billiard curve described by the motion of a point mass issued from q in direction v. Inside Q the mass moves with constant velocity, while reaching the boundary it bounces off according to the usual law of geometrical optic "the angle of incidence equals the angle of reflection". The billiard flow S_t obtained in this way is a special case of the so called *dispersing* (*Sinai*) *billiard* (cf. [Si1], [Si2], [DS]). Dispersing

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billiards, especially in bounded planar or toral domains, have been extensively studied in connection with some important problems in physics (see [Si1], [Si3], [CFS], [DS], [BSC], [Ch], [CvE], [Wo] and the references there). The motivation for the considerations in the present paper come from some problems in scattering theory (cf. for example [I], [Sj], [Burq], [PS]).

The natural phase space of the billiard is

$$M_1 = \{x = (q,v) \in \partial M : \langle n(q), v\rangle \geq 0\}.$$

For $x = (q,v) \in M_1$ denote by $\varphi(x)$ the *angle between* $n(q)$ and v; then $\cos\varphi(x) = \langle n(q), v\rangle$. Let $x = (q,v) \in M_1$ be such that the billiard trajectory $\gamma(x)$ has a common point with ∂Q and let $t > 0$ be the minimal number for which $S_t(x) \in \partial M$. Define $B(x) = S_t(x)$. Clearly, if $B(x) = (p,w)$, then $p \in \partial Q$ is the point at which the ray issued from q in direction v hits ∂Q and w is the reflection of v with respect to the *tangent plane* $T_p(\partial Q)$ to ∂Q at p. Let M' be the set of all $x \in M_1$ for which $B(x)$ is defined. The map $B : M' \longrightarrow M_1$ is called the *billiard ball map.*

In order to get a continuous flow, one has to consider the billiard flow on the *quotient space* $\tilde{M} = M/\sim$, where $\sim$ is the folowing equivalent relation on M: $(q,v) \sim (p,w)$ iff $q = p$ and either $v = w$ or $q = p \in \Gamma$ and v and w are symmetric with respect to the tangent plane $T_q(\Gamma)$ to Γ at q. Clearly $\tilde{M} = \tilde{\pi}(M_1)$, where $\tilde{\pi} : M \longrightarrow \tilde{M}$ is the natural projection. The projection of the billiard flow S_t on M is a continuous flow on $\tilde{M}$ which is also called billiard flow. We will use the same notation S_t for it. Also, avoiding the cumbersome notation $\tilde{\pi}(x)$, $S_t(\tilde{\pi}(x))$, etc., we will write x and $S_t(x)$ instead. It will be clear from the context whether we consider points in M or in $\tilde{M}$.

Throughout $\tilde{\rho}$ will be a metric on $\tilde{M}$ such that

$$\tilde{\rho}(\tilde{\pi}(q,v), \tilde{\pi}(p,w)) \leq \max\{\|q-p\|, \|v-w\|\} \qquad (q,v),(p,w) \in M.$$

Clearly such a metric exists. Given $x = (q,v) \in M$, denote by $(q_j(x), v_j(x))$ the jth reflection point of the forward trajectory $\{S_t(x) : t \geq 0\}$, provided there are at least j reflections, and by $t_j(x)$ be the corresponding *time of reflection*, that is $S_{t_j(x)}(x) = (q_j(x), v_j(x))$. If $x \in M_1$, then $B^j(x) = (q_j(x), v_j(x))$. In this case we will assume that $q_0(x) = q$ and $v_0(x) = v$.

Let $\kappa_0 > 0$ be the *minimal sectional curvature* of $\Gamma = \partial K$ with respect to the outer unit normal field $n(q)$ of Γ. It follows from the strict convexity of K_j for each j that $\kappa_0 > 0$. Finally, introduce the notation

$$l_0 = \left[\frac{D^2}{a^2}\right] + 2 \;,\; \varphi_0 = \frac{\pi}{2}(1 - \frac{1}{l_0}) \;,\; \kappa = \min\{\kappa_0, 2\kappa_0\cos\varphi_0, \frac{2}{a}\}, \tag{2}$$

$$\delta = \frac{1}{(1+a\kappa)^{\frac{1}{s-1}}} \;,\; \delta_1 = \frac{1}{(1+a\kappa)^{\frac{1}{D(s-1)}}} \;,\; C = \frac{24(s-1)D^2(1+a\kappa)^{\frac{2}{s-1}}}{a\log(1+a\kappa)}. \tag{3}$$

Our main result in this paper is the following.

Theorem 1.1. *Let K be an obstacle of the form* (1) *and let $n \in \mathbf{N}$ and $x =$*

$(q_0, v_0), x' = (q'_0, v'_0) \in M_1$ be such that for each integer j with $0 \leq j \leq n$ the points $q_j(x)$ and $q_j(x')$ are well-defined and both belong to Γ_{i_j} for some i_j. Then

$$\|v_j(x) - v_j(x')\| \leq \frac{12D}{a}(\delta^j + \delta^{n-j}) \quad (0 \leq j \leq n), \tag{4}$$

and there exists τ with $|\tau| \leq \|q_0 - q'_0\| < D$ such that

$$\tilde{\rho}\left(S_t(x), S_{t+\tau}(x')\right) \leq C(\delta_1^t + \delta_1^{T-\tau}) \quad (0 \leq t \leq T), \tag{5}$$

where $T = t_n(x)$. Moreover, there exists another global constant $\tilde{C} > 0$, independent of x, x' and n such that for every $i = 0, 1, \ldots, n - l_0 + 1$ there exists $j = i, i+1, \ldots, i + l_0 - 1$ with

$$\|q_j(x) - q_j(x')\| < \tilde{C}(\delta^j + \delta^{n-j}) \quad \text{and} \quad |t_j(x) - t_j(x') - \tau| \leq \tilde{C}(\delta_2^j + \delta_2^{n-j}), \tag{6}$$

where $\delta_2 = \delta_1^a$.

The obstacle K is said to satisfy the *condition* (H) of M.Ikawa if the convex hull of every two convex components of K does not have common points with any other convex component of K. In the special case when the condition (H) is satisfied, a result similar to the above theorem can be easily derived from the papers of M. Ikawa [I] and J. Sjöstrand (see Sect. B in [Sj]).

In the proof of Theorem 1.1 (see Sect. 2 and 3) we essentially use some ideas from Ya. Sinai [Si1], [Si2], M. Ikawa [I] (see also [Burq]) and J. Sjöstrand [Sj]. Using the same tools, in Sect. 6 we prove a similar result which concerns the propagation of a single convex front – a situation which is important for some problems in scattering theory (cf. for example [I], [Burq] and [PS]).

The billiard ball map B defines a dynamical system on the set M_0 of those $x \in M_1$ so that for each $m \in \mathbf{N}$ both $B^m(x)$ and $B^{-m}(x)$ are defined. Clearly M_0 is a compact subset of ∂M and $B : M_0 \longrightarrow M_0$ is a bijection. It is also clear that in general B is not continuous.

A natural invariant measure for B on M_1 is the *Liouville measure* $d\nu = \cos\varphi dq dv$. However in the situation under consideration, M_0 has Lebesgue measure zero in M_1, i.e. in $Q \times \mathbf{S}^{d-1}$, so the Liouville measure is of little help when one considers the restriction of B on M_0. In the special case when K is the union of three congruent disks in $\mathbf{R}^2$ centered at the vertices of an equilateral triangle, an invariant measure on M_0 was constructed by Lopes and Markarian [LM] (see also [ChM]).

In some degenerate cases M_0 is finite, and when M_0 is infinite it is probably the union of a finite set and finitely many Cantor sets of Lebesgue measure zero. Though the points of discontinuity of B on M_1 form a set of measure zero, it is not clear whether in general the points of discontinuity of B on M_0 form a relatively small subset of M_0 (say, a subset of first Baire category). So, it is not quite clear how to define the topological entropy of $B : M_0 \longrightarrow M_0$.

On the other hand, the billiard flow S_t is always continuous, so its topological entropy is well-defined. It is reasonable to consider the billiard flow on its largest bounded invariant set

$$\tilde{L} = \{S_t(\tilde{\pi}(x)) : x \in M_0, t \in \mathbf{R}\}$$

(this is actually the non-wandering set of S_t in $\tilde{M}$), and define the *topological entropy* h of S_t to be the topological entropy $h(S_t) = h(S_1)$ of the flow $S_t : \tilde{L} \longrightarrow \tilde{L}$.

There is a natural way of coding the billiard trajectories using the boundary components Γ_i. Let

$$\Sigma = \prod_{j=-\infty}^{\infty} F_0,$$

where $F_0 = \{1, 2, \ldots, s\}$ and let $\sigma : \Sigma \longrightarrow \Sigma$ be the *Bernoulli shift* on Σ, i.e. $\sigma(\{i_j\}) = \{i'_j\}$, where $i'_j = i_{j+1}$ for each j. Given $x \in M_0$, for every integer j there exists (unique) $i_j \in \{1, 2, \ldots, s\}$ such that $B^j(x) \in \Gamma_{i_j}$. Define $f(x) = \{i_j\}$. In this way one gets a map $f : M_0 \longrightarrow \Sigma$ such that $f \circ B = \sigma \circ f$. In particular, $\Sigma_K = f(M_0)$ is an invariant subset for σ. It follows from Theorem 1.1 that f is invertible, so it provides a *coding* for the billiard on M_0. Using it, in Sect. 4 we prove the following.

Theorem 1.2. *For every obstacle K of the form* (1) *the topological entropy h of the billiard flow S_t does not exceed* $\frac{\log(s-1)}{a}$.

For bounded two-dimensional dispersing billiards such result was obtained by Chernov [Ch] using the estimate of the number of periodic points from [St]. In fact, Chernov [Ch] showed that $\liminf \frac{P_n(B)}{n} \geq h(B)$, where $P_n(B)$ is the number of periodic points of B with period n and $h(B)$ is the topological entropy of B (in this case it is well-defined). It was also proved in [Ch] that for bounded two-dimensional dispersing billiards the periodic points form a dense subset of the phase spece. An interesting (but probably difficult) question is whether the same holds in the situation considered above. Theorem 1.3 below shows that this is so when the condition (H) is satisfied.

Clearly in some cases we may have $h = 0$. A reasonable conjecture seems to be that $h > 0$ whenever the set M_0 is infinite. Let us mention that for dispersing billiards in bounded domains Q the entropy h is always positive (cf. Sinai [Si3] and [Ch]). For such billiards it is also known that a symbolic dynamics with an infinite alphabet always exists [BSC] (see also [GO]).

In our situation, in general the map f is not continuous and Σ_K is not compact, so f does not provide a symbolic dynamics for the billiard ball map B. However, in the special case when the condition (H) is satisfied, the billiard flow S_t is an Axiom A flow, Σ_K is compact and f is a homeomorphism, so it does provide symbolic dynamics for B. In fact, in this case it is easily seen that

$$\Sigma_K = \{\{i_j\} : i_j \neq i_{j+1} \text{ for all } j\},$$

so the restriction σ_K of the Bernoulli shift σ to Σ_K is a topological Markov chain with entropy $\log(s-1)$. Consequently, the billiard flow S_t is naturally isomorphic to a special flow over σ_K which gives an asymptotic for the ditribution of the closed billiard trajectories. For $n = 2$ the latter was obtained by T. Morita [Mor] using a result of Parry and Pollicott [PP]. In fact, combining Theorem 1.1 above (or the earlier result of Ikawa and Sjöstrand) and a modification of the argument in Sect. 3 of [Mor] (see Sect. 5 below), one gets a generalization of Morita's result to all dimensions $n \geq 2$.

Theorem 1.3. *Let K sastisfy the condition (H) and let h be the topological entropy of the billiard flow $S_t : \tilde{L} \longrightarrow \tilde{L}$. Then the flow S_t on $\tilde{L}$ is naturally isomorphic to a special flow over σ_K such that the corresponding closed orbits have the same periods. Moreover*

$$\#\{\gamma : e^{hT_\gamma} \leq \lambda\}.\frac{\log \lambda}{\lambda} \to 1 \qquad \text{as } \lambda \to \infty,$$

where γ runs over the set of all prime closed trajectories of S_t and T_γ is the period of γ, and $\frac{\log(s-1)}{D} \leq h \leq \frac{\log(s-1)}{a}$.

2. Pseudo-billiard trajectories

Let K be an arbitrary obstacle in $\mathbf{R}^n$ of the form (1). Throughout we will assume that $0 \in K$ and will denote by U_0 the *open ball with centre* 0 *and radius* $2D$. Clearly $K \subset U_0$.

By a *pseudo-billiard trajectory* we mean a curve of the form $\gamma = \cup_{j=0}^{k}[q_j, q_{j+1}]$ such that:

(i) $q_0 \neq q_1$, $q_k \neq q_{k+1}$ and for each $j = 1, \ldots, k-1$, $q_j \in \Gamma_{i_j}$, where $i_j = 1, \ldots, s$ and $i_{j+1} \neq i_j$ for each j;

(ii) for each $j = 1, \ldots, k$ either q_j lies on the segment $[q_{j-1}, q_{j+1}]$ (then q_j will be called an *intersection point* or a *tangent point* for the trajectory) or $[q_{j-1}, q_j]$ and $[q_j, q_{j+1}]$ satisfy the law of reflection at q_j with respect to Γ_{i_j} (in which case q_j will be called a *reflection point* for the trajectory).

It is clear from this definition that in general a pseudo-billiard trajectory may intersect the interior of K (see Fig. 1). As we will see below, in some cases it is more convenient to consider certain pseudo-billiard trajectories instead of the corresponding proper billiard trajectories in the exterior Q of K.

We begin with the following simple but rather important lemma.

Lemma 2.1. *Let l_0 and φ_0 be defined by* (2). *Then for every pseudo-billiard trajectory $q_0, q_1, \ldots, q_{l_0+1}$ there exists $j = 1, 2, \ldots, l_0$ such that q_j is a reflection point for the trajectory and $\varphi_j < \varphi_0$, where φ_j is the angle between the segment $[q_j, q_{j+1}]$ and $n(q_j)$.*

Proof. Given an arbitrary pseudo-billiard trajectory $q_0, q_1, \ldots, q_{l_0+1}$, assume that $\varphi_j \geq \varphi_0$ for all $j = 1, \ldots, l_0$. Then, according to (2), $\psi_j = \frac{\pi}{2} - \varphi_j \leq \frac{\pi}{2[D^2/a^2]}$. Denote $e_j = q_{j+1} - q_j$; then $\|e_j\| \leq a$ for each $j = 1, \ldots, l_0$. Since for $1 \leq i < j \leq l_0$ the angle between e_i and e_j does not exceed

$$\psi_i + \psi_{i+1} + \ldots + \psi_{j-1} < (j-i)\frac{\pi}{2[D^2/a^2]} \leq \frac{\pi}{2},$$

it follows that $\langle e_i, e_j \rangle > 0$. Therefore

$$D^2 \geq \|q_{l_0} - q_1\|^2 = \|\sum_{j=1}^{l_0-1} e_j\|^2 \geq \sum_{j=1}^{l_0-1} \|e_j\|^2 \geq (l_0 - 1)a^2.$$

This yields $l_0 \leq \frac{D^2}{a^2} + 1 < \left[\frac{D^2}{a^2}\right] + 2$ which is a contradiction with (2). $\square$

Let $n \geq s$. An n-tuple

$$\alpha = (i_1, i_2, \ldots , i_n) \tag{7}$$

will be called a *configuration* of length n if $i_j = 1, \ldots , s$ for each j and $i_j \neq i_{j+1}$ for $j < n$.

Let (7) be a fixed configuration. Consider the open sets

$$U_\alpha = \{q \in U_0 : a < \mathrm{dist}(q, K_{i_1})\} \quad , \quad V_\alpha = \{p \in \mathbf{R}^d : a < \mathrm{dist}(p, K_{i_n})\}.$$

For $(q, p) \in U_\alpha \times V_\alpha$ define

$$F_\alpha(q; \hat{q}; p) = \|q - q_1\| + \sum_{i=1}^{n-1} \|q_i - q_{i+1}\| + \|q_n - p\|$$

for $\hat{q} = (q_1, \ldots , q_n) \in \hat{K} = K_{i_1} \times \ldots \times K_{i_n}$. Thus, one gets a map

$$F_\alpha : U_\alpha \times K_{i_1} \times \ldots \times K_{i_n} \times V_\alpha \longrightarrow \mathbf{R}.$$

For a fixed $(q, p) \in U_\alpha \times V_\alpha$ there exists $\hat{q} \in \hat{K}$ such that $F_\alpha(q, \hat{q}, p) = \min_{\tilde{q} \in \hat{K}} F_\alpha(q, \tilde{q}, p)$. In general $\hat{q}$ is not unique but the *pseudo-billiard* trajectory

$$\gamma^{(\alpha)}(q, p) = [q, q_1] \cup \left(\bigcup_{i=1}^{n-1} [q_i, q_{i+1}] \right) \cup [q_n, p] \tag{8}$$

is uniquelly determined. This follows for example from the argument in Sect. 3 of [St]. Set

$$q_0(q, p) = q \quad , \quad q_{n+1}(q, p) = p,$$

and proceeding by induction on j, denote by $q_j(q, p)$ the point on the segment $\gamma^{(\alpha)}(q, p) \cap K_{i_j}$ which is closest to $q_{j-1}(q, p)$ (see Fig. 1).

Figure 1

Define

$$v_j(q, p) = \frac{q_{j+1}(q, p) - q_j(q, p)}{\|q_{j+1}(q, p) - q_j(q, p)\|} \in \mathbf{S}^{d-1}.$$

It is easily seen that for each $1 \leq j \leq n$, either one of the segments $[q_{j-1}(q, p), q_j(q, p)]$ and $[q_j(q, p), q_{j+1}(q, p)]$ intersects the interior of K_{i_j}, in which case $q_j(q, p)$ will be called an *intersection point* of $\gamma^{(\alpha)}(q,p)$, or these two segments satisfy the law of reflection at $q_j(q,p)$ with respect to $\Gamma_{i_j} = \partial K_{i_j}$. In the latter case $q_j(q,p)$ will be called a *reflection point* of $\gamma^{(\alpha)}(q,p)$. Clearly a reflection point may be a point of *transversal reflection* or a *tangent point*.

It is easy to derive from this definition that each $q_j(q,p)$ is a continuous function of $(q,p) \in U_\alpha \times V_\alpha$, and if $\gamma(q,p)$ is not tangent to K_{i_j} at $q_j(q,p)$, then $q_j(q,p)$ is differentiable with respect to q and p in a neighbourhood of (q,p).

Let $\xi_t^{(\alpha)}(q,p)$ be the shift of q along the trajectory $\gamma^{(\alpha)}(q,p)$ after time t. Denote by $\omega_t^{(\alpha)}(q,p) \in \mathbf{S}^{d-1}$ the direction of $\gamma^{(\alpha)}(q,p)$ at $\xi_t^{(\alpha)}(q,p)$. At reflection points $\xi_t^{(\alpha)}(q,p)$, one has to identify pairs of directions symmetric with respect to the corresponding tangent plane. In others words, as for the billiard flow S_t, one has to consider $S_t^{(\alpha)}(q,p) = (\xi_t^{(\alpha)}(q,p), \omega_t^{(\alpha)}(q,p))$ as a point in $\tilde{M}$. Clearly $S_t^{(\alpha)}(q,p)$ is well-defined for $0 \le t \le T^{(\alpha)}(q,p)$, where $T^{(\alpha)}(q,p)$ is the *length of the trajectory* $\gamma^{(\alpha)}(q,p)$. For $t > T^{(\alpha)}(q,p)$ we define $\omega_t^{(\alpha)}(q,p) = v_m(q,p)$ and $\xi_t^{(\alpha)}(q,p) = p + (t - T^{(\alpha)}(q,p))v_m(q,p)$, while for $t < 0$ we set $\omega_t^{(\alpha)}(q,p) = -v + 2\langle v, n(q)\rangle n(q)$ and $\xi_t^{(\alpha)}(q,p) = q + tv$. Given $p \in V_\alpha$ and $q, q' \in U_\alpha$, set

$$t_p(q,q') = T^{(\alpha)}(q',p) - T^{(\alpha)}(q,p).$$

The following estimates and some of the arguments in their proofs are the main ingredients for the proof of Theorem 1.1.

Lemma 2.2. *Let $m \in \mathbf{N}$, α be a configuration of the form (7) and let $(q_0,p_0) \in U_\alpha \times V_\alpha$ be such that for each $j = 1, \ldots, m$, $q_j(q_0,p_0)$ is not a tangent point of $\gamma^{(\alpha)}(q_0,p_0)$ to Γ_{i_j}. Then*

$$\|\partial_q \xi^{(\alpha)}_{t+t_p(q_0,q)}(q,p_0)_{|q=q_0}\| < C_1 \delta^t \quad , \quad \|\partial_p \xi_t^{(\alpha)}(q_0,p)_{|p=p_0}\| < C_1 \delta^{T-t} \tag{9}$$

for $0 \le t \le T$, where $T = T^{(\alpha)}(q_0,p_0)$ and C_1 is given by

$$C_1 = \frac{8D(s-1)(1+a\kappa)^{\frac{1}{s-1}}}{a\log(1+a\kappa)}. \tag{10}$$

To prove Lemma 2.2, we first obtain a local estimate about $\frac{\|v_j(q_0,p_0) - v_j(q_0,p)\|}{\|q_0 - q\|}$. Clearly this will provide an estimate for $\|\partial_q v_j(q_0,p_0)\|$. In the same way one obtains an estimate for $\|\partial_p v_j(q_0,p_0)\|$.

Let

$$0 = \tau_0(q,p) < \tau_1(q,p) < \ldots < \tau_n(q,p) < \tau_{n+1}(q,p) = T^{(\alpha)}(q,p)$$

be such that $\xi_{\tau_j}^{(\alpha)}(q,p) = q_j(q,p)$ for each $j = 0, 1, \ldots, n, n+1$.

Fix for a moment $(q_0,p_0) \in U_\alpha \times V_\alpha$ such that **the trajectory** $\gamma^{(\alpha)}(q_0,p_0)$ **does not contain points tangent to** Γ. For $b > 0$, denote by U_b the ball with center q_0 and radius b and by V_b the ball with center p_0 and radius b. Take $b < \frac{a}{4}$ so small that for each $(q,p) \in U_b \times V_b$ the trajectory $\gamma(q,p)$ has no tangencies to Γ. We may as well assume that b is so small that for each $(q,p) \in U_\alpha \times V_\alpha$ we have

$$\tau_{j-1}(q_0,p_0) < \tau_j(q,p) < \tau_{j+1}(q_0,p_0) \quad (j = 1, \ldots, n). \tag{11}$$

Given $j = 1, \dots, n$, denote by $\varphi_j(q,p)$ the (smallest positive) angle between the vector $q_{j+1}(q,p) - q_j(q,p)$ and the unit normal $n(q_j(q,p))$ to Γ at $q_j(q,p)$. Let $j_1 < j_2 < \dots < j_k$ be the sequence of all $j = 1, \dots, n$ with $\varphi_j(q_0, p_0) < \varphi_0$. It follows from Lemma 2.1 that $j_{l+1} - j_l \leq l_0$ $(1 \leq l \leq k)$, therefore

$$k \geq \frac{n}{l_0}. \tag{12}$$

In what follows we also assume that $b > 0$ is chosen so small that

$$\varphi_{j_l}(q,p) < \varphi_0 \quad \text{for all } (q,p) \in U_b \times V_b \ , \quad l = 1, \dots, k. \tag{13}$$

Denote by X a sphere with center q_0 and radius $r \leq \frac{a}{2}$; then the sectional curvature of X is $\frac{1}{r} \geq \frac{2}{a}$. Denote by X' the set of those $q \in X$ so that there exists $p \in V_b$ for which the first segment $[q_0, q_1(q_0,p)]$ of $\gamma^{(\alpha)}(q_0,p)$ intersects X at q. Then X' is an open subset of X and $n_X(q) = \frac{q-q_0}{\|q-q_0\|}$ is the outer normal field to X'. Set $X'_t = \{\xi_t^{(\alpha)}(q_0,p) : p \in V_b\}$, and for $t \in [r,T]$, denote by $\kappa_t(p)$ the *minimal sectional curvature* of X'_t at $\xi_t^{(\alpha)}(q_0,p)$. When $S_t^{(\alpha)}(q_0,p)$ is a reflection point of $\gamma^{(\alpha)}(q_0,p)$, denote

$$\kappa_t^+(p) = \lim_{t' \downarrow t} \kappa_t(p) \quad , \quad \kappa_t^-(p) = \lim_{t' \uparrow t} \kappa_t(p).$$

Lemma 2.3. *We have*

$$\kappa_t(p) \geq \frac{\kappa}{1 + l_0 D \kappa} \tag{14}$$

for all $p \in V_b$ and $t \in [r,T]$.

Proof. Fix $p \in V_b$ and denote $t_l = \tau_{j_l}(q_0,p)$. Since $j_{l+1} - j_l \leq l_0$, we have $t_{l+1} \leq t_l + l_0 D$ for all $l = 1, \dots, k-1$.

Let $r \leq t < t_1(q_0,p)$. Then $\kappa_t(p) = \frac{\kappa'}{1+t\kappa'}$, where $\kappa' = \frac{1}{r}$ is the sectional curvature of X'. Since for every $z > 0$, the function $x \mapsto \frac{x}{1+zx}$ is increasing on $[0,\infty)$, $\kappa' \geq \frac{2}{a} \geq \kappa$ and $t \leq l_0 D$ imply $\kappa_t(p) \geq \frac{\kappa}{1+t\kappa} \geq \frac{\kappa}{1+l_0 D\kappa}$.

Assume $1 < j_1$. Then $t_2 \leq l_0 D$ and $\kappa_{t_1}^+ \geq \kappa_{t_1}^- \geq \frac{\kappa}{1+t_1\kappa}$ (cf. Proposition A.1 in the Appendix) imply that for $t \in [t_1, t_2)$ we have

$$\kappa_t(p) \geq \frac{\kappa_{t_1}^+}{1 + (t-t_1)\kappa_{t_1}^+} \geq \frac{\frac{\kappa}{1+t_1\kappa}}{1 + (t-t_1)\frac{\kappa}{1+t_1\kappa}} = \frac{\kappa}{1+t\kappa} \geq \frac{\kappa}{1+l_0 D\kappa}.$$

Proceeding in this way, one shows that (14) holds for all $t \in [r, t_1)$. Next, Proposition A.1, (13) and (2) yield $\kappa_{t_1}^+(p) \geq 2\kappa_0 \cos\varphi_{j_1}(q_0,p) \geq 2\kappa_0 \cos\varphi_0 \geq \kappa$. Now, repeating the above argument, one gets that (14) holds for all $t \in [t_1, t_2]$, etc. Using a simple induction, it follows that (14) holds for all $t \in [r,T]$. $\square$

Lemma 2.4. (Ikawa [I], Lemma 3.7) *Let X and Y be the boundaries of two disjoint compact strictly convex bodies in $\mathbf{R}^n$. Suppose X and Y are smooth and denote by n_X the unit outer normal field to X. Given $\eta \in X$, let $\eta(t)$ $(t \geq 0)$ be the shift of η along the billiard trajectory in the exterior of Y issued from $(\eta, n_X(\eta))$. Suppose $\eta, \zeta \in X$ are such that the trajectories $\{\eta(t) : t \geq 0\}$ and $\{\zeta(t) : t \geq 0\}$*

hit transversally Y *at* $\eta(t_1)$ *and* $\zeta(t_2)$*, respectively, and* $t_1 \leq t_2$*. Then* $\|\eta^{(i)}(t_2) - \zeta(t_2)\| \leq \|\eta(t_2) - \zeta(t_2)\|$*, where* $\eta^{(i)}(t_2) = \eta + t_2 n_X(\eta)$*.*

Figure 2

We also need the following lemma the first part of which is a local version of Lemma 3.6 of Ikawa [I]. The proof is simple and we omit it.

Lemma 2.5. *Let* Y *be a smooth convex surface in* $\mathbf{R}^d$ *with unit normal field* $n_Y(p)$*, let* $p_0 \in Y$ *and let* $\kappa' > 0$ *be such that the minimal sectional curvature of* Y *at* p *is greater than* κ' *for each* $p \in Y$*.*

(a) *There exists an open neighbourhood* W *of* p_0 *in* Y *such that whenever* $q, p \in W$ *we have*

$$\|(q + tn_Y(q)) - (p + tn_Y(p))\| \geq (1 + t\kappa')\|q - p\|.$$

(b) *If the neighbourhood* W *of* p_0 *is sufficiently small, then for all* $q, p \in W$ *and* $t \geq 0$ *we have* $\|q - p\| \leq \|(q + tn_Y(q)) - p\|$*.*

The following is the main part in the proof of the central Lemma 2.2.

Lemma 2.6. (a) *For* δ *given by (3) we have*

$$\|v_i(q_0, p) - v_i(q_0, p_0)\| < \frac{12}{a}\delta^{n-i}\|p_0 - p\| \quad , \quad 0 \leq i \leq n. \tag{15}$$

for each $p \in V_b$ *. Moreover, if* $i = j_l$ *for some* $l = 1, \ldots, k$*, then we have*

$$\|q_i(q_0, p) - q_i(q_0, p_0)\| < \frac{3}{\cos\varphi_0}\delta^{n-i}\|p_0 - p\|. \tag{16}$$

(b) *For every* $p \in V_b$ *we have*

$$|T(q_0, p_0) - T(q_0, p)| \leq \|p_0 - p\|, \tag{17}$$

and

$$|\tau_j(q_0, p_0) - \tau_j(q_0, p)| \leq \|q_j(q_0, p_0) - q_j(q_0, p)\| \leq D \tag{18}$$

for each $j = 1, \ldots, n$*.*

(c) *Let* δ_1 *be defined by (3). Then*

$$\tilde{\rho}\left(S_t^{(\alpha)}(q_0, p_0), S_t^{(\alpha)}(q_0, p)\right) \leq C_1 \delta_1^{T-t}\|p_0 - p\| \quad , \quad 0 \leq t, p \in V_b, \tag{19}$$

where $T = T^{(\alpha)}(q_0, p_0)$ *and* C_1 *is given by* (10).

Proof. Denote $\delta_0 = \frac{1}{1+a\kappa}$, and notice that for δ given by (3) we have $\delta = \delta_0^{\frac{1}{s-1}}$.

Let $l_1 \leq l_2 \leq \ldots \leq l_m$ be the sequence of all indices $l = 0, 1, \ldots, n$ such that $q_l(q_0, p_0)$ is a reflection point of $\gamma^{(\alpha)}(q_0, p_0)$. For convenience set $l_0 = 0$ and $l_{m+1} = n$. Clearly, if $n \geq 1$, then $l_{i+1} - l_i \leq s-1$ for each i, so $i \leq l_i \leq (s-1)i$ for each $i \geq 1$. In particular $n \geq (m+1)(s-1)$.

Given $1 \leq j \leq m$, denote $\eta_j = q_{l_j}(q_0, p_0)$, $\eta_j^{(r)} = \xi^{(\alpha)}_{\tau_{l_j}(q_0,p)}(q_0, p_0)$ and $\zeta_j = q_{l_j}(q_0, p)$ if $\tau_{l_j}(q_0, p_0) \leq \tau_{l_j}(q_0, p)$ and $\eta_j = q_{l_j}(q_0, p)$, $\eta_j^{(r)} = \xi^{(\alpha)}_{\tau_{l_j}(q_0,p_0)}(q_0, p)$ and $\zeta_j = q_{l_j}(q_0, p_0)$ otherwise. Let $\eta_j^{(i)}$ be the point symmetric to $\eta_j^{(r)}$ with respect to the tangent plane $T_{\eta_j}(\partial K)$.

Let us prove (b) first. Let $1 \leq j \leq m$. We may assume $\tau_{l_j}(q_0, p_0) \leq \tau_{l_j}(q_0, p)$; the other case is similar. Let $t = \tau_{l_j}(q_0, p_0)$, $t' = \tau_{l_j}(q_0, p) - t$. Then the points $q_{l_j}(q_0, p_0) = \xi_t^{(\alpha)}(q_0, p_0)$ and $q' = \xi_t^{(\alpha)}(q_0, p)$ lie on the (strictly) convex surface $Y = \{\xi_t^{(\alpha)}(q_0, p') : p' \in V_b\}$ and $q_{l_j}(q_0, p) = q' + t' n_Y(q')$, where $n_Y(q')$ is the outer unit normal to Y at q'. This clearly implies $t' \leq \|q_{l_j}(q_0, p_0) - q_{l_j}(q_0, p)\|$, which proves (18).

To prove (17), we may assume that $T = T^{(\alpha)}(q_0, p_0) \leq T^{(\alpha)}(q_0, p)$; the other case is similar. Then $t = T^{(\alpha)}(q_0, p) - T \geq 0$. Consider the strictly convex surface $Y = \{\xi_T^{(\alpha)}(q_0, p') : p' \in V_b\}$. Clearly $p_0 \in Y$ and $p'' = \xi_T^{(\alpha)}(q_0, p) \in Y$, while $p' = p'' + t n_Y(p'')$. The convexity of Y now implies $\|p_0 - p\| \geq \|p'' - p'\| = t$. Thus $|T - T^{(\alpha)}(q_0, p)| = t \geq \|p_0 - p\|$, which proves (17).

Next, we are going to prove (a). First notice that Lemma 2.4 yields $\|\eta_1^{(r)} - \zeta_1\| \geq \|\eta_1^{(i)} - \zeta_1\| \geq \frac{a}{2r}\|q_0 - q\|$. Let $1 \leq j < n$. Applying Lemmas 2.4 and 2.5 (see Fig. 3 for one possible configuration), one gets $\|\eta_{j+1}^{(r)} - \zeta_{j+1}\| \geq \|\eta_{j+1}^{(i)} - \zeta_{j+1}\| \geq (1 + a\kappa)\|\eta_j^{(r)} - \zeta_j\|$. Finally, for $l = n$, using Lemma 2.5 (b) and (a), we have

$$\|p_0 - p\| \geq (1 + a\kappa)\|\eta_m^{(r)} - \zeta_m\|. \tag{20}$$

A simple induction shows now that $\|p_0 - p\| \geq (1 + a\kappa)^{m-j}\|\eta_j^{(r)} - \zeta_j\|$ for $1 \leq j < m$. This yields

$$\|\eta_j^{(r)} - \zeta_j\| \leq \delta_0^{m-j}\|p_0 - p\|. \tag{21}$$

Consequently, for $j < m$,

$$\|\eta_{j+1}^{(i)} - \zeta_{j+1}\| \leq \|\eta_{j+1}^{(r)} - \zeta_{j+1}\| \leq \delta_0^{m-j-1}\|p_0 - p\|. \tag{22}$$

Without loss of generality we may assume that ζ_j and η_{j+1} are on the same trajectory (the case on Fig. 3; the other case is similar). Denote $L = \|\eta_{j+1}^{(i)} - \zeta_j\|$. Then $L = \|\zeta_{j+1} - \eta_j^{(r)}\|$ by construction and so one of the vectors $v_{l_j}(q_0, p_0)$ and $v_{l_j}(q_0, p)$ coincides with $\frac{1}{L}(\eta_{j+1}^{(i)} - \zeta_j)$ and the other one with $\frac{1}{L}(\zeta_{j+1} - \eta_j^{(r)})$.

Now combining (21) and (22) gives

$$\|v_{l_j}(q_0, p_0) - v_{l_j}(q_0, p)\| \leq \frac{1}{L}\|(\eta_{j+1}^{(i)} - \zeta_j) - (\zeta_{j+1} - \eta_j^{(r)})\| \leq \frac{1}{L}\left(\|\eta_j^{(r)} - \zeta_j\| + \|\eta_{j+1}^{(i)} - \zeta_{j+1}\|\right)$$

$$\leq \frac{1}{a}\delta_0^{m-j}\|p_0 - p\| + \frac{1}{a}\delta_0^{m-j-1}\|p_0 - p\| < \frac{4}{a}\delta_0^{m-j}\|p_0 - p\|.$$

Hence the inequality

$$\|v_{l_j}(q_0, p_0) - v_{l_j}(q_0, p)\| < \frac{4}{a}\delta_0^{m-j}\|p_0 - p\| \tag{23}$$

holds for any $j < m$. It also holds for $j = m$; this follows trivially from (20).

Figure 3

To prove (15), consider an arbitrary $i = 0, 1, \dots, n$. The case $m = 0$ is trivial, so we will assume that $m \geq 1$. There exists $j = 0, 1 \dots, m$ with $l_j \leq i < l_{j+1}$. Then $n - i \leq l_{m+1} - l_j \leq (m + 1 - j)(s - 1)$. ¿From this, (23) and (3) one gets

$$\begin{aligned} \|v_i(q_0, p_0) - v_i(q_0, p)\| &= \|v_{l_j}(q_0, p_0) - v_{l_j}(q_0, p)\| < \frac{4}{a}\delta_0^{m-j}\|p_0 - p\| \\ &< \frac{4}{a\delta_0}\delta_0^{\frac{n-i}{s-1}}\|p_0 - p\| = \frac{12}{a}\delta^{n-i}\|p_0 - p\|, \end{aligned}$$

which proves (15).

Figure 4

Finally, let $l = r_i$ for some $i = 1, \dots, k$. Then $\varphi_l < \varphi_0$ and also $l = l_j$ for some $j = 1, \dots, m$. Consider the triangle $\eta_j \zeta_j \eta_j^{(r)}$ (see Fig. 4). We have $\psi \geq \frac{\pi}{2} - \varphi_0$. Clearly if $\psi > \frac{\pi}{2}$, then by (21) $\|\eta_j - \zeta_j\| < \|\eta_j^{(r)} - \zeta_j\| < \delta_0^{m-j}\|p_0 - p\|$. Let $\psi \leq \frac{\pi}{2}$. Then, using (21) again, we get

$$\|q_l(q_0, p_0) - q_l(q_0, p_0)\| = \|\eta_j - \zeta_j\| = \frac{\sin\psi'}{\sin\psi}\|\eta_j^{(r)} - \zeta_j\| \leq \frac{1}{\cos\varphi_0}\delta_0^{m-j}\|p_0 - p\|.$$

As above, $n - l = n - l_j \leq (m + 1 - j)(s - 1)$, therefore

$$\|q_l(q_0, p_0) - q_l(q_0, p_0)\| \leq \frac{1}{\delta_0 \cos\varphi_0}\delta_0^{\frac{n-l}{s-1}}\|p_0 - p\| \leq \frac{3}{\cos\varphi_0}\delta^{n-l}\|p_0 - p\|,$$

which proves (16).

It remains to prove (c). Given $t \in (0, T]$, take r with $0 < r < \min\{t, \frac{a}{2}\}$ and, as before, denote by X the sphere with centre q_0 and radius r. Then the inequalities in (a) hold. There exists $j = 0, 1, \ldots, m$ such that

$$t' = \tau_{l_j} \leq t < t'' = \tau_{l_{j+1}}. \tag{24}$$

Set $l = l_j$. Without loss of generality we may assume that $\eta_j = q_l(q_0, p_0)$ and $\zeta_j = q_l(q_0, p)$; the other case is considered similarly. With this assumption, (11) implies $t' \leq \tau_l(q_0, p) < t''$. Then there exists $p' \in [p_0, p]$ such that $\tau_{l_j}(q_{)}, p') = t$.

Figure 5

First, suppose that $j < n$. Set $\tilde{t} = t - t'$, $v' = v_{l_{j-1}}(q_0, p_0)$, $\eta' = \eta_j + \tilde{t}v'$, $w = v_{l_j}(q_0, p)$, $w'' = v_{l_j}(q_0, p_0)$, $\eta'' = \eta_j + \tilde{t}w''$, $\zeta' = q_{l_j}(q_0, p')$, $w' = v_{l_j}(q_0, p')$ (see Fig. 5). Using (22) and (24) for the pairs p_0, p' and p', p, we get

$$\begin{aligned}\|\xi_t^{(\alpha)}(q_0, p_0) - \xi_t^{(\alpha)}(q_0, p)\| &\leq \|\xi_t^{(\alpha)}(q_0, p_0) - \xi_t^{(\alpha)}(q_0, p')\| + \|\xi_t^{(\alpha)}(q_0, p') - \xi_t^{(\alpha)}(q_0, p)\| \\ &= \|\eta' - \zeta'\| + \|\zeta' - \eta''\| \leq \delta_0^{m-j}\|p_0 - p'\| + \delta_0^{m-j}\|p' - p\| = \delta_0^{m-j}\|p_0 - p\|.\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{\rho}(S_t^{(\alpha)}(q_0, p_0), S_t^{(\alpha)}(q_0, p)) &\leq \tilde{\rho}(S_t^{(\alpha)}(q_0, p_0), S_t^{(\alpha)}(q_0, p')) + \tilde{\rho}(S_t^{(\alpha)}(q_0, p'), S_t^{(\alpha)}(q_0, p)) \\ &= \tilde{\rho}((\eta', v'), (\zeta', v'')) + \tilde{\rho}((\zeta', w'), (\eta'', w'')) \\ &\leq \frac{4}{a}\delta_0^{m-j}\|p_0 - p'\| + \frac{4}{a}\delta_0^{m-j}\|p' - p\| = \frac{4}{a}\delta_0^{m-j}\|p_0 - p\|.\end{aligned}$$

Notice that $a \leq t_{l+1} - t_l \leq D$. As in the proof of (a) above we have $n - l = l_{m+1} - l_j \leq (m + 1 - j)(s - 1)$. Moreover $t_l \leq t \leq t_{l+1}$ implies $t + (n - l)D > T$, so $n - l > \frac{T-t}{D}$. Therefore $m + 1 - j > \frac{n-l}{s-1} > \frac{T-t}{(s-1)D}$, which implies $\tilde{\rho}(S_t^{(\alpha)}(q_0, p_0), S_t^{(\alpha)}(q_0, p)) < \frac{4}{a\delta_0}\delta_0^{\frac{T-t}{(s-1)D}}\|p_0 - p\| < \frac{12}{a}\delta_1^{T-t}\|p_0 - p\|$. This proves (19) in the case when $j < m$.

Finally, consider the case when $j = m$, i.e. $t \geq \tau_{l_m}(q_0, p_0)$. Let $l = l_m$. If $t \leq \tau_l(q_0, p)$, then (19) follows trivially from the above argument. So, we will assume that $t \geq \max\{\tau_l(q_0, p_0), \tau_l(q_0, p)\}$. Notice that in this case (23) implies $\|\omega_t^{(\alpha)}(q_0, p_0) - \omega_t^{(\alpha)}((q_0, p)\| = \|v_m(q_0, p_0) - v_m(q_0, p)\| < \frac{4}{a}\|p_0 - p\|$. So, in order to prove (19), it remains to show that

$$\|\xi_t^{(\alpha)}(q_0, p_0) - \xi_t^{(\alpha)}(q_0, p)\| < C_1\delta_1^{T-t}\|p_0 - p\|. \tag{25}$$

Again without loss of generality one may assume that $\eta_m = q_l(q_0,p_0)$ and $\zeta_m = q_l(q_0,p)$. Then for $t' = t - \tau_l(q_0,p)$ we have $\xi_t^{(\alpha)}(q_0,p_0) = \eta_m + t'v_l(q_0,p_0)$ and $\xi_t^{(\alpha)}(q_0,p) = \zeta_m + t'v_l(q_0,p)$, and clearly $\|\xi_t^{(\alpha)}(q_0,p_0) - \xi_t^{(\alpha)}(q_0,p)\| \leq \|p_0 - p\|$ holds for $t \leq T' = \min\{T, T^{(\alpha)}(q_0,p)\}$. In this case we have $T - t \leq 2D$, so $3\delta_1^{T-t} \geq 3\delta_1^{2D} = \frac{3}{(1+a\kappa)^{\frac{2}{s-1}}} \geq \frac{3}{1+a\kappa} \geq 1$. Therefore (cf. (10))

$$\|\xi_t^{(\alpha)}(q_0,p_0) - \xi_t^{(\alpha)}(q_0,p)\| \leq \|p_0 - p\| \leq 3\delta_1^{T-t}\|p_0 - p\| < C_1\delta_1^{T-t}\|p_0 - p\|.$$

Let $t > T'$. Consider the points $p' = \xi_{T'}^{(\alpha)}(q_0,p_0)$ and $p'' = \xi_{T'}^{(\alpha)}(q_0,p_0)$. For $v' = v_l(q_0,p_0)$ and $v'' = v_l(q_0,p)$, according to (23) we have $\|v' - v''\| < \frac{4}{a}\|p_0 - p\|$. Then

$$\begin{aligned}
\|\xi_t^{(\alpha)}(q_0,p_0) &- \xi_t^{(\alpha)}(q_0,p)\| = \|(p' + (t-T')v') - (p'' + (t-T')v'')\| \\
&\leq \|p' - p''\| + (t-T')\|v' - v''\| \leq \|p_0 - p\| + (t-T')\frac{4}{a}\|p_0 - p\| \\
&= \left(1 + \frac{4(t-T')}{a}\right)\|p_0 - p\|.
\end{aligned}$$

Using the fact that $\delta_1^{-1} = (1+a\kappa)^{\frac{1}{D(s-1)}}$, one easily checks that $\lambda(u) = \frac{4D(s-1)}{a\log(1+a\kappa)}\delta_1^{-u} - \frac{4u}{a} - 1 \geq 0$ for all $u \geq 0$. Thus, $1 + \frac{4u}{a} \leq C_2\delta^{-u}$ for all $u \geq 0$, where

$$C_2 = \frac{4D(s-1)}{a\log(1+a\kappa)}.$$

Consequently, $\|\xi_t^{(\alpha)}(q_0,p_0) - \xi_t^{(\alpha)}(q_0,p)\| < C_2\delta_1^{T'-t}\|p_0 - p\|$ for all $t \geq T'$. Since $T' \leq T - D$, it follows that (cf. (10) again)

$$\begin{aligned}
\|\xi_t^{(\alpha)}(q_0,p_0) - \xi_t^{(\alpha)}(q_0,p)\| &< C_2\delta_1^{T-t-D}\|p_0 - p\| = C_2(1+a\kappa)^{\frac{1}{s-1}}\delta_1^{T-t}\|p_0 - p\| \\
&\leq C_1\delta_1^{T-t}\|p_0 - p\|.
\end{aligned}$$

This proves (25) which implies (19). $\square$

Recall that for $q, q' \in U_\alpha$ and $p \in V_\alpha$, $t_p(q,q') = T^{(\alpha)}(q,p) - T^{(\alpha)}(q',p)$. This clearly implies $t_p(q,q') + t_p(q',q'') = t_p(q,q'')$.

Corollary 2.7. (a) *For each $q \in U_b$ we have $\|v_i(q_0,p_0) - v_i(q,p_0)\| < \frac{12}{a}\delta_1^i\|q_0 - q\|$ for $1 \leq i \leq n$, and if $i = j_l$ for some $l = 1, \ldots, k$, then*

$$\|q_i(q_0,p_0) - q_i(q,p_0)\| < \frac{3}{\cos\varphi_0}\delta_1^i\|p_0 - p\|.$$

(b) *For every $q \in U_b$ we have $|T(q_0,p_0) - T(q,p_0)| \leq \|q_0 - q\|$, and*

$$|\tau_j(q_0,p_0) - \tau_j(q,p_0)| \leq \|q_j(q_0,p_0) - q_j(q,p_0)\| \leq D$$

for $j = 1, \ldots, n$.

(c) *Let δ_1 be defined by (3). Then for every $q \in U_b$ we have*

$$\tilde{\rho}(S_t^{(\alpha)}(q_0,p_0), S_{t+\tau}^{(\alpha)}(q,p_0)) \leq C_1\delta_1^t\|q_0 - q\| \quad , \ 0 \leq t \leq T, \tag{26}$$

where $T = T^{(\alpha)}(q_0,p_0)$, $\tau = t_{p_0}(q_0,q)$ and C_1 is given by (10).

Proof. Parts (a) and (b) follow immediately from the corresponding parts of Lemma 2.6, changing the roles of q and p and replacing the configuration α by $-\alpha = (i_n, i_{n-1}, \ldots, i_2, i_1)$.

(c) Notice that $T = T^{(-\alpha)}(p_0, q_0)$. Changing the roles of q and p and using Lemma 2.6 (c) with α replaced by $-\alpha$, one gets $\tilde{\rho}(S_t^{(-\alpha)}(p_0, q_0), S_t^{(-\alpha)}(p_0, q)) \leq C_1 \delta_1^{T-t} \|q_0 - q\|$ for all $t \geq 0$ and $q \in U_b$. Fix $q \in U_b$ for a moment and denote $q' = S_T^{(-\alpha)}(p_0, q)$. Notice that $S_t^{(-\alpha)}(p_0, q_0) = S_{T-t}^{(\alpha)}(q_0, p_0)$, while $S_t^{(-\alpha)}(p_0, q) = S_{T-t+\tau}^{(\alpha)}(q, p_0)$. Therefore $\tilde{\rho}(S_{T-t}^{(\alpha)}(q_0, p_0), S_{T-t+\tau}^{(\alpha)}(q, p_0)) \leq C_1 \delta_1^{T-t} \|q_0 - q\|$ for all $t \geq 0$, and so $\tilde{\rho}(S_t^{(\alpha)}(q_0, p_0), S_{t+\tau}^{(\alpha)}(q, p_0)) \leq C_1 \delta_1^t \|q_0 - q\|$ for $0 \leq t \leq T$.

Proof of Lemma 2.2. The second inequality in (9) follows from (19), while the first follows from (26). □

3. Exponential instability of the billiard

Our objective in this section is to prove Theorem 1.1.

Given $x = (q, v) \in M'$, let $(q_j(x), v_j(x)) = B^j(x)$, provided $B^j(x)$ is defined. Denote by $t_j(x)$ the corresponding *times of reflection*, that is $S_{t_j(x)}(x) = B^j(x)$.

Let $x = (q_0, v_0) \in M_1$, $x' = (q'_0, v'_0) \in M_1$ and $n \in \mathbf{N}$ be as in Theorem 1.1. Then for each $j = 0, 1, \ldots, n$ there exists $i_j \in \{1, 2, \ldots, s\}$ with $q_j(x), q_j(x') \in \Gamma_{i_j}$. Define α by (7). Since $q_0 = q_0(x), q'_0 = q_0(x') \in \Gamma_{i_0}$, it is clear that both q_0 and q'_0 belong to the set U_α (cf. Sect. 2). In fact the whole segment $[q_0, q'_0]$ is in U_α. Also, the points

$$p_0 = q_n(x) \quad , \quad p'_0 = q_n(x')$$

are in Γ_{i_n}, so $[p_0, p'_0] \subset V_\alpha$.

It follows from the assumptions about x and x' that $S_t^{(\alpha)}(q_0, p_0) = S_t(x)$ for $0 \leq t \leq T^{(\alpha)}(q_0, p_0) = t_n(x)$ and $S_t^{(\alpha)}(q'_0, p'_0) = S_t(x')$ for $0 \leq t \leq T^{(\alpha)}(q'_0, p'_0) = t_n(x')$. Moreover

$$\|q_0 - q'_0\| \leq \operatorname{diam}(K_{i_0}) < D \quad , \quad \|p_0 - p'_0\| \leq \operatorname{diam}(K_{i_n}) < D.$$

Given $q \in U_\alpha$, $p \in V_\alpha$, denote by Σ_q (resp. Σ_p) the set of those $p \in V_\alpha$ (resp. $q \in U_\alpha$) such that $\gamma^{(\alpha)}(q, p)$ is tangent to Γ at $q_j(q, p)$ for at least one $j = 1, \ldots, m$. Clearly Σ_q is closed in V_α. Using the strict convexity of the components of ∂K and some arguments from ch.3 in [PS], it can be easily shown that Σ_q is always contained in a finite disjoint union of submanifolds of V_α of dimension $d-1$. In particular Σ_q has Lebesgue measure zero in V_α. Moreover, for arbitrary points $p, p' \in V_\alpha$, Thom's transversality theorem (cf. for example ch. 2 in [GG]) shows that almost all (with respect to the C^∞ Whitney topology) smooth curves $p(r), 0 \leq r \leq 1$, with $p(0) = p$ and $p(1) = p'$ are transversal to Σ_q and therefore have only finitely many common points with Σ_q. The sets Σ_p have similar properties.

Given $p, p' \in V_\alpha$, denote by $L(p, p')$ the *infimum of the lengths of smooth curves* $p(r), 0 \leq r \leq 1$, in V_α such that $p(0) = p$ and $p(1) = p'$. In the same way one defines $L(q, q')$ for $q, q' \in U_\alpha$. It follows easily from the definitions of U_α and V_α that in general

$$L(q, q') < 11D \quad , \quad L(p, p') < 11D \quad (q, q' \in U_\alpha; p, p' \in V_\alpha).$$

Since $[q_0, q_0'] \subset U_\alpha$ and $[p_0, p_0'] \subset V_\alpha$, we have

$$L(q_0, q_0') = \|q_0 - q_0'\| \quad , \quad L(p_0, p_0') = \|p_0 - p_0'\|.$$

Lemma 3.1. *For all $q, q' \in U_\alpha$ and $p, p' \in V_\alpha$ we have*

$$|T^{(\alpha)}(q,p) - T^{(\alpha)}(q,p')| \le L(p,p') \quad , \quad |T^{(\alpha)}(q,p) - T^{(\alpha)}(q',p)| \le L(q,q'),$$

and therefore $|T^{(\alpha)}(q,p) - T^{(\alpha)}(q',p')| \le L(p,p') + L(q,q')$.

Proof. Consider a smooth curve $p(r), 0 \le r \le 1$, in V_α with $p(0) = p$ and $p(1) = p'$ such that $F = \{r \in [0,1] : p(r) \in \Sigma_q\}$ is finite. It then follows from Lemma 2.6 (b) that the function $I = [0,1] \setminus F \ni r \mapsto T^{(\alpha)}(q, p(r))$ is differentiable and $\|\frac{d}{dr}T^{(\alpha)}(q,p(r))\| \le \|p'(r)\|$ for each $r \in I$. Since $T^{(\alpha)}(q,p(r))$ is continuous on $[0,1]$, it follows that

$$|T^{(\alpha)}(q,p) - T^{(\alpha)}(q,p')| \le \int_0^1 \|\frac{d}{dr}T^{(\alpha)}(q,p(r))\|dr \le \int_0^1 \|p'(r)\|dr.$$

This holds for every curve $p(r)$ connecting p and p' and transversal to Σ_q, therefore $|T^{(\alpha)}(q,p) - T^{(\alpha)}(q,p')| \le L(p,p')$.

In the same way one shows that $|T^{(\alpha)}(q,p) - T^{(\alpha)}(q',p)| \le L(q,q')$. Consequently $|T^{(\alpha)}(q,p) - T^{(\alpha)}(q',p')| \le |T^{(\alpha)}(q,p) - T^{(\alpha)}(q,p')| + |T^{(\alpha)}(q,p') - T^{(\alpha)}(q',p')| \le L(p,p') + L(q,q')$, which completes the proof of the assertion.

Lemma 3.2. *Let $q, q' \in U_\alpha$ and $p, p' \in V_\alpha$ be arbitrary points and let $\tau = t_{p'}(q,q')$. Then*

$$\|v_i(q,p) - v_i(q',p')\| \le \frac{12}{a}(\delta^{n-i}L(p,p') + \delta^i L(q,q')) \quad (0 \le i \le n), \tag{27}$$

and

$$\tilde{\rho}\left(S_t^{(\alpha)}(q,p), S_{t+\tau}^{(\alpha)}(q',p')\right) \le C_1 \left(\frac{L(p,p')}{\delta_1^{L(p,p')}} \delta_1^{T-t} + \frac{L(q,q')}{\delta_1^{L(q,q')}} \delta_1^t \right) \tag{28}$$

for all $0 \le t \le T - L(p,p') - L(q,q')$, where $T = T^{(\alpha)}(q,p)$.

Proof. Consider an arbitrary curve $p(r), 0 \le r \le 1$, in V_α with $p(0) = p, p(1) = p'$ which is transversal to Σ_q. Then $F = \{r \in [0,1] : p(r) \in \Sigma_q\}$ is finite. For $r \in I = [0,1] \setminus F$ it follows from Lemma 2.6 (a) that

$$\|\frac{d}{dr}v_i(q,p(r))\| \le \frac{12}{a}\delta^{n-i}\|p'(r)\|.$$

Therefore $\|v_i(q,p) - v_i(q,p')\| \le \int_0^1 \|\frac{d}{dr}v_i(q,p(r))\|dr \le \frac{12}{a}\delta^{n-i}\int_0^1 \|p'(r)\|dr$, which yields

$$\|v_i(q,p) - v_i(q,p')\| \le \frac{12}{a}\delta^{n-i}L(p,p') \quad (0 \le i \le n). \tag{29}$$

Let again $r \in I = [0,1] \setminus F$. Denote

$$T' = \inf\{T^{(\alpha)}(q,p(r)) : 0 \le r \le 1\}.$$

It then follows from Lemma 3.1 that $0 \leq T - T' \leq L(p, p')$.

Let $0 \leq t \leq T'$ and let (r', r'') be an open interval contained in I. Fix for a moment an arbitrarily small $\epsilon > 0$. For every $r \in (r', r'')$ there exists $b = b(r) > 0$ so that (19) holds with $q_0 = q$ and $p_0 = p(r)$. Hence there exists a finite sequence $r' < r_0 < r_1 < \ldots < r_k < r''$ such that $\|r' - r_0\| < \epsilon$, $\|r'' - r_k\| < \epsilon$,

$$\left| \sum_{i=0}^{k-1} \|p(r_{i+1}) - p(r_i)\| - \int_{r'}^{r''} \|p'(r)\| dr \right| < \epsilon \tag{30}$$

and $\tilde{\rho}\left(S_t^{(\alpha)}(q, p(r_i)), S_t^{(\alpha)}(q, p(r_{i+1})\right) < C_1 \delta_1^{T'-t} \|p(r_i) - p(r_{i+1})\|$ for each $i = 0, 1, \ldots, k-1$. Set $l_i = \sum_{j=0}^{i-1} \|p(r_j) - p(r_{j+1})\|$. Then $\tilde{\rho}\left(S_t^{(\alpha)}(q, p(r_0)), S_t^{(\alpha)}(q, p(r_k)\right) \leq \sum_{i=0}^{k-1} \tilde{\rho}\left(S_t^{(\alpha)}(q, p(r_i)), S_t^{(\alpha)}(q, p(r_{i+1}))\right) \leq C_1 \delta_1^{T'-t} \sum_{i=0}^{k-1} \|p(r_j) - p(r_{j+1})\|$. Combining this with (30) gives $\tilde{\rho}\left(S_t^{(\alpha)}(q, p(r_0)), S_t^{(\alpha)}(q, p(r_k)\right) \leq C_1 \delta^{T'-t} \left(\int_{r'}^{r''} \|p'(r)\| dr - \epsilon\right)$, and letting $\epsilon \to 0$, one gets $\tilde{\rho}\left(S_t^{(\alpha)}(q, p(r')), S_t^{(\alpha)}(q, p(r''))\right) \leq C_1 \delta_1^{T'-t} \int_{r'}^{r''} \|p'(r)\| dr$. Summing up over all connected components (r', r'') of I, one obtains

$$\tilde{\rho}\left(S_t^{(\alpha)}(q, p), S_t^{(\alpha)}(q, p')\right) \leq C_1 \delta_1^{T'-t} \int_0^1 \|p'(r)\| dr$$

for all $t \geq 0$. From this and $\delta^{T'-t} \leq \delta^{-L(p,p')} \delta^{T-t}$ it follows that

$$\tilde{\rho}\left(S_t^{(\alpha)}(q, p), S_t^{(\alpha)}(q, p')\right) \leq C_1 \delta_1^{-L(p,p')} \delta_1^{T-t} L(p, p') \quad , \quad t \geq 0. \tag{31}$$

Next, fix a curve $q(r), 0 \leq r \leq 1$, in U_α which is transversal to Σ_p and such that $q(0) = q, q(1) = q'$. Then $E = \{r \in [0, 1] : q(r) \in \Sigma_p\}$ is finite and $J = [0, 1] \setminus E$ is open in $[0, 1]$.

As above, this time using Corollary 2.7 (a), one gets

$$\|v_i(q, p') - v_i(q, p')\| \leq \frac{12}{a} \delta^i \int_0^1 \|q'(r)\| dr,$$

for all $i = 0, 1, \ldots, m$ which yields $\|v_i(q, p') - v_i(q', p')\| \leq \frac{12}{a} \delta^i L(q, q')$ for $0 \leq i \leq m$. This and (29) clearly imply (27).

We are now going to get an estimate for $\tilde{\rho}(S_t^{(\alpha)}(q, p'), S_{t+\tau}^{(\alpha)}(q', p'))$ similar to (31). This is a bit more subtle than (31).

It follows from Lemma 3.1 that for

$$T'' = \inf\{T^{(\alpha)}(q(r), p') : 0 \leq r \leq 1\}$$

we have $T \geq T'' \geq T^{(\alpha)}(q, p') - L(q, q') \geq T - L(p, p') - L(q, q')$. Let $0 \leq t \leq T''$ and let (r', r'') be an open interval contained in I. Fix for a moment an arbitrarily small $\epsilon > 0$. For every $r \in (r', r'')$ there exists $b = b(r) > 0$ so that (26) holds with $q_0 = q(r)$ and $p_0 = p'$. Hence there exists a finite sequence $r' < r_0 < r_1 < \ldots < r_k < r''$ such that $\|r' - r_0\| < \epsilon$, $\|r'' - r_k\| < \epsilon$,

$$\left| \sum_{i=0}^{k-1} \|q(r_{i+1}) - q(r_i)\| - \int_{r'}^{r''} \|q'(r)\| dr \right| < \epsilon \tag{32}$$

and

$$(33)\ \tilde\rho\left(S_t^{(\alpha)}(q(r_i),p'),S^{(\alpha)}_{t+t_{p'}(q(r_i),q(r_{i+1}))}(q(r_{i+1}),p')\right) < C_1\delta_1^t\|q(r_i)-q(r_{i+1})\|$$

for each $i=0,1,\dots,k-1$. Introduce the notation

$$s_i = t_{p'}(q(r_0),q(r_i)) \quad , \quad s'_i = t_{p'}(q(r_i),q(r_{i+1})) \quad , \quad l_i=\sum_{j=0}^{i-1}\|q(r_j)-q(r_{j+1})\|.$$

Clearly $|s_i|\le\|q(r_0)-q(r_i)\|\le l_i$ for all i.

We will prove by induction on i that

$$\tilde\rho\left(S_t^{(\alpha)}(q(r_0),p'),S^{(\alpha)}_{t+s_i}(q(r_i),p')\right) < C_1\delta_1^{-l_{i-1}}\delta_1^t l_i. \tag{34}$$

It follows from (33) that (34) holds for $i=1$. Assume that (34) holds for some i with $1\le i\le k-1$. Using (33) with t replaced by $t+s_i$ and the fact that $s_i+s'_i=s_{i+1}$, we get

$$\begin{aligned}&\tilde\rho\left(S_{t+s_i}^{(\alpha)}(q(r_0),p'),S^{(\alpha)}_{t+s_{i+1}}(q(r_{i+1}),p')\right)\le C_1\delta_1^{t+s_i}\|q(r_i)-q(r_{i+1})\|\\ &\le C_1\delta_1^{s_i}\delta_1^t\|q(r_i)-q(r_{i+1})\|\le C_1\delta_1^{-l_i}\delta_1^t\|q(r_i)-q(r_{i+1})\|.\end{aligned}$$

This and (34) for i imply

$$\begin{aligned}&\tilde\rho\left(S_t^{(\alpha)}(q(r_0),p'),S^{(\alpha)}_{t+s_{i+1}}(q(r_{i+1}),p')\right)\le\tilde\rho\left(S_t^{(\alpha)}(q(r_0),p'),S^{(\alpha)}_{t+s_i}(q(r_i),p')\right)\\ &\qquad+\tilde\rho\left(S_{t+s_i}^{(\alpha)}(q(r_0),p'),S^{(\alpha)}_{t+s_{i+1}}(q(r_{i+1}),p')\right)\\ &\le C_1\delta_1^{-l_{i-1}}\delta_1^t\sum_{j=0}^{i-1}\|q(r_j)-q(r_{j+1})\|+C_1\delta_1^{-l_i}\delta^t\|q_i-q_{i+1}\|\le C_1\delta_1^{-l_i}l_{i+1}.\end{aligned}$$

This proves (34) for $i+1$. Thus, (34) holds for all $i=1,\dots,k$. From (34) with $i=n$ and (32) it follows that $\tilde\rho\left(S_t^{(\alpha)}(q(r_0),p'),S^{(\alpha)}_{t+s_k}(q(r_n),p')\right)<C_1\delta_1^{-l_{k-1}}\delta^t l_k<C_1\delta_1^{-(\int_{r'}^{r''}\|q'(r)\|dr+\epsilon)}\delta^t\left(\int_{r'}^{r''}\|q'(r)\|dr+\epsilon\right)$. Now letting $\epsilon\to0$, we get

$$\tilde\rho\left(S_t^{(\alpha)}(q(r'),p'),S^{(\alpha)}_{t+t_{p'}(q(r'),q(r''))}(q(r''),p')\right)\le C_1\delta_1^{-\int_{r'}^{r''}\|q'(r)\|dr}\delta_1^t\int_{r'}^{r''}\|q'(r)\|dr.$$

Finally, applying the latter to each of the finitely many connected components (r',r'') of I and summing up, one obtains

$$\tilde\rho\left(S_t^{(\alpha)}(q,p'),S^{(\alpha)}_{t+t_{p'}(q,q')}(q',p')\right)\le C_1\delta_1^{-\int_0^1\|q'(r)\|dr}\delta_1^t\int_0^1\|q'(r)\|dr.$$

This clearly yields

$$\tilde\rho\left(S_t^{(\alpha)}(q,p'),S^{(\alpha)}_{t+t_{p'}(q,q')}(q',p')\right)\le C_1\delta_1^{-L(q,q')}L(q,q')\delta_1^t \tag{35}$$

for all t with $0\le t\le T''$.

Let $0 \leq t \leq T - L(p,p') - L(q,q')$. Then $0 \leq t \leq T''$, and so (35) and (31) imply

$$\tilde{\rho}\left(S_t^{(\alpha)}(q,p), S_{t+\tau}^{(\alpha)}(q',p')\right) \leq \tilde{\rho}\left(S_t^{(\alpha)}(q,p), S_t^{(\alpha)}(q,p')\right) + \tilde{\rho}\left(S_t^{(\alpha)}(q,p'), S_{t+t_{p'}(q,q')}^{(\alpha)}(q',p')\right)$$

$$\leq C_1 \delta_1^{-L(p,p')} \delta_1^{T-t} L(p,p') + C_1 \delta_1^{-L(q,q')} L(q,q') \delta_1^t.$$

This proves the lemma. □

We go on with the proof of Theorem 1.1. Using (27) with $q = q_0$, $q' = q_0'$, $p = p_0$, $p' = p_0'$, and the fact that $L(q_0, q_0') = \|q_0 - q_0'\| < D$ and $L(p_0, p_0') = \|p_0 - p_0'\| < D$, one immediately gets (4).

To prove (5), define $\tau = t_{p'}(q_0, q_0')$. Then clearly $|\tau| \leq \|q_0 - q_0'\| < D$. Notice that $\pi(S_t(x)) = \xi_t^{(\alpha)}(x)$ for $0 \leq t \leq T$ and $\pi(S_{t+\tau}(x')) = \xi_{t+\tau}^{(\alpha)}(x')$ for $0 \leq t + \tau \leq T^{(\alpha)}(q_0', p_0')$. Since $T^{(\alpha)}(q_0', p_0') - \tau = T^{(\alpha)}(q_0, p_0') \leq T - D$, it follows that $\pi(S_{t+\tau}(x')) = \xi_{t+\tau}^{(\alpha)}(x')$ holds for $D \leq t \leq T - D$.

Let $D \leq t \leq T - 2D$. Then $t \leq T - L(q_0, q_0') - L(p_0, p_0')$ and so (28) holds. This and the previous remark imply

$$\tilde{\rho}\left(S_t(x), S_{t+\tau}(x')\right) \leq \frac{C_1 D}{\delta_1^D}(\delta_1^{T-t} + \delta_1^t). \tag{36}$$

Next, assume that $0 \leq t \leq D$. Then, using (36) with $t = D$, we get

$$\tilde{\rho}\left(S_t(x), S_{t+\tau}(x')\right) \leq \tilde{\rho}\left(S_t(x), S_D(x)\right) + \tilde{\rho}\left(S_D(x), S_{D+\tau}(x')\right) + \tilde{\rho}\left(S_{D+\tau}(x'), S_{t+\tau}(x')\right)$$

$$\leq 2D + \frac{C_1 D}{\delta_1^D}(\delta_1^{T-D} + \delta_1^D) < 2D + 2C_1 D < 3DC_1 = \frac{3DC_1}{\delta_1^D}\delta_1^D < \frac{3DC_1}{\delta_1^D}(\delta_1^{T-t} + \delta_1^t).$$

In the same way, for $T - 2D \leq t \leq T$ one obtains

$$\tilde{\rho}(S_t(x), S_{t+\tau}(x')) \leq \tilde{\rho}\left(S_t(x), S_D(x)\right) + \tilde{\rho}\left(S_D(x), S_{D+\tau}(x')\right) + \tilde{\rho}\left(S_{D+\tau}(x'), S_{t+\tau}(x')\right)$$

$$\leq 2D + \frac{C_1 D}{\delta_1^D}(\delta_1^{2D} + \delta_1^{T-2D}) < 2D + 2C_1 D \delta_1^D < 3DC_1 \delta_1^D = \frac{3DC_1}{\delta_1^D}\delta_1^{2D} < \frac{3DC_1}{\delta_1^D}(\delta_1^{T-t} + \delta_1^t).$$

Hence for each $t \in [0, T]$ we have $\tilde{\rho}\left(S_t(x), S_{t+\tau}(x')\right) \leq \frac{3DC_1}{\delta_1^D}(\delta_1^{T-t} + \delta_1^t)$. Since the constant C determined by (3) satisfies $C = \frac{3DC_1}{\delta_1^D}$, (5) follows.

It remains to prove (6). Given i, it follows from Lemma 2.1 that there exists $j = i, i+1, \ldots, i + l_0 - 1$ with $\varphi_j(x) < \varphi_0$, where $\varphi_j(x)$ is the angle between the vector $v_{j+1}(x) - v_j(x)$ and the unit normal $n(q_j(x))$. Fix j with this property. The strict convexity of the components of K implies the existence of a global constant $A > 0$ such that if $\xi, \xi' \in \Gamma_i$ for some $i = 1, \ldots, s$, then

$$\|\xi - \xi'\| \leq A \|n(\xi) - n(\xi')\|. \tag{37}$$

Set

$$\epsilon = \frac{12D}{a} \min\{\delta^{j-1} + \delta^{n-j+1}, \delta^j + \delta^{n-j}\}.$$

It is sufficient to consider the case $\epsilon < \frac{1}{2}\cos\varphi_0$. Denote $v = v_j(x) - v_{j-1}(x)$, $w = v_{j+1}(x) - v_j(x)$, $\xi = q_j(x)$, $v' = v_j(x') - v_{j-1}(x')$, $w' = v_{j+1}(x') - v_j(x')$,

$\xi' = q_j(x')$. Then $\|v - v'\| < \epsilon$ and $\|w - w'\| < \epsilon$. Notice that $\|w - v\| = \langle w - v, n(\xi)\rangle = 2\cos\varphi_j(x) \geq \cos\varphi_0$. Therefore

$$\|n(\xi) - n(\xi')\| = \|\frac{w-v}{\|w-v\|} - \frac{w'-v'}{\|w'-v'\|}\| \leq \|\frac{(w-v)-(w'-v')}{\|w-v\|}\|$$

$$+\|w'-v'\|\left|\frac{1}{\|w-v\|} - \frac{1}{\|w'-v'\|}\right| \leq \frac{2\epsilon}{2\cos\varphi_0} + \frac{2\epsilon}{2\cos\varphi_0(2\cos\varphi_0 - \epsilon)} < \frac{3\epsilon}{\cos^2\varphi_0}.$$

This and (37) imply

$$\|q_j(x) - q_j(x')\| = \|\xi - \xi'\| \leq \frac{3A\epsilon}{\cos^2\varphi_0} \leq \frac{36AD}{a\cos^2\varphi_0}\left(\delta^j + \delta^{n-j}\right),$$

which proves the first inequality in (6).

To prove the second inequality in (6), we may assume that $t_j(x) \geq t_j(x') + \tau$; the other case is similar. Set $\eta = \pi(S_{t_j(x)+\tau}(x'))$, and use (5) with $t = t_j(x)$. This gives $\|\xi - \eta\| = \|\pi(S_{t_j(x)}(x)) - \pi(S_{t_j(x)+\tau}(x'))\| \leq C\left(\delta_1^{t_j(x)} + \delta_1^{T-t_j(x)}\right)$ $\leq C(\delta_1^{ja} + \delta_1^{(n-j)a}) = C(\delta_2^j + \delta_2^{n-j})$, where $\delta_2 = \delta_1^a = \delta^{\frac{a}{D}}$. Since $\cos\varphi_j(x') = \frac{1}{2}\|w' - v'\| > \cos\varphi_j(x) - \epsilon > \cos\varphi_0 - \frac{1}{2}\cos\varphi_0 = \frac{1}{2}\cos\varphi_0$, we have

$$\|\eta - \xi'\| \leq \frac{\|\xi - \eta\|}{\sin(\frac{\pi}{2} - \varphi_j(x'))} = \frac{\|\xi - \eta\|}{\cos\varphi_j(x')} \leq \frac{2C}{\cos\varphi_0}(\delta_2^j + \delta_2^{n-j}).$$

On the other hand, in this case we clearly have $|t_j(x) - t_j(x') - \tau| = \|\eta - \xi'\|$, so this proves the second inequality in (6). □

4. Estimate of the entropy

In this section Theorem 1.2 is proved. In what follows we use the Bowen-Dinaburg definition of topological entropy (see for example Sect. 7.2 in [W]).

Consider the Bernoulli shift $\sigma : \Sigma \longrightarrow \Sigma$ and the map $f : M_0 \longrightarrow \Sigma_K \subset \Sigma$ defined in Sect. 1. Here $\Sigma_K = f(M_0)$ is an invariant subset for the Bernoulli shift σ but Σ_K is not necessarily closed in Σ. Define the metric ω_0 on $F_0 = \{1, 2, \dots s\}$ by $\omega_0(i, i') = 0$ if $i = i'$ and 1 otherwise. Then

$$\omega(\{i_j\}, \{i'_j\}) = \sum_{j=-\infty}^{\infty} \delta_1^{|j|+1}\omega_0(i_j, i'_j)$$

defines a metric on Σ which generates the Tichonov (product) topology on Σ. Here δ_1 is the constant defined by (3). It follows easily from Theorem 1.1 that $g = f^{-1} : \Sigma_K \longrightarrow M_0$ is a continuous map. However in general it is not Lipshitz.

Consider the topological Markov chain $\sigma_A : \Sigma_A \longrightarrow \Sigma_A$ determined by the matrix $A = \{a_{ij}\}_{i,j=1}^s$, $a_{ij} = 0$ if $i = j$ and 1 otherwise. That is,

$$\Sigma_A = \{\{b_i\} \in \Sigma : a_{b_i b_{i+1}} = 1 \text{ for all } i\}, \tag{38}$$

and σ_A is the restriction of σ to Σ_A. It is known (cf. Theorem 7.13 in [W] for example) that $h(\sigma_A) = \log\lambda$, λ being the maximal eigenvalue of the matrix A. In our case the maximal eigenvalue of A is $s - 1$, so $h(\sigma_A) = \log(s-1)$.

It follows from the definition of Σ_K that it is contained in Σ_A, so $\overline{\Sigma_K} \subset \Sigma_A$. Consequently, $h(\sigma, \Sigma_K) = h(\sigma, \overline{\Sigma_K}) \leq h(\sigma_A) = \log(s-1)$.

Let $\epsilon = \delta_1^k$ for some $k \in \mathbf{N}$. We assume that k is so big that $2C\epsilon < a$. Here δ_1 and C are the constants defined by (3).

Set $\epsilon' = 2C\epsilon^a + \epsilon$. Consider an arbitrary $n \in \mathbf{N}$ and choose a subset E of Σ_K of minimal cardinality which is $(n+1, \epsilon)$ spanning for σ on Σ_K, i.e. $|E| = r_{n+1}(\epsilon, \Sigma_K, \sigma)$. Define $m = \left[\frac{(n+k)D}{\epsilon}\right] + 1$. Then clearly $\frac{1}{m} < \frac{\epsilon}{(n+k)D}$. For each $y \in g(E)$ set $E'_y = \{S_{\frac{j}{m}}(B^{-k}(y)) : 0 \leq j \leq m\}$, and let $E' = \cup_{y \in g(E)} E'_y$. Clearly

$$|E'| \leq |E| \left(\frac{(n+k)D}{\epsilon} + 2\right) = r_{n+1}(\epsilon, \Sigma_K, \sigma) \left(\frac{(n+k)D}{\epsilon} + 2\right). \tag{39}$$

Let $m_n = [na] + 1$. We claim that E' is (m_n, ϵ')-spanning for S_1 on $\tilde{L}$. Indeed, consider an arbitrary $z \in \tilde{L}$. Then there exists $x = (q, v) \in M_0$ such that $z = \tilde{\pi}(q + t'v, v)$ for some $t' \geq 0$. For x there exists $y \in M_0$ such that $g(y) \in E$ and

$$\omega(\sigma^j(f(x)), \sigma^j(f(y))) < \epsilon = \delta_1^k \quad , \quad j = 0, 1, \dots, n.$$

This implies $i_l = i'_l$ for all $l = -k, -k+1, \dots, -1, 0, 1, \dots, k+n$, where $f(x) = (i_l)$ and $f(y) = (i'_l)$. Applying Theorem 1.1 to $\hat{x} = B^{-k}(x)$ and $\hat{y} = B^{-k}(y)$, we get that there exists τ with $|\tau| \leq D$ such that

$$\tilde{\rho}(S_t(\hat{x}), S_{t+\tau}(\hat{y})) \leq C(\delta_1^t + \delta_1^{T-t}) \quad (0 \leq t \leq T),$$

where $T = t_{n+2k+1}(\hat{x})$. For $t = t_k(\hat{x})$ we have $S_t(\hat{x}) = x$ and therefore

$$\tilde{\rho}(x, S_{t_k(\hat{x})+\tau}(\hat{y})) < C(\delta_1^{T-t_k(\hat{x})} + \delta_1^{t_k(\hat{x})}.$$

Take j_0 such that $\frac{j_0}{m} \leq t_k(\hat{x}) + \tau < \frac{j_0+1}{m}$. Then $z = S_{\frac{j_0}{r}}(\hat{y}) \in E'$. For $0 \leq t \leq na$ we have

$$\begin{aligned}
\tilde{\rho}(S_t(x), S_t(z)) &= \tilde{\rho}(S_{t+t_k(\hat{x})}(\hat{x}), S_{t+\frac{j_0}{m}}(\hat{y})) \\
&\leq \tilde{\rho}(S_{t+t_k(\hat{x})}(\hat{x}), S_{t+t_k(\hat{x})+\tau}(\hat{y})) + \tilde{\rho}(S_{t+t_k(\hat{x})+\tau}(\hat{y}), S_{t+\frac{j_0}{m}}(\hat{y})) \\
&< C\left(\delta_1^{T-t_k(\hat{x})-t} + \delta_1^{t+t_k(\hat{x})}\right) + \left|(t + \frac{j_0}{m}) - (t + t_k(\hat{x}) + \tau)\right| \\
&< C\left(\delta_1^{t_{m+k}(x)-t} + \delta_1^{t_k(\hat{x})}\right) + \frac{1}{m}.
\end{aligned}$$

Here we have taken into account that $T - t_k(\hat{x}) = t_{n+2k+1}(\hat{x}) - t_k(\hat{x}) = t_{n+k+1}(x)$. Since $t_k(\hat{x}) \geq ka$ and $t_{n+k+1}(x) - t \geq (n+k+1)a - [na] > ka$ for $0 \leq t \leq na$, it follows from the above estimates that $\tilde{\rho}(S_t(x), S_t(z)) < 2C\delta_1^{ka} + \frac{1}{m} < \epsilon'$ for $0 \leq t \leq na$. Thus, for all integers $j = 0, 1, \dots, [na]$ we have $\tilde{\rho}(S_j(x), S_j(z)) < \epsilon'$. This means that E' is (m_n, ϵ')-spanning for the map S_1 on $\tilde{L}$, and according to (39),

$$r_{m_n}(\epsilon', \tilde{L}, S_1) \leq |E'| \leq r_{n+1}(\epsilon, \Sigma_K, \sigma) \left(\frac{(n+k)D}{\epsilon} + 2\right).$$

Since $\frac{m_n}{n} \to a$ as $n \to \infty$, it now follows that $ar(\epsilon', \tilde{L}, S_1) \leq r(\epsilon, \Sigma_K, \sigma)$, which implies $ah(S_1) \leq h(\sigma, \Sigma_K) \leq \log(s-1)$. Therefore $h(S_1) \leq \frac{1}{a}\log(s-1)$ which proves the theorem. $\square$

5. Billiards without singularities

Throughout we assume that K is of the form (1) and satisfies the condition (H). Let ρ be the metric on $M \subset \mathbf{R}^n \times \mathbf{R}^n$ defined by

$$\rho((q,v),(p,w)) = \max\{\|q-p\|, \|v-w\|\}.$$

Under the condition (H) one can add the following to the statement of Theorem 1.1.

Theorem 5.1. *Let K have the form* (1) *and satisfy the condition* (H). *Then there exists a global constant $C_2 > 0$ with the following property. If $n \in \mathbf{N}$ and $x = (q_0, v_0), x' = (q_0', v_0') \in M_1$ are such that for each integer j with $1 \leq j \leq n$ the points $q_j(x)$ and $q_j(x')$ are well-defined and both belong to Γ_{i_j} for some i_j, then*

$$\rho(B^j(x), B^j(x')) < C_2(\delta_1^j + \delta_1^{n-j}) \quad (0 \leq j \leq n-1) \tag{40}$$

and there exists τ with $|\tau| \leq \|q_0 - q_0'\| < D$ such that

$$|t_j(x) - t_j(x') - \tau| \leq C_2(\delta_2^j + \delta_2^{n-j}), \tag{41}$$

where $\delta_2 = \delta_1^a$ and δ_1 is defined by (3).

The inequality (40) in the above theorem follows from [I] (see also [Burq]) and [Sj]. Here we derive both (40) and (41) from Theorem 1.1.

Proof. It follows from the condition (H) that there exists $\psi_0 \in (0, \frac{\pi}{2})$ such that for any three successive reflection points q_1, q_2, q_3 of a billiard trajectory in Q the angle between the vectors $q_3 - q_2$ and $n(q_2)$ is less than or equal to ψ_0. Using this and (4), it follows from the last argument in Sect. 3 that there exists a constant $C_2 > 0$ such that both (40) and (41) hold. $\Box$

In what follows we use the notation from the beginning of Sect. 4 and assume that K satisfies the condition (H). Then the map $f : M_0 \longrightarrow \Sigma_K$ is clearly continuous. Thus, Σ_K is compact and f is a homeomorphism between M_0 and Σ_K. In fact, it follows from Theorem 5.1 that $g = f^{-1} : \Sigma_K \longrightarrow M_0$ is a Lipshitz map. We claim that

$$\Sigma_K = \Sigma_A, \tag{42}$$

Σ_A being given by (38). Clearly $\Sigma_K \subset \Sigma_A$. Let $\xi = \{i_j\}$ be a periodic element of Σ_A. It then follows from Sect. 2 of [St] that there exists a periodic point $x \in M_0$ of the billiard ball map B such that $f(x) = \xi$. Hence $\xi \in \Sigma_K$. Since the periodic elements of Σ_A are dense in Σ_A and both Σ_K and Σ_A are compact, (42) follows.

Proof of Theorem 1.3. The above remark shows that $f : M_0 \longrightarrow \Sigma_K = \Sigma_A$ is a homeomorphism which conjugates the billiard ball map $B : M_0 \longrightarrow M_0$ and the topological Markov chain $\sigma_K : \Sigma_K \longrightarrow \Sigma_K$. Consider the map $\lambda : \Sigma_K \longrightarrow \mathbf{R}$, defined by $\lambda(\xi) = t_1(g(\xi)) = \|q_1(g(\xi)) - q_0(g(\xi))\|$, and let

$$\Sigma_K^\lambda = \{(\xi, t) \in \Sigma_K \times \mathbf{R} : 0 \leq t \leq \lambda(\xi)\}.$$

Then the map $\Lambda : \Sigma_K^\lambda \longrightarrow L$, $\Lambda(\xi, t) = S_t(g(\xi))$, is a homeomorphism which defines a conjugacy between the special flow over σ_K on Σ_K^λ and the billiard flow $S_t : L \longrightarrow L$.

Let h be the topological entropy of S_t, i.e. $h = h(S_1)$. It follows from Theorem 1.2 that $h \leq \frac{1}{a}\log(s-1)$. Since $B : M_0 \longrightarrow M_0$ and $\sigma_K : \Sigma_K \longrightarrow \Sigma_K$ are conjugate, we have $h(B) = h(\sigma_K) = \log(s-1)$. Now applying a straightforward modification of the argument in Sect. 4, one gets $\frac{1}{D}\log(s-1) \leq h$. We omit the details.

It remains to establish the asymptotic $\#\{\gamma : e^{hT_\gamma} \leq \lambda\}.\frac{\log\lambda}{\lambda} \to 1$ as $\lambda \to \infty$. As mentioned in [Mor] (see Remark 3.1 there), this follows from Parry and Pollicott [PP] and the following lemma.

Lemma 5.2. *There do not exist $c > 0$ and functions $\mu : \Sigma_K \longrightarrow \mathbf{R}$ and $m : \Sigma_K \longrightarrow \mathbf{Z}$ such that*

$$\lambda = \mu \circ \sigma_K - \mu + cm. \tag{43}$$

Proof of Lemma 5.2. Suppose there exist $c > 0$ and functions μ and m which satisfy (43). This implies that every periodic billiard trajectory in Q has a period (length) of the form ck, k being an integer. Using an appropriate dilation in $\mathbf{R}^d$, we may assume that $c = 1$. This means that every periodic billiard trajectory in Q has integer length. We are now going to show that this is impossible when $s \geq 3$.

In what follows we consider configurations

$$\alpha = (i_0, i_1, \dots, i_{n-1}, i_n = i_0) \tag{44}$$

such that $i_j \in \{1, 2, 3\}$ for all j. Clearly for every α of the form (44) there exists a unique $x = x(\alpha) \in M_0$ with $\pi(x) \in \Gamma_{i_0}$ such that $B^n(x) = x$ and $B^j(x) \in \Gamma_{i_j}$ for all $j = 0, 1, \dots, n$. Let $q_1 \in \Gamma_1$, $q_2 \in \Gamma_2$ be such that $[q_1, q_2]$ is orthogonal to Γ_1 at q_1 and to Γ_2 at q_2, i.e. $d_0 = \|q_1 - q_2\| = \mathrm{dist}(K_1, K_2)$. By assumption, we have $d_0 \in \mathbf{N}$. Given $k \in \mathbf{N}$, define

$$\alpha_k = (\underbrace{1, 2, 1, 2, \dots, 1, 2}_{2k \text{ terms}}, 3, 1),$$

and let

$$q_1^{(k)} \in \Gamma_1, q_2^{(k)} \in \Gamma_2, \dots, q_{4k-1}^{(k)} \in \Gamma_1, q_{4k}^{(k)} \in \Gamma_2, q_{4k+1}^{(k)} \in \Gamma_3, q_{4k+2}^{(k)} \in \Gamma_1$$

be the successive reflection points of the corresponding billiard trajectory. Denote by T_k the length of this trajectory. Cutting the last two terms of α_k, one gets the configuration

$$\beta_k = (\underbrace{1, 2, 1, 2, \dots, 1, 2}_{2k \text{ terms}}).$$

Clearly the billiard trajectories

$$q_1^{(k)}, q_2^{(k)}, \dots, q_{4k-1}^{(k)}, q_{4k}^{(k)} \quad \text{and} \quad \underbrace{q_1, q_2, \dots, q_1, q_2}_{2k \text{ terms}}$$

both follow the configuration β_k. It then follows from Theorem 5.1 that

$$\|q_{2k-1}^{(k)} - q_1\| < C\delta^{2k} \ , \ \|q_{2k}^{(k)} - q_2\| < C\delta^{2k} \ , \|q_{2k+3}^{(k)} - q_1\| < C\delta^{2k-4}. \tag{45}$$

for some global constant $C > 0$ independent of k.

Next, consider the folowing periodic configuration which is a cyclic permutation of α_{k-1}:

$$\alpha'_{k-1} = (\underbrace{1,2,1,2,\dots,1,2}_{2k-2\text{ terms}},3,\underbrace{1,2,1,2,\dots,1,2}_{2k-2\text{ terms}},1).$$

Clearly the successive reflection points of the corresponding periodic billiard trajectory are

$$q^{(k-1)}_{2k-1}, q^{(k-1)}_{2k}, \dots, q^{(k-1)}_{4k-5}, q^{(k-1)}_{4k-4}, q^{(k-1)}_{4k-3}, q^{(k-1)}_{1}, q_2(k-1), \dots, q^{(k-1)}_{2k-3}, q^{(k-1)}_{2k-2}, q^{(k-1)}_{2k-1},$$

and the length of this trajectory is T_{k-1}. Another billiard trajectory which follows the same configuration α'_{k-1} is

$$q^{(k)}_{2k+3}, q^{(k)}_{2k+4}, \dots, q^{(k)}_{4k-1}, q^{(k)}_{4k}, q^{(k)}_{4k+1}, q^{(k)}_{1}, q^{(k)}_{2}, \dots, q^{(k)}_{2k-3}, q^{(k)}_{2k-2}, q^{(k)}_{2k-1}.$$

Denote the lenght of the latter by T'_{k-1}. It then follows from Theorem 5.1 that

$$|T_{k-1} - T'_{k-1}| \leq \|q^{(k-1)}_{2k-1} - q^{(k)}_{2k+3}\| + \|q^{(k-1)}_{2k-1} - q^{(k)}_{2k-1}\|.$$

Using the latter and (45) for k and $k-1$, one gets $|T_{k-1} - T'_{k-1}| < 4C\delta^{2k-4}$. In fact, T_{k-1} is the total minimum of the length function corresponding to the periodic configuration α'_{k-1} (cf. Sect. 3 in [St]), so $T_{k-1} < T'_{k-1}$. Thus,

$$T_{k-1} < T'_{k-1} < T_{k-1} + 4C\delta^{2k-4}. \tag{46}$$

On the other hand, we have $T_k = T'_{k-1} + \|q^{(k)}_{2k-1} - q^{(k)}_{2k}\| + \|q^{(k)}_{2k} - q^{(k)}_{2k+1}\|$, and (45) and (46) imply $T_{k-1} + 4d_0 < T_k < T_{k-1} + 4d_0 + 4C\delta^{2k-4}$. Since d_0, T_{k-1} and T_k are integers by assumption, the above is clearly impossible for large k – contradiction. This proves the assertion. □

6. Propagation of a single convex wavefront

Let again K be an obstacle of the form (1) and let L be a convex domain in $\mathbf{R}^d$ with smooth boundary ∂L such that $L \cap K_1 = \emptyset$ and ∂L divides $\mathbf{R}^d$ into two connected subsets. This implies that the orthogonal projection $H : \mathbf{R}^d \setminus L \longrightarrow \partial L$ is well-defined and smooth. Notice that $L \cap K$ may be non-empty. Define

$$a' = \min\{\mathrm{dist}(L, K_1), a\} \quad , \quad D' = \mathrm{diam}(K \cup L'),$$

where L' is the orthogonal projection of K_1 on L, and determine l'_0, φ'_0, κ', etc. as in (2) and (3) replacing a and D by a' and D', respectively.

The following theorem provides information about the propagation of the wave front issued from the surface ∂L in direction of the unit outward normal field n_L to ∂L. Given $q \in \partial L \setminus K$, consider the billiard trajectory $\gamma(q, n_L(q))$ in $M = Q \times \mathbf{S}^{d-1}$ isuued from $x = (q, n_l(q))$. As before, we shall denote by $q_j(x)$ the successive reflection points of $\gamma(x)$, and by $v_j(x)$ and $t_j(x)$ the corresponding velocities and times of reflection. Here $j = 1, 2, \dots$. For later convenience set $q_0(x) = q$, $v_0 = n_l(q)$ and $t_0(x) = 0$.

Theorem 6.1. *Let K and L be as above. Let $n \in \mathbf{N}$ and $q, q' \in \partial L$ be such that for $x = (q, n_L(q))$ and $y = (q', n_L(q')$ we have $q_1(x), q_2(y) \in \Gamma_1$ and for each integer j*

with $2 \leq j \leq n+1$ the points $q_j(x)$ and $q_j(y)$ are well-defined and both belong to Γ_{i_j} for some i_j. Then

$$\|v_i(x) - v_i(y)\| \leq \frac{12D'}{a'} {\delta'}^{n-i} \quad (0 \leq i \leq n) \tag{47}$$

and

$$\tilde{\rho}(S_t(x), S_t(y)) \leq C' {\delta'}^{T-t} \qquad (0 \leq t \leq T), \tag{48}$$

where $T = t_n(x)$. Moreover, for every $i = 0, 1, \ldots, n - l_0$ there exists $j = i, i+1, \ldots, i + l_0 - 1$ such that

$$\|q_j(x) - q_j(y)\| < C' {\delta'}_1^{n-j} \quad \text{and} \quad |t_j(x) - t_j(y)| \leq C' {\delta'}_1^{n-j}, \tag{49}$$

where ${\delta'}_1 = {\delta'}^{\frac{1}{D'}}$. If K satisfies the condition (H), then $C' > 0$ can be taken such that (49) hold for each $j = 0, 1, \ldots, m$.

Proof. We use modifications of parts of the arguments from Sect. 2 and 3. First, determine the configuration α by (7), where $i_1 = 1$ and $i_2, \ldots, i_n$ are as in Theorem 6.1. Define the set V_α as in Sect. 2, replacing a by a', and consider the function

$$G_\alpha : L \times K_{i_1} \times \ldots \times K_{i_n} \times V_\alpha \longrightarrow \mathbf{R}$$

defined by

$$G_\alpha(\hat{q}, p) = \sum_{i=0}^{n-1} \|q_i - q_{i+1}\| + \|q_n - p\|,$$

where $\hat{q} = (q_0, q_1, \ldots, q_n)$. Given $p \in V_\alpha$, there exists $\hat{q} \in L \times K_{i_1} \times \ldots \times K_{i_n}$ such that $G_\alpha(\hat{q}, p)$ is minimal. Again, $\hat{q}$ is not unique, but q_0 is unique, $q_0 \in \partial L$ and the pseudo-billiard trajectory $\gamma^{(\alpha)}(q_0, p)$ (see (8)) is uniquely determined. Now define $q_0(p) = q_0$ and determine the points $q_j(p)$ inductively so that $q_j(p)$ is the closest point of the segment $\gamma^{(\alpha)}(q_0, p) \cap K_{i_j}$ to $q_{j-1}(p)$. Using the notation from Sect. 2, define $v_j(p) = v_j(q_0(p), p)$, $S_t^{(\alpha)}(p) = S_t^{(\alpha)}(q_0(p), p)$, $\xi_t^{(\alpha)}(p) = \xi_t^{(\alpha)}(q_0(p), p)$, etc. Now for $X = \partial L$ one proves the estimates (15), (16), (17), (18) and (19) with $q_0 = q_0(p)$ using exactly the same argument.

Next, we use the arguments from the first parts of the proofs of Lemmas 3.1 and 3.2 with $p = q_{n+1}(x)$, $p' = q_{n+1}(y)$ to get

$$\|v_i(x) - v_i(y)\| \leq \frac{12}{a'} {\delta'}^{n-i} \|q_{n+1}(x) - q_{n+1}(y)\| \leq \frac{12D'}{a'} {\delta'}^{n-i}$$

for all $i = 0, 1, \ldots, n$. This proves (47). In the same way one gets

$$\tilde{\rho}\left(S_t^{(\alpha)}(p), S_t^{(\alpha)}(p')\right) \leq C_1 \delta_1^{-D'} \delta_1^{T-t} \|q_{n+1}(x) - q_{n+1}(y)\| \leq \frac{C_1 D'}{\delta_1^{D'}} \delta_1^{T-t},$$

which proves (48). Finally, the inequalities (49) can be derived in the same way as in the proof of the corresponding part of Theorem 1.1. We omit the details. □

Appendix. Curvature of convex wavefronts

Let X and Y be smooth hypersurfaces in $\mathbf{R}^d$ endowed with smooth unit normal fields $n_X(x)$ and $n_Y(y)$, respectively. We assume that both X and Y are convex with respect to the given normal fields.

In what follows we use the notation $l(q,v) = \{q + tv : t \geq 0\}$, where $q, v \in \mathbf{R}^d$, $v \neq 0$.

Let $q_0 \in X$ be such that the ray $l(q_0, n_X(q_0))$ hits transversally Y at some point p_0. Set $t_0 = \|p_0 - q_0\|$. Consider an arbitrary smooth parametrization $q = q(u), u \in U$ of X near q_0. Here U is some open subset of $\mathbf{R}^{d-1}$. Let $q_0 = q(u_0)$.

For every $u \in U$ close to u_0, the ray $l(q(u), n_X(u))$ hits transversally Y at some point $p(u)$. Denote by $f(u)$ the reflected direction of $n_X(u)$ at $p(u)$, i.e. $f(u) = n_X(u) - 2\langle n_X(u), n_Y(u)\rangle n_Y(u)$, where $n_X(u) = n_X(q(u))$, $n_Y(u) = n_Y(p(u))$. Then for the angle $\varphi(u)$ between $f(u)$ and $n_Y(u)$ we have $\cos\varphi(u) = \langle f(u), n_Y(u)\rangle$.

Let $\gamma(u)$ be the (billiard) trajectory formed by the segment $[q(u), p(u)]$ and the ray $l(p(u), n_X(u))$. For $t \geq 0$, denote by $S_t(u)$ the shift of $q(u)$ along $\gamma(u)$ after time t. Set $X_t = \{S_t(u) : u \in U\}$. Clearly for $t < t_0$ and $t > t_0$, X_t is a smooth (local) hypersurface. Denote by $\kappa(t)$ the *minimal sectional curvature* of X_t at the point $S_t(u_0)$ and set $\kappa_-(t_0) = \lim_{t\nearrow t_0}\kappa(t)$ $\kappa_+(t_0) = \lim_{t\searrow t_0}\kappa(t)$. Finally, let k_0 be the *minimal sectional curvature* of Y at p_0 and let $\varphi_0 = \varphi(u_0)$.

The following fact is a folklore. We prove it for the sake of completeness.

Proposition A.1. $\kappa_+(t_0) \geq \kappa_-(t_0) + 2\cos\varphi_0 k_0$.

Proof. Fix for a moment t' and t'' with $0 \leq t' < t_0 < t''$ and set $q'(u) = S_{t'}(u)$, $q''(u) = S_{t''}(u)$, $X' = X_{t'}$ and $X'' = X_{t''}$.

Given $a = (a_1, \dots, a_{d-1}) \in \mathbf{R}^{d-1}$, consider the vectors $v' = \sum_{i=1}^{d-1} a_i \frac{\partial q'}{\partial u_i}(u_0) \in T_{q'(u_0)}X'$, $v'' = \sum_{i=1}^{d-1} a_i \frac{\partial q''}{\partial u_i}(u_0) \in T_{q''(u_0)}X''$, $w = \sum_{i=1}^{d-1} a_i \frac{\partial p}{\partial u_i}(u_0) \in T_{p(u_0)}Y$.

We are going to express the sectional curvatures of X', X'' and Y at the points $q'(u_0)$, $q''(u_0)$ and $p(u_0)$ with respect to the sections determined by q', q'' and w, respectively.

Let I' and II' be the first and the second fundamental forms of X' at $q'(u_0)$. Then $I'(v') = \sum_{i,j=1}^{d-1} v'_i v'_j g'_{ij}(u_0)$ and $II'(v') = \sum_{i,j=1}^{d-1} v'_i v'_j b'_{ij}(u_0)$, where $g'_{ij}(u) = \left\langle \frac{\partial q'}{\partial u_i}(u), \frac{\partial q'}{\partial u_j}(u)\right\rangle$ $b'_{ij}(u) = \left\langle n_X(u), \frac{\partial^2 q'}{\partial u_i u_j}(u)\right\rangle$.

The corresponding quantities for X'' will be denoted by I'', II'', g''_{ij}, b''_{ij} and those for Y by $I^{(Y)}$, $II^{(Y)}$, $g_{ij}^{(Y)}$ and $b_{ij}^{(Y)}$.

We have $p(u) = q'(u) + (t(u) - t')n_X(u)$ and $q''(u) = p(u) + (t'' - t(u))f(u)$. ¿From these equalities one derives with some standard computations (see Sect. 10.1 in [PS] for more details) that $I''(v'') = I^{(Y)}(w) - \langle n_Y(u), w\rangle^2 + O(t'' - t_0)$ and $II''(q'') = 2\cos\varphi_0 II^{(Y)}(w) + II'(v') - O(t_0 - t') + O(t'' - t_0)$. On the other hand, $I^{(Y)}(w) = I'(v') + \langle n_X(u_0), w\rangle^2 + O(t_0 - t')$. Combining the last three equalities gives

$$
\begin{aligned}
-\frac{II''(v'')}{I''(v'')} &= \frac{-2\cos\varphi_0 II^{(Y)}(w) - II'(q')}{I^{(Y)}(w) - \langle n_Y(u_0), w\rangle^2 + O(t''-t_0)} + O(t_0-t') + O(t''-t_0)\\
&\geq 2\cos\varphi_0\left(-\frac{II^{(Y)}(w)}{I^{(Y)}(w)+O(t''-t_0)}\right) - \frac{II'(v')}{I'(v')+O(t_0-t')+O(t''-t_0)}\\
&\quad +O(t_0-t')+O(t''-t_0).
\end{aligned}
$$

Now letting $t' \nearrow t_0$ and $t'' \searrow t_0$, we get

$$
-\frac{II''(v'')}{I''(v'')} \geq -2\cos\varphi_0\frac{II^{(Y)}(w)}{I^{(Y)}(w)} - \frac{II'(v')}{I'(v')} \geq 2\cos\varphi_0 k_0 + \kappa_-(t_0).
$$

This holds for every $v'' \in T_{q''(u_0)}X''$, so the assertion follows. $\square$

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