Localization for Schrödinger operators with effective barriers

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December 1995

Abstract

We consider two Schrödinger operators $H_0 = -\Delta + V_0$ and $H = -\Delta + V$ in $L^2(\mathbf{R}^d)$, the dimension d being arbitrary. We assume that the resolvent set $\rho(H_0)$ of H_0 is known and that the potentials $V_0(x)$ and $V(x)$ coincide on a sequence of ring shaped (but not necessarily spherical) regions U_n, $n = 1, 2, \ldots$. Under certain assumptions on the geometry of the U_n and their complements we prove the following result for the operator family $H(\lambda) = H + \lambda W$, where W is a continuous and compactly supported potential of fixed sign and σ_{ac} and σ_c denote the absolutely continuous and continuous spectrum, respectively: (i) $\sigma_{ac}(H(\lambda)) \cap \rho(H_0) = \emptyset$ for every $\lambda \in \mathbf{R}$, (ii) $\sigma_c(H(\lambda)) \cap \rho(H_0) = \emptyset$ for almost every $\lambda \in \mathbf{R}$ with respect to Lebesgue measure. We also give an explicit decay estimate for eigenfunctions, thus establishing localization for $H(\lambda)$ in arbitrary spectral gaps of H_0. The result can be applied to potentials with a sequence of wide barriers as well as to perturbations of periodic potentials.

1 The Results

The entry of questions from the physics of disordered media has considerably changed and enriched the spectral theory of Schrödinger operators in the last two decades. It has been found that potentials with random, quasiperiodic, or other irregular types of potential asymptotics lead to "non-classical" spectral phenomena like *localization* (a notion coming from physics and now generally accepted to mean dense pure point spectrum with exponentially decaying eigenfunctions) or singular continuous spectrum. This shift of attention is stressed, for

*Research supported by NSF-Grant DMS-9401417

example, in the recent review [15]. For general information on random and quasiperiodic potentials see the books [13] and [1].

One-dimensional disordered media are now well understood from the viewpoint of spectral theory, but fundamental questions remain open in higher dimensions, central being the problem of proving or disproving localization in energy regions far from the spectral boundaries.

In this work we give a result on localization for multi-dimensional Schrödinger operators, whose potentials exhibit a series of wider and wider "barriers". Here, the term barriers is to be understood in the general sense of regions which — at certain energies — do not allow quantum mechanical transport. An example to keep in mind is a potential which coincides with a given periodic potential in certain areas but is arbitrary in other areas. Under certain assumptions on the geometry of these areas we can prove localization in all the spectral gaps of the periodic comparison potential. This gives examples of multi-dimensional Schrödinger operators exhibiting localization in energy regions far away from their spectral boundaries.

Another feature of our results is that they need only a minimal amount of "randomness" of the potential. In fact, we work with only a single real random parameter, a coupling constant of a local perturbation. Our potentials are asymptotically deterministic, thus illustrating that irregularity of the potential is more basic for getting localization than actual randomness.

We are now going to state our results explicitly:

The real-valued potentials V_0 and V in $\mathbb{R}^d$ are assumed to have negative parts $(V_0)_-$ and V_- in the Kato class K_d and positive parts $(V_0)_+$ and V_+ in the local Kato class $K_{d,loc}$. The corresponding self-adjoint Schrödinger operators $H_0 = -\Delta + V_0$ and $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ are defined in the usual way by form methods (e.g. [14]).

H_0 takes the role of a comparison operator for which we assume to have knowledge on the location of its spectrum $\sigma(H_0)$ respectively resolvent set $\rho(H_0) = \mathbf{C} \backslash \sigma(H_0)$. A typical situation to have in mind is that of a gap existing in $\sigma(H_0)$. H compares to H_0 in that the potential V coincides with V_0 on certain areas to be described in the following.

Let A_n, $n = 1, 2, \ldots$, be open and bounded subsets of $\mathbb{R}^d$ with $A_n \subset A_{n+1}$ for all n and $\bigcup_n A_n = \mathbb{R}^d$. Also, let $S_n := \partial A_n$ be the boundary of A_n and

$$\delta_n := \min \left\{ \operatorname{dist}(S_n, \{x \in \mathbb{R}^d : V_0(x) \neq V(x)\}), \frac{1}{2} \operatorname{dist}(S_n, S_{n-1} \cup S_{n+1}) \right\} > 0.$$

The latter definition says that $V_0(x)$ and $V(x)$ coincide for values of x in the regions $U_n := \{x \in \mathbb{R}^d : \operatorname{dist}(x, S_n) < \delta_n\}$ and also that the U_n are pairwise disjoint.

Our main new result is ($|\cdot|$ denotes Lebesgue measure, $\overline{M}$ the closure of a set M)

Theorem 1 *Assume that $\delta_n \to \infty$ and*

$$\sum_n \left| A_{n+1} \setminus \overline{A}_{n-1} \right| e^{-\gamma \delta_n} < \infty \tag{1}$$

for every $\gamma > 0$. Then there is a subset S of $\mathbb{R} \setminus \sigma(H_0)$ such that $|(\mathbb{R} \setminus \sigma(H_0)) \setminus S| = 0$ and for every $E \in S$, every multiplication operator Q by a bounded and measurable function of *compact support and every positive integer N one has that*

$$\sup_{\eta \neq 0} \|(H - E - i\eta)^{-1} Q\| < \infty \tag{2}$$

and

$$\sup_{\eta \neq 0} \|\chi_{\mathbb{R}^d \setminus \overline{A}_N} (H - E - i\eta)^{-1} Q\| \leq C \exp\left(-\gamma \sum_{n=1}^{N} \tilde{\delta}_n\right), \tag{3}$$

where $\tilde{\delta}_n := \min_{k \geq n} \delta_k$ and $C = C(Q, E)$ and $\gamma = \gamma(Q, E) > 0$ are constants which do not depend on N.

The first assumption of Theorem 1 says that we need to have a sequence of regions growing in size where $V_0(x) = V(x)$. We think of the regions with $V(x) = V_0(x)$ as "quantum mechanically effective" barriers for an electron having energy $E \in \mathbb{R} \setminus \sigma(H_0)$. The case of classical barriers, i.e. $V(x) \geq E_0$ for some $E_0 \in \mathbb{R}$ and all $x \in \bigcup_n U_n$, is included here by choosing $V_0(x) := \max\{V(x), E_0\}$ and $E \in (-\infty, E_0)$.

Condition (1) imposes a restriction on the size of the regions with $V_0(x) \neq V(x)$, but still allows them to grow much more quickly (in n) than the U_n's. This leaves enough freedom for V in order to include many examples where H has a lot of spectrum in gaps of $\sigma(H_0)$ (for example, whole gaps can be filled with spectrum by choosing $V = 0$ in $\mathbb{R}^d \setminus \bigcup_n U_n$ and constructing suitable Weyl sequences to show that for this choice $[0, \infty) \subset \sigma(H)$). However, from (2) we can conclude that no absolutely continuous spectrum appears in $\rho(H_0)$, and, after slightly randomizing H, that there is no continuous spectrum at all. This is the content of the next Theorem, which also provides an estimate on the decay of eigenfunctions resulting from (3). Here σ_{ac} denotes the absolutely continuous spectrum, σ_c the continuous spectrum, i.e. the union of σ_{ac} and the singular continuous spectrum σ_{sc}.

Theorem 2 *Let $H = -\Delta + V$ have the properties given in Theorem 1 and in addition let $V \in L^{(d+1)/2}_{loc}(\mathbb{R}^d)$. Also, let $W : \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous, compactly supported, $W \not\equiv 0$ and of fixed sign (i.e. either ≥ 0 or ≤ 0). Then for the family $H_\lambda = H + \lambda W$, $\lambda \in \mathbb{R}$, there exists a subset M_0 of $\mathbb{R}$ such that $|\mathbb{R} \setminus M_0| = 0$ and*

(i) $\sigma_{ac}(H(\lambda)) \cap \rho(H_0) = \emptyset$ for every $\lambda \in \mathbb{R}$,

(ii) $\sigma_c(H(\lambda)) \cap \rho(H_0) = \emptyset$ for every $\lambda \in M_0$,

(iii) for every $\lambda \in M_0$ and every eigenfunction φ of H_λ to an eigenvalue in $\mathbb{R} \setminus \sigma(H_0)$ there exist C and $\gamma > 0$ such that

$$|\varphi(x)| \leq C \exp\left(-\gamma \sum_{n=1}^{N-1} \tilde{\delta}_n\right)$$

for every N and $x \in \mathbb{R}^d \setminus \overline{A}_N$. Here $\tilde{\delta}_n$ is defined as in Theorem 1.

Part (i) of the above result is not new and mainly listed for completeness. It was proven in [19] under (essentially) weaker conditions (see also [2] for a related result). For example, if S_n is a sphere of radius R_n, then $\sigma_{ac}(H) \cap \rho(H_0) = \emptyset$ follows from the results in [19] under the assumption

$$\liminf_{n\to\infty} R_n^{d-1} e^{-\gamma\delta_n} = 0 \quad \text{for every } \gamma > 0, \tag{4}$$

and this result applies to the H_λ as well as to H. Condition (1) would now read $\sum_n (R_{n+1}^d - R_{n-1}^d) \exp(-\gamma\delta_n) < \infty$ for every $\gamma > 0$. This implies (4) using

$$R_{n+1}^d - R_{n-1}^d \geq (R_{n+1} - R_{n-1}) R_n^{d-1}$$

and $\delta_n \to \infty$.

Results of type (ii) were first proven for dimension $d = 1$. In [10] Kirsch, Molchanov and Pastur study Schrödinger operators on the half-line $[0, \infty)$ defined by $H = -d^2/dx^2 + V$ and a boundary condition at 0 (discrete one-dimensional Schrödinger operators are studied in [9]). If $V \geq 0$ and $V(x) \geq V_n$ on $[x_n - h_n, x_n + h_n]$ for sequences x_n, h_n and V_n such that $V_n \to \infty$ and $h_n^2 V_n \to \infty$ ("high" barriers), then it is known from [16] that $\sigma_{ac}(H) = \emptyset$ for every such operator H. Under an additional summability assumption similar to (1) it is shown in [10] that $\sigma_c(H) = \emptyset$ for almost every choice of the boundary condition at 0. They also give a related result for "wide" rather than "high" barriers: If $V(x) \geq E_0$ on a series of growing intervals and a summability condition controls the size of the regions with $V(x) < E_0$, then $\sigma_c(H) \cap (-\infty, E_0) = \emptyset$ holds for almost every boundary condition.

Several extensions of the KMP-results were given in [20] (still only treating $d = 1$): (a) Absence of continuous spectrum was proven in arbitrary spectral gaps of a comparison operator $H_0 = -d^2/dx^2 + V_0$ providing regions which act as quantum mechanically effective barriers. (b) the operators were allowed to be unbounded from below, (c) the whole line model $-d^2/dx^2 + V + \lambda W$ with varying λ was considered (instead of varying boundary conditon), (d) decay estimates for eigenfunctions were given, which yield exponential decay in suitable situations.

Theorem 2(iii) provides these decay estimates in the multi-dimensional case. Let, for example, S_n be spheres of radius $R_n = n^\alpha$ and let $\delta_n = \varepsilon n^\beta$, where $\varepsilon > 0$, $\beta > 0$ and $\alpha \geq \beta + 1$. Then $\tilde{\delta}_n = \delta_n$ and $\sum_{n=1}^{N-1} \delta_n \geq C\varepsilon N^{\beta+1}$. So, for $n^\alpha < |x| \leq (n+1)^\alpha$ one has

$$|\varphi(x)| \leq \tilde{C} \exp\left(-\tilde{\gamma} n^{\beta+1}\right) \leq C \exp\left(-\gamma (n+1)^{\beta+1}\right) \leq C \exp(-\gamma |x|^{(\beta+1)/\alpha}),$$

that is superpolynomial decay. In the case $\alpha = \beta + 1$, i.e. when the size of the regions with $V(x) = V_0(x)$ grows at the same rate than the size of the complementary regions, the eigenfunctions decay exponentially.

The proof of Theorem 1 is given in Section 2, which constitutes the main part of this paper. Some of the basic ideas (e.g. resolvent expansion methods, the use of the Borel-Cantelli lemma to overcome a small denominator problem) are taken from the proof of the corresponding one-dimensional result in [10]. Our argument, however, is more functional analytic in spirit, allowing for the generalization to $d > 1$ and arbitrary spectral gaps of H_0. The latter is also made possible by using Combes-Thomas estimates instead of explicit decay estimates for eigenfunctions to eigenvalues below the bottom of the essential spectrum. A lemma of de Branges provides the existence of boundary values of localized resolvents, replacing the Weyl-Titchmarsh m-functions used in a similar context in $d = 1$. Finally, we add an iteration procedure, which leads to the decay estimate (3).

For Schrödinger operators in $L^2(\mathbb{R}^d)$, general results relating estimates of the form (2) to localization were given in [3] (after being studied for other types of operators in [17] and [6]). In [3] they were applied to a proof of localization at low energies for some classes of random potentials. Provided with Theorem 1, we use them to prove Theorem 2 in Section 3.

Results related to ours were recently also given by R. Montcho [12], who studies the multidimensional high barriers case and gives applications to Schrödinger operators with spatially growing randomness, including absence of diffusion.

Acknowledgement: I acknowledge many fruitful discussions with J.-M. Combes, W. Kirsch, B. Simon and P. Stollmann on the subject of this work. I am also grateful for the hospitality at the Université de Toulon et du Var and the California Institute of Technology, where part of this work was done.

2 Proof of Theorem 1

Our proof will rely on the following result which essentially goes back to de Branges [5] (χ_A denotes the characteristic function of a set A):

Lemma 3 *If $H = -\Delta + V$ with $V_- \in K_d$ and $V_+ \in K_{d,loc}$, then there exists a subset $S_0 \subset \mathbb{R}$ of full measure, i.e. $|\mathbb{R} \setminus S_0| = 0$, such that*

$$\sup_{\neq 0} \left\| \chi_A (H - E - i\eta)^{-1} \chi_B \right\| < \infty \tag{5}$$

for every $E \in S_0$ and every pair of compact subsets A and B of $\mathbb{R}^d$.

Proof: First of all, by countability and monotonicity arguments it is enough to prove (5) for fixed compact sets A and B and almost every $E \in J_0$, where $J_0 = [a, b]$ is a bounded interval. We also may assume $A = B$ since $\|\chi_A(H - E - i\eta)^{-1}\chi_B\| \leq \|\chi_{A\cup B}(H - E - i\eta)^{-1}\chi_{A\cup B}\|$.

Let $J := [a-1, b+1]$, $\chi = \chi_A$ and $E_H(M)$ be the spectral projection on M corresponding to H. Then

$$\begin{aligned} & \|\chi(H - E - i\eta)^{-1}\chi\| \\ & \quad \leq \ \|\chi E_H(\mathbb{R} \setminus J)(H - E - i\eta)^{-1}\chi\| + \|\chi E_H(J)(H - E - i\eta)^{-1}\chi\|. \end{aligned} \tag{6}$$

Since $\mathrm{dist}(E + i\eta, \mathbb{R} \setminus J) \geq 1$ for all $E \in J_0$ and $\eta > 0$, the first term is bounded uniformly by 1. $\chi E_H(J)$ is a Hilbert-Schmidt operator (e.g. [14, Theorem B.9.1]). Therefore $\phi(z) = \chi E_H(J)(H - z)^{-1}\chi$ is a trace class valued analytic function in the upper half plane with non-negative imaginary part. A result of de Branges ([5, Lemma 1], for a proof see also [8, p.149-150]) implies the existence of $\lim_{\eta\downarrow 0} \phi(E + i\eta)$ in Hilbert-Schmidt norm for almost every E. In particular, we have boundedness of the second term on the r.h.s. of (6) for almost every E (note that $\|\phi(\bar{z})\| = \|\phi(z)\|$). #

Proceeding now with the proof of Theorem 1 we note that it is enough to prove (2) and (3) for almost every $E \in I$, where I is a fixed compact subinterval of $\mathbb{R} \setminus \sigma(H_0)$ ($\mathbb{R} \setminus \sigma(H_0)$ is a countable union of such intervals).

For the compact subset $J := \{x + iy | x \in I, |y| \leq 1\}$ of $\mathbf{C} \setminus \sigma(H_0)$ we will use the following Combes-Thomas type estimate taken from [19, Lemma 3.1]:

Lemma 4 *For any compact subset J of $\mathbf{C}\setminus\sigma(H_0)$ there exist constants $C = C(J)$ and $\gamma_0 = \gamma_0(J) > 0$ such that for all $\varphi, \tilde{\varphi} \in L^\infty$ with $\|\varphi\|_\infty \leq 1$, $\|\tilde{\varphi}\|_\infty \leq 1$ and* $\mathrm{dist}(\mathrm{supp}\,\varphi, \mathrm{supp}\,\tilde{\varphi}) \geq \delta$ *and all $z \in J$, $i = 1, \ldots, d$:*

$$\|\tilde{\varphi}(H_0 - z)^{-1}\varphi\|^2 \leq Ce^{-\gamma_0\delta}, \tag{7}$$

$$\|\tilde{\varphi}\partial_i(H_0 - z)^{-1}\varphi\|^2 \leq Ce^{-\gamma_0\delta}. \tag{8}$$

For γ_0 as in Lemma 4 and some $\gamma_1 \in (0, \gamma_0/2)$ define $\alpha_n := e^{-\gamma_1\delta_n} + e^{-\gamma_1\delta_{n+1}}$. We also define the operators H_n to be the self-adjoint realizations of $-\Delta + V$ on $L^2(A_{n+1} \setminus \overline{A}_n)$ with Dirichlet boundary conditions (i.e. H_n is the self-adjoint operator corresponding to the form $\int(\overline{\nabla f}\nabla g + \overline{f}Vg)$ on $C_0^\infty(A_{n+1} \setminus \overline{A}_n)$. Also, let $\Delta_n(E) := \mathrm{dist}(E, \sigma(H_n))$ and with S_0 from Lemma 3 define

$$S := \{E \in I \cap S_0 \,|\, \Delta_n(E) \geq \alpha_n \text{ for } n \geq \text{a suitable } n_1(E)\}. \tag{9}$$

Lemma 5 $|S| = |I|$.

Proof: Let Ω be open and bounded in $\mathbb{R}^d$ and $H_\Omega = -\Delta + V$ in $L^2(\Omega)$ with Dirichlet boundary conditions. Since V_- is infinitesimally formbounded with respect to $-\Delta_\Omega$ we find from the variational characterization of the eigenvalues λ_k of H_Ω and λ_k^0 of $-\Delta_\Omega$ that

$$\lambda_k \geq (1 - \varepsilon)\lambda_k^0 - C_\varepsilon. \tag{10}$$

¿From this and Weyl's law for the λ_k^0 we get that the number of k's such that $\lambda_k \leq \lambda$ is bounded from above by $C_\lambda|\Omega|$, C_λ being independent of Ω. Applying this to $\Omega = A_{n+1} \setminus \overline{A}_n$ we get that

$$\text{Number of eigenvalues of } H_n \text{ in } (-\infty, \max\{I\} + 2) \leq C|A_{n+1} \setminus \overline{A}_n|. \tag{11}$$

For $M_n := \{E \in I | \Delta_n(E) < \alpha_n\}$ we have by (11) that

$$|M_n| \leq C\alpha_n|A_{n+1} \setminus \overline{A}_n|.$$

Thus, assumption (1) gives $\sum |M_n| < \infty$. The Borel-Cantelli argument implies that $\Delta_n(E) \geq \alpha_n$ for almost every $E \in I$ and $n \geq n_1(E)$. Lemma 5 follows from this and Lemma 3. #

We will now prove (2) and (3) for $E \in S$, where we can assume $\eta \leq 1$. To do this, we will first construct certain smooth cut-off functions, which separate the regions with $V(x) = V_0(x)$ from regions with $V(x) \neq V_0(x)$. Here we have to pay a price for aiming to treat boundaries S_n of general open and bounded subsets A_n of $\mathbb{R}^d$ (rather than assuming the S_n to be smooth surfaces). This price comes in having to use *regularized distance functions* ([18, p.170f]):

There exist constants $c_1 > 0$, $c_2 > 0$ and B_α (α an arbitrary multi-index) such that every closed set $F \subset \mathbb{R}^d$ admits a C^∞-function $\Theta_F : \mathbb{R}^d \setminus F \longrightarrow (0, \infty)$ with the following properties:

$$c_1 \operatorname{dist}(x, F) \le \Theta_F(x) \le c_2 \operatorname{dist}(x, F) \tag{12}$$

for $x \in \mathbb{R}^d \setminus F$, and

$$|\partial^\alpha \Theta_F(x)| \le B_\alpha(\operatorname{dist}(x, F))^{1-|\alpha|} \tag{13}$$

for $x \in \mathbb{R}^d \setminus F$ and all α. Note that c_1, c_2 and the B_α are independent of F.

With c_1 from (12) we choose $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(x) = 0$ if $|x| \ge c_1$ and $\chi(x) = 1$ if $|x| \le c_1/2$. Let

$$\tilde{\chi}_n(x) := \chi\left(\frac{\Theta_{S_n}(x)}{\delta_n}\right)$$

and

$$\chi_n(x) := \chi\left(\frac{4c_2\Theta_{S_n}(x)}{c_1\delta_n}\right).$$

Extending the $\tilde{\chi}_n$ and χ_n by 1 to S_n we get C_0^∞-functions which by (12) have the properties

$$\tilde{\chi}_n(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x, S_n) \ge \delta_n \\ 1 & \text{if } \operatorname{dist}(x, S_n) \le c_1\delta_n/2c_2 \end{cases} \tag{14}$$

and

$$\chi_n(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x, S_n) \ge c_1\delta_n/4c_2 \\ 1 & \text{if } \operatorname{dist}(x, S_n) \le c_1^2\delta_n/8c_2^2 \end{cases} \tag{15}$$

Now let $u_\eta := (H - E - i\eta)^{-1}Qf$ for an arbitrarily given $f \in L^2(\mathbb{R}^d)$. We will estimate the size of u_η in the regions with $V(x) = V_0(x)$ and $V(x) \ne V_0(x)$ separately.

For $E \in I$ one gets

$$\begin{aligned} \chi_n u_\eta &= \chi_n \tilde{\chi}_n u_\eta = \chi_n (H_0 - E - i\eta)^{-1}(H_0 - E - i\eta)\tilde{\chi}_n u_\eta \\ &= \chi_n (H_0 - E - i\eta)^{-1}(\tilde{\chi}_n Qf + (\Delta\tilde{\chi}_n)u_\eta - 2\nabla \cdot (\nabla\tilde{\chi}_n)u_\eta). \end{aligned} \tag{16}$$

The latter holds since form methods (cf. [7]) show that $\tilde{\chi}_n u_\eta \in D(H_0)$ and $H_0\tilde{\chi}_n u_\eta = H\tilde{\chi}_n u_\eta = \tilde{\chi}_n H u_\eta + (\Delta\tilde{\chi}_n)u_\eta - 2\nabla \cdot (\nabla\tilde{\chi}_n)u_\eta$. By the compact support of Q we have $\tilde{\chi}_n Q = 0$ for $n \ge n_0$ and therefore

$$\chi_n u_\eta = \chi_n (H_0 - E - i\eta)^{-1}((\Delta\tilde{\chi}_n) - 2\nabla \cdot (\nabla\tilde{\chi}_n))u_\eta, \quad n \ge n_0. \tag{17}$$

Here, and at several later instances, Q-dependence enters our considerations, eventually leading to the Q-dependence of the constants in (3). We will not explicitely state this dependence in the proof.

By (14) and (15) we have $\operatorname{dist}(\operatorname{supp}\chi_n, \operatorname{supp}(|\nabla\tilde{\chi}_n| + |\Delta\tilde{\chi}_n|)) \geq C\delta_n$ and (13) implies $\|\nabla\tilde{\chi}_n\|_\infty \leq C$, $\|\Delta\tilde{\chi}_n\|_\infty \leq C$. Therefore (7) and (the adjoint operator version of) (8) imply

$$\left\|\chi_n(H_0 - E - i\eta)^{-1}((\Delta\tilde{\chi}_n) - 2\nabla\cdot(\nabla\tilde{\chi}_n))\right\|^2 \leq Ce^{-\gamma_0\delta_n} \tag{18}$$

for all $n \in \mathbb{N}$, $E \in I$ and $\eta \in (0,1]$. Thus (17) implies

$$\|\chi_n u_\eta\|^2 \leq Ce^{-\gamma_0\delta_n}\int_{U_n}|u_\eta|^2. \tag{19}$$

Next we estimate the size of u_η on the regions $A_{n+1}\setminus\overline{A}_n$ using the operators H_n. Reasoning similar to (16) and (17) yields that on $A_{n+1}\setminus\overline{A}_n$ one has

$$\begin{aligned}(1-\chi_n-\chi_{n+1})u_\eta &= (H_n - E - i\eta)^{-1}(H_n - E - i\eta)(1-\chi_n-\chi_{n+1})u_\eta \\ &= (H_n - E - i\eta)^{-1}(-(\Delta\chi_n + \Delta\chi_{n+1}) + 2\nabla\cdot(\nabla\chi_n + \nabla\chi_{n+1}))u_\eta\end{aligned} \tag{20}$$

for $n \geq n_0$. By definition of $\Delta_n(E)$ we have

$$\|(H_n - E - i\eta)^{-1}\| \leq \frac{1}{\Delta_n(E)}. \tag{21}$$

If $c_0 < \inf\sigma(H)$, then $\|(H_n - c_0)^{-1}\nabla\| \leq C$ uniformly in n by form methods. Thus, the resolvent equation yields

$$\begin{aligned}&\|(H_n - E - i\eta)^{-1}\nabla\| \\ &= \|(H_n - c_0)^{-1}\nabla + (E + i\eta - c_0)(H_n - E - i\eta)^{-1}(H_n - c_0)^{-1}\nabla\| \\ &\leq C\left(1 + \frac{1}{\Delta_n(E)}\right).\end{aligned} \tag{22}$$

The argument leading to (19) can also be applied to the terms $(\Delta\chi_n)u_\eta$, $(\Delta\chi_{n+1})u_\eta$, $(\nabla\chi_n)u_\eta$ and $(\nabla\chi_{n+1})u_\eta$. One gets

$$\|(\Delta\chi_n + \Delta\chi_{n+1})u_\eta\|^2 \leq Ce^{-\gamma_0\delta_n}\int_{U_n}|u_\eta|^2 + Ce^{-\gamma_0\delta_{n+1}}\int_{U_{n+1}}|u_\eta|^2 \tag{23}$$

and

$$\|(\nabla\chi_n + \nabla\chi_{n+1})u_\eta\|^2 \leq Ce^{-\gamma_0\delta_n}\int_{U_n}|u_\eta|^2 + Ce^{-\gamma_0\delta_{n+1}}\int_{U_{n+1}}|u_\eta|^2. \tag{24}$$

Evaluating (20) by using (21) to (24) gives

$$\begin{aligned}&\int_{A_{n+1}\setminus\overline{A}_n}|(1-\chi_n-\chi_{n+1})u_\eta|^2 \\ &\leq C\left(1 + \frac{1}{\Delta_n(E)}\right)^2\left(e^{-\gamma_0\delta_n}\int_{U_n}|u_\eta|^2 + e^{-\gamma_0\delta_{n+1}}\int_{U_{n+1}}|u_\eta|^2\right)\end{aligned} \tag{25}$$

for $n \geq n_0$. Merging (19) with (25) and summing yields for $N \geq n_0$

$$\begin{aligned}\int_{\mathbb{R}^d \setminus \overline{A}_N} |u_\eta|^2 &\leq 2 \sum_{n \geq N} \left(\int_{U_n} |\chi_n u_\eta|^2 + \int_{A_{n+1} \setminus \overline{A}_n} |(1 - \chi_n - \chi_{n+1}) u_\eta|^2 \right) \\ &\leq \sum_{n \geq N} b_n(E) \int_{U_n} |u_\eta|^2, \end{aligned} \tag{26}$$

where

$$b_n(E) := C \left(1 + \frac{1}{\Delta_n(E)^2} + \frac{1}{\Delta_{n-1}(E)^2} \right) e^{-\gamma_0 \delta_n}. \tag{27}$$

Since $E \in S$ we have for $n \geq n_1(E)$ that

$$\begin{aligned} b_n(E) &\leq C \left(1 + \frac{1}{\alpha_n^2} + \frac{1}{\alpha_{n-1}^2} \right) e^{-\gamma_0 \delta_n} \\ &\leq C \left(1 + 2e^{2\gamma_1 \delta_n} \right) e^{-\gamma_0 \delta_n} \\ &\leq Ce^{-(\gamma_0 - 2\gamma_1)\delta_n} \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{28}$$

In particular, we have the rough estimate $b_n(E) < 1/2$ for $n \geq n_0(E)$. Using (26) we get that

$$\int_{\mathbb{R}^d} |u_\eta|^2 \leq \int_{\overline{A}_{n_0(E)}} |u_\eta|^2 + \frac{1}{2} \sum_{n \geq n_0(E)} \int_{U_n} |u_\eta|^2 \leq \int_{\overline{A}_{n_0(E)}} |u_\eta|^2 + \frac{1}{2} \int_{\mathbb{R}^d} |u_\eta|^2,$$

i.e.

$$\int_{\mathbb{R}^d} |u_\eta|^2 \leq 2 \int_{\overline{A}_{n_0(E)}} |u_\eta|^2.$$

This holds for every f in the definition of u_η, thus implying

$$\|(H - E - i\eta)^{-1} Q\|^2 \leq 2 \|\chi_{\overline{A}_{n_0(E)}} (H - E - i\eta)^{-1} Q\|^2.$$

Since $E \in S \subset S_0$ we can apply Lemma 3 to conclude (2).

In order to prove (3) we first note that (28) actually showed that

$$b_n(E) \leq Ce^{-\gamma_2 \delta_n} \quad \text{for } n \geq n_1(E), \tag{29}$$

where $\gamma_2 := \gamma_0 - 2\gamma_1$. Therefore, by (26) we have for $n \geq n_1(E)$

$$\begin{aligned}\int_{\mathbb{R}^d \setminus \overline{A}_N} |u_\eta|^2 &\leq \sum_{n \geq N} Ce^{-\gamma_2 \delta_n} \int_{U_n} |u_\eta|^2 \\ &\leq Ce^{-\gamma_2 \tilde{\delta}_N} \int_{\mathbb{R}^d \setminus \overline{A}_{N-1}} |u_\eta|^2. \end{aligned}$$

Iterating this estimate $N - n_1(E) + 1$ times (as long as (29) applies) and using that $\tilde{\delta}_n \to \infty$ we get

$$\begin{aligned} \int_{\mathbb{R}^d \setminus \overline{A}_N} |u_\eta|^2 &\leq C^{N-n_1(E)+1} \exp\left(-\gamma_2 \sum_{n=n_1(E)}^{N} \tilde{\delta}_n\right) \|u_\eta\|^2 \\ &\leq C(E) \exp\left(-\gamma_3 \sum_{n=1}^{N} \tilde{\delta}_n\right) \|u_\eta\|^2 \end{aligned} \tag{30}$$

for some $\gamma_3 = \gamma_3(E) > 0$ and all $N \geq n_1(E)$. From (2) we know that $\|u_\eta\|^2 \leq C(E)\|f\|^2$ uniformly in f and η. Thus, (3) follows from (30). (Note that (3) for $N < n_1(E)$ is an immediate consequence of (2).) This completes the proof of Theorem 1.

3 Proof of Theorem 2

It is enough to prove Theorem 2 for $W \geq 0$, since from this it follows for negative W by applying the result to $-W$.

Let I be a fixed compact interval in $\mathbb{R} \setminus \sigma(H_0)$. Then we have the following:

(i) $W^{1/2}(H - z)^{-1}W^{1/2}$ is compact for every z with $\operatorname{Im} z \neq 0$, and

(ii) there exists a measurable subset I_0 of I such that $|I_0| = |I|$ and

$$\sup_{\eta \neq 0} \|(H - E - i\eta)^{-1} W^{1/2}\| < \infty$$

for every $E \in I_0$.

Statement (i) follows for example from Theorem B.9.3 of [14] and (ii) follows with $I_0 := I \cap S$ from Theorem 1. Theorem 3.2 of [3] says that under these assumptions

$$\sigma_{ac}(\tilde{H}(\lambda)) \cap I = \emptyset \quad \text{for every } \lambda \in \mathbb{R}, \tag{31}$$

and there exists $M(I) \subset \mathbb{R}$ with $|\mathbb{R} \setminus M(I)| = \emptyset$ and

$$\sigma_{sc}(\tilde{H}(\lambda)) \cap I = \emptyset \quad \text{for every } \lambda \in M(I), \tag{32}$$

where $\tilde{H}(\lambda)$ is the restriction of $H(\lambda)$ to the closure of

$$\{f(H(\lambda))W^{1/2}\phi \, : \, f \in L^\infty(\mathbb{R}^d), \, \phi \in L^2(\mathbb{R}^d)\}. \tag{33}$$

Under the assumption $V \in L^{(d+1)/2}_{loc}(\mathbb{R}^d)$, the set (33) is dense in $L^2(\mathbb{R}^d)$ (see [3, Appendix 2]) and therefore $\tilde{H}(\lambda) = H(\lambda)$. (31) directly implies part (i) of Theorem 2.

Let $A := ((\mathbb{R} \cap \rho(H_0)) \setminus S) \cup \{\text{eigenvalues of} H = H(0)\}$. We have $|A| = 0$. If $E_\lambda(\cdot)$ denotes the spectral family of $H(\lambda)$, then it follows from spectral averaging (e.g. Corollary 4.2 in [3]) that $W^{1/2}E_\lambda(A)W^{1/2} = 0$ for every $\lambda \in M_1$, where $M_1 \subset \mathbb{R}$ is such that $|\mathbb{R} \setminus M_1| = \emptyset$. This implies

$$\|E_\lambda(A)W^{1/2}\varphi\|^2 = \langle W^{1/2}E_\lambda(A)W^{1/2}\varphi, \varphi\rangle = 0$$

for every $\varphi \in L^2(\mathbb{R}^d)$ and therefore

$$E_\lambda(A)f(H(\lambda))W^{1/2}\varphi = 0$$

for every $\varphi \in L^2(\mathbb{R}^d)$ and $f \in L^\infty(\mathbb{R}^d)$. This shows $E_\lambda(A) = 0$ for every $\lambda \in M_1$ by the denseness observed above. By the definition of A this has two consequences for $\lambda \in M_1$ (with $\sigma_p(H(\lambda))$ and $\sigma_p(H)$ denoting the set of eigenvalues of $H(\lambda)$ and H, respectively):

(a) $\sigma_p(H(\lambda)) \cap \rho(H_0) \subset S$,

(b) $\sigma_p(H(\lambda)) \cap \sigma_p(H) = \emptyset$.

Write $\mathbb{R} \cap \rho(H_0) = \bigcup_n I_n$ as a countable union of compact intervals and let $M_0 := M_1 \cap \bigcap_n M(I_n)$. Then $|\mathbb{R} \setminus M_0| = 0$ and part (ii) of Theorem 2 holds for $\lambda \in M_0$ by (32).

To prove (iii) let $\lambda \in M_0$, E an eigenvalue of $H(\lambda)$ in $\rho(H_0)$ and φ be the corresponding eigenfunction. By (b) above, E is not an eigenvalue of H and therefore the dominated convergence theorem and the spectral theorem imply $\lim_{\eta\to 0}(H - E - i\eta)^{-1}(H - E)\varphi = \varphi$ in $L^2(\mathbb{R}^d)$. But $(H - E)\varphi = -\lambda W\varphi$, i.e.

$$\varphi = -\lim_{\eta\to 0} \lambda(H - E - i\eta)^{-1}W\varphi,$$

and thus for every $N \in \mathbb{N}$:

$$\|\chi_{\mathbb{R}^d\setminus\overline{A}_N}\varphi\| \le |\lambda| \sup_{\eta>0} \|\chi_{\mathbb{R}^d\setminus\overline{A}_N}(H - E - i\eta)^{-1}W\|\|\varphi\|. \tag{34}$$

By (a) we have $E \in S$. Therefore Theorem 1 with $Q = W$ implies

$$\|\chi_{\mathbb{R}^d\setminus\overline{A}_N}\varphi\| \le C \exp\left(-\gamma \sum_{n=1}^{N} \tilde{\delta}_n\right) \quad \text{for every } N \in \mathbb{N}. \tag{35}$$

This L^2-decay estimate can be turned into a pointwise estimate by using the subsolution estimate $|\varphi(x)|^2 \le C\int_{|x-y|\le r} |\varphi(y)|^2\,dy$ with C only depending on r, E and the local K_d-norm of V_- ([4, p.18]). Since $\inf \delta_n > 0$, from (35) we get for $x \in \mathbb{R}^d \setminus \overline{A}_N$ that

$$|\varphi(x)|^2 \le C \int_{\mathbb{R}^d\setminus\overline{A}_{N-1}} |\varphi(y)|^2 \le C \exp\left(-\gamma \sum_{n=1}^{N-1} \tilde{\delta}_n\right).$$

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