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Irreducible representations of the Heisenberg algebra in Krein spaces

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Abstract. The representations of the Heisenberg algebra in Krein spaces, more generally in weakly complete inner product spaces, are classified under general regularity and irreducibility conditions. Besides the Fock representation, two other types appear, one with negative, the other with two sided discrete spectrum of the number operator.

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1. Introduction.

The Heisenberg algebra is the algebraic structure at the basis of Quantum Mechanics (QM), and it is of great interest not only from a physical point of view, but also as a basic mathematical structure. It is the Lie algebra generated by the so–called canonical variables, q_i, p_i, $i = 1 \ldots n$, satisfying the following commutation relations (CCR in unbounded form) [1] [2] [3]

$$[q_i, p_j] = i\,\delta_{i,j} \quad , \quad [q_i, q_j] = 0 = [p_i, p_j] \tag{1.1}$$

The foundations of QM rely on the analysis of the representations of such an algebra and great clarification of the mathematical structures of QM has been obtained by this analysis [4] [5] [6]

¿From a technical point of view, non trivial problems arise because the commutation relations (1.1) imply that q_i and p_i cannot be represented by bounded operators (in fact, the corresponding group is not compact); as a consequence, domain questions become important, the concept of irreducibility is delicate (see below), etc. For these reasons, one usually considers the algebra $\mathcal{A}$ generated by the Weyl exponentials $\exp i\alpha q$, $\exp i\beta p$, $\alpha, \beta \in \mathbb{R}^n$ (briefly, the Weyl algebra). This amounts to consider, instead of the Lie algebra (1.1), the associated group (the Heisenberg group), and to analyse its representations. A systematic classification of such representations in a Hilbert space, under general regularity conditions, is available [5][6]. It should however be remarked that the choice of the Weyl algebra, generated by rather special functions of the canonical variables q_i, p_i, gives rise to problems, e.g. for the algebraic setting of the dynamics, since in general the time evolution does not leave the Weyl algebra stable [7]. Actually, most of the standard treatments of QM models deal with the canonical variables directly, and therefore it is of some interest to consider the problem of analysing the representations of the Heisenberg algebra directly.

This strategy is actually the most convenient one if one wants to consider the case of the representations of the CCR's in indefinite metric spaces, in particular in Krein spaces, since in this case q_i and p_i are not selfadjoint operators, the construction of the Weyl exponentials is in general problematic and requires domain and regularity assumptions, and the advantage is not significant since the Weyl exponentials need not to be bounded operators.

A notable case of an analysis of the Hilbert space representations of the CCR's based on the Heisenberg algebra is the theorem by Rellich and Dixmier [8] [9] [10], which overcomes the difficulties mentioned above by assuming that $N = 1/2\,(p^2 + q^2 -$

1) is essentially selfadjoint on a dense domain invariant under the closure of p and q. The result of this note is a generalization of the theorem by Rellich and Dixmier, which covers the case of representations of the Heisenberg algebra in Krein spaces, toghether with a discussion of irreducibility. The main motivation of this note is that of a preparatory result for the classification of the representation of the CCR's in Krein space, when the number of degrees of freedom is infinite, a problem which is relevant for realistic gauge quantum field theories [11] [12].

Crucial technical points of our approach are: i) a proper position of the irreducibility problem in a situation in which the algebra consists of unbounded operators, ii) regularity assumptions in the same spirit of Rellich and Dixmier. The result is that the spectrum of $N = 1/2\,(p^2 + q^2 - 1)$ is discrete (but not semibounded, in general), and in terms of it one has a classification of the irreducible representations of the Heisenberg algebra in Krein spaces (more generally, in weakly complete non–degenerate inner product spaces). Up to isometries between indefinite spaces, such representations are completely described by the action of the algebra on the (weakly) dense domain generated by the eigenvector of N.

2. General setting

As mentioned in the Introduction, in order to analyse the representations of the Heisenberg algebra $\mathcal{H}$ in Krein spaces [13] [14], we have to specify domain and regularity conditions and a proper definition of irreducibility. For simplicity, in the following we restrict our attention to the Heisenberg algebra for one degree of freedom, the extension to a finite number of degrees of freedom being straightforward.

We briefly recall that a Krein space $\mathcal{K}$ is an indefinite, non–degenerate, inner product space, with the properties
i) it is weakly complete
ii) there is an operator J, self–adjoint with respect to the inner product, $J^\dagger = J$ and with $J^2 = 1$, such that $< \cdot, J\cdot >$ is a positive inner product (giving a Hilbert space structure to $\mathcal{K}$)

Only property i) is needed in the following analysis, which remains therefore valid for any (sequentially) weakly complete non–degenerate inner product space $\mathcal{V}$ (see Proposition 11 below). We recall that (sequential) weak completeness means that if $\forall x \in \mathcal{K}$ $< x, y_n >$ is a convergent sequence, then there exists $y \in \mathcal{K}$ such that $< x, y_n >$ converges to $< x, y >$, $\forall x \in \mathcal{K}$. The uniqueness of the weak limit is guaranteed by the non–degeneracy of the inner product ($< x, y >= 0 \;\; \forall x \in \mathcal{K} \Rightarrow y = 0$).

Definition 1. *A representation of a $*$ algebra $\mathcal{A}$ in a Krein space $\mathcal{K}$ is a $*$–morphism ρ of $\mathcal{A}$ into the algebra of operators acting on a dense invariant domain $\mathcal{D} \subset \mathcal{K}$, namely*

$$\rho(\mathcal{A})\,\mathcal{D} \subset \mathcal{D}$$

and, $\forall x, y \in \mathcal{D}$, $A \in \mathcal{A}$

$$< x, \rho(A)y >=< \rho(A^*)x, y > \tag{2.1}$$

Eq.(2.1) implies that all the operators $\rho(A)$, $A \in \mathcal{A}$, have a densely defined adjoint $\rho(A)^\dagger$, with respect to the indefinite inner product, and $\rho(A^*) \subset \rho(A)^\dagger$. All the operators $\rho(A)$ are therefore closable. We shall denote by $\mathcal{D}_c$ the common domain of the closure $\overline{\rho(A)}$ of $\rho(A)$, $A \in \mathcal{A}$,

$$\mathcal{D}_c = \bigcap_{A \in \mathcal{A}} \mathcal{D}_{\overline{\rho(A)}} \supset \mathcal{D} \tag{2.2}$$

In the following, for simplicity, we will denote $\rho(A)$ by A. In general, the domain $\mathcal{D}_c$ is not invariant under the closures $\bar{A}$, $A \in \mathcal{A}$. Any invariant (non–trivial) subspace $\mathcal{D}' \subset \mathcal{D}_c$, $\bar{A}\mathcal{D}' \subset \mathcal{D}'$ $\forall A \in \mathcal{A}$, provides a representation of the algebra in the sense of Definition 1. In fact, eq.(2.1) implies $\bar{A} \subset (A^*)^\dagger$, and the invariance of $\mathcal{D}'$ implies, $\forall x \in \mathcal{D}'$, $\bar{B}x \in \mathcal{D}_{\bar{A}} \subset \mathcal{D}_{(A^*)^\dagger}$, so that $\forall y \in \mathcal{D}$, $x_n \in \mathcal{D}'$ with $x_n \to x$ and Bx_n convergent,

$$< y, (AB)x_n > = < A^*y, Bx_n > \to$$

$$< A^*y, \bar{B}x > = < y, (A^*)^\dagger \bar{B}x > = < y, \bar{A}\bar{B}x >$$

By Definition 1, the representation of $\mathcal{H}$ in $\mathcal{K}$ is also generated by the representative of the so–called creation and annihilation operators

$$a^* \equiv 1/\sqrt{2}\,(q - ip) \qquad , \qquad a \equiv 1/\sqrt{2}\,(q + ip)$$

leaving $\mathcal{D}$ invariant and satisfying $[a, a^*] = 1$ on $\mathcal{D}$. By the Heisenberg relations, $N = a^*a$ on $\mathcal{D}$.

Remark 1. Domain conditions are crucial because of the unboundedness of the operators a, a^*; moreover, for the analysis of the representations it is essential to consider the closures of the operators, since otherwise the restrictions of the same representation to different domains would appear as different, in general inequivalent representations.

Remark 2. In a separable Krein space the (strong, Hilbert space) closure of an operator A (with densely defined adjoint) coincides with its weak closure (the weak topologies defined by the indefinite and by the positive inner product coincide). We recall that an operator A on a (separable) Krein space is weakly closable if $x_n \to x$ weakly, and Ax_n weakly convergent implies that the limit of Ax_n only depends on x; equivalently, $x_n \to 0$ weakly and Ax_n weakly convergent imply $Ax_n \to 0$ weakly. Operators with densely defined adjoints are weakly closable, and, by taking subsequences and means, it is easy to show that closure and weak closure coincide. By the same argument, in non–separable Krein spaces the weak closure by sequences is always contained in the strong closure. The notion of weak closure remains valid for all sequentially weakly complete inner product spaces, and is actually all what is needed in the following. In such spaces, the extension of the algebraic relations to invariant domains $\mathcal{D}' \subset \mathcal{D}_c$ follows by the same argument as above.

According to the discussion given above, we will consider representations of the Heisenberg algebra which satisfy an additional *regularity condition.*

Definition 2. *A representation of the Heisenberg algebra on a domain $\mathcal{D}$ will be called regular iff it satisfies the following condition:*

A. (*Regularity condition*). *The operator $N = 1/2\,(q^2 + p^2 - 1)$ is the weak derivative, on $\mathcal{D}$, of a one parameter group $U(s)$, $s \in \mathbb{R}$:*

$$\frac{d}{ds} < y, U(s)x > |_{s=0} = i < y, Nx > \qquad \forall y \in \mathcal{K},\ x \in \mathcal{D} \tag{2.3}$$

with

$$U(s)\mathcal{D} \subset \mathcal{D} \tag{2.4}$$

$$U(s)U(t) = U(s+t) \qquad \text{on } \mathcal{D} \tag{2.5}$$

Condition A implies

$$< U(s)x, U(s)y > = < x, y > \qquad \forall x, y \in D \tag{2.6}$$

and therefore $U(-s) \subset U(s)^\dagger$, so that $U(s)$, $s \in \mathbb{R}$, are closable operators. Moreover, weakly on $\mathcal{D}$,

$$\frac{d}{ds}(U(s)\,a\,U(-s)) = iU(s)\,[a^* a\,, a]\,U(-s) = -i\,U(s)\,a\,U(-s)$$

so that

$$U(s)\,a\,U(-s) = e^{-is}\,a \quad , \qquad U(s)\,a^*\,U(-s) = e^{is}\,a^* \qquad \text{on } \mathcal{D} \tag{2.7}$$

Therefore $U(s)$ implements the one–parameter group of gauge automorphisms of $\mathcal{H}$

$$\gamma^s(a) = e^{-is}\,a \qquad , \qquad \gamma^s(a^*) = e^{is}\,a^* \tag{2.8}$$

Condition A, eq.(2.3), also implies that $U(s)$ is weakly continuous at the origin, and then, by the group law, for all $s \in \mathbb{R}$. Eqs.(2.3)–(2.5) immediately imply

$$\frac{d}{ds}U(s) = iNU(s) \qquad \forall s \in \mathbb{R}$$

as a weak derivative, on $\mathcal{D}$.

We denote by $\mathcal{H}_{ext}$ the $*$ algebra generated by $\mathcal{H}$ and the group $U(s)$, $s \in \mathbb{R}$, $U(s)^* = U(-s)$, with the relations (2.7). By Def.2, a regular representation of $\mathcal{H}$ on a domain $\mathcal{D} \subset \mathcal{K}$ defines a representation of $\mathcal{H}_{ext}$ on $\mathcal{D}$. In the following, we will denote by $\mathcal{D}_c$ the common domain of the closure of the operators of $\mathcal{H}_{ext}$ (see eq.(2.2)); we will omit the bar which denotes the closure, when no ambiguity may arise.

Remark 3. The regularity condition plays the rôle of the Rellich–Dixmier condition (essential self–adjointness of N on $\mathcal{D}$), and in fact, in the ordinary Hilbert space case, the stability of $\mathcal{D}$ under $U(s)$ implies the essential self–adjointness of its generator N on $\mathcal{D}$. A stronger form of the Rellich–Dixmier condition is here justified because, without positivity of the inner product, the $U(s)$ need not to be continuous everywhere defined operators.

The next issue is a proper notion of irreducibility. As it is well known, the concept of irreducibility is delicate in the case of algebras of unbounded operators. Algebraic reducibility is a very weak property for algebras of *unbounded* operators, but also the existence of a proper closed subspace $\mathcal{K}_1 \subset \mathcal{K}$ containing a common linear domain $\mathcal{D}_1$ invariant under $\mathcal{H}$ (*topological reducibility*) [15] is too weak, in particular for the Heisenberg algebra; in fact, in the ordinary irreducible (Schrödinger) representation of the Weyl algebra in $L^2(\mathbb{R})$, it is satisfied by the Heisenberg algebra, taken on the dense domain of C^∞ functions $f(x)$ of fast decrease, with all derivatives vanishing at $x = 0$; the subdomain of function with positive support is in fact invariant under the Heisenberg algebra, and its closure is a proper subspace of $L^2(\mathbb{R})$ (see the Appendix for more details).

A more relevant notion, which uses the regularity properties of the representation, is that of topological irreducibility *for the extended algebra* $\mathcal{H}_{ext}$:

Definition 3. *A regular representation of* $\mathcal{H}$ *is called domain irreducible if there is no non–trivial domain* $\mathcal{D}_1 \subset \mathcal{D}_c$ *(the common dense domain of the weak closures of the elements of* $\mathcal{H}_{ext}$ *) such that*
i) $\mathcal{D}_1$ *is invariant under* $\mathcal{H}_{ext}$
ii) $\mathcal{D}_1$ *is not dense in* $\mathcal{K}$

The above definition recognizes the crucial rôle played by the "integrated" elements $U(s)$, very close to the rôle of the one–parameter group generated by the Nelson elliptic operator Δ in the classification of the Hilbert space representations of unbounded Lie algebras [15] [16].

It is immediate to see that domain irreducibility in the above sense is equivalent to cyclicity of all vectors x belonging to subspaces of $\mathcal{D}_c$ invariant under $\mathcal{H}_{ext}$, i.e. $\mathcal{H}_{ext}\, x$ is dense in $\mathcal{K}$. In fact, if x belongs to an invariant domain, by applying polynominals of elements of $\mathcal{H}_{ext}$ to x one gets an invariant subspace of $\mathcal{D}_c$, which must be dense, by Def.3.

An alternative definition of irreducibility is the one related to the triviality of the commutant, and it is often used in the analysis of unbounded operator algebras [15]. Given the representation of $\mathcal{H}$ on the domain $\mathcal{D}$, one could formulate commutant irreducibility by asking that any (weakly closable) operator B which leaves $\mathcal{D}$ invariant and commutes on $\mathcal{D}$ with $\mathcal{H}$ and $U(s)$, is a multiple of the identity.

The relation between this notion and Definition 3 does not appear as trivial as in the bounded positive case, because a spectral theory of unbounded operators in Krein spaces is not available. One of the points is that reduction may take place only after the operators have been extended by closure, and in fact a more relevant notion of commutant irreducibility must be given in terms of the extended domain $\mathcal{D}_c$

Definition 4. *A regular representation of* $\mathcal{H}$ *is said to be commutant irreducible if whenever a (densely defined) weakly closable operator* B *has the properties*
i) B *leaves invariant a non–trivial domain* $\mathcal{D}_1 \subset \mathcal{D}_c$, *invariant under* $\mathcal{H}_{ext}$
ii) B *commutes with* $\mathcal{H}_{ext}$ *on* $\mathcal{D}_1$
then B *is a multiple of the identity.*

It is easy to see that commutant irreducibility in the sense of Definition 4 implies domain irreducibility (Definition 3).

In sections 3 and 4 we analyze irreducible representations in the sense of Def.4. In Sect.5 the same classification will be obtained (Proposition 12) assuming only domain irreducibility (Def.3), but using additional regularity assumptions on $U(s)$.

3. Regular irreducible representations of the Heisenberg algebra in Krein spaces

The general setting discussed in the previous section allows for the following characterization of the regular (Def.2) irreducible (Def.4) representations (Def.1) of the Heisenberg algebra $\mathcal{H}$.

Theorem 5. *The regular irreducible representations of the Heisenberg algebra $\mathcal{H}$ in a Krein space $\mathcal{K}$ fall in the following classes, up to isometries defined on dense subspaces:*

1) $\pi_{\mathbb{Z}^+}$: it is characterized by the existence of a vector $\psi_0 \in \mathcal{D}_c$ (Fock vector) satisfying

$$a\,\psi_0 = 0 \tag{3.1}$$

The operators $U(s)$, $s \in \mathbb{R}$ have a common complete set of eigenvectors ψ_n, $n \in \mathbb{N}$ with

$$U(s)\,\psi_n = e^{ins}\,\psi_n \tag{3.2}$$

2) $\pi_{\mathbb{Z}^-}$: it is characterized by the existence of a vector $\psi_{-1} \in \mathcal{D}_c$ (anti–Fock vector) satisfying

$$a^*\psi_{-1} = 0 \tag{3.3}$$

The operators $U(s)$ have a common complete set of eigenvectors ψ_n, $n = -1, -2, \ldots$ satisfying eq.(3.2).

3) $\pi^\theta_{\mathbb{Z}}$, $\theta \in (0,1)$: it is characterized by the existence of a vector $\psi_\theta \in \mathcal{D}_c$ satisfying

$$U(s)\,\psi_\theta = e^{i\theta s}\,\psi_\theta \tag{3.4}$$

The operators $U(s)$ have a common complete set of eigenvectors ψ_n, $n \in \mathbb{Z}$, with

$$U(s)\,\psi_n = e^{i(\theta+n)s}\,\psi_n \tag{3.5}$$

("Dirac sea states").

In all the cases, the common eigenvectors of the (closure of) operators $U(s)$, $s \in \mathbb{R}$, are unique (i.e there are no multiplicities), and there are no other eigenvalues.

The proof follows from a few lemmas:

Lemma 6. *The closures of $U(s)$, $s \in \mathbb{R}$, have common eigenvectors ψ_n, $n \in \mathbb{Z}$,*

$$U(s)\psi_n = e^{i(n+\theta)s}\,\psi_n \tag{3.6}$$

with θ fixed in $[0,1)$; the linear span of such vectors is dense in $\mathcal{K}$.

Proof. By eq.(2.4), the operator $U(2\pi)$ leaves $\mathcal{D}$ invariant, and by eq.(2.8) it commutes with $\mathcal{H}$ (and obvously with $U(s)$, $\forall s \in \mathbb{R}$). Therefore, by the irreducibility of the representation, $U(2\pi)$ is a multiple of the identity, and actually a phase, by eq.(2.6):

$$U(2\pi) = e^{2\pi i\theta} \quad , \qquad \theta \in [0,1) \tag{3.7}$$

Thus, the one–parameter group

$$V(s) \equiv U(s)\,e^{-is\theta} \quad , \qquad s \in \mathbb{R} \tag{3.8}$$

is a $U(1)$ group:

$$V(2\pi) = 1 \tag{3.9}$$

and, $\forall \psi \in \mathcal{D}$, the vector valued function $V(s)\psi$ is periodic and weakly continuous, by condition A. Hence, the Riemann sums

$$\psi_n^{(\varepsilon)} \equiv \frac{\varepsilon}{2\pi} \sum_{s_k = k\varepsilon,\ 0\le k<2\pi/\varepsilon} e^{-ins_k}\,V(s_k)\,\psi \tag{3.10}$$

converge weakly, as $\varepsilon \to 0$, to the (weak) integral

$$\psi_n \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{-ins}\,V(s)\,\psi\,ds \tag{3.11}$$

which defines a vector in $\mathcal{K}$, by weak completeness. Moreover, $\forall t \in \mathbb{R}$, $V(t)\,\psi_n^{(\varepsilon)}$ also converges weakly as $\varepsilon \to 0$:

$$V(t)\,\psi_n^{(\varepsilon)} = \frac{\varepsilon}{2\pi} \sum_{s_k = k\varepsilon,\ 0\le k<2\pi/\varepsilon} e^{-ins_k}\,V(t+s_k)\,\psi =$$

$$= \frac{\varepsilon}{2\pi} \sum_{s_k} e^{-ins_k} \, e^{int} \, V(s_k) \, \psi \, \rightarrow \, e^{int} \, \psi_n \tag{3.12}$$

(where the periodicity of $V(s)$ has been used). Thus, all the vectors ψ_n are in the domain of the closures of $V(t)$, and

$$V(s) \, \psi_n = e^{ins} \psi_n \tag{3.13}$$

Furthermore, the vector valued weakly continuous periodic functions $V(s) \, \psi$ can be expressed (pointwise in s) by their (weakly convergent) Fourier series,

$$V(s) \, \psi = \sum_{n \in \mathbb{Z}} e^{ins} \, \psi_n \tag{3.14}$$

For $s = 0$, eq.(3.14) expresses all $\psi \in \mathcal{D}$ as weak limits of linear combinations of the vectors ψ_n, so that their linear span is dense in $\mathcal{K}$.

Lemma 7. *The vectors ψ_n constructed in Lemma 6 are eigenvectors of the closure of $a^* a$:*

$$a^* a \, \psi_n = (n + \theta) \, \psi_n \tag{3.15}$$

Proof. With the notations of the proof of Lemma 5, by condition A, in the sense of weak derivatives:

$$(a^* a - \theta) \, \psi_n^{(\varepsilon)} = \frac{-i}{2\pi} \, \varepsilon \sum_{s_k} e^{-ins_k} \, \frac{d}{dt} V(t) \, V(s_k) \, \psi \, |_{t=0} =$$

$$= \frac{-i}{2\pi} \, \varepsilon \sum_{s_k} e^{-ins_k} \, V(s_k) \, \frac{d}{dt} V(t) \, \psi \, |_{t=0}$$

$$\rightarrow \frac{-i}{2\pi} \int_0^{2\pi} ds \, e^{-ins} \, \frac{d}{ds} V(s) \, \psi \, = n \, \frac{1}{2\pi} \int_0^{2\pi} ds \, e^{-ins} \, V(s) \, \psi$$

where an integration by part has been done in the last step.

Lemma 8. *The linear span $\mathcal{D}_1$ of all the vectors ψ_n constructed in Lemma 6, as ψ varies on $\mathcal{D}$, is contained in $\mathcal{D}_c$ (the common domain of the weak closures of the elements of $\mathcal{H}_{ext}$), and it is left invariant by $\mathcal{H}_{ext}$. The Heisenberg relations hold on $\mathcal{D}_1$, and each vector in $\mathcal{D}_1$ is cyclic for $\mathcal{H}_{ext}$. For any $\psi \in \mathcal{D}$ there is an integer $\bar{n}$ such that the vectors $(a^*)^k \psi_{\bar{n}}$, $a^l \psi_{\bar{n}}$ generate a dense subspace of $\mathcal{D}_1$.*

Proof. By eq.(2.7),

$$a\,\psi_n^{(\varepsilon)} = \frac{\varepsilon}{2\pi}\sum_{s_k} e^{-ins_k}\, a\, V(k\varepsilon)\,\psi =$$

$$= \frac{\varepsilon}{2\pi}\sum_{s_k} e^{-ins_k}\, e^{is_k} V(s_k)\, a\,\psi = (a\psi)_{n-1}^{(\varepsilon)} \tag{3.16}$$

and similarly

$$a^*\psi_n^{(\varepsilon)} = (a^*\psi)_{n+1}^{(\varepsilon)} \tag{3.17}$$

Since the r.h.s. of eqs.(3.16),(3.17) converge by Lemma 6, the vectors ψ_n belong to the domain of the weak closure of a and a^*, and

$$\bar{a}\,\psi_n = (a\,\psi)_{n-1} \qquad , \qquad \bar{a}^*\,\psi_n = (a^*\psi)_{n+1} \tag{3.18}$$

The same treatment applies to all the products of a and a^*, so that $\mathcal{D}_1 \subset \mathcal{D}_c$ and $\mathcal{H}_{ext} D_1 \subset \mathcal{D}_1$. By irreducibility, $\mathcal{H}_{ext}\chi$ is dense in $\mathcal{K}$, $\forall \chi \in \mathcal{D}_1$. The Heisenberg commutation relations hold on $\mathcal{D}_1$, as a consequence of their validity on $\mathcal{D}$, by eq.(3.18):

$$(aa^* - a^*a)\,\psi_n = a\,(a^*\psi)_{n+1} - a^*(a\psi)_{n-1} = ((aa^* - a^*a)\,\psi)_n = \psi_n \tag{3.19}$$

¿From eqs.(3.19) and (3.15) one easily gets

$$a\,a^*\,\psi_n = (n+1+\theta)\,\psi_n \tag{3.20}$$

(an equation which could have been derived directly as in Lemma 7). For each ψ in $\mathcal{D}$, by eq.(3.14) there is at least one $\psi_{\bar{n}}$ which does not vanish, and is then cyclic. Given any product of operators a, a^* applied to $\psi_{\bar{n}}$, by using eqs.(3.18),(3.15),(3.20), one can write it as a constant times $(a^*)^k\,\psi_{\bar{n}}$, if the number of a^* operators exceeds, by k, that of the a's, or, otherwise, as $a^k\,\psi_{\bar{n}}$. The ciclicity of $\psi_{\bar{n}}$ implies therefore the density in $\mathcal{K}$ of the linear span of the vectors $\{(a^*)^k\,\psi_{\bar{n}}\,,\ a^l\psi_{\bar{n}}\,,\ k, l \in \mathbb{N}\}$.

Proof of Theorem 5. By Lemma 8, the analysis of regular irreducible representations of $\mathcal{H}$ can be done on the domain $\mathcal{D}_1$. Given a vector $\psi_{\bar{n}} \in \mathcal{D}_1$, we consider first the case in which at least one of the sequences $a^k\,\psi_{\bar{n}}$ terminates. If, for some $k > 0$,

$$a^k\,\psi_{\bar{n}} = 0 \qquad\qquad a^{k-1}\psi_{\bar{n}} \neq 0 \tag{3.21}$$

then, by eqs.(3.15),(3.18) one has

$$0 = a^*a\,a^{k-1}\,\psi_{\bar{n}} = a^*a\,(a^{k-1}\,\psi)_{\bar{n}-k+1} = (\bar{n}-k+1+\theta)\,a^{k-1}\,\psi_{\bar{n}}$$

i.e.

$$\bar{n} - k + 1 + \theta = 0 \tag{3.22}$$

which implies $\theta = 0$, since $\theta \in [0, 1)$. Similarly,

$$(a^*)^l\, \psi_{\bar{n}} = 0 \qquad\qquad (a^*)^{l-1}\, \psi_{\bar{n}} \neq 0 \tag{3.23}$$

implies, by eqs.(3.18),(3.20),

$$\bar{n} + l + \theta = 0 \tag{3.24}$$

and therefore again $\theta = 0$ Eqs.(3.22) and (3.24) are incompatible, and therefore only one sequence can terminate.

Case 1. Eq.(3.21) holds. Then, putting

$$\psi_0 \equiv a^{k-1}\, \psi_{\bar{n}} = (a^{k-1}\, \psi)_0 \tag{3.26}$$

one has

$$a\, \psi_0 = 0 \tag{3.25}$$

and $\mathcal{D}_1$ is generated by the vectors $(a^*)^l\, \psi_0$, $l > 0$. (*Fock representation*). By using eqs.(3.18),(3.20),(3.25) one gets

$$< (a^*)^l \psi_0, (a^*)^m \psi_0 > = \begin{cases} 0 \quad \text{for } \; l \neq m \\ m! \, < \psi_0, \psi_0 > \quad\quad \text{for } \; l = m \end{cases} \tag{3.27}$$

Since the linear span of the vectors $(a^*)^m\, \psi_0$ is dense in $\mathcal{K}$ (Lemma 8), the non–degeneracy of the inner product implies $< \psi_0, \psi_0 > \neq 0$, and eq.(3.27) fully determines the inner product in $\mathcal{K}$, which turns out to be definite. Without loss of generality we can normalize $< \psi_0, \psi_0 > \equiv \eta_0 = \pm 1$, and the Krein metric operator is then

$$J = \eta_0 1$$

Apart from the sign of the inner product, the representation is the usual Fock representation in a Hilbert space.

Case 2. Eq.(3.23) holds. Then, putting

$$\psi_{-1} \equiv (a^*)^{l-1}\, \psi_{\bar{n}} = ((a^*)^{l-1}\, \psi)_{-1} \tag{3.28}$$

one has

$$a^*\, \psi_{-1} = 0 \tag{3.29}$$

and $\mathcal{D}_1$ is generated by the vectors $a^l\,\psi_0$, $l>0$. (*"Anti–Fock representation"*). By using eqs.(3.15),(3.18),(3.28) one gets

$$< a^l\psi_{-1}, a^m\psi_{-1} >= \delta_{l,m}\, m!\,(-1)^m \;\; < \psi_{-1}, \psi_{-1} > \tag{3.30}$$

The non–degeneracy of the inner product implies $< \psi_{-1}, \psi_{-1} > \neq 0$, and the inner product in $\mathcal{K}$ is fully determined by eq.(3.30), and it is not definite.

The metric operator which defines the Hilbert space structure of $\mathcal{K}$ is not uniquely determined by eq.(3.30). There is however a distinguished Krein structure associated to eq.(3.30), defined by the metric operator

$$J = \eta_1\,(-1)^N \tag{3.31}$$

where $\eta_1 \equiv < \psi_{-1}, \psi_{-1} >= \pm 1$.

In general, the representation defined (on $\mathcal{D}_1$) by eq.(3.30) may live in a Krein space obtained by closing $\mathcal{D}$ with a (weak) topology different from that defined by the above metric operator. However, all such Krein spaces contain dense domains $\mathcal{D}_1$, on which the corresponding representations are related by (indefinite space) isometries. Thus, any representation of type 2 is isometrically equivalent to the representation defined by eq.(3.30) on the Krein space given by the metric (3.31).

Case 3. None of the sequences $a^k\,\psi_{\bar n}$, $(a^*)^l\,\psi_{\bar n}$ terminates. Then one of the sequences contains a vector ψ_θ with

$$a^* a\,\psi_\theta = \theta\,\psi_\theta \tag{3.32}$$

and $\mathcal{D}_1$ is generated by the vectors $a^l\,\psi_\theta$, $(a^*)^k\,\psi_\theta$, $l, k \in \mathbb{N}$. (*"Dirac sea states"*). Different eigenvalues of $U(s)$ define orthogonal eigenspaces (with the indefinite inner product), and, using the commutation relations of Lemma 8, one has

$$< (a^*)^l\psi_\theta, (a^*)^m\psi_\theta >= \delta_{l,m}\,(\theta+1)(\theta+2)\ldots(\theta+m) \;\; < \psi_\theta, \psi_\theta > \tag{3.33}$$

$$\begin{aligned} < a^l\psi_\theta, a^m\psi_\theta >&= \delta_{l,m}\,\theta(\theta-1)\ldots(\theta-m) \;\; < \psi_\theta, \psi_\theta >= \\ &= \delta_{l,m}\,(-1)^{m-1}\,\theta(1-\theta)\ldots(m-1-\theta) \;\; < \psi_\theta, \psi_\theta > \end{aligned} \tag{3.34}$$

Again $< \psi_\theta, \psi_\theta > \neq 0$ follows from non–degeneracy of the inner product, which is fully determined by eqs.(3.33),(3.34), and is not definite. Again, the metric operator is not uniquely determined, and a distinguished Krein structure is defined by

$$J\,(a^*)^m\,\psi_\theta = \eta_0\,(a^*)^m\,\psi_\theta \qquad\qquad J\,a^l\,\psi_\theta = \eta_0\,(-1)^{l-1}\,a^l\,\psi_\theta \tag{3.35}$$

with $\eta_0 \equiv <\psi_\theta, \psi_\theta> = \pm 1$. The same arguments as in case 2 then apply to the equivalence of representations in different Krein spaces.

In order to control all the common eigenvectors of $U(s)$, $s \in \mathbb{R}$, assume that a vector χ, in the domain of the closure of all the operators $U(s)$, satisfies

$$U(s)\,\chi = \lambda(s)\chi$$

By weak continuity, and eq.(2.5),

$$\lambda(s) = e^{i\mu s}$$

¿From eq.(2.6) (extended to the closure of the operators), it follows that χ is orthogonal to all the eigenvectors of $U(s)$ with different eigenvalues, and therefore, by the density of the domains $\mathcal{D}_1$ constructed above, $\exp i\mu s$ is, in the three different cases, one of the eigenvalues classified above,

$$e^{i\mu s} = e^{i(n+\lambda)s}$$

For each $\psi \in \mathcal{D}$ the set of eigenvectors constructed in Lemma 8 generates a dense subspace; since, by the argument given above, χ is orthogonal to all the such eigenvectors, except for the eigenvalue $\exp i(\bar{n}+\theta)s$, it follows that χ is a multiple of the corresponding eigenvector, which is therefore unique. It also follows that the all the subspaces of $\mathcal{D}_1$ introduced in Lemma 8 actually coincide with $\mathcal{D}_1$.

4. Properties of the regular irreducible representations of the Heisenberg algebra.

We list some properties of the regular irreducible representation of $\mathcal{H}$ in Krein spaces, and show how the result generalize to weakly complete inner product space.

We recall that the characterization of Theorem 5 amounts to the construction of a dense subspace of $\mathcal{D}_1$ of $\mathcal{K}$, where the representation is given explicitly. The indefinite inner product is completely determined on $\mathcal{D}_1$, and the representations are therefore classified up to isometries of such domains. Different positive scalar product, and therefore different weak topologies and different completions, are compatible with the representations on $\mathcal{D}_1$, but the simple choice of Hilbert product given in the proof of Theorem 5 is always possible and gives rise to very simple spectral properties of the operators $U(s)$:

Proposition 9. *In the regular irreducible representations of the Heisenberg algebra, with the Krein structures introduced in the proof of Theorem 5, the operators $U(s)$ which implement the gauge transformations (2.8) are unitary operators with simple pure point spectrum.*

Proof. All the Krein metric operators J introduced in the proof of Theorem 5 commute with $U(s)$; eq.(2.6) then implies that $U(s)$ are unitary (with respect to the positive scalar product). From Theorem 5 it follows immediately that the spectrum is pure point, without multiplicities.

The logic of Theorem 5 is that, *if* regular irreducible representations exist, they are of the types 1)–3); to complete the classification, it is necessary to show that indeed all the representations constructed in Theorem 5 fulfill the regularity and irreducibility conditions. Algebraic relations and regularity are obvious, on suitable domains (e.g., the linear span of the vectors ψ_n), and for the irreducibilty we have:

Proposition 10. *All the representations characterized in Theorem 5 are commutant irreducible (Def.4), and therefore also domain irreducible (Def.3).*

Proof. The proof follows from the absence of multiplicities for the common eigenvalues of $U(s)$, $s \in \mathbb{R}$. If a closable operator B leaves a domain $\mathcal{D}' \subset \mathcal{D}_c$ invariant, with $\mathcal{H}_{ext}\, \mathcal{D}' \subset \mathcal{D}'$, then for any $\psi \in \mathcal{D}'$ one can construct vectors $\psi_n^{(\varepsilon)}$ as in Lemma 6, eq.(3.10), which converge weakly to vectors ψ_n, eq.(3.12), still belonging to an invariant domain contained in $\mathcal{D}_c$; if B commutes with $\mathcal{H}_{ext}$, then

$$B\,\psi_n^{(\varepsilon)} \to (B\,\psi)_n \qquad \text{weakly} \tag{4.1}$$

We can fix n so that ψ_n is non–vanishing; since B is (weakly) closable, ψ_n belongs to the domain of its weak closure $\bar{B}$, and

$$\bar{B}\,\psi_n = (B\psi)_n \tag{4.2}$$

Both ψ_n and $(B\psi)_n$ are eigenvectors of $U(s)$, $\forall s \in \mathbb{R}$, as in Lemma 6, eq.(3.6). Since there are no multiplicities, eq.(4.2) implies

$$B\,\psi_n = \lambda\,\psi_n \tag{4.3}$$

Since B commutes with $\mathcal{H}$, eq.(4.3) also holds, with the same λ, for the vectors $\{a^k\psi_{\bar{n}}, (a^*)^l\psi_{\bar{n}}\}$, generating a dense domain. B is closable, and therefore $B = \lambda 1$

The same analysis as in Theorem 5 can be done under the weaker assumption that $\mathcal{D}$ is a dense subspace of a sequentially weakly complete non–degenerate inner product space $\mathcal{V}$, rather than a Krein space. The assumption is indeed weaker, since there exist weakly complete inner product space which are not Krein spaces.

Proposition 11. *The regular, commutant irreducible representations of the Heisenberg algebra in sequentially weakly complete spaces are classified by Theorem 5, up to isometries defined on dense subspaces.*

Proof. The proof of Theorem 5 only uses the representation of $\mathcal{H}$ on a weakly dense domain $\mathcal{D}$ and the regularity condition A on $\mathcal{D}$, and constructs the common eigenvectors of $U(s)$ by taking weak limits and extending weakly closable operators.

Absence of multiplicities for the eigenvalues of $U(s)$ and irreducibility of the representations follow in the same way. On the dense domain $\mathcal{D}_1$ constructed in the proof of Theorem 5, the representations are clearly identical to those in Krein spaces; their classification has therefore nothing to do with the completion of $\mathcal{D}_1$, which can always be embedded in the distinguished Krein spaces constructed in the proof of Theorem 5.

Weak completeness is however necessary, in general, for the construction of the domain $\mathcal{D}_1$ starting from the domain $\mathcal{D}$ of a given representation; the domain $\mathcal{D}$ is in fact not unique, and may be any dense subspace, with the only constraints of invariance and validity of the regularity condition.

5. An alternative derivation.

The same results as in Theorem 5 can be obtained, in any complete non degenerate inner product space $\mathcal{V}$, under the following assumptions:

i) Regularity (Def.2)

ii) Domain Irreducibility (Def.3)

iii) Temperedness of $U(s)$, $s \in \mathbb{R}$: $\forall x \in \mathcal{D}$, there exists a polynomial $P_x(n)$ such that $\forall y \in \mathcal{V}$ there exists a constant C_y, such that

$$| < y, U(2\pi n)x > | \leq C_y\, P_x(n) \qquad \forall n \in \mathbb{Z} \tag{5.1}$$

In fact we have

Proposition 12. *Under the assumptions i), ii), iii), the operator $U(2\pi n)$ is a multiple of the identity,*

$$U(2\pi n) = e^{2\pi i\theta}\, 1 \tag{5.2}$$

Proof. By eq.(2.7), $U(2\pi n)$ commutes with the algebra $\mathcal{H}$ on $\mathcal{D}$, but eq.(5.2) does not immediately follow, as in the standard case (bounded operators in Hilbert spaces), and a crucial rôle is played by the regularity and temperedness conditions (a counterexample, if regularity does not hold, was given in Sect.2). A substitute for the spectral theory of the group $U(2\pi n)$ is necessary, and can be constructed if condition iii) is satisfied. The Fourier transform of $U(2\pi n)$,

$$T(t) = \sum_{n=-\infty}^{\infty} U(2\pi n)\, e^{int}$$

is in fact well defined as an operator valued distribution on $\mathcal{D}$ by the following argument: define

$$T_N(t) \equiv \sum_{n=-N}^{N} U(2\pi n)\, e^{int}$$

and, for all $f(t)$ infinitely differentiable on the circle, $t \in [0, 2\pi)$

$$T_N(f) \equiv \int_0^{2\pi} T_N(t)\, f(t)\, dt = \sum_{n=-N}^{N} c_f^{(n)}\, U(2\pi n) \tag{5.3}$$

with $c_f^{(n)}$ the Fourier coefficients of f. Since such coefficients decrease faster than any polynomial, the regularity condition iii) immediately implies that $T_N(f)$ converges weakly on $\mathcal{D}$, as $N \to \infty$. By the weak completness of $\mathcal{V}$, the limit defines an operator which maps $\mathcal{D}$ in $\mathcal{V}$. Furthermore, since all the elements $A \in \mathcal{H}$ are closable,

$$A\, T_N(f)\, x = T_N(f)\, A\, x \to T(f)\, A\, x \qquad \forall x \in \mathcal{D}$$

implies

$$A\, T_N(f)\, x \to A\, T(f)\, x \tag{5.4}$$

and therefore

$$[A, T(f)] = 0 \qquad \text{on } \mathcal{D} \tag{5.5}$$

The operator valued distribution $T(f)$ satisfies on $\mathcal{D}$

$$T(1) = \sum_{n=-\infty}^{\infty} U(2\pi n) \int_0^{2\pi} e^{int}\, dt = U(0) = 1$$

If $f_1(t)$ and $f_2(t)$ have disjoint support, $f_1(t)\, f_2(t) = 0$, then

$$T(f_1)\, T(f_2) = 0 \tag{5.6}$$

In fact, on $\mathcal{D}$,

$$T(f_1)\,T(f_2) = \sum_{n,m} U(2\pi n)\,U(2\pi m)\,c_{f_1}^{(n)}\,c_{f_2}^{(m)} =$$

$$= \sum_{k,m} U(2\pi k)\,c_{f_1}^{(k-m)}\,c_{f_2}^{(m)} \equiv \sum_k U(2\pi k)\,c^{(k)}$$

with

$$c^{(k)} = \sum_m c_{f_1}^{(k-m)}\,c_{f_2}^{(m)} = \sum_m \int_0^{2\pi} e^{ikt}\,e^{-imt}\,f_1(t)\,dt \int_0^{2\pi} e^{imt'}\,f_2(t')\,dt' =$$

$$\sum_m \int_0^{2\pi} e^{ikt}\,e^{-im(t-t')}\,f_1(t)\,f_2(t')\,dt\,dt' = \int_0^{2\pi} e^{ikt}\,f_1(t)\,f_2(t)\,dt = 0$$

In order to prove that $U(2\pi n)$ is a multiple of the identity we consider two possibilities:
i) The support of T consists of only one point, denoted by t_0. Let us fix $x \in \mathcal{D}$; the bound (5.1) and the support property of T implie, $\forall y \in \mathcal{D}$,

$$< y, U(2\pi n) e^{-int_0} x >= \sum_{k=0}^{N} a_k(y)\,n^k \tag{5.7}$$

Then, by putting

$$x_k \equiv \sum_{n=0}^{N} A_{kn}\,U(2\pi n)\,e^{-int_0} x \quad , \qquad k = 0, 1 \ldots N \tag{5.8}$$

with A_{kn} the inverse matrix of $B_{nj} \equiv n^j$,

$$\sum_{n=0}^{N} A_{kn}\,n^j = \delta_{kj}$$

one has

$$a_k(y) =< y, x_k >$$

and therefore

$$U(2\pi n)\,x = \sum_{j=0}^{N} n^j\,e^{int_0}\,x_j \tag{5.9}$$

The lowest integer N for which eq.(5.7) holds will be called the degree of x. The subspace $\mathcal{D}_M \subset \mathcal{D}$ of vectors of degree less or equal to some M is invariant under

$\mathcal{H}_{ext}$, and non–empty for sufficiently large M; we have therefore constructed operators O_j, $j = 0 \dots M$, defined on $\mathcal{D}_M$ by $O_j x = x_j$; by eq.(5.8), the operators O_j are given by finite linear combinations of operators $U(2\pi n)$, so that they are closable and leave $\mathcal{D}_M$ invariant. The operator O_M does not vanish, if M is chosen small enough, and, if $M > 0$, it satisfies, by the group law for $U(s)$,

$$O_M^2 = 0 \tag{5.10}$$

$O_M \mathcal{D}_M$ is invariant under $\mathcal{H}_{ext}$ and is therefore dense, by domain irreducibility; this is incompatible with eq.(5.10); all vectors in $\mathcal{D}$ are therefore of degree 0, i.e.

$$U(2\pi n)x = e^{int_0} x \qquad \forall x \in \mathcal{D} \tag{5.11}$$

ii) The support contains at least two points. There are then two functions, f_1, f_2, with $T(f_i) \neq 0$, $i = 1, 2$, and $f_1 f_2 = 0$. Then, $T(f_1)\mathcal{D}$, by eq.(5.4) a subspace of $\mathcal{D}_c$, must be dense, since it is invariant under $\mathcal{H}$ and the representation is domain irreducible. But then $T(f_2)T(f_1) = 0$ is incompatible with $T(f_2) \neq 0$, since all $T(f)$ are closable.

The proof of Theorem 5 then proceeds as in Sect.3

Appendix

We briefly discuss here:
1. a class of "counterexamples" concerning the notion of irreducibility, and its relation with regularity, for the Heisenberg algebra in a Hilbert space;
2. an example of a weakly complete inner product space which is *not* a Krein space.

1. The (ordinary) regular representation of the Weyl algebra in the Hilbert space $L^2(\mathbb{R})$,

$$e^{i\alpha q}\, e^{i\beta p}\, \psi(x) = e^{i\alpha x}\, \psi(x + \beta)$$

defines (by taking strong derivatives) representations of the Heisenberg algebra with reducibility and regularity properties which depend substantially on the domains which are considered. In fact, regularity and (topological) irreducibility hold if an invariant domain of analytic vectors for q and p is chosen, e.g. the domain generated by the eigenvectors of N, but very different situations arise for different domains.

Consider, e.g., the dense domain $\mathcal{D}_0$ of C^∞ functions $f(x)$ of fast decrease, vanishing toghether with all their derivatives at $x = 0$. $\mathcal{D}_0$ is clearly invariant under the Heisenberg algebra, but the representation of the Heisenberg algebra on $\mathcal{D}_0$ does *not* satisfy the regularity condition, since the operators $U(s)$ exist but do not leave $\mathcal{D}_0$ invariant. The subdomains $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$, consisting of function with positive, resp. negative, support are invariant, and their closure are proper orthogonal subspaces of $L^2(\mathbb{R})$. The representation is therefore *domain reducible.* It is also *commutant reducible*, since the projector on $L^2(0,\infty)$ commutes with the Heisenberg algebra on $\mathcal{D}_0$ (not only in the weak sense, since the domain is left invariant by the projector).

If the representation of $\mathcal{H}$ is defined, by taking strong derivatives, on a domain $\mathcal{D} \supset \mathcal{D}_0$ invariant under $\mathcal{H}$ and $U(s)$, $s \in \mathbb{R}$, then the regularity condition is satisfied. However, domain irreducibility does *not* hold for *the Heisenberg algebra* on $\mathcal{D}$, since the previously considered domains are obviously still invariant under $\mathcal{H}$. Domain irreducibility *holds for the extended algebra* $\mathcal{H}_{ext}$; in fact, invariance of a domain $\mathcal{D}'$ under $U(s)$ implies, using the construction given in the proof of Theorem 5, that one, and therefore every, eigenvector of the operators $U(s)$ belongs to the closure of $\mathcal{D}'$, and $\mathcal{D}'$ is therefore dense.

Identical conclusions hold for commutant reducibility, since one may still consider the same projectors as above, as operators defined on $\mathcal{D}_0$, commuting there with $\mathcal{H}$. On the other hand, commutativity with $\mathcal{H}_{ext}$ on an invariant domain (Def.4) implies triviality as in the proof of Proposition 10.

It may also be remarked that the operator N is *not* essentially self–adjoint on $\mathcal{D}_0$, and that the extended algebra (with the validity of relations (2.7)) can be constructed only on the basis of *one* choice for its self–adjoint extension (namely by taking functions continuous, toghether with their derivative, at $x = 0$). All other choices lead to different extensions of $\mathcal{H}$ (with algebraic relations different from (2.7)), and in general to different spectra of N.

Very similar examples can be constructed by using any sequence of intervals (rather than $(-\infty, 0)$ and $(0,\infty)$) to define invariant subspaces, with very similar results. Our conclusion is that both the *regularity* property and the use of the *extended algebra* are essential in order to avoid pathologies in the discussion of the representations of the Heisenberg algebra, even in positive (Hilbert) spaces.

In indefinite spaces, domain questions become even more critical. E.g., on different domains in the same Krein space one may define different, and in general inequivalent, positive scalar products, so that the same domains can also be seen as dense subspaces of *different Krein spaces.* On the other hand, since a sufficiently general spectral theory is not available in indefinite spaces, all *regularity* properties must be expressed in terms of domains. Roughly speaking, in order to fix the representation one must start with a domain which is large enough for the control of the regularity condition; no extension is then needed, and the main rôle of the Hilbert structure is that of giving rise to complete spaces.

2. In order to construct a weakly complete inner product space which is not a Krein space, we start with the space V_0 of *finite* sequences a_n, $n \in \mathbb{Z}$, with the inner product

$$< a_n \, , \, b_m > \equiv \sum_n \bar{a}_{-n} b_n$$

On the space V_0 it is easy to define positive scalar products which majorize the indefinite inner product (see [13]), and therefore V_0 can be completed to a Krein space (inequivalent scalar products defining different Krein completions).

However, not all weakly complete inner product spaces which contain V_0 as a dense subspace are Krein spaces. Consider in fact the space $V \supset V_0$ consisting of *all* sequences which terminate on the right, $a_n = 0$ for n large enough. The space V is weakly complete, because i) weak convergence implies convergence of each term of the sequences, ii) by completeness of the l^2 spaces, the limit sequences x_n belong to all the spaces l^2 with *any* weight function $p_n > 0$ on $n \in \mathbf{N}$,

$$\sum_{n=0}^{\infty} |x_n|^2 \, p_n < \infty$$

and therefore x_n vanishes for n large enough.

Moreover, V is not a Krein space, since in any Krein space containing V_0 there are always sequences which are infinite on the right, and have weakly convergent truncations; one such sequence is immediately constructed, e.g., as $1/(c_k \, k^2)$, $k > 0$, with c_k the Hilbert norm of the sequence $\delta_{n,k}$, in the given Krein space. The argument also shows that V is not a Banach space, so that it cannot be obtained from V_0 by completion with respect to a norm.

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