On the free-energy of the Hopfield model*

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Abstract

The general theory of inhomogeneous mean-field systems of Ref. [18] provides a variational expression for the (almost sure) limiting free energy density of the Hopfield model

$$H_{N,p}^{\{\boldsymbol{\xi}\}}(S) = -\frac{1}{2N}\sum_{i,j=1}^{N}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu S_i S_j$$

for Ising spins S_i and p random patterns $\boldsymbol{\xi}^\mu = (\xi_1^\mu, \xi_2^\mu, \cdots, \xi_N^\mu)$ under the assumption that

$$\lim_{N\to\infty} N^{-1}\sum_{i=1}^{N}\delta_{\boldsymbol{\xi}_i} = \lambda \quad , \quad \boldsymbol{\xi}_i = (\xi_i^1, \xi_i^2, \cdots, \xi_i^p)$$

exists (almost surely) in the space of probability measure over p copies of $\{-1,1\}$. Including an "external field" term $-\sum_{\mu=1}^{p} h^\mu \sum_{i=1}^{N}\xi_i^\mu S_i$, we give a number of general properties of the free-energy density and compute it for a): $p=2$ in general, and b): p arbitrary when λ is uniform and at most the two components h^{μ_1} and h^{μ_2} are non-zero, obtaining the (almost sure) formula

$$f(\beta,\mathbf{h}) = \frac{1}{2}f^{cw}(\beta, h^{\mu_1}+h^{\mu_2}) + \frac{1}{2}f^{cw}(\beta, h^{\mu_1}-h^{\mu_2})$$

for the free energy, where f^{cw} denotes the limiting free energy density of the Curie-Weiß model with unit interaction constant. In both cases, we obtain explicit formulae for the limiting (almost sure) values of the so-called overlap parameters

$$m_N^\mu(\beta,\mathbf{h}) = N^{-1}\sum_{i=1}^{N}\xi_i^\mu < S_i >$$

in terms of the Curie-Weiß magnetizations.

For the general i.i.d. case with $Prob\{\xi_i^\mu = \pm 1\} = (1/2)\pm\epsilon$, we obtain the lower bound $1+4\epsilon^2(p-1)$ for the temperature T_c separating the trivial free regime where the overlap vector is zero, from the non-trivial regime where it is non-zero. This lower bound is exact for: $p=2$, or $\epsilon=0$, or $\epsilon=\pm 1/2$. For $p=2$ we identify an intermediate temperature region between $T_* = 1-4\epsilon^2$ and $T_c = 1+4\epsilon^2$ where the overlap vector is homogeneous (i.e. all its components are equal) and non-zero. T_* marks the transition to the non-homogeneous regime where the components of the overlap vector are distinct. We conjecture that the homogeneous non-zero regime exists for $p \geq 3$ and that $T_* = \max\{1-4\epsilon^2(p-1), 0\}$.

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*The three figures mentioned in the text are not included in the present version.

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1 Introduction

The Hopfield model for N classical Ising spins $\{S_i : \ i = 1, 2, \cdots, N\}$, is specified by the choice of p configurations $\{\boldsymbol{\xi}^\mu = (\xi_1^\mu, \xi_2^\mu, \cdots, \xi_N^\mu) : \ \mu = 1, 2, \cdots, p\}$ which determine the interaction constants

$$J_{i,j}^{\{\boldsymbol{\xi}\}} = -\frac{1}{2N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \quad , \quad i, j = 1, 2, \cdots, N$$

entering the Hamiltonian

$$H_{N,p}^{\{\boldsymbol{\xi}\}}(S) = \sum_{i,j=1}^{N} J_{i,j}^{\{\boldsymbol{\xi}\}} S_i S_j \ . \tag{1.1}$$

The statistical mechanics of this random mean-field model was studied in the late seventies by Pastur and Figotin [1], although it gained notoriety and its name after Hopfield [2] proposed it as a model for an associative memory [3]. In this context the p configurations $S = \boldsymbol{\xi}^\mu$ are called patterns and they become fixed points of the "linear threshold dynamics"

$$S_i(t+1) = sign\left(-2 \sum_{j=1}^{N} J_{i,j}^{\{\boldsymbol{\xi}\}} S_j(t)\right)$$

as $N \to \infty$.

The equilibrium thermodynamics of the model assuming the $\{\xi_i^\mu\}$ are independent identically distributed random variables with $Prob\{\xi_i^\mu = \pm 1\} = 1/2$, and p independent of N, was analyzed by many authors after the work of Pastur and Figotin (e.g. Refs. [4, 5, 6]). The analysis of the case of p growing with N but $p/N \to \alpha$ as $N \to \infty$ followed. One distinguishes the subextensive regime $\alpha = 0$ (or regime of perfect memory) where satisfactory and complete results are available [7, 8, 9, 10, 11, 12], from the extensive regime $\alpha > 0$ where only partial results are known [13, 14, 15]. The present state of affairs, excluding the general "self-averaging" property of the model obtained in Ref. [15], is clearly exposed in Ref. [16].

Let

$$f_{N,p}^{\{\boldsymbol{\xi}\}}(\beta) = \ (\beta N)^{-1} \ln Z_{N,p}^{\{\boldsymbol{\xi}\}}(\beta)$$

be minus the free-energy density associated with the partition function

$$Z_{N,p}^{\{\boldsymbol{\xi}\}}(\beta) = \sum_{S \in \{\pm 1\}^p} \exp\{-\beta H_{N,p}^{\{\boldsymbol{\xi}\}}(S)\} \ .$$

The strongest result on the thermodynamic limit under the i.i.d. assumption in the subextensive case (which includes the case of finite p) is in Ref. [11]: 1) $f_{N,p}^{\{\boldsymbol{\xi}\}}(\beta)$ converges almost surely (i.e. with probability one) as $N \to \infty$ to the limiting negative free-energy density of the Curie-Weiß model with interaction constant 1 at the same reciprocal temperature. 2) Let $a^{\pm}(\beta)$ denote the largest (+), resp. smallest (-), solution of the equation $a = \tanh(\beta a)$. Upon adding $-h \sum_{i=1}^{N} \xi_i^\nu S_i$ to the Hamiltonian, the measures induced on the so-called overlap parameters

$$m_N^\mu(\beta; \{\boldsymbol{\xi}\}) = N^{-1} \sum_{i=1}^{N} \xi_i^\mu < S_i >$$

(here $< \cdot >$ denotes the thermal expectation) converge as $h \to \pm 0$ weakly and almost surely to the point measure sitting at $a^{\pm}(\beta)$ times the ν-th unit vector in $\mathbb{R}^{\mathbf{N}}$.

The purpose of this note is, firstly, to point out an alternative treatment of the case of finitely many patterns using the methods of Refs. [17, 18]. These papers provide a general and efficent treatment of the equilibrium thermodynamics of arbitrary mean-field systems. The last section of the second paper [18], gives a general result which can be applied directly to the case of finite p. Moreover, these results are obtained under an ergodic hypothesis on the random process $\{\xi_i^\mu\}$ which is weaker than the i.i.d. assumption. Secondly, we establish a number of properties of the limiting free-energy density, compute it in some particular cases and give a lower bound for the critical temperature marking the transition from zero to non-zero overlap in the general i.i.d. case.

2 Basic results

We consider more general models where the configuration space $\mathcal{S}$ of the spin is a finite discrete set of reals. The positive integer p is fixed and supressed in the notation. Let σ denote the "spin-function" (identity function) on $\mathcal{S}$, i.e. $\sigma(s) = s$ $(s \in \mathcal{S})$. We let $\mathcal{X}$ be the p-fold direct product of $\mathcal{S}$ provided with the product topology. By collecting the ξ_i^μ's with fixed i to $\boldsymbol{\xi_i} = (\xi_i^1, \xi_i^2, \cdots, \xi_i^p) \in \mathcal{X}$, for $i = 1, 2, \cdots, N$, we obtain a random process on the spin sites taking values in $\mathcal{X}$. It is crucial for the application of the theory of Ref. [17, 18], that the randomness can be localized in the spin sites, in the present case

$$J_{i,j}^{\{\boldsymbol{\xi}\}} = -\frac{1}{2N}\langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_j \rangle \quad , \quad i, j = 1, 2, \cdots, N \ .$$

Here and in the following $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^p$.

Given a vector $\mathbf{h} = (h^1, h^2, \cdots, h^p)$ with real components h^μ, we add the term

$$-\sum_{\mu=1}^{p} h^\mu \sum_{i=1}^{N} \xi_i^\mu S_i = -\sum_{i=1}^{N} \langle \mathbf{h}, \boldsymbol{\xi}_i \rangle S_i \quad ,$$

to the Hamiltonian (1.1) and denote by $f_N^{\{\boldsymbol{\xi}\}}(\beta, \mathbf{h})$ the corresponding (negative) free-energy density. This incorporates p "external magnetic fields" of strength h^μ in the "direction of the pattern μ", which permit the control of the overlap parameters:

$$\frac{\partial f_N^{\{\boldsymbol{\xi}\}}}{\partial h^\mu}(\beta, \mathbf{h}) = m_N^\mu(\beta, \mathbf{h}; \{\boldsymbol{\xi}\}) \ . \tag{2.1}$$

We write $\delta_{\mathbf{x}}$ for the point measure sitting at $\mathbf{x} \in \mathcal{X}$. The only hypothesis on a realization of the random process $\{\xi_i^\mu\}$ used in Ref. [18] is:

$$\text{weak - }\lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} \delta_{\boldsymbol{\xi}_i} = \lambda \tag{2.2}$$

in the space of probability measures on $\mathcal{X}$. This ergodicity condition is satisfied almost surely if $\{\xi_i^\mu\}$ are i.i.d., or more generally if $\{\boldsymbol{\xi}_i\}$ is a stationary $\mathcal{X}$-valued Markov process.

Any probability measure ϕ on $\mathcal{S}$, is specified by the numbers $\{\phi(\{s\}) : \ s \in \mathcal{S}\}$; we denote by

$$S(\phi) = -\sum_{s\in\mathcal{S}} \phi(\{s\}) \ln(\phi(\{s\}))$$

its entropy relative to the uniform probability measure (i.e. normalized counting measure) on $\mathcal{S}$, denoted by ω.

We write Λ for the support in $\mathcal{X}$ of the probability measure λ. Consider the set Γ_λ of functions $\mathbf{x} \mapsto \phi_\mathbf{x}$ on Λ taking values in $\mathcal{M}(\mathcal{S})$ the probability measures on $\mathcal{S}$. If $\Phi = \{\mathbf{x} \mapsto \phi_\mathbf{x}\}$ is such a family, then we let $\mathbf{m}(\Phi)$ be the vector in $\mathbb{R}^p$ with components

$$m^\mu(\Phi) \stackrel{def}{=} \sum_{\mathbf{x}\in\Lambda} \lambda(\{\mathbf{x}\})\, x^\mu < \sigma >_{\phi_\mathbf{x}} \;\; , \;\; \mu = 1, 2, \cdots, p\ . \tag{2.3}$$

We stress that $\mathbf{m}(\cdot)$ depends on λ although we do not make this dependence explicit in order to not overload the notation.

Theorem [18]: *If (2.2) holds (resp. holds almost surely, i.e. with probability* 1*), then*

$$\lim_{N\to\infty} f_N^{\{\boldsymbol{\xi}\}}(\beta, \mathbf{h}) = \max_{\Phi\in\Gamma_\lambda} \left\{ \frac{1}{2} \parallel \mathbf{m}(\Phi) \parallel^2 + \langle \mathbf{h}, \mathbf{m}(\Phi)\rangle + \beta^{-1} \sum_{\mathbf{x}\in\Lambda} \lambda(\{\mathbf{x}\})\, S(\phi_\mathbf{x}) \right\} \tag{2.4}$$

(resp. almost surely).

For a family $\Phi = \{\mathbf{x} \mapsto \phi_\mathbf{x}\} \in \Gamma_\lambda$ over Λ, we will write

$$S_\lambda(\Phi) = \sum_{\mathbf{x}\in\Lambda} \lambda(\{\mathbf{x}\})\, S(\phi_\mathbf{x})\ .$$

This is a positive, strictly concave function on Γ_λ, the convex sum of two families being defined pointwise : $(\alpha\Phi + (1-\alpha)\Psi)_\mathbf{x} = \alpha\phi_\mathbf{x} + (1-\alpha)\psi_\mathbf{x}$. S_λ assumes its maximal value $\ln(\mid \mathcal{S} \mid)$ only for the family having as constant value on Λ the uniform probability measure ω on $\mathcal{S}$. The minimal value 0 of S_λ is assumed exactly on those families Φ for which for every $\mathbf{x} \in \Lambda$ one has $\phi_\mathbf{x} = \delta_s$ for some $s \in \mathcal{S}$ which may depend on $\mathbf{x}$.

We write $f(\beta, \mathbf{h})$ for the non-random limiting (negative) free-energy density. The solution of the variational problem of (2.4) poses a rather unmanageable problem in the general case. However, in the classical Ising spin case $\mathcal{S} = \{1, -1\}$ a probability measure ϕ on $\mathcal{S}$ is uniquely determined by the expectation value $< \sigma >_\phi$, since $\phi(\{1\}) = 1 - \phi(\{-1\}) = \frac{1+<\sigma>_\phi}{2}$. Thus, the entropy of ϕ is a function of this expectation

$$S(\phi) = -\phi(\{1\}) \ln(\phi(\{1\})) - \phi(\{-1\}) \ln(\phi(\{-1\})) = \eta\left(\frac{1\pm < \sigma >_\phi}{2}\right)\ ,$$

where $\eta(t) = -t\ln(t) - (1-t)\ln(1-t)$ for $t \in [0, 1]$ with the usual understanding that $0 \ln(0) = 0$. It follows that the functional to be maximized in (2.4) depends only on the set $\{< \sigma >_{\phi_\mathbf{x}} \colon \mathbf{x} \in \Lambda\}$ of expectations, and one obtains a "gap" equation in terms of these expectations which is the multidimensional analogue of the equation $a = \tanh(\beta(Ja + h))$ for the Curie-Weiß model. For $\phi \in \mathcal{M}(\mathcal{S})$, we write $\widehat{\phi}$ for the flipped measure determined by $< \sigma >_{\widehat{\phi}} = - < \sigma >_\phi$; alternatively $\widehat{\phi}(\{\pm 1\}) = \phi(\{\mp 1\})$. Notice that $S(\widehat{\phi}) = S(\phi)$. We then obtain the following result for the classical Ising spin case.

Proposition 1 *Let* $\mathcal{S} = \{1, -1\}$.

1. *If* $\Phi = \{\mathbf{x} \to \phi_\mathbf{x}\}$ *is a maximizing family for the variational problem (2.4) then*

   $$< \sigma >_{\phi_\mathbf{x}} = \tanh\left(\beta\langle \mathbf{m}(\Phi) + \mathbf{h}, \mathbf{x}\rangle\right) \tag{2.5}$$

   for every $\mathbf{x} \in \Lambda$.

2. *The solutions of the gap equation (2.5) have the following symmetry properties:*

(a) *If* $\Phi = \{\mathbf{x} \to \phi_{\mathbf{x}}\}$ *is a solution then* $<\sigma>_{\phi_{-\mathbf{x}}} = -<\sigma>_{\phi_{\mathbf{x}}}$ *for every* $\mathbf{x} \in \Lambda$ *such that* $-\mathbf{x} \in \Lambda$*; that is* $\widehat{\phi_{\mathbf{x}}} = \phi_{-\mathbf{x}}$ *on* $\Lambda \cap (-\Lambda)$.

(b) *If* $\mathbf{h} = \mathbf{0}$*, then* $\Phi = \{\mathbf{x} \to \phi_{\mathbf{x}}\}$ *is a solution iff* $\widehat{\Phi} = \{\mathbf{x} \to \widehat{\phi_{\mathbf{x}}}\}$ *is a solution.*

In their seminal paper Ref. [4], Amit, Gutfreund and Sompolinsky use replica methods to obtain

$$f(\beta, \mathbf{h}) = \max_{\mathbf{m}} F^{AGS}(\mathbf{m}) \ , \tag{2.6}$$

where

$$F^{AGS}(\mathbf{m}) := (-1/2) \parallel \mathbf{m} \parallel^2 + \beta^{-1} \sum_{\mathbf{x}\in\Lambda} \lambda(\mathbf{x}) \ln\left(2cosh\left(\beta\langle \mathbf{m}+\mathbf{h}, \mathbf{x}\rangle\right)\right) \quad , \quad \mathbf{m} \in \mathbb{R}^p \ .$$

Roughly speaking, this is obtained by writing the partition function as a Gaussian integral and using the method of the largest term and a self-averaging assumption. These arguments have been made rigorous for finitely many patterns in [6] for the i.i.d. case with symmetric distribution and for $\mathbf{h} = \mathbf{0}$. Presumably the work of Bovier, Gayrard and Picco can be used to provide a proof of (2.6) with the same assumption in the subextensive regime. We obtain a proof of (2.6) using only the ergodicity condition (2.2) by studying the relationship between F^{AGS} and F.

For $\mathbf{m} \in \mathbb{R}^p$, we define a family $\Phi[\mathbf{m}] \in \Gamma_\lambda$ by

$$<\sigma>_{\Phi[\mathbf{m}]_{\mathbf{x}}} = \tanh\left(\beta\langle \mathbf{m}+\mathbf{h}, \mathbf{x}\rangle\right) \quad , \quad \mathbf{x} \in \Lambda \ .$$

The map $\mathbb{R}^p \ni \mathbf{m} \to \Phi[\mathbf{m}]$ is inyective. A straightforward computation using the formula $\eta\left(\frac{1+\tanh(u)}{2}\right) = \ln(2\cosh(u)) - u\tanh(u)$, gives

$$F(\Phi[\mathbf{m}]) = F^{AGS}(\mathbf{m}) + (1/2) \parallel \mathbf{m} - \mathbf{m}(\Phi[\mathbf{m}]) \parallel^2 \ . \tag{2.7}$$

The first and second derivatives of F^{AGS} are:

$$\left(\nabla F^{AGS}\right)(\mathbf{m}) = -\mathbf{m} + \mathbf{m}(\Phi[\mathbf{m}]) \ ;$$

$$\frac{\partial^2 F^{AGS}}{\partial m^\mu \partial m^\nu}(\mathbf{m}) = -\delta_{\mu,\nu} + \beta \sum_{\mathbf{x}\in\Lambda} \lambda(\mathbf{x}) x^\mu x^\nu \cosh(\beta\langle \mathbf{m}+\mathbf{h}, \mathbf{x}\rangle)^{-2} \ .$$

Thus, $\mathbf{m}_o$ is a critical point of F^{AGS} iff

$$\mathbf{m}_o = \mathbf{m}(\Phi[\mathbf{m}_o]) \ . \tag{2.8}$$

Notice that if $\mathbf{h} = \mathbf{0}$, then $\mathbf{m} = \mathbf{0}$ is a critical point.

We compute first and second derivatives of F with respect to $<\sigma>_{\Phi_{\mathbf{x}}}$:

$$\begin{aligned}
\frac{\partial F}{\partial <\sigma>_{\Phi_{\mathbf{x}}}}(\Phi) &= \beta^{-1}\lambda(\mathbf{x}) \left(\beta\langle \mathbf{m}(\Phi)+\mathbf{h}, \mathbf{x}\rangle - \tanh^{-1}(<\sigma>_{\Phi_{\mathbf{x}}})\right) \\
&= \beta^{-1}\lambda(\mathbf{x}) \left(\tanh^{-1}\left(<\sigma>_{\Phi[\mathbf{m}(\Phi)]_{\mathbf{x}}}\right) - \tanh^{-1}(<\sigma>_{\Phi_{\mathbf{x}}})\right) \ ;
\end{aligned}$$

$$\frac{\partial^2 F}{\partial <\sigma>_{\Phi_{\mathbf{x}}} \partial <\sigma>_{\Phi_{\mathbf{y}}}}(\Phi) = \lambda(\mathbf{x})\lambda(\mathbf{y})\langle \mathbf{x}, \mathbf{y}\rangle - \beta^{-1}\delta_{\mathbf{x},\mathbf{y}}\lambda(\mathbf{x}) \left(1 - <\sigma>_{\Phi_{\mathbf{x}}}^2\right)^{-1} \ . \tag{2.9}$$

Thus, Φ^o is a critical point of F iff (c.f. (2.5))

$$<\sigma>_{\Phi[\mathbf{m}(\Phi^o)]_{\mathbf{x}}} = <\sigma>_{\Phi^o_{\mathbf{x}}} \ , \quad \text{for every } \mathbf{x} \in \Lambda \iff \Phi^o = \Phi[\mathbf{m}(\Phi^o)], \tag{2.10}$$

Proposition 2 *The maps*

$$\mathbf{m} \mapsto \Phi[\mathbf{m}] \ \ , \ \ \Phi \mapsto \mathbf{m}(\Phi)$$

restricted to the critical points of F^{AGS} and of F respectively are each others inverses, and define a biyective correspondence between critical points of F^{AGS} and critical points of F. One has

$$F^{AGS}(\mathbf{m}) = F(\Phi[\mathbf{m}]) \ \ , \ \ F^{AGS}(\mathbf{m}(\Phi)) = F(\Phi)$$

if $\mathbf{m}$ (resp. Φ) is a critical point of F^{AGS} (resp. F). In particular, the maximizers of F^{AGS} and F are in biyective correspondence and the maximal value of the functionals is equal.

Proof: We have remarked that $\mathbf{m} \mapsto \Phi[\mathbf{m}]$ is inyective. Suppose Φ^1 and Φ^2 are critical points of F. Then $\Phi^j = \Phi[\mathbf{m}(\Phi^j)]$ for $j = 1, 2$. If $\mathbf{m}(\Phi^1) = \mathbf{m}(\Phi^2)$, then we conclude that $\Phi^1 = \Phi^2$. Thus, $\Phi \mapsto \mathbf{m}(\Phi)$ is inyective when restricted to the critical points of F. Using (2.8) and (2.10), we obtain: $\mathbf{m}_o$ is critical for $F^{AGS} \Leftrightarrow \mathbf{m}_o = \mathbf{m}(\Phi[\mathbf{m}_o]) \Rightarrow \Phi[\mathbf{m}_o] = \Phi[\mathbf{m}(\Phi[\mathbf{m}_o])] \Leftrightarrow \Phi[\mathbf{m}_o]$ is critical for F; and: Φ^o is critical for $F \Leftrightarrow \Phi^o = \Phi[\mathbf{m}(\Phi^o)] \Rightarrow \mathbf{m}(\Phi^o) = \mathbf{m}(\Phi[\mathbf{m}(\Phi^o)]) \Rightarrow \mathbf{m}(\Phi^o)$ is critical.

The rest of the claim follows from (2.7). $\square$

We write $f^{cw}(\beta, h; J)$ for the limiting negative free-energy density of the Curie-Weiß model with interaction constant $J \geq 0$ at the reciprocal temperature β and external field h:

$$-\frac{J}{2N} \sum_{i,j=1}^{N} S_i S_j - h \sum_{i=1}^{N} S_i \ ; \tag{2.11}$$

and $m^{cw}(\beta, h; J)$ for the corresponding limiting mean magnetization. We recall the following familiar facts which will be often used without mentioning them.

$$f^{cw}(\beta, h; J) = \max_{t \in [-1,1]} g(\beta, h, t; J)$$

where

$$g(\beta, h, t; J) = \frac{J}{2} t^2 + ht + \beta^{-1} \eta \left(\frac{1+t}{2} \right) \ . \tag{2.12}$$

For $\beta J \leq 1$ or $h \neq 0$, there is a unique maximizer $t = m^{cw}(\beta, h; J)$ for g. For $\beta J > 1$, and $h = 0$ there are two maximizers $m^{cw}(\beta, 0^{\pm}; J)$ with $-m^{cw}(\beta, 0^-; J) = m^{cw}(\beta, 0^+; J) > 0$. For all $\beta J \leq 1$, one has $m^{cw}(\beta, 0; J) = 0$. One has $\lim_{\beta \downarrow 1/J} m^{cw}(\beta, 0^{\pm}; J) = m^{cw}(1/J, 0; J) = 0$. $\beta \mapsto f^{cw}(\beta, h; J)$ is a strictly convex, strictly decreasing, and differentiable function with $-\beta^2 \frac{\partial f^{cw}}{\partial \beta}(\beta, h; J) = \eta \left(\frac{1+m^{cw}(\beta,h;J)}{2} \right)$. $h \mapsto f^{cw}(\beta, h; J)$ is strictly convex, even, and differentiable except for $h = 0$ when $\beta J > 1$. For $\beta J > 1$, the left and right derivatives $(\frac{\partial f^{cw}}{\partial h})_{\pm}(\beta, 0; J)$ are given by $m^{cw}(\beta, 0^{\pm}; J)$.

Proposition 3 *Let $\mathcal{S} = \{1, -1\}$.*

1. $0 < \beta \mapsto f(\beta, \mathbf{h})$ *is strictly convex, and strictly decreasing.* $0 < \beta \mapsto \beta f(\beta, \mathbf{0})$ *is non-decreasing. Moreover,* $f(\beta, \mathbf{0}) \geq f^{cw}(\beta, 0; 1)$.

2. $\mathbf{h} \mapsto f(\beta, \mathbf{h})$ *is convex and invariant under inversion. Moreover, if λ is uniform, i.e. $\lambda(\{\mathbf{x}\}) = 2^{-p}$, then $\mathbf{h} \mapsto f(\beta, \mathbf{h})$ is invariant under permutations of the components of $\mathbf{h}$ and invariant under a change of sign of any component of $\mathbf{h}$.*

3. *For all $\beta > 0$ and $\mathbf{h} \in \mathbb{R}^p$ one has:*

$$\begin{aligned} \beta^{-1}\ln(2) &\leq \max_{\mu\in\{1,2,\cdots,p\}} f^{cw}\left(\beta, \langle \mathbf{J}(\mu), \mathbf{h}\rangle \,;\, \| \mathbf{J}(\mu) \|^2\right) \\ &\leq f(\beta,\mathbf{h}) \\ &\leq \sum_{\mathbf{x}\in\Lambda} \lambda(\{\mathbf{x}\}) f^{cw}\left(\beta, \langle \mathbf{x},\mathbf{h}\rangle; 2^p\lambda(\{\mathbf{x}\})\right) \ ; \end{aligned} \tag{2.13}$$

where

$$J^{\nu}(\mu) \stackrel{def}{=} \sum_{\{\mathbf{x}\in\Lambda:\, x^{\mu}=1\}} x^{\nu}(\lambda(\{\mathbf{x}\}) + \lambda(\{-\mathbf{x}\})) \ , \quad \nu = 1,2,\cdots,p \ .$$

4. *Suppose $\beta \leq p^{-1}$. There exists a unique family $\Phi^o \in \Gamma_\lambda$ maximizing (2.4). When $\mathbf{h} = \mathbf{0}$, Φ^o is the constant family taking the value ω on Λ; and one has $\mathbf{m}(\Phi^o) = \mathbf{0}$ and $f(\beta,\mathbf{0}) = \beta^{-1}\ln(2)$. Moreover, if $\beta < p^{-1}$, then*

   (a) *$\mathbf{h} \mapsto f(\beta,\mathbf{h})$ is differentiable with*

   $$\mathbf{m}(\beta,\mathbf{h}) \stackrel{def}{=} \lim_{N\to\infty} \mathbf{m}_N(\beta,\mathbf{h},\boldsymbol{\xi}) = (\nabla_h f)(\beta,\mathbf{h}) = \mathbf{m}(\Phi^o)$$

   the limit existing (almost surely) under condition (2.2) (almost surely).

   (b) *$\beta \mapsto f(\beta,\mathbf{h})$ is differentiable, and if (2.2) holds (almost surely) the limiting entropy density exists (almost surely) and equals $S_\lambda(\Phi^o) = -\beta^{-2}\frac{\partial f}{\partial\beta}(\beta,\mathbf{h})$.*

5. *If $\beta_1 < \beta_2$ and $f(\beta_2,\mathbf{0}) = \beta_2^{-1}\ln(2)$, then $f(\beta_1,\mathbf{0}) = \beta_1^{-1}\ln(2)$. The set $\{\beta > 0 : f(\beta,\mathbf{0}) = \beta^{-1}\ln(2)\}$ is not empty and equal to the interval $(0,\beta_c]$ where $p^{-1} \leq \beta_c \leq 1$.*

The bound p^{-1} on the critical value of β is attained in special cases: for $p = 2$ when λ is supported by two points (see section IV); and for general p when $Prob\{\xi_i^{\mu} = 1\}$ is either 0 or 1 (see section V).

The bounds of (2.13) suffice to compute f when $\mathbf{h} = \mathbf{0}$ and $\beta 2^p \lambda(\{\mathbf{x}\}) \leq 1$, for all $\mathbf{x} \in \Lambda$. It is known that $f^{cw}(\beta,0;J) = \beta^{-1}\ln(2)$ for $\beta J \leq 1$; thus under the condition on β and λ, the upper bound of (2.13) gives $f(\beta,\mathbf{0}) = \beta^{-1}\ln(2)$. This happens for example when λ is equidistributed on Λ and $\beta \leq 2^{-p} \mid \Lambda \mid (\leq 1)$.

The lower bound $\max_{\mu\in\{1,2,\cdots,p\}} f^{cw}\left(\beta, \langle \mathbf{J}(\mu),\mathbf{h}\rangle \,;\, \| \mathbf{J}(\mu) \|^2\right)$ is exact if $\mathbf{h}$ has only one non-zero component and λ is uniform; leading to the value $f^{cw}(\beta, \| \mathbf{h} \|; 1)$ for f.

The following result shows that the upper bound of (2.13) is attained when λ is uniform, and at most two components of $\mathbf{h}$ are non-zero.

Proposition 4 *Let $\mathcal{S} = \{1,-1\}$ and suppose λ is uniform and $\mathbf{h}$ has at most two non-zero components say h^{μ_1} and h^{μ_2} then*

$$f(\beta,\mathbf{h}) = \frac{1}{2} f^{cw}(\beta, h^{\mu_1} + h^{\mu_2}; 1) + \frac{1}{2} f^{cw}(\beta, h^{\mu_1} - h^{\mu_2}; 1) \ ; \tag{2.14}$$

in particular $f(\beta,\mathbf{0}) = f^{cw}(\beta,0;1)$. $0 < \beta \mapsto f(\beta,\mathbf{h})$ is differentiable. If Φ and Ψ are maximizers of (2.4) then $S_\lambda(\Phi) = S_\lambda(\Psi) = -\beta^2\frac{\partial f}{\partial\beta}(\beta,\mathbf{h})$. If (2.2) holds true (almost surely) the limiting entropy density exists (almost surely) and is equal to $-\beta^2\frac{\partial f}{\partial\beta}(\beta,\mathbf{h})$.

Moreover

1. *If $\beta > 1$ (resp. $\beta \leq 1$) then for $\mid h^{\mu_1} \mid \neq \mid h^{\mu_2} \mid$ (resp. arbitrary h^{μ_1} and h^{μ_2}), one has:*

(a) *there is a unique family* $\Phi^o \in \Gamma_\lambda$ *maximizing (2.4); it is given by*

$$\phi^o_{\mathbf{x}} = \begin{cases} \phi^{cw}(\beta, h^{\mu_1} + h^{\mu_2}) & , \quad \text{if } x^{\mu_1} = x^{\mu_2} = 1 \\ \phi^{cw}(\beta, h^{\mu_1} - h^{\mu_2}) & , \quad \text{if } x^{\mu_1} = -x^{\mu_2} = 1 \\ \widehat{\phi}^{cw}(\beta, h^{\mu_1} - h^{\mu_2}) & , \quad \text{if } -x^{\mu_1} = x^{\mu_2} = 1 \\ \widehat{\phi}^{cw}(\beta, h^{\mu_1} + h^{\mu_2}) & , \quad \text{if } x^{\mu_1} = x^{\mu_2} = -1 \end{cases} .$$

where $\phi^{cw}(\beta, h)$ *is the unique probability measure on* $\mathcal{S}$ *such that* $<\sigma>_{\phi^{cw}(\beta,h)} = m^{cw}(\beta, h; 1)$. *One has* $m^\kappa(\Phi^o) = 0$ *for* κ *distinct from* μ_1 *and* μ_2, *and*

$$m^{\mu_1}(\Phi^o) = (1/2)\left(m^{cw}(\beta, h^{\mu_1} + h^{\mu_2}; 1) + m^{cw}(\beta, h^{\mu_1} - h^{\mu_2}; 1)\right) ,$$

$$m^{\mu_2}(\Phi^o) = (1/2)\left(m^{cw}(\beta, h^{\mu_1} + h^{\mu_2}; 1) + m^{cw}(\beta, h^{\mu_2} - h^{\mu_1}; 1)\right) .$$

(b) $(h^{\mu_1}, h^{\mu_2}) \mapsto f(\beta, \mathbf{h})$ *is differentiable and*

$$\begin{aligned} m^{\mu_1}(\beta, \mathbf{h}) &\stackrel{def}{=} \lim_{N\to\infty} m_N^{\mu_1}(\beta, \mathbf{h}; \{\boldsymbol{\xi}\}) = \left(\frac{\partial f}{\partial h^{\mu_1}}\right)(\beta, \mathbf{h}) = m^{\mu_1}(\Phi^o) \\ m^{\mu_2}(\beta, \mathbf{h}) &\stackrel{def}{=} \lim_{N\to\infty} m_N^{\mu_2}(\beta, \mathbf{h}; \{\boldsymbol{\xi}\}) = \left(\frac{\partial f}{\partial h^{\mu_2}}\right)(\beta, \mathbf{h}) = m^{\mu_2}(\Phi^o) \end{aligned}$$

the limits existing (almost surely) under condition (2.2) (almost surely).

2. *If* $\beta > 1$, *then*

(a) $(h^{\mu_1}, h^{\mu_2}) \mapsto f(\beta, \mathbf{h})$ *is not differentiable on the two lines* $\mid h^{\mu_1} \mid = \mid h^{\mu_2} \mid$.

(b) *for* $\mid h^{\mu_1} \mid = \mid h^{\mu_2} \mid \neq 0$ *there are* $2^{2^{(p-2)}}$ *families* Φ *maximizing (2.4). For each of these one has* $m^\kappa(\Phi) = 0$ *for* $\mu_1 \neq \kappa \neq \mu_2$ *and* $(m^{\mu_1}(\Phi), m^{\mu_2}(\Phi))$ *is one of the following two vectors:*

$$\left.\begin{array}{l} \left(\frac{m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^+;1)}{2}, \frac{m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^-;1)}{2}\right) \\ \left(\frac{m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^-;1)}{2}, \frac{m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^+;1)}{2}\right) \end{array}\right\} ,$$

when $h^{\mu_1} = h^{\mu_2}$;

$$\left.\begin{array}{l} \left(\frac{m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^+;1)}{2}, \frac{-m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^+;1)}{2}\right) \\ \left(\frac{m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^-;1)}{2}, \frac{-m^{cw}(\beta,2h^{\mu_1};1)+m^{cw}(\beta,0^-;1)}{2}\right) \end{array}\right\} ,$$

when $h^{\mu_1} = -h^{\mu_2}$.
Here $m^{cw}(\beta, 0^\pm; 1)$ *is the spontaneous magnetization of sign* $\pm$ *of the Curie-Weiß model for* $\beta > 1$ *and interaction constant* 1.

(c) *for* $h^{\mu_1} = h^{\mu_2} = 0$ *there are* $2^{2^{(p-1)}}$ *families* Φ *maximizing (2.4). For each of these one has* $m^\kappa(\Phi) = 0$ *for* $\mu_1 \neq \kappa \neq \mu_2$ *and* $(m^{\mu_1}(\Phi), m^{\mu_2}(\Phi))$ *is one of the following four vectors:*

$$\left(m^{cw}(\beta, 0^+; 1), 0\right) , \left(m^{cw}(\beta, 0^-; 1), 0\right) ,$$

$$\left(0, m^{cw}(\beta, 0^+; 1)\right) , \left(0, m^{cw}(\beta, 0^-; 1)\right) .$$

The families of 2.b) and 2.c) are given explicitely in the next section for arbitrary λ and $p = 2$.

For uniform λ and at most two non-zero field components Proposition 4 provides a rather complete picture. For subcritical temperatures (i.e. $\beta > 1$) the subgradients of f on the two lines $\mid h^{\mu_1} \mid = \mid h^{\mu_2} \mid$ can be described explicitely to obtain the phase diagramm of Fig. 1. There is a jump discontinuity in the μ_1-th and μ_2-th limiting overlap parameters on each line as indicated in the figure. The (h^{μ_1}, h^{μ_2})-plane spilts into the four open cones delimited by the two lines; in each cone the limiting behaviour of the overlap parameters as $\mathbf{h} = (h^{\mu_1}, h^{\mu_2}) \to (0,0)$ for $\beta > 1$ is different:

$$\lim_{\mathbf{h}\to\mathbf{0}\,;\,0<|h^{\mu_2}|<h^{\mu_1}} (m^{\mu_1}(\beta,\mathbf{h}), m^{\mu_2}(\beta,\mathbf{h})) = \Big(m^{cw}(\beta, 0^+; 1), 0\Big) \ ;$$

$$\lim_{\mathbf{h}\to\mathbf{0}\,;\,0<|h^{\mu_1}|<h^{\mu_2}} (m^{\mu_1}(\beta,\mathbf{h}), m^{\mu_2}(\beta,\mathbf{h})) = \Big(0, m^{cw}(\beta, 0^+; 1)\Big) \ ;$$

$$\lim_{\mathbf{h}\to\mathbf{0}\,;\,0<|h^{\mu_2}|<-h^{\mu_1}} (m^{\mu_1}(\beta,\mathbf{h}), m^{\mu_2}(\beta,\mathbf{h})) = \Big(m^{cw}(\beta, 0^-; 1), 0\Big) \ ;$$

$$\lim_{\mathbf{h}\to\mathbf{0}\,;\,0<|h^{\mu_1}|<-h^{\mu_2}} (m^{\mu_1}(\beta,\mathbf{h}), m^{\mu_2}(\beta,\mathbf{h})) = \Big(0, m^{cw}(\beta, 0^-; 1)\Big) \ .$$

In simple words, as $\mathbf{h} \to \mathbf{0}$, $\mathbf{m}$ follows the behaviour dictated by the field component with maximal modulus.

The simple picture that emerges for uniform λ and two non-zero components is due to the fact that f is given by the upper bound of (2.13) where all $< \sigma >_{\phi_{\mathbf{x}}}$ are decoupled. This is quantitatively wrong for three or more non-zero field components. If λ is uniform, $p = 3$, and all the three components of $\mathbf{h}$ are non-zero; then for all $\beta > 0$

$$f(\beta, \mathbf{h}) < 2^{-p} \sum_{\mathbf{x}\in\mathcal{X}} f^{cw}\left(\beta, \langle \mathbf{x}, \mathbf{h}\rangle; 1\right) \tag{2.15}$$

except for some particular non-zero values of the components.

3 Proofs

The theorem is a special case of a general result obtained in Ref. [18]. We will describe briefly the main ideas and key steps of the proof in the appendix. In this section we give the proof of the main consequences: Propositions 1,3 and 4.

Given a family $\Phi = \{\phi_{\mathbf{x}}\}$ of probability measures on $\mathcal{S} = \{1, -1\}$, we denote by $F(\Phi)$ (disregarding the β and $\mathbf{h}$ dependence) the functional (negative "free energy") to be maximized in (2.4):

$$F(\Phi) = 2^{-1} \parallel \mathbf{m}(\Phi) \parallel^2 + \langle \mathbf{h}, \mathbf{m}(\Phi) \rangle + \beta^{-1} S_\lambda(\Phi) \ .$$

Due to (2.3), F depends only on the values of Φ on the support of λ. We observe that Γ_λ equipped with the metric $d(\Phi, \Psi) = \max_{\mathbf{x}\in\Lambda} |< \sigma >_{\phi_{\mathbf{x}}} - < \sigma >_{\psi_{\mathbf{x}}}|$ is a compact metric space (isomorphic to $[-1,1]^{|\Lambda|}$ via the identification of $\phi \in \mathcal{M}(\mathcal{S})$ with $< \sigma >_\phi \in [-1,1]$). Moreover, F is continuous.

Proof of Proposition 1: Viewing F as a function of $\{< \sigma >_{\phi_{\mathbf{x}}} \colon \ \mathbf{x} \in \Lambda\}$, we have computed the gradient in Section II just before Proposition 2. Setting all first partial derivatives equal to 0 when $\lambda(\{\mathbf{x}\}) \neq 0$ gives the gap-equations (2.5) as the condition for critical point of F . We must show that the maximal value of F is not assumed in a family taking the boundary values where the derivatives do not exist; that is not for family Φ for which $<\sigma>_{\phi_{\mathbf{x}_o}} = \pm 1$ for some $\mathbf{x}_o \in \Lambda$.

Lemma 1 *If Φ is a maximizing family for (2.4), then $|<\sigma>_{\phi_{\mathbf{x}}}| < 1$ for all $\mathbf{x} \in \Lambda$.*

Proof: Take such a "boundary" family Φ with $<\sigma>_{\phi_{\mathbf{x}_o}} = \pm 1$ for some $\mathbf{x}_o \in \Lambda$, and define $\widetilde{\Phi}$ by changing only the value at $\mathbf{x}_o$; i.e. $<\sigma>_{\widetilde{\phi}_{\mathbf{x}_o}} = \pm a$ for an $a \in (-1,1)$ to be specified. We can compute the change in F and get

$$\ell(a) \stackrel{def}{=} F(\widetilde{\Phi}) - F(\Phi) = \lambda(\{\mathbf{x}_o\})\, g\,(\beta\,,\, W_{\pm},\, a;\, p\lambda(\{\mathbf{x}_o\})) - \frac{p}{2}\lambda(\{\mathbf{x}_o\})^2 - \lambda(\{\mathbf{x}_o\})W_{\pm}$$

where $W_{\pm} = \pm\langle \mathbf{x}_o, \mathbf{h} + \mathbf{m}(\Phi)\rangle - \lambda(\{\mathbf{x}_o\})p$, and g is given by (2.12). Now the maximal value of $g\,(\beta, \pm W\,,\,\cdot\,; p\lambda(\{\mathbf{x}_o\}))$ is certainly larger than its value at $a=1$ for any $\beta > 0$, so that there is a with $\ell(a) > \ell(1) = 0$. $\square$

The symmetry properties stated in 2. are inmediate. This completes the proof of Proposition 1.

Proof of 1. of Proposition 3: Clearly $f(\cdot, \mathbf{h})$ is convex and non-increasing as the maximum of strictly convex and strictly decreasing functions $\beta \mapsto F_\beta(\Phi)$. To prove the strict statements we use the following fact which follows (see the statements about $S_\lambda(\Phi)$ made after the Theorem) from Lemma 1:

(3.1) For any family Φ maximizing (2.4) one has $S_\lambda(\Phi) > 0$.

Now suppose $0 < \beta_1 < \beta_2$, and Φ_{β_2} is a maximizing family for F_{β_2}. Then,

$$f(\beta_2, \mathbf{h}) = F_{\beta_2}(\Phi_{\beta_2}) = F_{\beta_1}(\Phi_{\beta_2}) - \frac{\beta_2 - \beta_1}{\beta_1\beta_2} S_\lambda(\Phi_{\beta_2}) \leq F_{\beta_1}(\Phi_{\beta_2}) \leq f(\beta_1, \mathbf{h})\ .$$

There is equality in the first inequality iff $S_\lambda(\Phi_{\beta_2}) = 0$; which is impossible by (3.1). Thus, $f(\cdot, \mathbf{h})$ is strictly decreasing. Suppose $\beta_1 \neq \beta_2$, $\alpha \in (0,1)$, and let $\beta_\alpha = \alpha\beta_1 + (1-\alpha)\beta_2$. Choosing a maximizing family Φ_{β_α} for F_{β_α} we get as above

$$f(\beta_\alpha, \mathbf{h}) \leq f(\beta_1, \mathbf{h}) + \frac{(1-\alpha)(\beta_1 - \beta_2)}{\beta_1\beta_\alpha} S_\lambda(\Phi_{\beta_\alpha})\ ,$$

$$f(\beta_\alpha, \mathbf{h}) \leq f(\beta_2, \mathbf{h}) + \frac{\alpha(\beta_2 - \beta_1)}{\beta_2\beta_\alpha} S_\lambda(\Phi_{\beta_\alpha})\ .$$

Multiplying these by α respectively $(1-\alpha)$ and adding, we get

$$f(\beta_\alpha, \mathbf{h}) \leq \alpha f(\beta_1, \mathbf{h}) + (1-\alpha) f(\beta_2, \mathbf{h}) - \frac{\alpha(1-\alpha)(\beta_1 - \beta_2)^2}{\beta_1\beta_2\beta_\alpha} S_\lambda(\Phi_{\beta_\alpha})$$

and the strict convexity follows by (3.1).

Suppose $\beta_1 < \beta_2$, and let Φ be a maximizing family for F_{β_1}. Then $\beta_2 f(\beta_2, \mathbf{0}) \geq \beta_2 F_{\beta_2}(\Phi) = ((\beta_2 - \beta_1)/2) \parallel \mathbf{m}(\Phi) \parallel^2 + \beta_1 F_{\beta_1}(\Phi) \geq \beta_1 f(\beta_1, \mathbf{0})$. $\square$

Finally notice that for $\mathbf{h} = \mathbf{0}$ the Hamiltonian (1.1) is bounded above by the Hamiltonian $H^{\{\boldsymbol{\xi}\}}_{N,p=1}$ which is equivalent to the Hamiltonian (2.11) of the Curie-Weiß model at zero field an unit interaction by the change of variables $S_i \mapsto \xi_i^1 S_i$. Thus, $f(\beta, \mathbf{0}) \geq f^{cw}(\beta, 0; 1)$ and, since 1 is the critical temperature of this Curie-Weiß model, we deduce $\beta_c \leq 1$.

This completes the proof of 1. of Proposition 3.

Proof of 2. of Proposition 3: The convexity of $f(\beta, \cdot)$ can be verified in various ways; e.g. $f(\beta, \mathbf{h})$ is the maximum of a functions which are affine in $\mathbf{h}$. To show that $f(\beta, \cdot)$ is even one argues that if $\Phi = \Phi(\mathbf{h})$ maximizes F, then the flipped family $\widehat{\Phi}$ obtained by flipping each $\phi_{\mathbf{x}}$, i.e. $(\widehat{\phi})_{\mathbf{x}} = \widehat{\phi_{\mathbf{x}}}$ maximizes F for $-\mathbf{h}$ since $\mathbf{m}(\widehat{\Phi}) = -\mathbf{m}(\Phi)$ and $S(\widehat{\phi_{\mathbf{x}}}) = S(\phi_{\mathbf{x}})$.

Suppose π is a permutation together with arbitrary changes of sign of the components of vectors in $\mathbb{R}^p$. For a family Φ let $\widetilde{\Phi}$ be defined by $\widetilde{\phi}_{\mathbf{x}} = \phi_{\pi(\mathbf{x})}$ Notice that π maps $\mathcal{X}$ bijectively onto itself. Then if

$$\lambda(\{\mathbf{x}\}) = \lambda(\{\pi(\mathbf{x})\}) \quad , \quad \text{for all } x \in \mathcal{X} \; , \tag{3.2}$$

we have

$$\mathbf{m}(\widetilde{\Phi}) = \sum_{\mathbf{x} \in \mathcal{X}} \lambda(\{\mathbf{x}\})\mathbf{x} < \sigma >_{\phi_{\pi(\mathbf{x})}} = \sum_{\mathbf{y} \in \mathcal{X}} \lambda(\{\pi^{-1}(\mathbf{y})\})\pi^{-1}(\mathbf{y}) < \sigma >_{\phi_{\mathbf{y}}} = \pi^{-1}(\mathbf{m}(\Phi))$$

π^{-1} being the inverse of π. Thus

$$F(\widetilde{\Phi}, \mathbf{h}) = \frac{1}{2} \parallel \mathbf{m}(\Phi) \parallel^2 + \langle \pi(\mathbf{h}), \mathbf{m}(\Phi) \rangle + \beta^{-1} \sum_{\mathbf{x} \in \mathcal{X}} \lambda(\{\pi^{-1}(\mathbf{x})\}) S(\phi_{\mathbf{x}}) = F(\Phi, \pi(\mathbf{h})) \; .$$

Hence if the family Φ maximizes F for given $\mathbf{h}$, the family $\widetilde{\Phi}$ maximizes F for $\pi^{-1}(\mathbf{h})$ and the maximal values coincide. In particular, if λ is uniform then (3.2) holds for every π. More generally: if γ is any linear isometric bijection of $\mathbb{R}^p$ which maps $\mathcal{X}$ onto itself and such that $\lambda(\{\mathbf{x}\}) = \lambda(\{\gamma(\mathbf{x})\})$ holds true for every $\mathbf{x} \in \mathcal{X}$ then $f(\beta, \mathbf{h}) = f(\beta, \gamma^{-1}(\mathbf{h}))$. This completes the proof of 2. of Proposition 3.

For $\mu = 1, 2, \ldots, p$ we define the subsets $\mathcal{A}_\mu \subset \mathcal{X}$ by

$$\mathcal{A}_\mu = \{\mathbf{x} \in \mathcal{X} : x^\mu = 1\} \; .$$

For a subset $\mathcal{A}$ of $\mathcal{X}$ we write $-\mathcal{A}$ for the subset obtained by changing the sign of all the members of $\mathcal{A}$. We have $\mathcal{A}_\mu \cap (-\mathcal{A}_\mu) = \emptyset$, and $\mathcal{A}_\mu \cup (-\mathcal{A}_\mu) = \mathcal{X}$.

Lemma 2

1. *If Φ maximizes (2.4), then $< \sigma >_{\phi_{-\mathbf{x}}} = - < \sigma >_{\phi_{\mathbf{x}}}$ (i.e. $\phi_{-\mathbf{x}} = \widehat{\phi_{\mathbf{x}}}$), for all $\mathbf{x}$ for which $\mathbf{x} \in \Lambda$, and $-\mathbf{x} \in \Lambda$. A family with this property will be called reflexive.*

2. *If $\mathbf{h} = \mathbf{0}$, then Φ maximizes (2.4) iff $\widehat{\Phi}$ maximizes (2.4).*

Proof: This follows from Proposition 1, but we give a direct proof.

1. Define a new family $\tilde{\Phi}$ by $\tilde{\phi}_{\mathbf{x}} = \alpha_1(\mathbf{x})\phi_{\mathbf{x}} + \alpha_2(\mathbf{x})\widehat{\phi_{-\mathbf{x}}}$, where:

$$\alpha_1(\mathbf{x}) = \begin{cases} \frac{\lambda(\{\mathbf{x}\})}{\lambda(\{\mathbf{x}\}) + \lambda(\{-\mathbf{x}\})} & , \quad \text{if } \mathbf{x} \in \Lambda \cap (-\Lambda) \\ 1 & , \quad \text{otherwise} \end{cases}$$

$$\alpha_2(\mathbf{x}) = 1 - \alpha_1(\mathbf{x}) \; .$$

Then, $\alpha_1(-\mathbf{x})=\alpha_2(\mathbf{x})$, which implies that $\tilde{\Phi}$ is reflexive. Now, letting $\Lambda_o=\Lambda\setminus(\Lambda\cap(-\Lambda))$,

$$
\begin{aligned}
m^\mu(\tilde{\Phi}) &= \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})x^\mu<\sigma>_{\tilde{\phi}_{\mathbf{x}}}+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})x^\mu<\sigma>_{\tilde{\phi}_{\mathbf{x}}}\\
&= \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})x^\mu<\sigma>_{\phi_{\mathbf{x}}}+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})\alpha_1(\mathbf{x})x^\mu<\sigma>_{\phi_{\mathbf{x}}}\\
&+ \sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})\alpha_2(\mathbf{x})x^\mu(-)<\sigma>_{\phi_{-\mathbf{x}}}\\
&= \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})x^\mu<\sigma>_{\phi_{\mathbf{x}}}+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})\alpha_1(\mathbf{x})x^\mu<\sigma>_{\phi_{\mathbf{x}}}\\
&+ \sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{-\mathbf{x}\})\alpha_2(-\mathbf{x})x^\mu<\sigma>_{\phi_{\mathbf{x}}}\\
&= \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})x^\mu<\sigma>_{\phi_{\mathbf{x}}}+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})\alpha_1(\mathbf{x})x^\mu<\sigma>_{\phi_{\mathbf{x}}}\\
&+ \sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{-\mathbf{x}\})\alpha_1(\mathbf{x})x^\mu<\sigma>_{\phi_{\mathbf{x}}}\\
&= \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})x^\mu<\sigma>_{\phi_{\mathbf{x}}}+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}(\lambda(\{\mathbf{x}\})\\
&+ \lambda(\{-\mathbf{x}\}))\alpha_1(\mathbf{x})x^\mu<\sigma>_{\phi_{\mathbf{x}}}=m^\mu(\Phi)\quad\forall\mu=1,2\ldots,p.
\end{aligned}
$$

In the second equality we use the definition of $\widehat{\phi}$, in the third equality we perform $\mathbf{x}\to-\mathbf{x}$ in the third summand, and in the fourth equality we use $\alpha_1(-\mathbf{x})=\alpha_2(\mathbf{x})$. By strict concavity of the entropy and following the same steps as before after the inequality, we get

$$
\begin{aligned}
S_\lambda(\tilde{\Phi}) &= \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})S(\phi_{\mathbf{x}})+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})S\left(\alpha_1(\mathbf{x})\phi_{\mathbf{x}}+\alpha_2(\mathbf{x})\widehat{\phi_{-\mathbf{x}}}\right)\\
&\geq \sum_{\mathbf{x}\in\Lambda_o}\lambda(\{\mathbf{x}\})S(\phi_{\mathbf{x}})+\sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})\alpha_1(\mathbf{x})S(\phi_{\mathbf{x}})\\
&+ \sum_{\mathbf{x}\in\Lambda\cap(-\Lambda)}\lambda(\{\mathbf{x}\})\alpha_2(\mathbf{x})S(\phi_{-\mathbf{x}})\\
&= S_\lambda(\Phi)\quad,
\end{aligned}
$$

and the inequality is strict if for some $\mathbf{x}_o\in\Lambda\cap(-\Lambda)$ one has $<\sigma>_{\phi_{-\mathbf{x}_o}}\neq-<\sigma>_{\phi_{\mathbf{x}_o}}$. Thus, $F(\tilde{\Phi})\geq F(\Phi)$, with strict inequality if Φ is not reflexive.

2. $m^\mu(\widehat{\Phi})=-m^\mu(\Phi)$, and $S_\lambda(\widehat{\Phi})=S_\lambda(\Phi)$ so that $F(\widehat{\Phi})=F(\Phi)$ if $\mathbf{h}=\mathbf{0}$.

□

Proof of Proposition 4: Suppose $p\geq 2$, that λ is uniform and that $\mathbf{h}$ has at most two non-zero components. Due to the permutational symmetry, we may assume that h^1 and h^2 are the only components which can be non-zero. Since $\Lambda=-\Lambda=\mathcal{X}$ we may (due to Lemma 2) obtain the maximal value of F by varying over reflexive families . Such a family Φ is specified uniquely by its values on $\mathcal{A}_1$, since $\phi_{\mathbf{x}}=\widehat{\phi_{-\mathbf{x}}}$ for $\mathbf{x}\in-\mathcal{A}_1$. We partition $\mathcal{A}_1$ into two sets $\mathcal{B}_1=\mathcal{A}_1\cap\mathcal{A}_2$ and $\mathcal{B}_2=\mathcal{A}_1\cap(-\mathcal{A}_2)$ of equal measure $2^{-(p-2)}$. The proof proceeds in four steps.

First step: If the family Φ is reflexive and has the additional property of being constant on each of the two sets $\mathcal{B}_1$ and $\mathcal{B}_2$[1], then – letting $\overline{\phi}_j$ be the constant value of $\phi_{\mathbf{x}}$ on $\mathcal{B}_j$ $(j = 1, 2)$:

$$m^1(\Phi) = \frac{1}{2}\left(<\sigma>_{\overline{\phi}_1} + <\sigma>_{\overline{\phi}_2}\right) \quad ; \tag{3.3}$$

$$m^2(\Phi) = \frac{1}{2}\left(<\sigma>_{\overline{\phi}_1} - <\sigma>_{\overline{\phi}_2}\right) \quad ; \tag{3.4}$$

$$m^\mu(\Phi) = 0 \quad , \quad \text{for all } \mu \neq 1, 2 \ . \tag{3.5}$$

To prove this, partition $\mathcal{A}_1$ into four disjoint subsets $\mathcal{C}_1 = \mathcal{B}_1 \cap \mathcal{A}_\mu$, $\mathcal{C}_2 = \mathcal{B}_1 \cap (-\mathcal{A}_\mu)$, $\mathcal{C}_3 = \mathcal{B}_2 \cap \mathcal{A}_\mu$, and $\mathcal{C}_4 = \mathcal{B}_2 \cap (-\mathcal{A}_\mu)$. Now

$$\begin{aligned} m^\mu(\Phi) &= 2^{-(p-1)}\left(\sum_{\mathbf{x}\in\mathcal{C}_1} <\sigma>_{\phi_{\mathbf{x}}} - \sum_{\mathbf{x}\in\mathcal{C}_2} <\sigma>_{\phi_{\mathbf{x}}} + \sum_{\mathbf{x}\in\mathcal{C}_3} <\sigma>_{\phi_{\mathbf{x}}} - \sum_{\mathbf{x}\in\mathcal{C}_4} <\sigma>_{\phi_{\mathbf{x}}}\right) \\ &= 2^{-(p-1)}\Big((\mid \mathcal{C}_1 \mid - \mid \mathcal{C}_2 \mid) <\sigma>_{\overline{\phi}_1} + (\mid \mathcal{C}_3 \mid - \mid \mathcal{C}_4 \mid) <\sigma>_{\overline{\phi}_2}\Big) \ . \end{aligned}$$

When $\mu = 1$, $\mid \mathcal{C}_1 \mid = \mid \mathcal{B}_1 \mid = 2^{p-2}$, $\mid \mathcal{C}_3 \mid = \mid \mathcal{B}_2 \mid = 2^{p-2}$, and $\mid \mathcal{C}_2 \mid = \mid \mathcal{C}_4 \mid = 0$. When $\mu = 2$, $\mid \mathcal{C}_1 \mid = \mid \mathcal{B}_1 \mid = 2^{p-2}$, $\mid \mathcal{C}_4 \mid = \mid \mathcal{B}_2 \mid = 2^{p-2}$, and $\mid \mathcal{C}_2 \mid = \mid \mathcal{C}_3 \mid = 0$. If $\mu \neq 1, 2$, then all four sets have 2^{p-3} elements.

Second step: For a given reflexive family Φ we construct a new family $\widetilde{\Phi}$ which is reflexive and constant on $\mathcal{B}_j$ $(j = 1, 2)$. This involves two probability measures ρ_j on $\mathcal{B}_j$ $(j = 1, 2)$. We obtain a lower bound on $F(\widetilde{\Phi}) - F(\Phi)$ and then choose these measures in order to maximize this lower bound. Define $\widetilde{\Phi}$ by putting

$$\widetilde{\phi}_{\mathbf{x}} = \sum_{\mathbf{y}\in\mathcal{B}_j} \rho_j(\{\mathbf{y}\})\phi_{\mathbf{y}} =: \psi_j \quad , \quad \text{if } \mathbf{x} \in \mathcal{B}_j$$

for $\mathbf{x} \in \mathcal{A}_1$ and extending reflexively for $\mathbf{x} \in (-\mathcal{A}_1)$, i.e. $\widetilde{\phi}_{\mathbf{x}} = \widehat{\widetilde{\phi}_{-\mathbf{x}}}$ for $\mathbf{x} \in (-\mathcal{A}_1)$, that is $\widetilde{\phi}_{\mathbf{x}} = \widehat{\psi_j}$ for $\mathbf{x} \in -\mathcal{B}_j$. The family $\widetilde{\Phi}$ is reflexive and constant on each of the subsets $\mathcal{B}_1$, $\mathcal{B}_2$, $-\mathcal{B}_1$ and $-\mathcal{B}_2$. By the first step, $\mathbf{m}(\widetilde{\Phi})$ is given by equations (3.3), (3.4) and (3.5). The entropy part of F for $\widetilde{\Phi}$ is

$$\begin{aligned} &2^{-p}\left(\sum_{\mathbf{x}\in\mathcal{B}_1} S(\widetilde{\phi}_{\mathbf{x}}) + \sum_{\mathbf{x}\in\mathcal{B}_2} S(\widetilde{\phi}_{\mathbf{x}}) + \sum_{\mathbf{x}\in(-\mathcal{B}_1)} S(\widetilde{\phi}_{\mathbf{x}}) + \sum_{\mathbf{x}\in(-\mathcal{B}_2)} S(\widetilde{\phi}_{\mathbf{x}})\right) \\ &= 2^{-(p-1)}\left(\sum_{\mathbf{x}\in\mathcal{B}_1} S(\widetilde{\phi}_{\mathbf{x}}) + \sum_{\mathbf{x}\in\mathcal{B}_2} S(\widetilde{\phi}_{\mathbf{x}})\right) \\ &= \frac{1}{2}S\left(\sum_{\mathbf{y}\in\mathcal{B}_1} \rho_1(\{\mathbf{y}\})\phi_{\mathbf{y}}\right) + \frac{1}{2}S\left(\sum_{\mathbf{y}\in\mathcal{B}_2} \rho_2(\{\mathbf{y}\})\phi_{\mathbf{y}}\right) \\ &= \frac{1}{2}S(\psi_1) + \frac{1}{2}S(\psi_2) \ . \end{aligned}$$

Thus

$$2F(\widetilde{\Phi}) = \frac{1}{2}(<\sigma>_{\psi_1})^2 + \frac{1}{2}(<\sigma>_{\psi_2})^2 + \left(h^1 + h^2\right) <\sigma>_{\psi_1} + \left(h^1 - h^2\right) <\sigma>_{\psi_2}$$

[1]When $p = 2$, $\mathcal{B}_j$ is a singleton.

$$+\beta^{-1}\left(S(\psi_1)+S(\psi_2)\right)\ .$$

For real h and $\beta>0$, let $g(\beta,h,\cdot)=g(\beta,h,\cdot\,;1)$ be the function on $[-1,1]$ defined by (2.12) for $J=1$. Then

$$F(\tilde{\Phi})=\frac{1}{2}g\left(\beta,h^1+h^2,<\sigma>_{\psi_1}\right)+\frac{1}{2}g\left(\beta,h^1-h^2,<\sigma>_{\psi_2}\right)\ . \tag{3.6}$$

We now compute $F(\Phi)$. The entropy part is

$$2^{-p}\sum_{\mathbf{x}\in\mathcal{A}_1}S(\phi_{\mathbf{x}})+2^{-p}\sum_{\mathbf{x}\in\mathcal{A}_1}S(\widehat{\phi_{\mathbf{x}}})=2^{-(p-1)}\sum_{\mathbf{x}\in\mathcal{A}_1}S(\phi_{\mathbf{x}})$$

$$=2^{-(p-1)}\left(\sum_{\mathbf{x}\in\mathcal{B}_1}S(\phi_{\mathbf{x}})+\sum_{\mathbf{x}\in\mathcal{B}_2}S(\phi_{\mathbf{x}})\right)$$

using reflexivity and $S(\widehat{\phi})=S(\phi)$.

Now, by reflexivity

$$m^\mu(\Phi)=2^{-p}\left(\sum_{\mathbf{x}\in\mathcal{A}_1}x^\mu<\sigma>_{\phi_{\mathbf{x}}}+\sum_{\mathbf{x}\in(-\mathcal{A}_1)}x^\mu<\sigma>_{\phi_{\mathbf{x}}}\right)=2^{-(p-1)}\sum_{\mathbf{x}\in\mathcal{A}_1}x^\mu<\sigma>_{\phi_{\mathbf{x}}}\ .$$

Using the partition $\mathcal{A}_1=\mathcal{B}_1\cup\mathcal{B}_2$, we get

$$m^1(\Phi)=2^{-(p-1)}\left(\sum_{\mathbf{x}\in\mathcal{B}_1}<\sigma>_{\phi_{\mathbf{x}}}+\sum_{\mathbf{x}\in\mathcal{B}_2}<\sigma>_{\phi_{\mathbf{x}}}\right)\ ;$$

$$m^2(\Phi)=2^{-(p-1)}\left(\sum_{\mathbf{x}\in\mathcal{B}_1}<\sigma>_{\phi_{\mathbf{x}}}-\sum_{\mathbf{x}\in\mathcal{B}_2}<\sigma>_{\phi_{\mathbf{x}}}\right)\ .$$

Thus,

$$F(\Phi)=2^{-1}2^{-2(p-1)}\sum_{\mu=1}^{p}\left(\sum_{\mathbf{x}\in\mathcal{A}_1}x^\mu<\sigma>_{\phi_{\mathbf{x}}}\right)^2$$

$$+\frac{h^1+h^2}{2}\sum_{\mathbf{x}\in\mathcal{B}_1}2^{-(p-2)}<\sigma>_{\phi_{\mathbf{x}}}+\frac{h^1-h^2}{2}\sum_{\mathbf{x}\in\mathcal{B}_2}2^{-(p-2)}<\sigma>_{\phi_{\mathbf{x}}}$$

$$+(2\beta)^{-1}\left(\sum_{\mathbf{x}\in\mathcal{B}_1}2^{-(p-2)}S(\phi_{\mathbf{x}})+\sum_{\mathbf{x}\in\mathcal{B}_2}2^{-(p-2)}S(\phi_{\mathbf{x}})\right)\ .$$

Upon using the Lemma of the appendix (with $\mathcal{A}=\mathcal{A}_1$) to estimate the sum of quadratic terms, we get

$$2^{-1}2^{-2(p-1)}\sum_{\mu=1}^{p}\left(\sum_{\mathbf{x}\in\mathcal{A}_1}x^\mu<\sigma>_{\phi_{\mathbf{x}}}\right)^2\ \le\ 2^{-1}2^{-(p-1)}\sum_{\mathbf{x}\in\mathcal{A}_1}\left(<\sigma>_{\phi_{\mathbf{x}}}\right)^2$$

$$=\frac{1}{2}\sum_{\mathbf{x}\in\mathcal{B}_1}2^{-(p-2)}\frac{1}{2}\left(<\sigma>_{\phi_{\mathbf{x}}}\right)^2+\frac{1}{2}\sum_{\mathbf{x}\in\mathcal{B}_2}2^{-(p-2)}\frac{1}{2}\left(<\sigma>_{\phi_{\mathbf{x}}}\right)^2\ .$$

Thus,

$$F(\Phi)\le\frac{1}{2}\sum_{\mathbf{x}\in\mathcal{B}_1}2^{-(p-2)}\left(\frac{1}{2}\left(<\sigma>_{\phi_{\mathbf{x}}}\right)^2+(h^1+h^2)<\sigma>_{\phi_{\mathbf{x}}}+\beta^{-1}S(\phi_{\mathbf{x}})\right)$$

$$+\frac{1}{2}\sum_{\mathbf{x}\in\mathcal{B}_2} 2^{-(p-2)}\left(\frac{1}{2}(<\sigma>_{\phi_{\mathbf{x}}})^2+(h^1-h^2)<\sigma>_{\phi_{\mathbf{x}}}+\beta^{-1}S(\phi_{\mathbf{x}})\right)$$

$$=\frac{1}{2}\sum_{\mathbf{x}\in\mathcal{B}_1} 2^{-(p-2)}g\left(\beta,h^1+h^2,<\sigma>_{\phi_{\mathbf{x}}}\right)\ +\ \frac{1}{2}\sum_{\mathbf{x}\in\mathcal{B}_2} 2^{-(p-2)}g\left(\beta,h^1-h^2,<\sigma>_{\phi_{\mathbf{x}}}\right)\ .$$

Our bound is then

$$F(\widetilde{\Phi})-F(\Phi)\geq\frac{1}{2}\left[g\left(\beta,h^1+h^2,<\sigma>_{\psi_1}\right)-2^{-(p-2)}\sum_{\mathbf{x}\in\mathcal{B}_1}g\left(\beta,h^1+h^2,<\sigma>_{\phi_{\mathbf{x}}}\right)\right]$$

(3.7)

$$+\frac{1}{2}\left[g\left(\beta,h^1-h^2,<\sigma>_{\psi_2}\right)-2^{-(p-2)}\sum_{\mathbf{x}\in\mathcal{B}_2}g\left(\beta,h^1-h^2,<\sigma>_{\phi_{\mathbf{x}}}\right)\right]\ .$$

We now choose the measures ρ_j in order to make the r.h.s. of this inequality as large as possible. By definition, the value $<\sigma>_{\psi_j}$ lies in the interval

$$\left[\min_{\mathbf{x}\in\mathcal{B}_j}\{<\sigma>_{\phi_{\mathbf{x}}}\},\max_{\mathbf{x}\in\mathcal{B}_j}\{<\sigma>_{\phi_{\mathbf{x}}}\}\right]\ ;$$

let t_j be a number in this interval that maximizes $g(\beta,h^1+h^2,\cdot)$ for $j=1$ and $g(\beta,h^1-h^2,\cdot)$ for $j=2$. Choose ρ_j such that $<\sigma>_{\psi_j}=t_j$. Then the averages over $\mathcal{B}_j$ in the r.h.s. of (3.7) are not greater than $g\left(\beta,h^1+h^2,t_1\right)=g\left(\beta,h^1+h^2,<\sigma>_{\psi_1}\right)$ and $g\left(\beta,h^1-h^2,t_2\right)=g\left(\beta,h^1-h^2,<\sigma>_{\psi_2}\right)$ respectively. For this choice of measures, the r.h.s. of (3.7) is non-negative and as large as possible.

Third step: By the first and second steps, we conclude that if λ is uniform and only the components h^{μ_1} and h^{μ_2} of $\mathbf{h}$ can be non-zero; then the maximum of F over Γ_λ is equal to the maximum of F over the reflexive families which are constant on $\mathcal{B}_1=\mathcal{A}_{\mu_1}\cap\mathcal{A}_{\mu_2}$ and on $\mathcal{B}_2=\mathcal{A}_{\mu_1}\cap(-\mathcal{A}_{\mu_2})$. Thus, denoting by ψ_j the values on $\mathcal{B}_j$, using (3.6), and $\mathcal{B}_1\cap\mathcal{B}_2=\emptyset$, we get

$$f(\beta,\mathbf{h})=\max_{\psi_1\,,\psi_2\in\mathcal{M}(\mathcal{S})}\left\{\frac{1}{2}g\left(\beta,h^{\mu_1}+h^{\mu_2},<\sigma>_{\psi_1}\right)+\frac{1}{2}g\left(\beta,h^{\mu_1}-h^{\mu_2},<\sigma>_{\psi_2}\right)\right\}$$

$$=\frac{1}{2}\max_{\psi\in\mathcal{M}(\mathcal{S})}g\left(\beta,h^{\mu_1}+h^{\mu_2},<\sigma>_{\psi}\right)+\frac{1}{2}\max_{\psi\in\mathcal{M}(\mathcal{S})}g\left(\beta,h^{\mu_1}-h^{\mu_2},<\sigma>_{\psi}\right)\ ;$$

but the respective maxima are equal to $f^{cw}(\beta,h^{\mu_1}\pm h^{\mu_2};1)$.

It is known that the value of t in $[-1,1]$ maximizing $g(\beta,h,\cdot)$ is unique if $h\neq 0$ or if $\beta\leq 1$; we denote it by $t(\beta,h)$. It thus follows that a maximizing reflexive family which is constant on $\mathcal{B}_1$ and on $\mathcal{B}_2$ is unique if $\mid h^{\mu_1}\mid\neq\mid h^{\mu_2}\mid$ or if $\beta\leq 1$; denote this family by Φ^o. Now suppose that Φ is a maximizing family; by the Lemma, Φ must be reflexive. Consider the uniformization $\widetilde{\Phi}$ on $\mathcal{B}_j$ of Φ constructed in the second step. We have:

$$F(\Phi)\ \geq\ F(\widetilde{\Phi})\ \geq\ F(\Phi)$$

$$+\frac{1}{2}\left[g\left(\beta,h^{\mu_1}+h^{\mu_2},<\sigma>_{\psi_1}\right)-2^{-(p-2)}\sum_{\mathbf{x}\in\mathcal{B}_1}g\left(\beta,h^{\mu_1}+h^{\mu_2},<\sigma>_{\phi_{\mathbf{x}}}\right)\right]$$

$$+\frac{1}{2}\left[g\left(\beta,h^{\mu_1}-h^{\mu_2},<\sigma>_{\psi_2}\right)-2^{-(p-2)}\sum_{\mathbf{x}\in\mathcal{B}_2}g\left(\beta,h^{\mu_1}-h^{\mu_2},<\sigma>_{\phi_{\mathbf{x}}}\right)\right]\ ,$$

where both corrections in $[\cdot]$ on the r.h.s. are non-negative. It thus follows that $F(\Phi) = F(\widetilde{\Phi})$, that $\widetilde{\Phi} = \Phi^o$, and that the corrections are zero. From the latter fact, we conclude (the argument is made explicit below) that Φ is constant on $\mathcal{B}_j$ $(j = 1, 2)$, and thus $\Phi = \Phi^o$.

It is known that there are two values of t in $[-1, 1]$ maximizing $g(\beta, h, \cdot)$ when $\beta > 1$ and $h = 0$; we denote them by $t(\beta, 0^{\pm})$. Suppose that $h^{\mu_1} = h^{\mu_2} \neq 0$, and let Φ be a maximizing family which must be reflexive. The above inequality now implies that the correction term is zero, i.e.

$$g\left(\beta, 2h^{\mu_1}, <\sigma>_{\psi_1}\right) - 2^{-(p-2)} \sum_{\mathbf{x}\in\mathcal{B}_1} g\left(\beta, 2h^{\mu_1}, <\sigma>_{\phi_{\mathbf{x}}}\right) = 0 \ ,$$

$$g\left(\beta, 0, <\sigma>_{\psi_2}\right) - 2^{-(p-2)} \sum_{\mathbf{x}\in\mathcal{B}_2} g\left(\beta, 0, <\sigma>_{\phi_{\mathbf{x}}}\right) = 0 \ ;$$

and that $<\sigma>_{\psi_1} = t(\beta, 2h^{\mu_1})$ and $<\sigma>_{\psi_2} = t(\beta, 0^{\pm})$. From the first identity we conclude that $<\sigma>_{\phi_{\mathbf{x}}}$ is constant on $\mathcal{B}_1$ and equal to $t(\beta, 2h^{\mu_1})$; this then determines uniquely the value of Φ (and $\widetilde{\Phi}$) on $\mathcal{B}_1$. From the second identity we conclude that on $\mathcal{B}_2$, $<\sigma>_{\phi_{\mathbf{x}}}$ takes any of the two values $t(\beta, 0^{\pm})$. Since there are 2^{p-2} elements in $\mathcal{B}_2$, there are $2^{2^{(p-2)}}$ possibilities[2]. Going through the other cases $h^{\mu_1} = -h^{\mu_2} \neq 0$, and $h^{\mu_1} = h^{\mu_2} = 0$ one obtains the claims of 2. of the Proposition.

Fourth step: The rest of the claims of Proposition 4 follow from: 1) the (familiar) solution of the variational problem $\max\{g(\beta, h, t) : \ t \in [-1, 1]\}$; 2) the known differentiability properties of $h \mapsto f^{cw}(\beta, h; J)$ and of $\beta \mapsto f^{cw}(\beta, h; J)$; 3) Eq. (2.1) and the fact that the limit of derivatives of a sequence of differentiable convex functions exists and is equal to the derivative of the limiting function when this function exists and is differentiable; and 4) equations (3.3) and (3.4).

This completes the proof of Proposition 4.

We now give a proof of (2.15). We assume λ is uniform and $p = 3$. By Lemma 2, it suffices to consider reflexive families. A reflexive family Φ is specified by the values it takes on the four one-point sets $\mathcal{B}_1 = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$, $\mathcal{B}_2 = \mathcal{A}_1 \cap \mathcal{A}_2 \cap (-\mathcal{A}_3)$, $\mathcal{B}_3 = \mathcal{A}_1 \cap (-\mathcal{A}_2) \cap \mathcal{A}_3$, and $\mathcal{B}_4 = \mathcal{A}_1 \cap (-\mathcal{A}_2) \cap (-\mathcal{A}_3)$. To shorten the notation, write $<j> = <\sigma>_{\phi_j}$, where ϕ_j is the value of Φ on $\mathcal{B}_j$. We compute

$$m^1(\Phi) = \frac{1}{4}\left(<1> + <2> + <3> + <4>\right) \ ;$$

$$m^2(\Phi) = \frac{1}{4}\left(<1> + <2> - <3> - <4>\right) \ ;$$

$$m^3(\Phi) = \frac{1}{4}\left(<1> - <2> + <3> - <4>\right) \ ;$$

to obtain

$$F(\Phi) = \frac{1}{4} g(\beta, h^1 + h^2 + h^3, <1>) + \frac{1}{4} g(\beta, h^1 + h^2 - h^3, <2>)$$
$$+ \frac{1}{4} g(\beta, h^1 - h^2 + h^3, <3>) + \frac{1}{4} g(\beta, h^1 - h^2 - h^3, <4>)$$
$$- \frac{1}{32}\left(<1> - <2> - <3> + <4>\right)^2 \ .$$

[2]In particular, Φ is constant on $\mathcal{B}_2$ only in two of these possibilities.

Writing $K(\Phi) = \frac{1}{32}\left(<1> - <2> - <3> + <4>\right)^2$, we then have

$$f(\beta, \mathbf{h}) = \max_{\Phi \text{ reflexive}} F(\Phi)$$

$$\leq \frac{1}{4} f^{cw}(\beta, h^1 + h^2 + h^3; 1) + \frac{1}{4} f^{cw}(\beta, h^1 + h^2 - h^3; 1)$$

$$+ \frac{1}{4} f^{cw}(\beta, h^1 - h^2 + h^3; 1) + \frac{1}{4} f^{cw}(\beta, h^1 - h^2 - h^3; 1)$$

$$= 2^{-3} \sum_{\mathbf{x} \in \mathcal{X}} f^{cw}(\beta, \langle \mathbf{h}, \mathbf{x} \rangle; 1) \ .$$

Due to $K(\Phi) \geq 0$, and the fact that $f^{cw}(\beta, \cdot; 1)$ is even. There is equality iff for a maximizing family Φ^o, one has $K(\Phi^o) = 0$, and $< i >$ is equal to the corresponding Curie-Weiß magnetization. In terms of the Curie-Weiß magnetizations, $K(\Phi^o) = 0$ iff

$$m^{cw}(\beta, h^1 + h^2 + h^3) - m^{cw}(\beta, h^1 + h^2 - h^3) - m^{cw}(\beta, h^1 - h^2 + h^3) + m^{cw}(\beta, h^1 - h^2 - h^3) = 0 \ .$$

This is certainly satisfied if one or more of the components of $\mathbf{h}$ is zero; it is also satisfied for certain particular non-zero values of the three components. But in general the expression is not zero.

Proof of 3. of Proposition 3: The upper bound follows by using the Lemma of the appendix in the case $\mathcal{A} = \mathcal{X}$ to estimate $\| \mathbf{m}(\Phi) \|^2$ for any family Φ; one gets

$$F(\Phi) \leq 2^{-1} 2^p \sum_{\mathbf{x} \in \Lambda} \lambda(\{\mathbf{x}\})^2 \left(< \sigma >_{\phi_{\mathbf{x}}}\right)^2 + \sum_{\mathbf{x} \in \Lambda} \lambda(\{\mathbf{x}\}) \langle \mathbf{h}, \mathbf{x} \rangle < \sigma >_{\phi_{\mathbf{x}}}$$

$$+ \beta^{-1} \sum_{\mathbf{x} \in \Lambda} \lambda(\{\mathbf{x}\}) \eta \left(\frac{1 + < \sigma >_{\phi_{\mathbf{x}}}}{2} \right)$$

$$= \sum_{\mathbf{x} \in \Lambda} \lambda(\{\mathbf{x}\}) g\left(\beta, \langle \mathbf{h}, \mathbf{x} \rangle, < \sigma >_{\phi_{\mathbf{x}}}; 2^p \lambda(\{\mathbf{x}\})\right) \ .$$

Thus, the upper bound follows by the Theorem.

The best lower bound on f is obtained by computing F for any family Φ which is reflexive and constant on $\mathcal{A}_\mu$ where it takes the value ϕ; one has:

$$m^\nu(\Phi) = < \sigma >_\phi J^\nu(\mu) \ ;$$

and thus $F(\Phi) = g\left(\beta, \langle \mathbf{J}(\mu), \mathbf{h} \rangle, < \sigma >_\phi; \| \mathbf{J}(\mu) \|^2\right)$. Thus as before

$$f(\beta, \mathbf{h}) \geq f^{cw}\left(\beta, \langle \mathbf{J}(\mu), \mathbf{h} \rangle; \| \mathbf{J}(\mu) \|^2\right) \ .$$

The claim follows by taking the maximum w.r.t. μ.

This completes the proof of 3. of Proposition 3.

Proof of 4. of Proposition 3: This proceeds by applying the contraction principle. Define a map T on Γ_λ by:

$$< \sigma >_{T(\Phi)_{\mathbf{x}}} = \tanh\left(\beta \langle \mathbf{m}(\Phi) + \mathbf{h}, \mathbf{x} \rangle\right) = \tanh\left(\beta \langle \mathbf{h}, \mathbf{x} \rangle + \beta \sum_{\mathbf{y} \in \Lambda} \lambda(\mathbf{y}) \langle \mathbf{x}, \mathbf{y} \rangle < \sigma >_{\phi_{\mathbf{y}}}\right)$$

recalling that a probability measure on $\mathcal{S}$ is uniquely determined by the expectation of σ. T maps Γ_λ into itself because the range of tanh is $(-1, 1)$.

We have

$$|<\sigma>_{T(\Phi)_{\mathbf{x}}} - <\sigma>_{T(\Psi)_{\mathbf{x}}}| \leq | \int_{\beta\langle \mathbf{m}(\Psi)+\mathbf{h},\mathbf{x}\rangle}^{\beta\langle \mathbf{m}(\Phi)+\mathbf{h},\mathbf{x}\rangle} \cosh(u)^{-2} du \ |$$

$$\leq \beta \sum_{\mathbf{y}} \lambda(\mathbf{y}) \ | \ \langle \mathbf{x},\mathbf{y}\rangle (<\sigma>_{\Phi_{\mathbf{y}}} - <\sigma>_{\Psi_{\mathbf{y}}}) \ | \leq \ \beta p \, d(\Phi,\Psi) \ ,$$

with strict inequality for $\Phi \neq \Psi$ because at least one of the intervals of integration is not empty and on this interval the cosh appearing in the integrand is equal to 1 for finitely many points. By taking the supremum over Λ, we conclude that

$$d(T(\Phi),T(\Psi)) \leq \beta p \, d(\Phi,\Psi) \tag{3.8}$$

with equality iff $\Phi = \Psi$. Thus (Ref. [19]; Problem 1, p.267) the gap equations have a unique solution Φ^o when $\beta p \leq 1$. One also concludes that

$$\Phi^o = \lim_{n\to\infty} T^n(\Phi) \tag{3.9}$$

for every $\Phi \in \Gamma_\lambda$.

By Proposition 1, the unique solution of (2.5) existing for $\beta p \leq 1$ gives a maximum of F. Thus, (2.4) has a unique maximizing family Φ^o for $\beta p \leq 1$[3]

If $\Omega \in \Gamma_\lambda$ denotes the constant family on Λ with value ω then $\mathbf{m}(\Omega) = \mathbf{0}$, so that Ω is indeed the only maximizer for $\mathbf{h} = \mathbf{0}$. The formula for $f(\beta, \mathbf{0})$ follows by evaluating $F(\Omega)$ using $S(\omega) = \ln(2)$.

To prove differentiability of $f(\beta,\cdot)$ we will have to work with different $\mathbf{h}$'s and thus use an index $\mathbf{h}$ to distinguish the various quantities. The same estimate leads to

$$|<\sigma>_{T_{\mathbf{h}}(\Phi)_{\mathbf{x}}} - <\sigma>_{T_{\mathbf{k}}(\Phi)_{\mathbf{x}}}| \leq \beta p \parallel \mathbf{h} - \mathbf{k} \parallel_\infty$$

where $\parallel \mathbf{h} \parallel_\infty = \max_\mu | h^\mu |$, and the inequality is strict as soon as $\mathbf{h} \neq \mathbf{k}$. Thus

$$d(T_{\mathbf{h}}(\Phi),T_{\mathbf{k}}(\Phi)) \leq \beta p \ \parallel \mathbf{h} - \mathbf{k} \parallel_\infty \ . \tag{3.10}$$

By the triangle inequality, (3.8), and (3.10), we have

$$d(T_{\mathbf{h}}(\Phi),T_{\mathbf{k}}(\Psi)) \leq \beta p \ \parallel \mathbf{h} - \mathbf{k} \parallel_\infty + \beta p d(\Phi,\Psi) \ . \tag{3.11}$$

We write $\Phi^o(\mathbf{h})$ for the unique maximizer corresponding to the field vector $\mathbf{h}$. For any integer n, and family Φ we have by the triangle inequality,

$$d(\Phi^o(\mathbf{h}),\Phi^o(\mathbf{k})) \leq d(\Phi^o(\mathbf{h}),T^n_{\mathbf{h}}(\Phi)) + d(T^n_{\mathbf{k}}(\Phi),\Phi^o(\mathbf{k})) + d(T^n_{\mathbf{h}}(\Phi),T^n_{\mathbf{k}}(\Phi))$$

Using (3.11), to estimate the last summand on the r.h.s., we get

$$d(\Phi^o(\mathbf{h}),\Phi^o(\mathbf{k})) \leq d(\Phi^o(\mathbf{h}),T^n_{\mathbf{h}}(\Phi)) + d(T^n_{\mathbf{k}}(\Phi),\Phi^o(\mathbf{k})) + \frac{\beta p(1-\beta^n p^n)}{1-\beta p} \parallel \mathbf{h} - \mathbf{k} \parallel_\infty \ .$$

[3] An alternative direct proof of this statement proceeds as follows. The Hessian (2.9) of F is $B - \beta^{-1}D$ with $B_{\mathbf{x},\mathbf{y}} = \lambda(\mathbf{x})\lambda(\mathbf{y})\langle \mathbf{x},\mathbf{y}\rangle$, and $D_{\mathbf{x},\mathbf{y}} = \delta_{\mathbf{x},\mathbf{y}}\lambda(\mathbf{x})(1- <\sigma>^2_{\phi_{\mathbf{x}}})^{-1} \ \geq \ \delta_{\mathbf{x},\mathbf{y}}\lambda(\mathbf{x}) \overset{def}{=} D^o_{\mathbf{x},\mathbf{y}}$. It follows that the quadratic form associated with the Hessian is bounded above by the quadratic form associated to $B - \beta^{-1}D^o$ which is independent of Φ. Moreover, for β sufficently small, the matrix $D^o - \beta B$ is positive definite. Thus for β sufficently small, the map $\{<\sigma>_{\phi_{\mathbf{x}}}\} \mapsto F(\Phi)$ is strictly concave. It follows that $\Gamma_\lambda \ni \Phi \mapsto F(\Phi)$ is strictly concave and there is a unique maximizing family.

Upon taking the limit $n \to \infty$ using (3.9), we conclude that

$$d(\Phi^o(\mathbf{h}), \Phi^o(\mathbf{k})) \leq \frac{\beta p}{1-\beta p} \parallel \mathbf{h} - \mathbf{k} \parallel_\infty . \tag{3.12}$$

A direct estimate on (2.3) using (3.12) now produces

$$\parallel \mathbf{m}(\Phi^o(\mathbf{h})) - \mathbf{m}(\Phi^o(\mathbf{k})) \parallel \leq \frac{\beta p^{3/2}}{1-\beta p} \parallel \mathbf{h} - \mathbf{k} \parallel_\infty . \tag{3.13}$$

We can now prove differentiability of $f(\beta, \cdot)$ as follows. Let $\mathbf{e}$ be an arbitrary unit vector in $\mathrm{I\!R}^p$. By virtue of convexity of $f(\beta, \cdot)$, the left and right derivatives $(f_{\mathbf{e}})_\pm$ in direction $\mathbf{e}$ exist and satisfy

$$\epsilon^{-1}[f(\beta, \mathbf{h}) - f(\beta, \mathbf{h} - \epsilon\mathbf{e})] \leq (f_{\mathbf{e}})_- \leq (f_{\mathbf{e}})_+ \leq \epsilon^{-1}[f(\beta, \mathbf{h} + \epsilon\mathbf{e}) - f(\beta, \mathbf{h})] \tag{3.14}$$

for all $\epsilon > 0$. Since $\Phi^o(\mathbf{k})$ is the (unique) maximizer of $F_{\mathbf{k}}$, we can estimate the r.h.s. of this inequality

$$\begin{aligned}
\epsilon^{-1}[f(\beta, \mathbf{h} + \epsilon\mathbf{e}) - f(\beta, \mathbf{h})] &= \epsilon^{-1}[F_{\mathbf{h}+\epsilon\mathbf{e}}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e})) - F_{\mathbf{h}}(\Phi^o(\mathbf{h}))] \\
&\leq \epsilon^{-1}[F_{\mathbf{h}+\epsilon\mathbf{e}}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e})) - F_{\mathbf{h}}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e}))] \\
= \epsilon^{-1}[\langle \mathbf{h} + \epsilon\mathbf{e}, \mathbf{m}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e}))\rangle - \langle \mathbf{h}, \mathbf{m}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e}))\rangle] &= \langle \mathbf{e}, \mathbf{m}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e}))\rangle .
\end{aligned}$$

Proceeding analogously with the l.h.s. of (3.14), we get

$$\epsilon^{-1}[f(\beta, \mathbf{h}) - f(\beta, \mathbf{h} - \epsilon\mathbf{e})] \geq \langle \mathbf{e}, \mathbf{m}(\Phi^o(\mathbf{h} - \epsilon\mathbf{e}))\rangle .$$

Thus,

$$\langle \mathbf{e}, \mathbf{m}(\Phi^o(\mathbf{h} - \epsilon\mathbf{e}))\rangle \leq (f_{\mathbf{e}})_- \leq (f_{\mathbf{e}})_+ \leq \langle \mathbf{e}, \mathbf{m}(\Phi^o(\mathbf{h} + \epsilon\mathbf{e}))\rangle ,$$

and $(f_{\mathbf{e}})_- = (f_{\mathbf{e}})_+ = \langle \mathbf{e}, \mathbf{m}(\Phi^o(\mathbf{h}))\rangle$ follows upon taking the limit $\epsilon \downarrow 0$ using (3.13).

The claim about the limit of the overlap parameter vector follows by the same arguments as those used in the fourth step of the proof of Proposition 4.

The proof of differentiability of $f(\cdot, \mathbf{h})$ is analogous. By convexity in β,

$$-\beta^{-1}(\beta-\epsilon)^{-1}S_\lambda(\Phi^o_{\beta-\epsilon}) \leq (\frac{\partial f}{\partial \beta})_-(\beta, \mathbf{h}) \leq (\frac{\partial f}{\partial \beta})_+(\beta, \mathbf{h}) \leq -\beta^{-1}(\beta+\epsilon)^{-1}S_\lambda(\Phi^o_{\beta+\epsilon}) .$$

Now $S_\lambda(\cdot)$ is continuous, and the continuity of Φ^o_β in β follows from the anlogues of (3.10), (3.11), and (3.12):

$$d(T_\beta(\Phi), T_{\beta'}(\Phi)) \leq p \mid \beta - \beta' \mid (1 + \parallel \mathbf{h} \parallel_\infty) ;$$

$$d(T_\beta(\Phi), T_{\beta'}(\Psi)) \leq p \mid \beta - \beta' \mid (1 + \parallel \mathbf{h} \parallel_\infty) + \beta' p d(\Phi, \Psi) ;$$

$$d(\Phi^o(\beta), \Phi^o(\beta')) \leq \frac{p\,(1 + \parallel \mathbf{h} \parallel_\infty)}{1-\beta p} \mid \beta - \beta' \mid .$$

Proof of 5. of Proposition 3: Since $f(\beta, \mathbf{h}) \geq \beta^{-1}\ln(2)$ by 3. of the proposition, the first claim follows from the non-decreasing property of $\beta \mapsto f(\beta, \mathbf{0})$ (1. of the proposition). The rest of the claims follow from 1. and 4. of the proposition and continuity of $f(\cdot, \mathbf{h})$.

4 The case of two patterns

If $p = 2$ then $\mathcal{X} = \{(+,+),(+,-),(-,+),(-,-)\}$ in obvious notation. A complete detailed description can be given because the $2^p = 4$ equations (2.5) decouple. The limiting negative free-energy density can be described in terms of $f^{cw}(\beta, h; J)$ the limiting negative free energy density at reciprocal temperature β of a Curie-Weiß model with interaction constant $J \geq 0$ and external field h given by (2.11). We write a generic $m^{cw}(\beta, h; J)$ for the Curie-Weiß magnetization, but when $h = 0$ we distinguish

$$m^{cw}(\beta, 0^{\pm}; J) = \begin{cases} 0 & , \quad \text{when } \beta J \leq 1 \\ \text{spontaneous magnetization of sign } \pm & , \quad \text{when } \beta J > 1 \end{cases} .$$

We write accordingly $\phi^{cw}(\beta, h; J)$ for the unique probability measure on $\mathcal{S}$ whose spin-expectation is $m^{cw}(\beta, h; J)$.

The dependence on the measure λ enters only through the parameters

$$a = \lambda(\{(+,+)\}) + \lambda(\{(-,-)\}) \ ; \quad b = \lambda(\{(+,-)\}) + \lambda(\{(-,+)\}) .$$

After a lenghty but straightforward discussion of the solutions of (2.5) using the knowledge about the Curie-Weiß model, we obtain the following results:

$$f(\beta, \mathbf{h}) = a f^{cw}(\beta, h^1 + h^2; 2a) + b f^{cw}(\beta, h^1 - h^2; 2b) \ ;$$

this is differentiable in the whole (h^1, h^2)-plane except for the two lines $|\ h^1\ | = |\ h^2\ |$ when $2\beta a > 1$ (for the line $h^2 = -h^1$) or $2\beta b > 1$ (for the line $h^2 = h^1$)[4]. One has $(\nabla f)(\beta, \mathbf{h}) = \mathbf{m}(\beta, \mathbf{h})$ with

$$m^1(\beta, \mathbf{h}) = a\, m^{cw}(\beta, h^1 + h^2; 2a) + b\, m^{cw}(\beta, h^1 - h^2; 2b) \ ;$$

$$m^2(\beta, \mathbf{h}) = a\, m^{cw}(\beta, h^1 + h^2; 2a) - b\, m^{cw}(\beta, h^1 - h^2; 2b) .$$

The families maximizing (2.4) can be described completely. When $\mathbf{h} = \mathbf{0}$, and $\beta \leq \min\{(2a)^{-1}, (2b)^{-1}\}$ the maximizing family is unique and $\phi^o_{\mathbf{x}}$ is the uniform probability measure.

When $\mathbf{h} = \mathbf{0}$, and $\beta > \max\{(2a)^{-1}, (2b)^{-1}\}$ there are four maximizing families

$$\phi^{(1)}_{(++)} = \phi^{cw}(\beta, 0^+; 2a) \ , \quad \phi^{(1)}_{(--)} = \phi^{cw}(\beta, 0^-; 2a) \ ,$$

$$\phi^{(1)}_{(+-)} = \phi^{cw}(\beta, 0^+; 2b) \ , \quad \phi^{(1)}_{(-+)} = \phi^{cw}(\beta, 0^-; 2b) \ ;$$

$$\phi^{(2)}_{(++)} = \phi^{cw}(\beta, 0^+; 2a) \ , \quad \phi^{(2)}_{(--)} = \phi^{cw}(\beta, 0^-; 2a) \ ,$$

$$\phi^{(2)}_{(+-)} = \phi^{cw}(\beta, 0^-; 2b) \ , \quad \phi^{(2)}_{(-+)} = \phi^{cw}(\beta, 0^+; 2b) \ ;$$

$$\phi^{(3)}_{(++)} = \phi^{cw}(\beta, 0^-; 2a) \ , \quad \phi^{(3)}_{(--)} = \phi^{cw}(\beta, 0^+; 2a) \ ,$$

$$\phi^{(3)}_{(+-)} = \phi^{cw}(\beta, 0^+; 2b) \ , \quad \phi^{(3)}_{(-+)} = \phi^{cw}(\beta, 0^-; 2b) \ ;$$

$$\phi^{(4)}_{(++)} = \phi^{cw}(\beta, 0^-; 2a) \ , \quad \phi^{(4)}_{(--)} = \phi^{cw}(\beta, 0^+; 2a) \ ,$$

[4]Notice that the critical values $\beta = (1/2a)$ and $\beta = (1/2b)$ of β are in the interval $(\frac{1}{2}, \infty)$. The bound $\beta = p^{-1}$ of Proposition 3 is attained for the degenerate cases $a = 1$ or $b = 1$.

$$\phi^{(4)}_{(+-)} = \phi^{cw}(\beta, 0^-; 2b) \ , \quad \phi^{(4)}_{(-+)} = \phi^{cw}(\beta, 0^+; 2b) \ ;$$

The associated $\mathbf{m}^{(j)} = \mathbf{m}(\Phi^{(j)})$ are:

$$\mathbf{m}^{(1)} = \Big(am^{cw}(\beta, 0^+; 2a) + bm^{cw}(\beta, 0^+; 2b) \ , \ am^{cw}(\beta, 0^+; 2a) + bm^{cw}(\beta, 0^-; 2b) \Big) \ ;$$

$$\mathbf{m}^{(2)} = \Big(am^{cw}(\beta, 0^+; 2a) + bm^{cw}(\beta, 0^-; 2b) \ , \ am^{cw}(\beta, 0^+; 2a) + bm^{cw}(\beta, 0^+; 2b) \Big) \ ;$$

$$\mathbf{m}^{(3)} = \Big(am^{cw}(\beta, 0^-; 2a) + bm^{cw}(\beta, 0^+; 2b) \ , \ am^{cw}(\beta, 0^-; 2a) + bm^{cw}(\beta, 0^-; 2b) \Big) \ ;$$

$$\mathbf{m}^{(4)} = \Big(am^{cw}(\beta, 0^-; 2a) + bm^{cw}(\beta, 0^-; 2b) \ , \ am^{cw}(\beta, 0^-; 2a) + bm^{cw}(\beta, 0^+; 2b) \Big) \ .$$

When $\mathbf{h} = \mathbf{0}$, and $\min\{(2a)^{-1}, (2b)^{-1}\} < \beta \le \max\{(2a)^{-1}, (2b)^{-1}\}$ there are two maximizing families: $\phi^{(1)} \equiv \phi^{(2)}$ and $\phi^{(3)} \equiv \phi^{(4)}$ when $a > b$; $\phi^{(1)} \equiv \phi^{(3)}$ and $\phi^{(2)} \equiv \phi^{(4)}$, for $b > a$. Accordingly, for $a > b$,

$$\mathbf{m}^{(1)} = \mathbf{m}^{(2)} = -\mathbf{m}^{(3)} = -\mathbf{m}^{(4)} = am^{cw}(\beta, 0^+; 2a)\,(1 \ , \ 1) \ ,$$

and for $b > a$

$$\mathbf{m}^{(1)} = \mathbf{m}^{(3)} = -\mathbf{m}^{(2)} = -\mathbf{m}^{(4)} = bm^{cw}(\beta, 0^+; 2b)\,(1 \ , \ -1) \ ,$$

Notice that in the case $a > b$ (resp. $b > a$) the two components of the overlap are equal (resp. the negative of each other).

When $| h^1 | \neq | h^2 |$ or if $\beta \le \min\{(2a)^{-1}, (2b)^{-1}\}$, the family Φ^o where the supremum in (2.4) is attained is unique:

$$\phi^{o}_{(++)} = \phi^{cw}(\beta, h^1 + h^2; 2a) \ , \quad \phi^{o}_{(--)} = \phi^{cw}(\beta, -h^1 - h^2; 2a) \ ,$$

$$\phi^{o}_{(+-)} = \phi^{cw}(\beta, h^1 - h^2; 2b) \ , \quad \phi^{o}_{(-+)} = \phi^{cw}(\beta, -h^1 + h^2; 2b) \ .$$

If $a > b$, $h^1 = -h^2 \neq 0$, and $(2a)^{-1} < \beta \le (2b)^{-1}$ there are two maximizing families:

$$\phi^{1}_{(++)} = \phi^{cw}(\beta, 0^+; 2a) \ , \quad \phi^{1}_{(--)} = \phi^{cw}(\beta, 0^-; 2a) \ ,$$

$$\phi^{1}_{(+-)} = \phi^{cw}(\beta, 2h^1; 2b) \ , \quad \phi^{1}_{(-+)} = \phi^{cw}(\beta, -2h^1; 2b) \ ;$$

$$\phi^{2}_{(++)} = \phi^{cw}(\beta, 0^-; 2a) \ , \quad \phi^{2}_{(--)} = \phi^{cw}(\beta, 0^+; 2a) \ ,$$

$$\phi^{2}_{(+-)} = \phi^{cw}(\beta, 2h^1; 2b) \ , \quad \phi^{2}_{(-+)} = \phi^{cw}(\beta, -2h^1; 2b) \ .$$

If $a < b$, $h^1 = h^2 \neq 0$, and $(2b)^{-1} < \beta \le (2a)^{-1}$ there are two maximizing families:

$$\phi^{1}_{(++)} = \phi^{cw}(\beta, 2h^1; 2a) \ , \quad \phi^{1}_{(--)} = \phi^{cw}(\beta, -2h^1; 2a) \ ,$$

$$\phi^{1}_{(+-)} = \phi^{cw}(\beta, 0^+; 2b) \ , \quad \phi^{1}_{(-+)} = \phi^{cw}(\beta, 0^-; 2b) \ ;$$

$$\phi^{2}_{(++)} = \phi^{cw}(\beta, 2h^1; 2a) \ , \quad \phi^{2}_{(--)} = \phi^{cw}(\beta, -2h^1; 2a) \ ,$$

$$\phi^{2}_{(+-)} = \phi^{cw}(\beta, 0^-; 2b) \ , \quad \phi^{2}_{(-+)} = \phi^{cw}(\beta, 0^+; 2b) \ .$$

We now consider the particular case where the ξ_i^μ are i.i.d. One then has an asymmetry parameter $\epsilon \in [-1/2, 1/2]$, such that

(4.1) $$Prob\{\xi_i^\mu = \pm 1\} = \frac{1}{2} \pm \epsilon \ .$$

In this case the characteristic parameters a and b determining the measure λ are given by

$$a = \frac{1}{2} + 2\epsilon^2 \quad , \quad b = \frac{1}{2} - 2\epsilon^2 \; .$$

At zero field $\mathbf{h} = \mathbf{0}$, and for $\epsilon \neq 0$, one has two distinct critical temperatures

$$T_c = (\beta_c)^{-1} = 1 + 4\epsilon^2 \quad , \quad T_* = (\beta_*)^{-1} = 1 - 4\epsilon^2 \; ,$$

and three distinct regimes:

1. *The homogeneous zero overlap regime*: For $T \geq T_c$, one has $f(\beta, \mathbf{0}) = \beta^{-1}\ln(2)$ and there is a unique maximizing family. The associated overlap parameter is $\mathbf{m} = \mathbf{0}$.

2. *The homogeneous non-zero overlap regime*: For $T_* \leq T < T_c$ one has $f(\beta, \mathbf{0}) = af^{cw}(\beta, 0; 2a) + b\beta^{-1}\ln(2)$, there are two maximizing families Φ^1 and Φ^2. The two associated overlap vectors $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ are the negative of each other ($-\mathbf{m}^{(1)} = \mathbf{m}^{(2)}$), and have equal non-zero components ($(m^{(j)})^1 = (m^{(j)})^2$).

3. *The inhomogeneous overlap regime*: For $T < T_*$, there are four maximizing families and associated to them four distinct overlap vectors. Each of these vectors has the property that its two components are unequal.

Thus, T_c is the transition temperature for the regime of zero overlap to non-zero overlap, and T_* ($< T_c$) is the transition temperature from the regime of homogeneous overlap (i.e., the two components of the overlap vectors are equal) to the non-homogeneous overlap (i.e., the two components of the overlap vectors are distinct). The "phase diagramm" is given in Fig. 2.

5 The general i.i.d. case

In the previous section dealing with $p = 2$ patterns and the general i.i.d. case, we have identified two critical temperatures $T_c > T_*$ which separate three different regimes. In this last section we collect the scanty information that we presently have for arbitrary finite p. Ref. [20] describes results obtained (via the methods of Amit et al.) assuming pair-correlated patterns; here also three distinct regimes appear[5].

We suppose that the ξ_i^μ are i.i.d. with distribution (4.1), and we consider the case $\mathbf{h} = \mathbf{0}$ omitting it from the notation. Since the Hamiltonian(1.1) is invariant under a change of sign of each ξ_i^μ, we conclude that $f(\beta)$ is an even function of ϵ. In view of 5. Proposition 3, we have

$$\beta_c = \sup\{\beta > 0 : f(\beta) = \beta^{-1}\ln(2)\} \; .$$

Alternatively, β_c is the supremum over the set of positive β such that F_β admits a unique maximizer Φ^o (which is then the uniform family assigning $<\sigma>= 0$ at each $\mathbf{x} \in \mathcal{X}$). The following upper bound is exact for every ϵ when $p = 2$; and for every p when $\epsilon = 0, \pm(1/2)$.

Proposition 5 $\beta_c^{-1} = T_c \geq 1 + 4\epsilon^2(p-1)$.

[5] A numerical study of the critical points of the functional F^{AGS} has been performed by Fontanari and Theumann. We only have a conference report [21] available and not their paper in the Journal de Physique. They distinguish a number of metastable "phases" below T_c arising from local maxima of F^{AGS}.

Proof: Notice that for $\epsilon = 0$ we already have proved $T_c = 1$. So we assume that $\epsilon \neq 0$.

For $\mathbf{x} \in \mathcal{X}$, let $r(\mathbf{x}) = (1/2)(\sum_{\mu=1}^{p} x^\mu + p)$ which is just the number of $+1$ in $\mathbf{x}$, so that $r(\mathbf{x}) \in \{0, 1, \cdots, p\}$. For $r = 0, 1, \cdots, p$ consider the subsets $\mathcal{X}_r$ of $\mathcal{X}$ of those $\mathbf{x}$ with $r(\mathbf{x}) = r$:

$$\mathcal{X}_r = \{\mathbf{x} \in \mathcal{X} : \sum_{\mu=1}^{p} x^\mu = 2r - p\} .$$

Then $\mid \mathcal{X}_r \mid = \begin{pmatrix} p \\ r \end{pmatrix}$, and

$$\lambda(\mathbf{x}) = (\frac{1}{2} + \epsilon)^{r(\mathbf{x})} (\frac{1}{2} - \epsilon)^{p - r(\mathbf{x})} .$$

Let

$$\lambda(r) = (\frac{1}{2} + \epsilon)^{r} (\frac{1}{2} - \epsilon)^{p - r} \begin{pmatrix} p \\ r \end{pmatrix}$$

denote the binomial distribution associated to the distribution of the ξ_i^μ's.

Suppose that the family $\Phi \in \Gamma_\lambda$ is uniform on each $\mathcal{X}_r$, i.e. $< \sigma >_{\Phi_\mathbf{x}}$ is constant on $\mathcal{X}_r$ for every r, then letting $\mathbf{s}$ be the vector in $\mathbb{R}^{p+1}$ with components $s_r =< \sigma >_{\Phi_\mathbf{x}}$, $\mathbf{x} \in \mathcal{X}_r$, we obtain

$$m^\mu(\Phi) = \sum_{r=0}^{p} (\frac{1}{2} + \epsilon)^{r} (\frac{1}{2} - \epsilon)^{p-r} s_r \sum_{\mathbf{x} \in \mathcal{X}_r} x^\mu .$$

Since all $\mathbf{x} \in \mathcal{X}_r$ arise from any one vector in $\mathcal{X}_r$ by permuting the components, the sum $\sum_{\mathbf{x} \in \mathcal{X}_r} x^\mu$ is independent of μ. The evaluation gives

$$\sum_{\mathbf{x} \in \mathcal{X}_r} x^\mu = \begin{pmatrix} p-1 \\ r-1 \end{pmatrix} - \begin{pmatrix} p-1 \\ r \end{pmatrix} = \frac{2r - p}{p} \begin{pmatrix} p \\ r \end{pmatrix} .$$

The computation of $S_\lambda(\Phi)$ is direct and we obtain

$$F(\Phi) = \frac{1}{2p} \left[\sum_{r=0}^{p} \lambda(r)(2r - p) s_r \right]^2 + \beta^{-1} \sum_{r=0}^{p} \lambda(r) \eta \left(\frac{1 + s_r}{2} \right) =: \widetilde{F}(\mathbf{s}) .$$

It follows that

$$f(\beta) \geq \max_{\Phi} F(\Phi) = \max_{\mathbf{s}} \widetilde{F}(\mathbf{s}) = \widetilde{f}(\beta)$$

where the first maximum is over the families which are uniform on each $\mathcal{X}_r$, the second maximum is over $p + 1$ dimensional vectors $\mathbf{s}$ with components in the interval $[-1, 1]$.

We will solve the variational problem for $\widetilde{F}$ and show that

$$\{\beta > 0 : \ \widetilde{f}(\beta) = \beta^{-1} \ln(2)\} = (0, \widetilde{\beta_c}] ,$$

with $\widetilde{\beta_c} = (1 + 4\epsilon^2(p - 1))^{-1}$. The inequality $f(\beta) \geq \widetilde{f}(\beta) \geq \widetilde{F}(\mathbf{0}) = \beta^{-1} \ln(2)$ then implies that $\beta_c \leq \widetilde{\beta_c}$ which is the claim.

The critical points of $\widetilde{F}$ are the solutions of

$$s_r = \tanh \left\{ \beta \frac{2r - p}{\sqrt{p}} \sum_{t=0}^{p} \lambda(t) \frac{2t - p}{\sqrt{p}} s_t \right\} \quad r = 0, 1, \ldots, p \tag{5.1}$$

Introduce the function $G : \mathbb{R} \to \mathbb{R}$:

$$G(m) = -\frac{1}{2p} m^2 + \frac{1}{\beta} \sum_{r=0}^{p} \lambda(r) \ln \left[2 \cosh \left(\beta m \frac{2r - p}{p} \right) \right] .$$

The condition $dG(m)/dm = 0$ is

$$m = \sum_{r=0}^{p} \lambda(r)(2r-p)\tanh\left(\beta m \frac{2r-p}{p}\right) \tag{5.2}$$

For $m \in \mathbb{R}$, let $\mathbf{s}(m)$ be the vector in $\mathbb{R}^{p+1}$ with components $s(m)_r = \tanh\left(\beta m \frac{2r-p}{p}\right)$. For $\mathbf{s} \in \mathbb{R}^{p+1}$, let $m(\mathbf{s}) = \sum_{r=0}^{p} \lambda(r)(2r-p)s_r$. The analogue of (2.7) is

$$\widetilde{F}(\mathbf{s}(m)) = G(m) + (1/2p)(m - m(\mathbf{s}(m)))^2 .$$

Moreover (5.1) and (5.2) are respectively equivalent to $\mathbf{s} = \mathbf{s}(m(\mathbf{s}))$ and $m = m(\mathbf{s}(m))$. Thus we have the analogue of Proposition 2: the maps $m \mapsto \mathbf{s}(m)$ and $\mathbf{s} \mapsto m(\mathbf{s})$ are each others inverses when restricted to critical points of G respectively $\widetilde{F}$; and $\widetilde{F}(\mathbf{s}) = G(m(\mathbf{s}))$ at critical points. In particular the maximizers of $\widetilde{F}$ and of G are in biyective correspondence and

$$\max_{\mathbf{s}} \widetilde{F}(\mathbf{s}) = \max_{m} G(m)$$

Let us now look at solutions of (5.2); to this end we analyze the function $h(m) := \sum_{r=0}^{p} \lambda(r)(2r-p)\tanh\left(\beta m \frac{2r-p}{p}\right)$ given by the r.h.s. of (5.2). The result of this analysis is the following lemma the proof of which is elementary

Lemma 3 *h is strictly increasing with $0 < h'(m) \le h'(0)$ $\forall m \in \mathcal{R}$. Moreover h is strictly concave (resp. strictly convex) for $m > 0$ (resp. $m < 0$).*

This result establishes that there are at most three solutions of the gap-equation $m = h(m)$ (say $m_o = 0, \pm a_o$), since a strictly convex or concave function can intersect a straight line at at most two points and, in the present case $m = 0$ is a solution.

Finally let us compute the value of $\widetilde{\beta_c}$, in which $m_o = 0$ ceases to be a maximizer. The second derivative of G at $m = 0$ is

$$G''(0) = -p^{-1} + \beta p^{-2} \sum_{r=0}^{p} \lambda(r)(2r-p)^2 ,$$

so that $\widetilde{\beta_c}$ is determined by the condition $\widetilde{\beta_c} \sum_{r=0}^{p} \lambda(r)(2r-p)^2 = p$. With the well known first and second moments of the binomial distribution we compute that $\widetilde{\beta_c}$ has the claimed value. □

To attempt to define the second critical reciprocal temperature consider the set

$$\mathcal{H} = \{\beta > 0 : \ m^\mu(\Phi) \text{ is independent of } \mu \in \{1, 2, \cdots, p\} \text{ for each maximizer } \Phi \text{ of } F_\beta\}$$

of reciprocal temperatures where every maximizing family Φ has homgeneous $\mathbf{m}(\Phi)$. The first problem is to show that this set is an interval. Our conjecture is that this is the case and that $\beta_* = \sup \mathcal{H}$ is given by

$$(\beta_*)^{-1} = T_* = \max\{1 - 4\epsilon^2(p-1)) , 0\} .$$

This would imply that for given $\epsilon \neq 0$, there is a critical value of p given by $p_c = \sqrt{1 + (2\epsilon)^{-2}}$ (which is greater than 2); or alternatively, for a given value of p there is a critical value of ϵ given by $\epsilon_c = (4(p-1))^{-1/2}$ (which is smaller than 1/2 for $p \geq 3$), such that for $p \geq p_c$, or alternatively $\mid \epsilon \mid \geq \epsilon_c$ the homogeneous regime persists and the non-homogeneous regime does not appear (i.e. $T_* = 0$). The "phase diagramm" for $p \geq 3$ would then be that of Fig. 3.

This subject is presently under investigation.

6 Appendix

The following result is used in the proof of Proposition 4 in the particular case $\mathcal{A} = \mathcal{A}_1$; and in the proof of 3. of Proposition 3 with $\mathcal{A} = \mathcal{X}$. In both cases the hypothesis is easily verified.

Lemma *If $\mathcal{A} \subset \mathcal{X}$ satisfies*

$$| \mathcal{A} \cap \mathcal{A}_\mu \cap \mathcal{A}_\nu | + | \mathcal{A} \cap (-\mathcal{A}_\mu) \cap (-\mathcal{A}_\nu) | = | \mathcal{A} \cap \mathcal{A}_\mu \cap (-\mathcal{A}_\nu) | + | \mathcal{A} \cap (-\mathcal{A}_\mu) \cap \mathcal{A}_\nu |$$

for all $\mu \neq \nu \in \{1, 2, \cdots, p\}$; then

$$\sum_{\mu=1}^{p} \left[\sum_{\mathbf{x} \in \mathcal{A}} \lambda(\{\mathbf{x}\}) x^\mu < \sigma >_{\phi_\mathbf{x}} \right]^2 \leq | \mathcal{A} | \sum_{\mathbf{x} \in \mathcal{A}} [\lambda(\{\mathbf{x}\}) < \sigma >_{\phi_\mathbf{x}}]^2 \ .$$

Proof: This is just Bessel's inequality in disguise. Using the $| \mathcal{A} |$ elements of $\mathcal{A}$ we define p vectors $\{e^\mu\}_{\mu=1}^p$ in $\mathrm{I\!R}^{|\mathcal{A}|}$ with components:

$$(e^\mu)_\mathbf{x} = x^\mu \quad , \quad \mathbf{x} \in \mathcal{A} \ .$$

We have

$$\begin{aligned}
\langle e^\mu, e^\nu \rangle &= \sum_{\mathbf{x} \in \mathcal{A}} x^\mu x^\nu = \sum_{\mathbf{x} \in \mathcal{A} \cap \mathcal{A}_\mu \cap \mathcal{A}_\nu} x^\mu x^\nu + \sum_{\mathbf{x} \in \mathcal{A} \cap \mathcal{A}_\mu \cap (-\mathcal{A}_\nu)} x^\mu x^\nu \\
&+ \sum_{\mathbf{x} \in \mathcal{A} \cap (-\mathcal{A}_\mu) \cap \mathcal{A}_\nu} x^\mu x^\nu + \sum_{\mathbf{x} \in \mathcal{A} \cap (-\mathcal{A}_\mu) \cap (-\mathcal{A}_\nu)} x^\mu x^\nu = | \mathcal{A} \cap \mathcal{A}_\mu \cap \mathcal{A}_\nu | \\
&- | \mathcal{A} \cap \mathcal{A}_\mu \cap (-\mathcal{A}_\nu) | - | \mathcal{A} \cap (-\mathcal{A}_\mu) \cap \mathcal{A}_\nu | + | \mathcal{A} \cap (-\mathcal{A}_\mu) \cap (-\mathcal{A}_\nu) | \\
&= | \mathcal{A} | - 2 | \mathcal{A} \cap \mathcal{A}_\mu \cap (-\mathcal{A}_\nu) | - 2 | \mathcal{A} \cap (-\mathcal{A}_\mu) \cap \mathcal{A}_\nu |
\end{aligned}$$

The last equality following from the fact that $\mathcal{A} = [\mathcal{A} \cap \mathcal{A}_\mu \cap \mathcal{A}_\nu] \cup [\mathcal{A} \cap \mathcal{A}_\mu \cap (-\mathcal{A}_\nu)] \cup [\mathcal{A} \cap (-\mathcal{A}_\mu) \cap \mathcal{A}_\nu] \cup [\mathcal{A} \cap (-\mathcal{A}_\mu) \cap (-\mathcal{A}_\nu)]$ is a partition into disjoint sets. If $\mu = \nu$, the two negative summands vanish, and we get $\langle e^\mu, e^\mu \rangle = | \mathcal{A} |$. Thus $\{| \mathcal{A} |^{-1/2} \ e^\mu\}_{\mu=1}^p$ is an orthonormal system in $\mathrm{I\!R}^{|\mathcal{A}|}$. Letting $W \in \mathrm{I\!R}^{|\mathcal{A}|}$ be the vector with components $W_\mathbf{x} = \lambda(\{\mathbf{x}\}) < \sigma >_{\phi_\mathbf{x}}$, one has $\langle W, e^\mu \rangle = \sum_{\mathbf{x} \in \mathcal{A}} \lambda(\{\mathbf{x}\}) x^\mu < \sigma >_{\phi_\mathbf{x}}$, and the claim follows by Bessel's inequality. □

The following intends to provide a synoptic description of the results of [17, 18] which provide a general theory of what were called inhomogeneous mean-field systems (classical, quantal, or mixed). The reader will obtain a better view (along with some generalizations) consulting the expository paper by R.F. Werner [22]. Such an inhomogeneous mean-field system is specified by:

- a unital C^*-algebra $\mathcal{A}$ (the observables of the single "particle" system);
- a compact space $\mathcal{X}$ (the points in $\mathcal{X}$ are the the parameters (random or not) specifying the interaction constants);
- a sequence $\{h_N : \ N = 1, 2, \cdots\}$ of elements h_N of the N-fold tensor-product $\mathcal{B}_N$ of the $C(\mathcal{X}) \otimes \mathcal{A}$[6] with itself;

[6] $C(\mathcal{X})$ denotes the (unital C^*-algebra of) continuous complex-valued functions on $\mathcal{X}$ (with the supremum norm); $C(\mathcal{X}) \otimes \mathcal{A}$ is isomorphic to $C(\mathcal{X}, \mathcal{A})$ the continuous $\mathcal{A}$-valued functions on $\mathcal{X}$.

- a sequence $\{x_N\}$ of points x_N in the N-fold direct product $\mathcal{X}_N$ of $\mathcal{X}$ with itself (the evaluation of h_N at the point x_N is the "Hamiltonian per particle" for N "particles").

The key observation of [17, 18] is that a mean-field model is characterized by the approximate permutational symmetry of the sequence $\{h_N\}$, that is to say

$$\| h_N - sym_N(h_{N-1} \otimes \mathbf{1}) \|_N \rightarrow 0 \tag{6.1}$$

as $N \to \infty$ where $\| \cdot \|_N$ denotes the norm of the N-fold tensor-product algebra $\mathcal{B}_N$, and sym_N denotes symmetrization with respect to the $N!$ permutations of the N components $C(\mathcal{X}) \otimes \mathcal{A}$ of $\mathcal{B}_N$. That is to say, in a mean-field model the energy density for N "particles" is obtained, up to small corrections, by taking the energy density for $N-1$ "particles" and symmetrizing with respect to permutations. A realization of the mean-field model specified by $(\mathcal{A}, \mathcal{X}, \{h_N\})$ is obtained by a specific choice ot the parameters, that is to say by a sequence $\{x_N\}$ of points $x_N \in \mathcal{X}_N$. Then $h_N(x_N)$ which is the element of the N-fold tensor-product $\mathcal{A}_N$ of $\mathcal{A}$ with itself obtained by evaluating h_N at x_N is the Hamiltonian per "particle".

Now select a reference state ρ on $\mathcal{A}$, let ρ_N denote the N-fold product state on $\mathcal{A}_N$. Denote by $\rho_N^{Nh_N(x_N)}$ the exponential perturbation of ρ_N by $Nh_N(x_N)$. In the familiar quantum case the positive trace-class operator associated to this perturbation is $\exp\{\ln(\rho_N) + Nh_N(x_N)\}$. In the familiar classical case ($\mathcal{A} = C(\mathcal{Y})$ for a compact space $\mathcal{Y}$) the perturbed state corresponds to the measure $\exp\{Nh_N(x_N; y_1, y_2, \cdots, y_N)\}d\rho(y_1, y_2, \cdots, y_N)$. The generalization to the general C^*-algebraic setting is due to Araki, and has proved to be a very useful concept in statistical mechanics. The perturbed state is not normalized, and the number

$$F(\rho_N, h_N(x_N)) = -\ln(\rho_N^{-Nh_N(x_n)}(\mathbf{1}))$$

is the increment in free-energy (for $\beta = 1$) of the renormalized perturbed state with respect to ρ_N (the non-interacting system with single "particle" state ρ). When available, one can choose ρ to be the normalized trace (quantum case) or a normalized counting measure (classical case) and then $F(\rho_N, h_N(x_N))$ is the free-energy up to a constant.

In order to obtain thermodynamic stability the sequence of points

$$x_N = (x_N(1), x_N(2), \cdots, x_N(N))$$

has to behave properly. Under the ergodicity condition

$$weak - lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} \delta_{x_N(j)} = \lambda \tag{6.2}$$

on the space of probability measures on $\mathcal{X}$, the following Gibbs variational principle is obtained in [18] (Theorem II.2):

$$\begin{aligned} &\lim_{N\to\infty} (\beta N)^{-1} F(\rho_N, \beta h_N(x_N)) \\ &= \inf\{\lim_{N\to\infty} \phi(h_N) + \beta^{-1} S_M((\lambda \otimes \rho)_\infty | \phi)\} \\ &= \inf\{U(\varphi) + \beta^{-1} S(\lambda \otimes \rho | \varphi)\} . \end{aligned} \tag{6.3}$$

Here, the first infimum is over the the states ϕ of the infinite tensor-product algebra $\mathcal{B}_\infty = C(\mathcal{X}_\infty) \otimes \mathcal{A}_\infty$ which are invariant under arbitrary finite permutations of the components of the tensor-product and whose restriction to $C(\mathcal{X}_\infty)$ are equal to the infinite product measure obtained from λ. For such states ϕ, it is inmediate that the limit $\lim_{N\to\infty} \phi(h_N)$

exists. $S_M((\lambda\otimes\rho)_\infty|\phi)$ denotes the limiting value of the relative entropy per "particle" $N^{-1}S((\lambda\otimes\rho)_N|(\phi)_N)$ where $(\lambda\otimes\rho)_N$ is the N-fold product state on $\mathcal{B}_N$ associated to $\lambda\otimes\rho$, and $(\phi)_N$ is the restriction of ϕ to the first (or any) N components of $\mathcal{B}_\infty$; the existence of this limit is again well-known.

The second infimum is over the states φ on $C(\mathcal{X})\otimes\mathcal{A}$ which give the probability measure λ when restricted to $C(\mathcal{X})$. $U(\varphi)=\lim_{N\to\infty}\varphi_N(h_N)$, and $S(\lambda\otimes\rho,\varphi)$ is the relative entropy of φ with respect to $\lambda\otimes\rho$. This second variational problem can be further simplified when the algebra $\mathcal{A}$ is separable. In this case every state γ of $C(\mathcal{X})\otimes\mathcal{A}$ can be uniquely decomposed as

$$\gamma=\int_{\mathcal{X}}^{\oplus}\lambda_\gamma(dx)\gamma_x$$

with a regular Borel probability measure λ_γ on $\mathcal{X}$ and a function $x\to\gamma_x$ on $\mathcal{X}$ with values in the state-space of $\mathcal{A}$. Then $S(\lambda\otimes\rho|\varphi)=\int\lambda(dx)S(\rho|\varphi_x)$, and the integral decomposition can be used to rewrite $U(\varphi)$ (we will do this for the Hopfield model below). It is this third variational expression which we quoted here as the Theorem. Results about weak limit points of the sequence $\{\rho_N^{Nh_N(x_N)}\}$ are obtained in [18]. Here, we have not included a discussion of them as applied to the Hopfield model.

The three variational expressions match the three levels of large-deviation theory (see [23]) when $\mathcal{A}$ is abelian, and the above result can be viewed as a non-commutative extension of the classical result of Cramér for weakly independent random variables. This was in fact one of the main motivations leading to the series of papers [24, 17, 18].

The proof is based on the two conditions (6.1) and (6.2) and two key ingredients: a characterization of the relative free-energy F as the Legendre transform of the relative entropy (due to Petz), and direct estimates of the relative entropy $S((\lambda\otimes\rho)_N|(\lambda\otimes\rho)_N^{Nh_N})$. Upon rewritting the proof for the abelian case $\mathcal{A}=C(\mathcal{Y})$ ($\mathcal{Y}$ is any compact space) one sees that it does not involve any direct large-deviation estimates: the two conditions suffice to perform the relative entropy estimates. We believe that herein lies the efficency of the method. The large-deviation estimates of [6, 11, 12] become rather unmanageable in the i.i.d. case if the single distribution is not symmetric.

In the Hopfield-like model of the theorem given in the present paper, which is classical with configuration space $\mathcal{S}$, we have $\mathcal{A}=C(\mathcal{S})$, and $\mathcal{X}$ is the p-fold direct product of $\mathcal{S}$ (we could have chosen the p-fold direct product of any compact set of reals). Choose the reference state (i.e. probability measure) ρ to be the uniform normalized measure ω on $\mathcal{S}$. Condition (2.2) is (6.2). Let $h_2\in[C(\mathcal{X})\otimes C\mathcal{S})]\otimes[C(\mathcal{X})\otimes C\mathcal{S})]=C(\mathcal{X}\times\mathcal{S}\times\mathcal{X}\times\mathcal{S})$ be given by

$$h_2(\mathbf{x}_1,S_1,\mathbf{x}_2,S_2)=-4^{-1}\sum_{j=1}^{2}\sum_{k=1}^{2}\langle\mathbf{x}_j,\mathbf{x}_k\rangle S_jS_k-2^{-1}\sum_{j=1}^{2}\langle\mathbf{h},\mathbf{x}_j\rangle S_j$$

where $S_1=\sigma\otimes\mathbf{1}$, and $S_2=\mathbf{1}\otimes\sigma$; then

$$\sup_{\boldsymbol{\xi}\,,\,S}\left\{N^{-1}H_{N,p}^{\{\boldsymbol{\xi}\}}(S)-sym_N\left(h_2\otimes\underbrace{\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{N-2}\right)(\boldsymbol{\xi}_{\mathbf{1}},S_1,\boldsymbol{\xi}_{\mathbf{2}},S_2,\cdots,\boldsymbol{\xi}_{\boldsymbol{N}},S_N)\right\}\le\frac{p}{N}\,,$$

which is the condition for an inhomogeneous mean-field model. Using the integral decomposition of a state (i.e. probability measure), we have

$$\varphi=\int_{\mathcal{X}}^{\oplus}\lambda(d\mathbf{x})\phi_{\mathbf{x}}$$

if the restriction of φ to $C(\mathcal{X})$ is λ. With this and the above estimate

$$U(\varphi)=(\varphi\otimes\varphi)(h_2)=-2^{-1}\parallel\mathbf{m}(\Phi)\parallel^2-\langle\mathbf{h},\mathbf{m}(\Phi)\rangle\,,$$

$$S(\lambda \otimes \omega|\varphi) = \int_{\mathcal{X}} \lambda(d\mathbf{x}) \left(-S(\phi_{\mathbf{x}}) - \sum_{s\in\mathcal{S}} \phi_{\mathbf{x}}(s) \ln(\omega(s)) \right) = -S_{\lambda}(\Phi) + \ln(|\ \mathcal{S}\ |) \ ,$$

with the notation we have used in the main body of the paper, i.e. $\Phi = \{\mathcal{X} \ni x \mapsto \phi_{\mathbf{x}} \in \mathcal{M}(\mathcal{S})\}$. Moreover, with $x_N = (\ \boldsymbol{\xi_1},\ \boldsymbol{\xi_2}, \cdots,\ \boldsymbol{\xi_N})$,

$$\beta^{-1} F(\rho_N, \beta h_N(x_N)) = \beta^{-1} \ln(|\ \mathcal{S}\ |) - f_{N,p}^{\{\boldsymbol{\xi}\}}(\beta) \ .$$

Inserting the last three expressions in the second variational expression of (6.3) gives (2.4).

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