THE THREE DIMENSIONAL POLYOMINOES OF MINIMAL AREA

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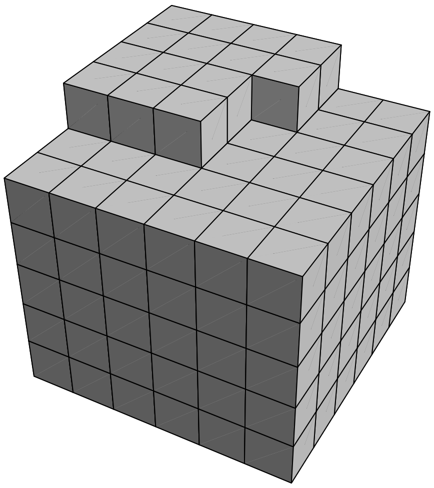
Abstract. The set of the three dimensional polyominoes of minimal area and of volume n contains a polyomino which is the union of a quasicube $j \times (j+\delta) \times (j+\theta)$, $\delta, \theta \in \{0,1\}$, a quasisquare $l \times (l+\epsilon)$, $\epsilon \in \{0,1\}$, and a bar k. This shape is naturally associated to the unique decomposition of $n = j(j+\delta)(j+\theta) + l(l+\epsilon) + k$ as the sum of a maximal quasicube, a maximal quasisquare and a bar. For n a quasicube plus a quasisquare, or a quasicube minus one, the minimal polyominoes are reduced to these shapes. The minimal area is explicitly computed and yields a discrete isoperimetric inequality. These variational problems are the key for finding the path of escape from the metastable state for the three dimensional Ising model at very low temperatures.



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1. INTRODUCTION

Suppose we are given n unit cubes. What is the best way to set them out, in order to obtain a shape having the smallest possible area?
A little thinking suggests the following answer: first build the greatest cube you can, say $j \times j \times j$. Then complete one of its side, or even two, if you can, to obtain a quasicube $j \times (j+\delta) \times (j+\theta)$, where $\delta, \theta \in \{0,1\}$. With the remaining cubes, build the greatest quasisquare possible, $l \times (l+\epsilon)$, $\epsilon \in \{0,1\}$, and put it on a side of the quasicube. With the last cubes, make a bar of length $k < l+\epsilon$ and stick it against the quasisquare.
Our main result is that this technique indeed yields a three dimensional polyomino of volume n and of minimal area, which is naturally associated to the unique decomposition of $n = j(j+\delta)(j+\theta)+l(l+\epsilon)+k$ as the sum of a maximal quasicube, a maximal quasisquare and a bar. We can compute easily the area of these shapes and we thus obtain a discrete isoperimetric inequality. However, the structure of the set of the minimal polyominoes having a fixed volume n depends heavily on n. When n is a quasicube plus a quasisquare, or a quasicube minus one, it is reduced to the polyominoes obtained by the preceding technique.
A remarkable consequence of this result is that there exists an infinite increasing sequence of minimal polyominoes. This fact is crucial for determining the path of escape from the metastable state for the three dimensional Ising model at very low temperatures [1,2]. The system will nucleate from one phase to another by creating a droplet which will grow through this sequence of minimal shapes. This question was our original motivation for solving the variational problems addressed here. The corresponding two–dimensional questions had already been handled [4,5,6]. In three dimensions, we needed a general large deviation framework [2,3] and the answer to precise global variational problems (like the previous ones), as well as local ones: what are the best ways (as far as area is concerned) to grow or to shrink a parallelepiped?
The first part of the paper deals with the two dimensional case. The two dimensional results are indeed necessary to handle the three dimensional situation, which is the subject of the second part.

2. THE TWO DIMENSIONAL CASE

We denote by (e_1, e_2) the canonical basis of the integer lattice $\mathbb{Z}^2$. A unit square is a square of area one, whose center belongs to $\mathbb{Z}^2$ and whose vertices belong to the dual lattice $(\mathbb{Z}+1/2)^2$. We do not distinguish between a unit square and its center: thus (x_1, x_2) denotes the unit square of center (x_1, x_2). A two dimensional polyomino is a finite union of unit squares. It is defined up to a translation. The set of all polyominoes is denoted by C. The area $|c|$ of the polyomino c is the number of its unit squares. We denote by C_n the set of all the polyominoes of area n. The perimeter $P(c)$ of a polyomino c is the number of unit edges of the dual lattice $(\mathbb{Z}+1/2)^2$ which belong only to one of the unit squares of c.

Notice that the perimeter is an even integer. For instance the perimeter of c in figure 2.1 is equal to 12 and its area is equal to 6.

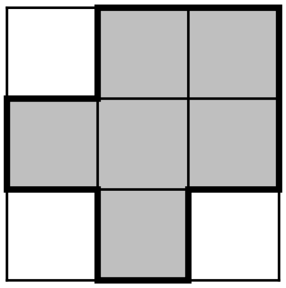

figure 2.1: a 2D polyomino

Our aim is to investigate the set M_n of the polyominoes of C_n having a minimal perimeter. This minimal perimeter is denoted by $P(n)$. We say that a polyomino c has minimal perimeter (or simply is minimal) if it belongs to the set $M_{|c|}$.

Theorem 2.1. *A polyomino c has minimal perimeter if and only if there does not exist a rectangle of area greater or equal than $|c|$ having a perimeter smaller than $P(c)$.*

Proof. The perimeter of a polyomino is the same as the perimeter of its smallest surrounding rectangle. $\square$

This characterization of minimal polyominoes gives very little insight into the possible shapes of the minimal polyominoes. Figure 2.2 shows the sets M_n for small values of n.

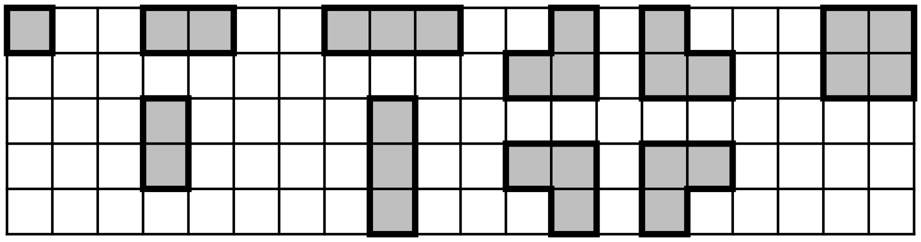

figure 2.2: the sets M_1, M_2, M_3, M_4

We now introduce some notations concerning polyominoes.

The lengths and the bars. Let c be a polyomino.
We define its horizontal and vertical lengths $l_1(c)$ and $l_2(c)$ by

$$l_1(c) = \mathrm{card}\{\, x_1 \in \mathbb{Z} : \exists\, x_2 \in \mathbb{Z} \quad (x_1, x_2) \in c \,\},$$
$$l_2(c) = \mathrm{card}\{\, x_2 \in \mathbb{Z} : \exists\, x_1 \in \mathbb{Z} \quad (x_1, x_2) \in c \,\}.$$

We define the horizontal and vertical bars $b_1(c,l)$ and $b_2(c,l)$ for l in $\mathbb{Z}$ by

$$b_1(c,l) = \{\, (x_1, x_2) \in c : x_2 = l \,\}, \quad b_2(c,l) = \{\, (x_1, x_2) \in c : x_1 = l \,\}.$$

The bars are one dimensional sections of the polyomino. An horizontal bar will also be called a row and a vertical bar a column. The extreme bars $b_1^*(c)$ and $b_2^*(c)$ are the bars associated with the lengths $l_2(c)$ and $l_1(c)$ i.e.

$$b_1^*(c) = b_1(c, l_2(c) - 1), \qquad b_2^*(c) = b_2(c, l_1(c) - 1).$$

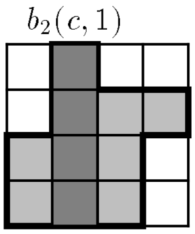

figure 2.3: a bar

Addition of polyominoes. We describe here a simple mechanism for adding polyominoes. For the sake of clarity, we need to work here with instances of the polyominoes having a definite position on the lattice $\mathbb{Z}^2$ i.e. we temporarily remove the indistiguishability modulo translations. Let c be a polyomino. By $c(x_1, x_2)$ we denote the unique polyomino obtained by translating c in such a way that

$$\begin{aligned} \min\{\, y_1 : \exists y_2 \quad (y_1, y_2) \in c(x_1, x_2)\,\} \;&=\; x_1, \\ \min\{\, y_2 : \exists y_1 \quad (y_1, y_2) \in c(x_1, x_2)\,\} \;&=\; x_2. \end{aligned}$$

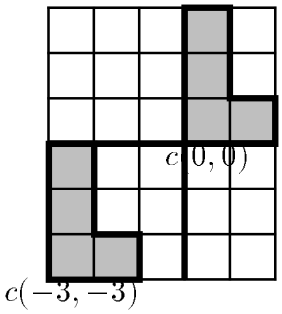

figure 2.4: translation

Remark. When we deal with polyominoes up to translations, we normally work with the polyominoes $c(0,0)$, for any c in C.

We define an operator $+_1$ from $C \times C$ to C by

$$\forall c, d \in C \qquad c +_1 d \;=\; c(0,0) \cup d(l_1(c), 0).$$

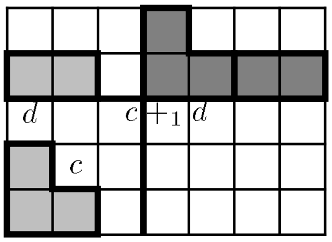

figure 2.5: operator $+_1$

Similarly the operator $+_2 : C \times C \to C$ is defined by $c +_2 d = c(0,0) \cup d(0, l_2(c))$. More generally, for i an integer, we put

$$c +_1^i d \;=\; c(0,0) \cup d(l_1(c), i), \quad c +_2^i d \;=\; c(0,0) \cup d(i, l_2(c)).$$

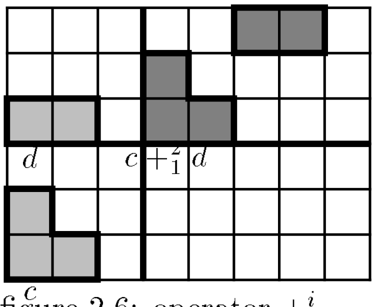

figure 2.6: operator $+_1^i$

Sometimes we will use the operator $+$ without specifying the direction: it will mean that this direction is in fact indifferent i.e. the statements hold for both operators $+_1$ and $+_2$. Finally, we define two operators on $C \times C$ with values in $\mathcal{P}(C)$, the subsets of C, by

$$c \oplus_1 d \;=\; \{\, c +_1^i d : l_2(d) + i \le l_2(c) \,\}, \quad c \oplus_2 d \;=\; \{\, c +_2^i d : l_1(d) + i \le l_1(c) \,\}.$$

Notice that $c \oplus_1 d$ (respectively $c \oplus_2 d$) is empty whenever $l_2(c) < l_2(d)$ (resp. $l_1(c) < l_1(d)$).

The basic polyominoes. We will concentrate mainly on particular simple shapes. First, we consider the rectangles. The rectangle of horizontal length l_1 and vertical length l_2 is denoted by $l_1 \times l_2$. A square is a rectangle $l_1 \times l_2$ with $l_1 = l_2$. A quasisquare is a rectangle $l_1 \times l_2$ with $|l_1 - l_2| \le 1$. The basic polyominoes are the polyominoes obtained by adding a bar to a rectangle (the length of the bar being smaller than the length of the side of the rectangle on which it is added). More precisely the set B of the basic polyominoes is

$$B \;=\; \{\, l_1 \times l_2 +_1 1 \times k : k < l_2 \,\} \cup \{\, l_1 \times l_2 +_2 k \times 1 : k < l_1 \,\}.$$

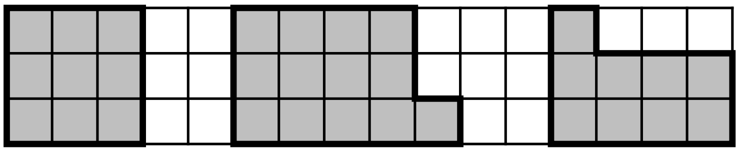

figure 2.7: basic polyominoes

When we add a bar $k \times 1$ or $1 \times k$ to a rectangle $l_1 \times l_2$, we will sometimes shorten the notation and simply write k: the direction of the bar will then be parallel to the side of the rectangle on which it is added. For instance $l_1 \times l_2 +_1 k$ will mean $l_1 \times l_2 +_1 1 \times k$. We are now ready to state the main result of this section.

Theorem 2.2. *For any n, the set M_n of the polyominoes of area n having a minimal perimeter contains a basic polyomino of the form*

$$(l+\epsilon) \times l +_2 k \times 1 \quad \textit{where} \quad \epsilon \in \{0,1\},\ 0 \le k < l+\epsilon,\ n = l(l+\epsilon)+k.$$

Remark. Notice that this statement also says that any integer n may be decomposed as $n = l(l+\epsilon)+k$, which is a purely arithmetical fact.

Proof. We choose an arbitrary polyomino c belonging to M_n (which is not empty!) and we apply to c a sequence of transformations in order to obtain a polyomino of the desired shape. The point is that the transformations never increase the perimeter of a polyomino nor change its area. Thus the perimeter remains constant during the whole sequence and the final polyomino still belongs to M_n. We first describe separately each transformation.

Projections p_1 and p_2. The projections are defined for any polyomino. Let c be a polyomino. The vertical projection p_2 consists in letting all the unit squares of c fall down vertically (along the direction of e_2, in the sense of $-e_2$) on a fixed horizontal line as shown on figure 2.8.

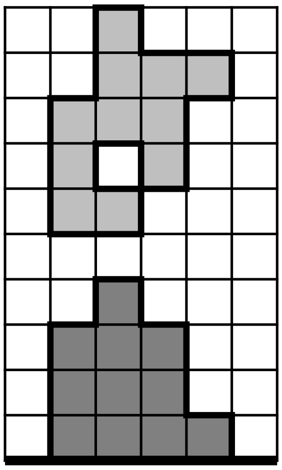

figure 2.8: vertical projection p_2

The horizontal projection p_1 is defined similarly, working with the vector e_1: we push horizontally towards the left all the unit squares against a fixed vertical line.

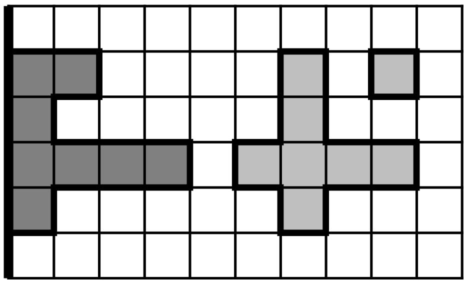

figure 2.9: horizontal projection p_1

Clearly, the projections don't change the area. They are projections in the sense that $p_1 \circ p_1 = p_1$, $p_2 \circ p_2 = p_2$. They decrease the perimeter. Consider for instance the vertical projection p_2. Focusing on two adjacent vertical bars, we see that the effect of the projection is to increase the number of vertical edges belonging simultaneously to both bars. Moreover, the projection p_2 decreases the number of horizontal edges of a bar which belong only to one unit square: after projection, this number is equal to 2.
The set $\mathcal{F} = p_2 \circ p_1(C)$ of all projected polyominoes is the set of Ferrers diagrams. Remark that for c in $\mathcal{F}$ we have $P(c) = 2(l_1(c) + l_2(c))$.

Fillings fill$(2 \to 1)$ **and fill**$(1 \to 2)$**.** These transformations are defined on the set $\mathcal{F}$ of the Ferrers diagrams. Let c belong to $\mathcal{F}$. The filling fill$(2 \to 1)$ proceeds as follows. While there remains a row below the top row (i.e. the extreme bar $b_1^*(c)$) which is strictly shorter than the length of the base row (that is the l_1–length of c), we remove the rightmost unit square of the top row (i.e. the square $(|b_1^*(c)| - 1, l_2(c) - 1)$ and we put it into the leftmost empty cell of the lowest incomplete row. The mechanism ends whenever there is a full rectangle below the top row.

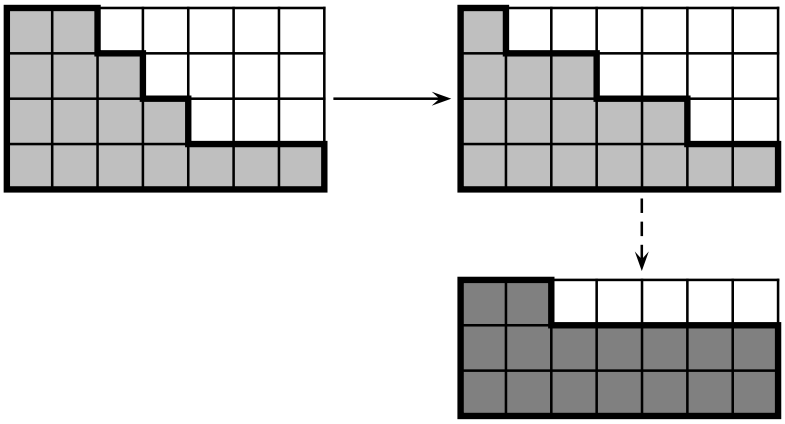

figure 2.10: filling$(2 \to 1)$

More precisely, let $l^* = \min\{\, l : l < l_2(c) : |b_1(c, l)| < l_1(c) \,\}$. If $l^* < l_2(c) - 1$ we take the square $(|b_1^*(c)| - 1, l_2(c) - 1)$ and we put it at $(|b_1(c, l^*)|, l^*)$. We do this until l^* equals $l_2(c)$.
The filling fill$(1 \to 2)$ is defined symmetrically: we use the top square of the rightmost column to fill the leftmost column which is shorter than the length $l_2(c)$.

Clearly, the fillings do not change the area and never increase the perimeter. They end with a basic polyomino (the addition of a rectangle and a bar).

Dividing. The domain of dividing is the set V of the basic "vertical" polyominoes

$$V \;=\; \{\, l_1 \times l_2 +_2 k \times 1 : k < l_1 \leq l_2 \,\}.$$

Let $c = l_1 \times l_2 +_2 k \times 1$ with $k < l_1 \leq l_2$ be an element of V.
Let $l_2 - l_1 = 2q + \epsilon$ where $\epsilon \in \{0,1\}$ be the euclidean division of $l_2 - l_1$ by 2.
The divided polyomino is then

$$\text{dividing}(c) \;=\; (l_1 \times l_1 +_2 l_1 \times q +_2 k \times 1) +_1 (q+\epsilon) \times l_1.$$

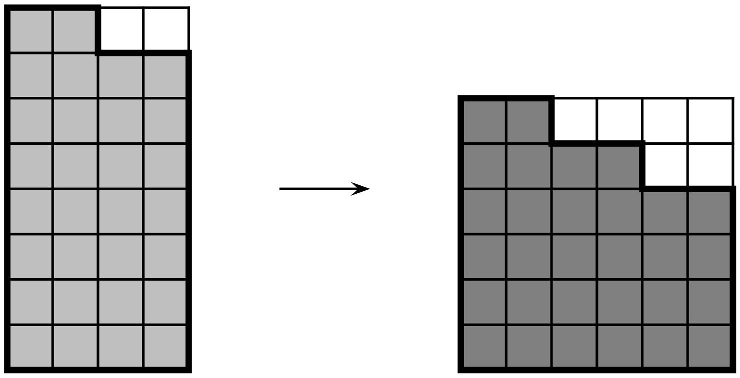

figure 2.11: dividing

We check easily that the dividing does not change the area nor increase the perimeter. Moreover it is the identity on the set $\{\,(l+\epsilon) \times l +_2 k \times 1 : k < l + \epsilon\,\}$.

The sequence of transformations. Let us start with a polyomino c belonging to M_n. We first apply the projections p_1 and p_2. Let $d = p_2 \circ p_1(c)$. We consider two cases according whether d is "vertical" or "horizontal". Let s_Δ be the symmetry with respect to the diagonal $x_1 = x_2$.

- If $l_1(d) \leq l_2(d)$ we set $e = d$.
- If $l_1(d) > l_2(d)$ we set $e = s_\Delta(d)$.

Now we have $l_1(e) \leq l_2(e)$. Next we apply the filling fill$(2 \to 1)$ to e and we obtain a polyomino f. Since the perimeter cannot decrease, the polyomino f is necessarily a basic "vertical" polyomino. Therefore we can apply the dividing to f. Let $g = \text{dividing}(f)$. Finally let $h = \text{fill}(2 \to 1)(g)$. Since the perimeter has not decreased during this last filling, h is a basic "vertical" polyomino. Because of the previous dividing operation, its associated rectangle is in fact a quasisquare. Thus h has the desired shape. $\square$

$$
\begin{array}{ll}
c & \\
\big\downarrow & p_2 \circ p_1 \\
d & \\
\big\downarrow & s_\Delta \text{ or nothing} \\
e \text{ with } l_1(e) \leq l_2(e) & \\
\big\downarrow & \text{filling}(2 \to 1) \\
f & \\
\big\downarrow & \text{dividing} \\
g & \\
\big\downarrow & \text{filling}(2 \to 1) \\
h &
\end{array}
$$

figure 2.12: the sequence of transformations

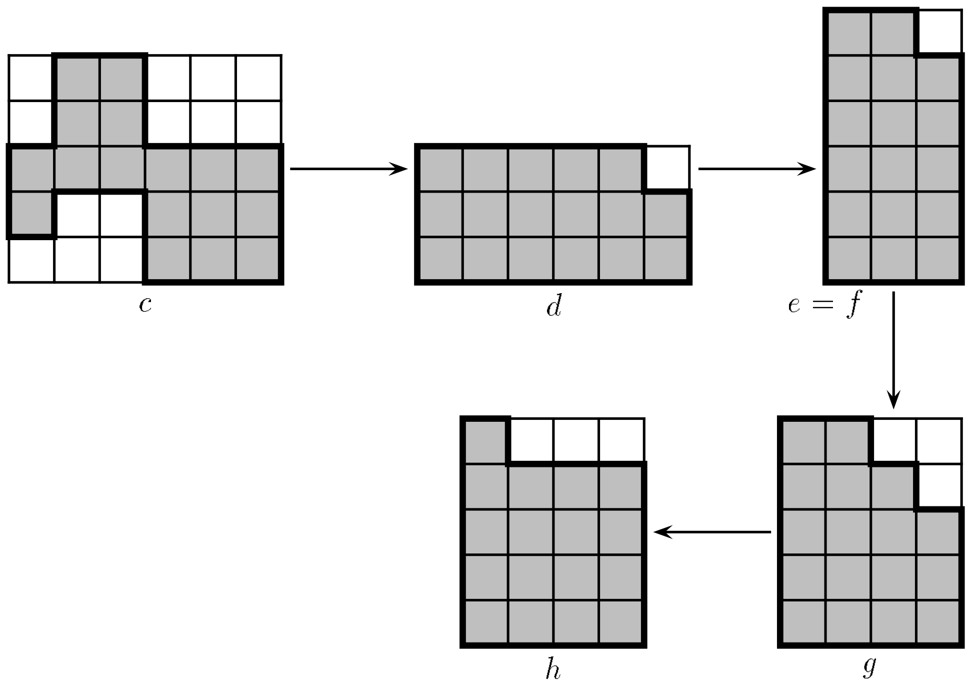

figure 2.13: an example

Figure 2.13 shows the action of the sequence of transformations. Notice that the starting polyomino c is not minimal: this is only to emphasize the role of the projections.

Proposition 2.3. *For each integer n there exists a unique 3–uple (l, k, ϵ) such that*

$$\epsilon \in \{0,1\}, \quad 0 \le k < l+\epsilon \quad \textit{and} \quad n = l(l+\epsilon)+k.$$

Proof. Fix a value of l. When ϵ and k vary in $\{0,1\}\times\{0\cdots l+\epsilon-1\}$ the quantity $l(l+\epsilon)+k$ takes exactly all the values in $\{l^2\cdots(l+1)^2-1\}$. Thus the decomposition exists and moreover l is unique, necessarily equal to $\lfloor\sqrt{n}\rfloor$. We remark finally that k is the remainder of the euclidean division of n by $l+\epsilon$. $\square$

Corollary 2.4. *The polyomino obtained at the end of the sequence of transformations does not depend on the polyomino initially chosen in the set M_n.*

Throughout the section, the decomposition of an integer n given by proposition 2.3 will be called "the decomposition" of the integer, without further detail. We can now easily compute the minimal perimeter.

Corollary 2.5. *The minimal perimeter $P(n)$ of a polyomino of area n is*

$$P(n) = \begin{cases} 2(2l+\epsilon) & \textit{if } k=0 \\ 2(2l+\epsilon)+2 & \textit{if } k>0 \end{cases}$$

where (l,k,ϵ) is the unique 3–uple satisfying $n = l(l+\epsilon)+k$, $\epsilon \in \{0,1\}$, $k < l+\epsilon$.

Corollary 2.6. *(discrete isoperimetric inequality)*
For any polyomino c, we have $P(c) \ge P(V(c))$, where $P(c)$ is the perimeter of c, $V(c)$ its volume and the function $n \mapsto P(n)$ is defined in corollary 2.5.

The canonical, standard and principal polyominoes. Proposition 2.3 and corollary 2.4 allow us to define a canonical polyomino m_n belonging to M_n. Let $n = l(l+\epsilon)+k$ be the decomposition of n. We set

$$m_n = \begin{cases} l \times l +_1 1 \times k & \text{if } \epsilon = 0 \\ (l+1)\times l +_2 k \times 1 & \text{if } \epsilon = 1 \end{cases}$$

The polyomino m_n is called the canonical polyomino of area n.

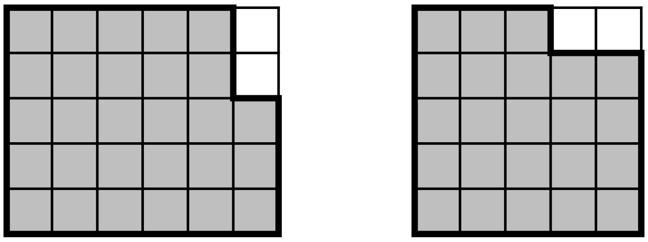

figure 2.14: the canonical polyominoes m_{28}, m_{23}

For c a polyomino, we denote by $\overline{c}$ its equivalence class modulo the planar isometries which leave the integer lattice $\mathbb{Z}^2$ invariant, that is modulo the symmetries s_Δ (with respect to the diagonal Δ), s_1 (with respect to the axis orthogonal to e_1), s_2 (with respect to the axis perpendicular to e_2). By $\overline{c}^{12}$ we denote the equivalence class modulo the two symmetries s_1 and s_2. If A is a subset of C, we put

$$\overline{A} = \bigcup_{c\in A} \overline{c}, \qquad \overline{A}^{12} = \bigcup_{c\in A} \overline{c}^{12}.$$

The set S_n of the standard polyominoes is

$$S_n \;=\; \begin{cases} \overline{l\times l \oplus_1 1\times k} & \text{if } \epsilon = 0 \\ \overline{(l+1)\times l \oplus_2 k\times 1} & \text{if } \epsilon = 1 \end{cases}$$

The set $\widetilde{M}_n$ of the principal polyominoes is

$$\widetilde{M}_n \;=\; \overline{l\times(l+\epsilon)\oplus_1 1\times k} \bigcup \overline{l\times(l+\epsilon)\oplus_2 k\times 1}.$$

The sets S_n and $\widetilde{M}_n$ coincide only when ϵ is zero. Clearly $\{m_n\}\subset S_n\subset \widetilde{M}_n\subset M_n$. Figure 2.15 shows that the inclusions might be strict.

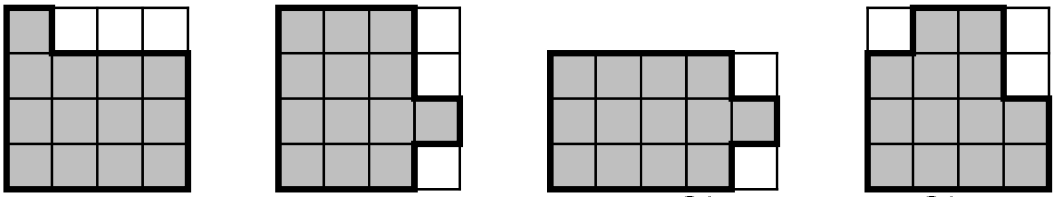

figure 2.15: elements of m_{13}, $S_{13}\setminus m_{13}$, $\widetilde{M}_{13}\setminus S_{13}$, $M_{13}\setminus\widetilde{M}_{13}$

In general, the set M_n is much larger than $\widetilde{M}_n$. It turns out that it is not the case for specific values of n.

Theorem 2.7. *The set of the minimal polyominoes M_n coincides with the set of the principal polyominoes $\widetilde{M}_n$ if and only if the integer n is of the form $l^2, l(l+1)-1, l(l+1), (l+1)^2-1$.*

Proof. Suppose first that M_n is equal to $\widetilde{M}_n$. This means that, modulo planar isometries and moves of the row $k\times 1$ or the column $1\times k$, the set M_n is reduced to the two polyominoes $(l+\epsilon)\times l +_2 k\times 1$ and $(l+\epsilon)\times l +_1 1\times k$ (which are isometric when $\epsilon=0$). If k is not equal to 0 or $l+\epsilon-1$, the polyomino $(l+\epsilon-1)\times 1 +_2^{-1} (l+\epsilon)\times(l-1) +_2 (k+1)\times 1$ belongs also to the set M_n.

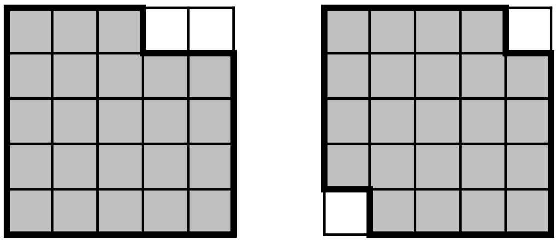

figure 2.16: two elements of M_{23}

Therefore k is equal to 0 or to $l+\epsilon-1$ and the integer n is of the form $l(l+\epsilon)$ or $l(l+\epsilon)-1$.
Conversely, we will examine for these particular values of n the possible actions of the sequence of transformations. That is, we will seek the antecedents of the final polyomino obtained at the end of the sequence. The main idea is that we started the sequence of transformations with a polyomino belonging to M_n so that the perimeter of the polyomino cannot change throughout the whole sequence.

- $n = l^2$. We have $\text{fill}(2\to 1)^{-1}(l\times l)\cap M_n = \{\, l\times l\,\}$ (if the filling has emptied a row to yield a square, there must have been a decrease of perimeter). Moreover,

$$\text{dividing}^{-1}(l\times l)\cap M_n = \{\, l\times l\,\}, \qquad (p_2 o p_1)^{-1}(l\times l)\cap M_n = \{\, l\times l\,\}.$$

Thus $M_{l^2} = \{\, l\times l\,\}$.

- $n = l(l+1)-1 = l^2+l-1$. We have

$$\begin{aligned} \text{fill}(2\to 1)^{-1}(l\times l +_2 (l-1)\times 1) &= \{\, l\times l +_2 (l-1)\times 1\,\} \\ \text{dividing}^{-1}(l\times l +_2 (l-1)\times 1) &= \{\, l\times l +_2 (l-1)\times 1\,\}, \\ s_\Delta^{-1}(l\times l +_2 (l-1)\times 1) &= \{\, l\times l +_1 1\times (l-1)\,\}, \end{aligned}$$

and also $(p_2 o p_1)^{-1}(\{\, l\times l +_2 (l-1)\times 1,\ l\times l +_1 1\times(l-1)\,\})\cap M_n = \widetilde{M}_{l^2+l-1}$ so that finally $M_{l(l+1)-1} = \widetilde{M}_{l(l+1)-1}$.

- $n = l(l+1)$. This case is similar to the previous one.
- $n = (l+1)^2-1 = l(l+1)+l$. We have

$$\begin{aligned} \text{fill}(2\to 1)^{-1}(l\times (l+1) +_2 l\times 1) &= \{\, l\times (l+1) +_2 l\times 1\,\} \\ \text{dividing}^{-1}(l\times (l+1) +_2 l\times 1) &= \{\, l\times (l+1) +_2 l\times 1\,\}, \\ (s_\Delta \circ p_1 \circ p_2)^{-1}(l\times (l+1) +_2 l\times 1) &= \widetilde{M}_{(l+1)^2-1}. \end{aligned}$$

We have thus checked that $M_n = \widetilde{M}_n$ for all these values of n. $\square$

As a by–product of this proof, we obtain the following corollary.

Corollary 2.8. *The set M_n is reduced to $\{ m_n \}$ if and only if n is of the form l^2. The set M_n coincides with S_n if and only if n is of the form $l^2, l(l+1)-1, l(l+1)$, in which case $S_n = \overline{m}_n$.*

Moves through the minimal polyominoes. We are interested in moving through the polyominoes by adding or removing one unit square at a time. How far is it possible to travel in this way through the minimal polyominoes?
Let us define three kernels q, q_-, q_+ on $C \times C$. First

$$\forall c, d \in C \qquad q_-(c,d) = \begin{cases} 1 & \text{if } d \subset c \text{ and } |c \setminus d| = 1 \\ 0 & \text{otherwise} \end{cases}$$

that is, $q_-(c,d) = 1$ if d may be obtained by removing a unit square from c, and $q_-(c,d) = 0$ otherwise. Next, we put $q_+(c,d) = q_-(d,c)$, that is $q_+(c,d) = 1$ if d may be obtained by adding a unit square to c, and $q_+(c,d) = 0$ otherwise. Finally we set $q(c,d) = q_-(c,d) + q_+(c,d)$ so that $q(c,d) = 1$ if the polyominoes differ by a unit square, and $q(c,d) = 0$ otherwise. Two polyominoes c, d are said to communicate if $q(c,d) = 1$. If Y is a subset of C and c is a polyomino, we set $q(c,Y) = 1$ if c communicates with at least one element of Y and $q(c,Y) = 0$ otherwise. The quantities $q_-(c,Y), q_+(c,Y)$ are defined similarly.

Definition 2.9. A corner of a polyomino c is a unit square of c having at least two edges belonging to the boundary of c.

Theorem 2.10. *Let l_1, l_2 be two integers such that the rectangle $l_1 \times l_2$ is minimal. Let $l_1 l_2 = l(l+\epsilon) + k$ be the decomposition of $l_1 l_2$. Any polyomino obtained by removing successively $m < k$ corners from $l_1 \times l_2$ is minimal.*

Proof. The removal of a corner cannot increase the perimeter of a polyomino. The perimeter of the canonical polyomino of area $l_1 l_2 - m$ (with $m < k$) is $2(2l+\epsilon)+2 = 2(l_1+l_2)$, so that a polyomino obtained after the removal of $m < k$ corners from $l_1 \times l_2$ is still minimal. $\square$

Theorem 2.11. *(characterization of the minimal polyominoes)*
All the minimal polyominoes can be obtained by removing successively m corner squares from a minimal rectangle $l_1 \times l_2$, where $m < k$, $l_1 l_2 = l(l+\epsilon) + k$.

Proof. Let c belong to M_n. The smallest rectangle $l_1 \times l_2$ surrounding c is necessarily minimal (by theorem 2.1). Let $l_1 l_2 = l(l+\epsilon) + k$ be the decomposition of $l_1 l_2$. Either $n = l(l+\epsilon)$ and c is a quasisquare or $n > l(l+\epsilon)$, in which case the result holds. $\square$

The next lemmas describe the way we can move starting from a canonical polyomino m_n.

Lemma 2.12. *Let ι be a planar isometry. For any n not of the form $l^2, l(l+1)$, $\iota(m_{n+1})$ is the unique polyomino of M_{n+1} which communicates with $\iota(m_n)$.*

Lemma 2.13. *For n of the form $l^2, l(l+1)$, we have*

$$\begin{aligned}\{\, c \in M_{n-1} : q_-(M_n, c) = 1 \,\} &= S_{n-1},\\ \{\, c \in M_{n+1} : q_+(M_n, c) = 1 \,\} &= \widetilde{M}_{n+1}.\end{aligned}$$

Lemma 2.14. *For n not of the form $l^2, l(l+1)$, we have*

$$\begin{aligned}\{\, c \in M_{n-1} : q_-(S_n, c) = 1 \,\} &\supset S_{n-1},\\ \{\, c \in M_{n+1} : q_+(S_n, c) = 1 \,\} &= S_{n+1},\\ \{\, c \in M_{n-1} : q_-(\widetilde{M}_n \setminus S_n, c) = 1 \,\} &\supset \widetilde{M}_{n-1} \setminus S_{n-1},\\ \{\, c \in M_{n+1} : q_+(\widetilde{M}_n \setminus S_n, c) = 1 \,\} &= \widetilde{M}_{n+1} \setminus S_{n+1}.\end{aligned}$$

Lemma 2.15. *The rectangle $l \times (l+2)$ is minimal but cannot grow and stay minimal. More precisely, we have $q_+(l \times (l+2), M_{l(l+2)+1}) = 0$.*

Proposition 2.16. *Except the quasisquares, no rectangle can grow and stay minimal.*

Proof. Let $l_1 \times l_2 = l_1 \times (l_1 + r)$ be a minimal rectangle. Such a rectangle can grow and stay minimal if and only if the decomposition of $l_1 \times l_2$ is $l_1(l_1 + r) = m(m + \epsilon)$, and $2l_1 + r = 2m + \epsilon$. Thus $l_1^2 + rl_1 - m(m + \epsilon) = 0$. Solving this equation, we get $2l_1 = -r + \sqrt{r^2 + 4m(m + \epsilon)}$ whence $2m + \epsilon = \sqrt{(2m + \epsilon)^2 + r^2 - \epsilon^2}$, implying finally $r = 0$ or $r = 1$. $\square$

Lemma 2.17. *Suppose n is not of the form $l(l+1)$.*
The polyominoes of M_{n+1} communicating with S_n belong to S_{n+1}.
The polyominoes of M_{n+1} communicating with $\widetilde{M}_n \setminus S_n$ belong to $\widetilde{M}_{n+1} \setminus S_{n+1}$.

Proof. This is just a restatement of partial results of lemmas 2.13 and 2.14. $\square$

A sequence $c_n, \cdots, c_m$ of polyominoes is increasing if $q_+(c_j, c_{j+1}) = 1$ for all j.

Lemma 2.18. *Suppose $n = l(l+1)$. Let c belong to S_n and suppose there is an increasing sequence of minimal polyominoes $c_n, \cdots, c_m$ such that $c_n = c$. Either c_{n+1} belongs to S_{n+1} or m is strictly less than $(l+1)^2$; in this last case, none of the polyominoes $c_{n+1}, \cdots, c_m$ is standard, and they are all principal.*

Proof. Suppose c_{n+1} is not standard i.e. $c_{n+1} \notin S_{n+1}$. Thus, we have $c_{n+1} = \iota((l+1) \times l +_1^i 1 \times 1)$ for some isometry ι and for some i, $0 \le i \le l-1$. Necessarily, for all k smaller than $\max(l-1, m-n)$, $c_{n+k} = \iota((l+1) \times l) +_1^i 1 \times k$ for some i, $0 \le i \le l-k$. None of these polyominoes is standard. Moreover lemma 2.15 implies that $m \le n + l = (l+1)^2 - 1$. $\square$

We eventually sum up several facts of interest in the next theorems.

Theorem 2.19. *The principal polyominoes can be completely shrinked through the minimal polyominoes: for any integer n and for any principal polyomino c in $\widetilde{M}_n$, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$ and $c_n = c$.*

Theorem 2.20. *The standard polyominoes can be grown or shrinked arbitrarily far through minimal polyominoes: for any integers n, m and for any standard polyomino c in S_m, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$ and $c_m = c$.*

Theorem 2.21. *Let $c_0, \cdots, c_n$ be an increasing sequence of minimal polyominoes starting from the empty polyomino ($c_0 = \emptyset$). If c_n is a standard polyomino (i.e. belongs to S_n) then all the polyominoes of the sequence are standard (i.e. $c_j \in S_j$ for all $j \leq n$).*

Proof. This result is a straightforward consequence of lemma 2.18. □

Proposition 2.22. *The infinite sequence $S_0, \cdots, S_n, \cdots$ of the sets of the standard polyominoes is the greatest sequence of subsets of the infinite sequence $M_0, \cdots, M_n, \cdots$ of the sets of the minimal polyominoes enjoying the properties stated in theorems 2.20 and 2.21.*

Proof. Let $S'_0, \cdots, S'_n, \cdots$ be a sequence included in $M_0, \cdots, M_n, \cdots$ for which the theorems 2.20 and 2.21 hold. Suppose there exists n such that $S'_n \not\subset S_n$. Let c belong to $S'_n \setminus S_n$. Let $l_1 \times l_2$ be the smallest rectangle surrounding c. A growing sequence of minimal polyominoes starting from c necessarily reaches $l_1 \times l_2$. By proposition 2.16, this rectangle can grow and stay minimal if and only if it is a quasisquare. Thus $l_1 \times l_2$ has to be a quasisquare. Suppose for instance $l_1 \times l_2 = l \times (l+1)$ (the other cases are similar). Since c can be obtained by growing the empty polyomino through minimal polyominoes, it contains necessarily the square l^2 i.e. $l^2 \subset c \subset l(l+1)$ and it follows that c is standard. □

Shrinking or growing a rectangle. We finally investigate the following problem. What is the best way to shrink or to grow a rectangle?
Let l_1, l_2, k be integers. We define

$$\begin{aligned} M(l_1 \times l_2, -k) &= \{\, c \in C_{l_1 l_2 - k} : c \subset l_1 \times l_2,\ P(c) \text{ minimal} \,\}, \\ M(l_1 \times l_2, +k) &= \{\, c \in C_{l_1 l_2 + k} : l_1 \times l_2 \subset c,\ P(c) \text{ minimal} \,\}. \end{aligned}$$

The set $M(l_1 \times l_2, -k)$ is the set of the polyominoes of minimal perimeter which can be obtained by removing k squares from $l_1 \times l_2$. Analogously, $M(l_1 \times l_2, +k)$ is the set of the polyominoes of minimal perimeter which can be obtained by adding k squares to $l_1 \times l_2$. A natural way to remove (add) k squares (for $k < l_1, k < l_2$) is to remove (add) a bar on a side of the rectangle; thus we define

$$\begin{aligned} S(l_1 \times l_2, -k) &= \overline{(l_1 - 1) \times l_2 \oplus_1 1 \times (l_2 - k)}^{12} \bigcup \overline{l_1 \times (l_2 - 1) \oplus_2 (l_1 - k) \times 1}^{12}, \\ S(l_1 \times l_2, k) &= \overline{\{\, l_1 \times l_2 \oplus_2 k \times 1,\, l_1 \times l_2 \oplus_1 1 \times k \,\}}^{12}. \end{aligned}$$

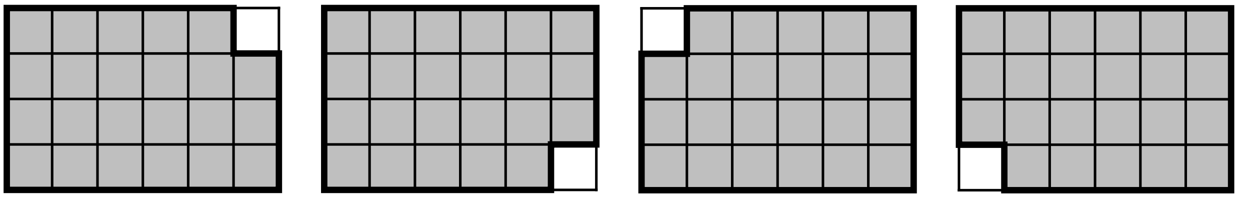

figure 2.17: the set $M(6 \times 4, -1)$

Figure 2.17 shows the set $M(6 \times 4, -1)$. Figure 2.18 shows the set $M(5 \times 5, -2)$ which contains the set S_{23}. In these cases, we see that $M(6 \times 4, -1) = S(6 \times 4, -1)$ anyway this does not occur in general : for instance $M(5 \times 5, -2) \supsetneq S(5 \times 5, -2)$.

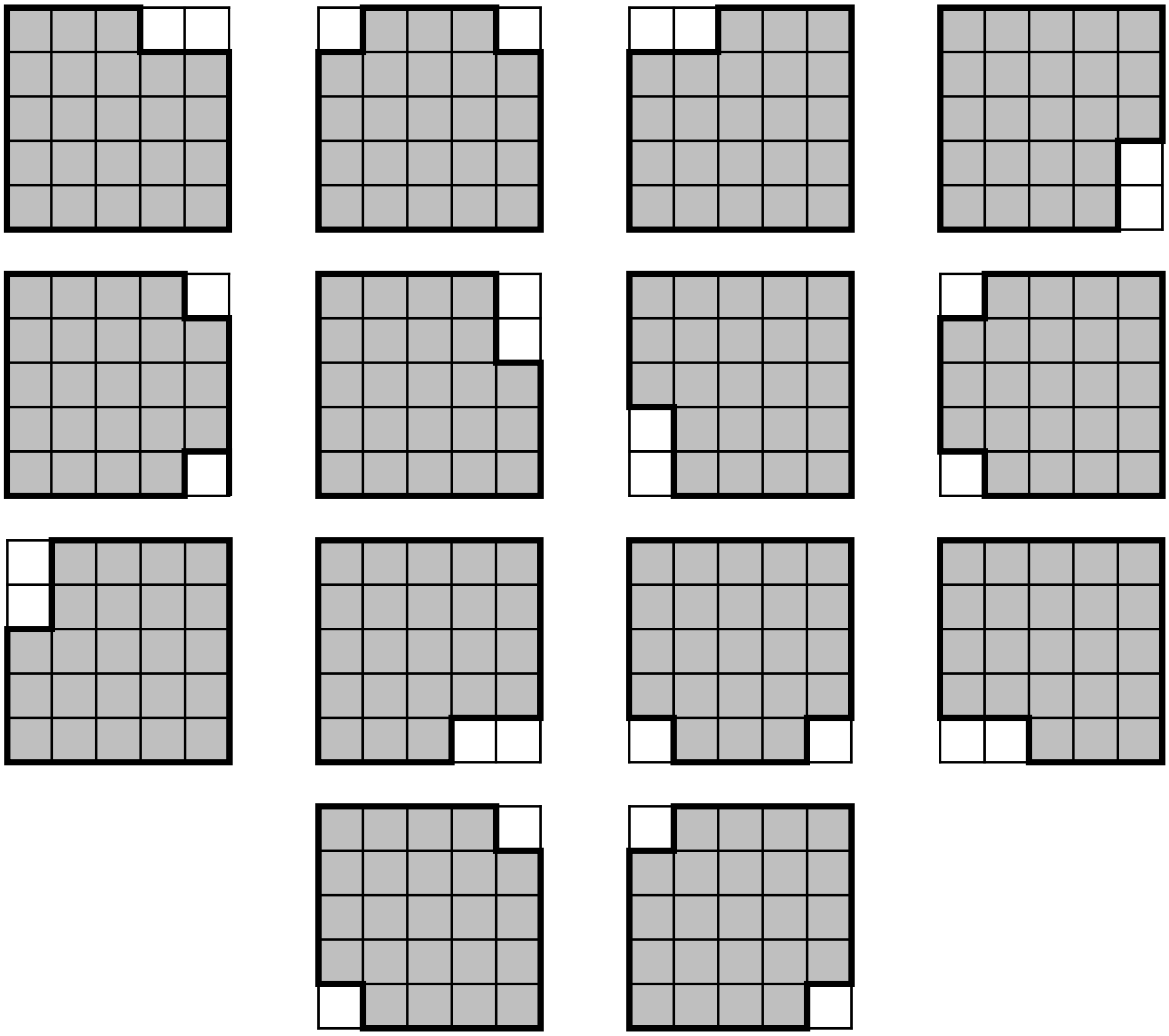

figure 2.18: the set $M(5 \times 5, -2) \supsetneq S_{23}$

Theorem 2.23. *Let l_1, l_2, k be positive integers such that $k < l_1$, $k < l_2$. The set $M(l_1 \times l_2, -k)$ is the set of the polyominoes obtained by removing successively k corner squares from $l_1 \times l_2$. In particular, $S(l_1 \times l_2, -k)$ is included in $M(l_1 \times l_2, -k)$.*

Proof. Such an operation leaves the perimeter unchanged. Moreover, the perimeter of a polyomino of $M(l_1 \times l_2, -k)$ is necessarily $2(l_1 + l_2)$ since there remains at least one square in each row and each column of the rectangle after the removal of k squares. □

Theorem 2.24. *Let l_1, l_2, k be positive integers such that $k < l_1$, $k < l_2$. The set $M(l_1 \times l_2, +k)$ is equal to the set $S(l_1 \times l_2, k)$.*

Proof. The perimeter of a polyomino of $M(l_1 \times l_2, +k)$ is greater or equal than $2(l_1 + l_2) + 2$ (since it contains $l_1 \times l_2$). The polyominoes of $S(l_1 \times l_2, k)$ have this perimeter, so that the minimal perimeter is exactly $2(l_1 + l_2) + 2$ and $S(l_1 \times l_2, k) \subset M(l_1 \times l_2, +k)$. Obviously, the polyominoes of $S(l_1 \times l_2, k)$ are the only ones satisfying the requirements. □

3. The three dimensional case

We denote by (e_1, e_2, e_3) the canonical basis of the integer lattice $\mathbb{Z}^3$. A unit cube is a cube of volume one, whose center belongs to $\mathbb{Z}^3$ and whose vertices belong to the dual lattice $(\mathbb{Z}+1/2)^3$. We do not distinguish between a unit cube and its center: thus (x_1, x_2, x_3) denotes the unit cube of center (x_1, x_2, x_3). A three dimensional polyomino is a finite union of unit cubes. It is defined up to a translation. The volume $V(c)$ of a polyomino c is the number of its unit cubes. We denote by $\mathcal{C}_n$ the set of the polyominoes of volume n and by $\mathcal{C}$ the set of all polyominoes. The area $A(c)$ of a polyomino c is the number of two dimensional unit squares belonging to the boundary of only one unit cube of c. Notice that the area is an even integer.

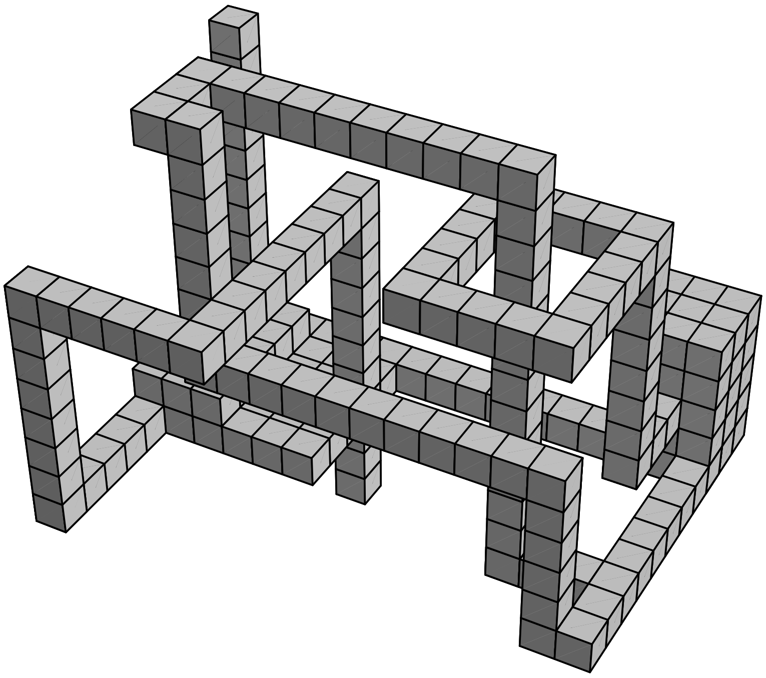

figure 3.1: a 3D polyomino

We wish to investigate the set $\mathcal{M}_n$ of the polyominoes of $\mathcal{C}_n$ having a minimal area. A polyomino c is said to be minimal if it belongs to the set $\mathcal{M}_{V(c)}$. Figure 3.2 shows elements of the sets $\mathcal{M}_n$ for small values of n.

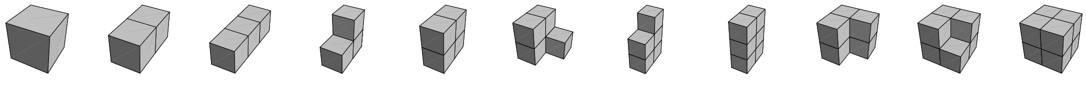

figure 3.2: the sets $\mathcal{M}_1, \cdots, \mathcal{M}_8$

When n becomes larger, the structure of $\mathcal{M}_n$ becomes very complex:

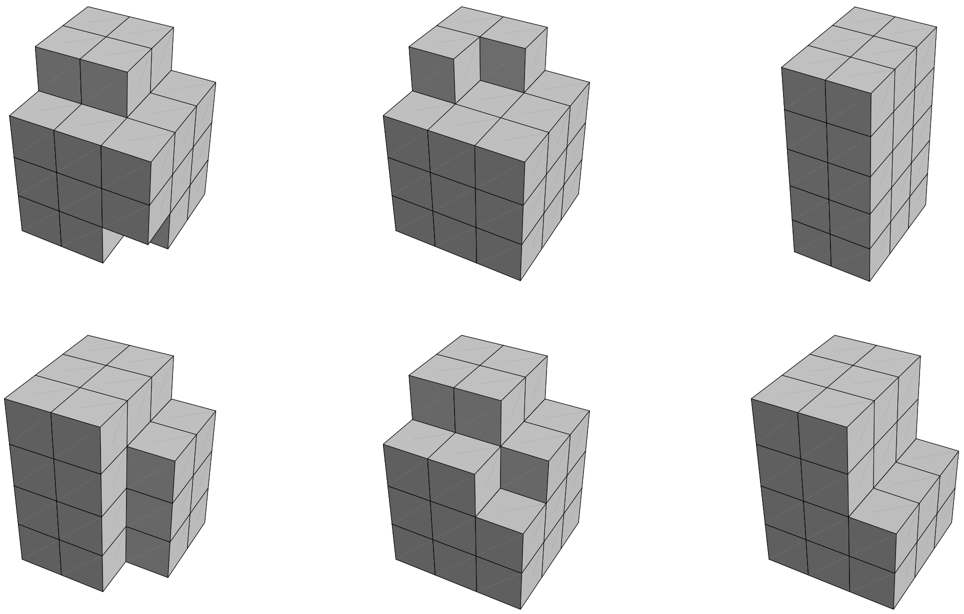

figure 3.3: some elements of $\mathcal{M}_{30}$

The lengths, the bars and the slices. Let c be a polyomino.
We define its lengths $j_1(c), j_2(c), j_3(c)$ along each axis by

$$\begin{aligned} j_1(c) &= \operatorname{card}\{\, x_1 \in \mathbb{Z} : \exists (x_2, x_3) \ \ (x_1, x_2, x_3) \in c \,\}, \\ j_2(c) &= \operatorname{card}\{\, x_2 \in \mathbb{Z} : \exists (x_1, x_3) \ \ (x_1, x_2, x_3) \in c \,\}, \\ j_3(c) &= \operatorname{card}\{\, x_3 \in \mathbb{Z} : \exists (x_1, x_2) \ \ (x_1, x_2, x_3) \in c \,\}. \end{aligned}$$

A three dimensional polyomino is said to be planar with normal vector e_i if its j_i–length is equal to one. Such a polyomino might effectively be seen as a two dimensional polyomino by transforming its unit cubes into unit squares (and keeping the orientation induced in the plane by the vector e_i). Conversely, given a vector e_i of the basis, we may see any two dimensional polyomino as a planar three dimensional polyomino with normal vector e_i. We simply transform the unit squares into unit cubes and rotate the polyomino so that its normal vector becomes e_i. This trick will be used several times in the sequel. Let α, β be two integers. We define the bars $b_1(c,\alpha,\beta), b_2(c,\alpha,\beta), b_3(c,\alpha,\beta)$ by

$$\begin{aligned} b_1(c,\alpha,\beta) &= \{\, (x_1, x_2, x_3) \in c : (x_2, x_3) = (\alpha, \beta) \,\}, \\ b_2(c,\alpha,\beta) &= \{\, (x_1, x_2, x_3) \in c : (x_1, x_3) = (\alpha, \beta) \,\}, \\ b_3(c,\alpha,\beta) &= \{\, (x_1, x_2, x_3) \in c : (x_1, x_2) = (\alpha, \beta) \,\}. \end{aligned}$$

The bars are one dimensional sections of the polyomino.
Let γ be an integer. We define the slices $s_1(c,\gamma), s_2(c,\gamma), s_3(c,\gamma)$ by

$$\begin{aligned} s_1(c,\gamma) &= \{ (x_1,x_2,x_3) \in c : x_1 = \gamma \}, \\ s_2(c,\gamma) &= \{ (x_1,x_2,x_3) \in c : x_2 = \gamma \}, \\ s_3(c,\gamma) &= \{ (x_1,x_2,x_3) \in c : x_3 = \gamma \}. \end{aligned}$$

The slices are two dimensional sections of the polyomino.

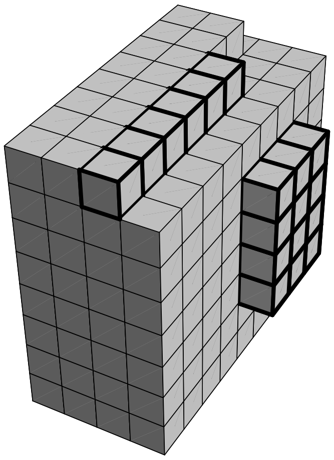

figure 3.4: the bar $b_1(c,2,7)$ and the slice $s_2(c,4)$

The extreme slices $s_1^*(c), s_2^*(c), s_3^*(c)$ are the slices associated to the lengths $j_1(c), j_2(c), j_3(c)$

$$s_1^*(c) = s_1(c, j_1(c)-1), \quad s_2^*(c) = s_2(c, j_2(c)-1), \quad s_3^*(c) = s_3(c, j_3(c)-1).$$

Addition of polyominoes. We describe here a simple mechanism for adding polyominoes. For the sake of clarity, we need to work here with instances of the polyominoes having a definite position on the lattice $\mathbb{Z}^3$ i.e. we temporarily remove the indistiguishability modulo translations. Let c be a polyomino. By $c(x_1,x_2,x_3)$ we denote the unique polyomino obtained by translating c in such a way that

$$\begin{aligned} \min\{ y_1 : \exists (y_2,y_3) \quad (y_1,y_2,y_3) \in c(x_1,x_2,x_3) \} &= x_1, \\ \min\{ y_2 : \exists (y_1,y_3) \quad (y_1,y_2,y_3) \in c(x_1,x_2,x_3) \} &= x_2, \\ \min\{ y_3 : \exists (y_1,y_2) \quad (y_1,y_2,y_3) \in c(x_1,x_2,x_3) \} &= x_3. \end{aligned}$$

Remark. When we deal with polyominoes up to translations, we normally work with the polyominoes $c(0,0,0)$, for any c in $\mathcal{C}$.

We define an operator $+_1$ from $\mathcal{C} \times (\mathcal{C} \cup C)$ to $\mathcal{C}$ (we recall that C is the set of the two dimensional polyominoes). First, on $\mathcal{C} \times \mathcal{C}$, we set

$$\forall c, d \in \mathcal{C} \qquad c +_1 d \;=\; c(0,0,0) \cup d(j_1(c), 0, 0).$$

Let now c belong to $\mathcal{C}$ and d belong to C (that is, d is a two dimensional polyomino). We define the three dimensional polyomino $c{+_1}d$ as follows. First, we transform d into a planar three dimensional polyomino d' by replacing its squares by unit cubes. We rotate d' so that its normal unit vector becomes e_1 (as if the two dimensional polyomino d was initially included in the plane (e_2, e_3)). Then we use the previous definition to set $c +_1 d = c +_1 d'$. The operators $+_2$ and $+_3$ from $\mathcal{C} \times (\mathcal{C} \cup C)$ to $\mathcal{C}$ are defined similarly. For instance, on $\mathcal{C} \times \mathcal{C}$, we set $c +_2 d = c(0,0,0) \cup d(0, j_2(c), 0)$ and $c +_3 d = c(0,0,0) \cup d(0,0,j_3(c))$.
More generally, given two integers α and β, we put for c, d in $\mathcal{C}$

$$\begin{aligned}
c +_1 (\alpha,\beta)\, d &= c(0,0,0) \cup d(j_1(c), \alpha, \beta),\\
c +_2 (\alpha,\beta)\, d &= c(0,0,0) \cup d(\alpha, j_2(c), \beta),\\
c +_3 (\alpha,\beta)\, d &= c(0,0,0) \cup d(\alpha, \beta, j_3(c)).
\end{aligned}$$

In the case d is a two dimensional polyomino, $c +_i (\alpha,\beta)\, d$ is defined analogously, working with the translated polyomino $d(\alpha, \beta)$ (this is a translation into the plane containing d).
Sometimes we will use the operator $+$ without specifying the direction: it will mean that this direction is in fact indifferent i.e. the statements hold for $+_1, +_2, +_3$ simultaneously.
Finally we define three operators $\oplus_1$, $\oplus_2$, $\oplus_3$ from $\mathcal{C} \times (\mathcal{C} \cup C)$ to $\mathcal{P}(\mathcal{C})$, the set of the subsets of $\mathcal{C}$, by

$$\begin{aligned}
c \oplus_1 d &= \{\, c +_1 (\alpha,\beta)\, d : j_2(d) + \alpha \leq j_2(c),\; j_3(d) + \beta \leq j_3(c) \,\},\\
c \oplus_2 d &= \{\, c +_2 (\alpha,\beta)\, d : j_1(d) + \alpha \leq j_1(c),\; j_3(d) + \beta \leq j_3(c) \,\},\\
c \oplus_3 d &= \{\, c +_3 (\alpha,\beta)\, d : j_1(d) + \alpha \leq j_1(c),\; j_2(d) + \beta \leq j_2(c) \,\}.
\end{aligned}$$

Notice that $c \oplus_1 d$ (respectively $c \oplus_2 d$, $c \oplus_3 d$) is empty whenever $j_2(d) > j_2(c)$ or $j_3(d) > j_3(c)$ (resp. $j_1(d) > j_1(c)$ or $j_3(d) > j_3(c)$, $j_1(d) > j_1(c)$ or $j_2(d) > j_2(c)$).

The basic polyominoes. We describe here some simple shapes of polyominoes of particular interest. Let j_1, j_2, j_3 be three integers. By $j_1 \times j_2 \times j_3$ we denote the parallelepiped whose lengths (with respect to the axis e_1, e_2, e_3) are j_1, j_2, j_3. A parallelepiped $j_1 \times j_2 \times j_3$ is a cube if $j_1 = j_2 = j_3$. It is a quasicube if $|j_1 - j_2| \leq 1$, $|j_2 - j_3| \leq 1, |j_3 - j_1| \leq 1$. Thus the quasicubes are the parallelepipeds $(j + \epsilon_1) \times (j + \epsilon_2) \times (j + \epsilon_3)$ where the ϵ_i's belong to $\{0, 1\}$. The basic three dimensional polyominoes are the polyominoes obtained by adding a two dimensional basic polyomino to a parallelepiped. More precisely, the set $\mathcal{B}$ of the basic polyominoes is

$$\begin{aligned}
\mathcal{B} \;=\; \{\, j_1 \times j_2 \times j_3 +_1 d : d \in B, d \subset j_2 \times j_3 \,\} &\cup \{\, j_1 \times j_2 \times j_3 +_2 d : d \in B, d \subset j_1 \times j_3 \,\}\\
&\cup \{\, j_1 \times j_2 \times j_3 +_3 d : d \in B, d \subset j_1 \times j_2 \,\}.
\end{aligned}$$

We are now in position to state the main result concerning the three dimensional minimal polyominoes.

Theorem 3.1. *For any integer n, the set $\mathcal{M}_n$ of the minimal polyominoes of volume n contains a basic polyomino of the form $j \times (j+\delta) \times (j+\theta) +_3 (l \times (l+\epsilon) +_2 k)$ (i.e. the addition of a quasicube, a quasisquare and a bar) where $\epsilon, \delta, \theta \in \{0,1\}$, $0 \leq k < l+\epsilon$, $l(l+\epsilon)+k < (j+\delta)(j+\theta)$, $n = j(j+\delta)(j+\theta) + l(l+\epsilon) + k$.*

Remark. This statement asserts also the existence of a decomposition of any integer n as $n = j(j+\delta)(j+\theta) + l(l+\epsilon) + k$ where the 6–uple $(j, l, k, \delta, \theta, \epsilon)$ satisfies the above conditions.

Proof. The proof is done in the same spirit as the corresponding two dimensional proof. We choose a polyomino belonging to $\mathcal{M}_n$ and we apply to it a sequence of transformations which regularize the shape of the polyomino until we get a polyomino of the desired form. These transformations leave the volume unchanged and never increase the area, so that the area is constant during the whole process and the final polyomino is still in $\mathcal{M}_n$. We first describe separately each transformation.

Projections p_1, p_2, p_3. These transformations are defined on the set $\mathcal{C}$ of all the polyominoes. Let c belong to $\mathcal{C}$. The projection p_3 consists in letting all the unit cubes of c fall (in the sense of $-e_3$) on a fixed plane orthogonal to e_3, as shown on figure 3.5.

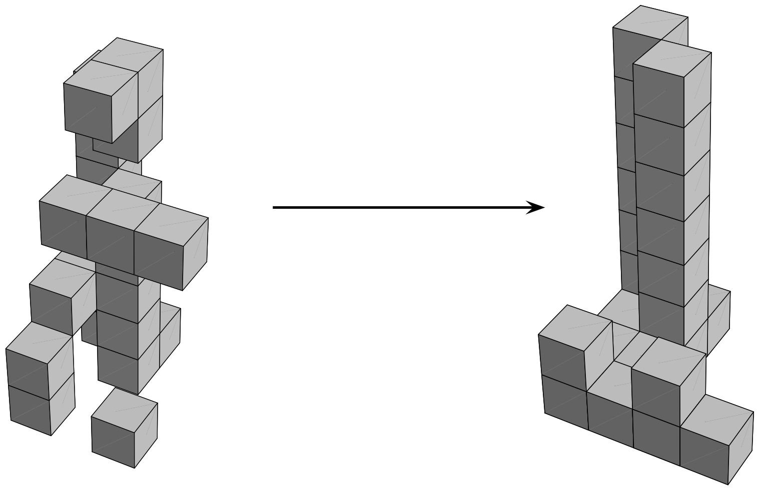

figure 3.5: projection p_3

The projections p_1 and p_2 are defined similarly, using the vectors e_1 and e_2 instead of e_3. The projections satisfy $p_i \circ p_i = p_i, 1 \leq i \leq 3$. Moreover they do not change the volume nor increase the area. A formal proof would rely on a tedious induction and would consist in counting the number of cubes in contact before and after the projections. It is obvious that the number of horizontal contacts is maximal after application of p_3. Moreover the number of vertical contacts between two adjacent columns is also maximal after application of p_3.

Slice–remodelling (sli-rem). This transformation is applied to a polyomino belonging to the set $\mathcal{G} = p_3 \circ p_2 \circ p_1(\mathcal{C})$ of all projected polyominoes. Let c belong to $\mathcal{G}$. We cut c into slices according to the direction of e_3 i.e. we consider all its intersections with the planes of equations $x_3 = \gamma$, which are the slices $s_3(c, \gamma)$. Such a slice $s_3(c, \gamma)$ may be seen as a two dimensional polyomino of area $|s_3(c, \gamma)|$ (we simply transform the unit cubes of $s_3(c, \gamma)$ into unit squares). We then replace $s_3(c, \gamma)$ by the associated two dimensional canonical polyomino $m_{|s_3(c,\gamma)|}$ (in which all the unit squares have been transformed into unit cubes), the orientation being specified by (e_1, e_2). We finally stack up all these new slices i.e. we build the polyomino sli–rem$(c) = m_{|s_3(c,0)|} +_3 m_{|s_3(c,1)|} +_3 \cdots +_3 m_{|s_3^*(c)|}$.

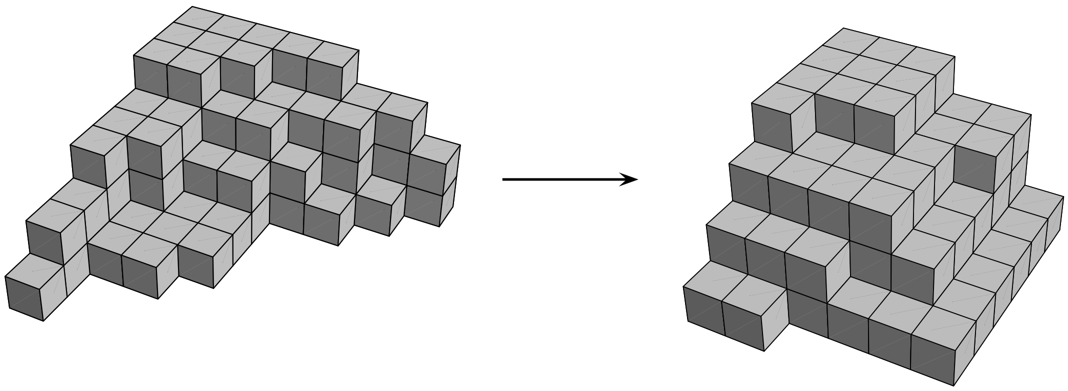

figure 3.6: slice–remodelling

The slice–remodelling does not change the volume nor increase the area. In fact, the number of horizontal contacts between two slices is maximal (equal to the number of cubes of the smallest slice) if the two slices are associated to two dimensional canonical polyominoes. Moreover the image of sli–rem is included in $\mathcal{G} = p_3 \circ p_2 \circ p_1(\mathcal{C})$.

Cube–moving. This transformation moves individually cubes of the polyomino. Let c belong to $\mathcal{G}$. Let (x_1, x_2, x_3) be the empty cell of smallest coordinate for the (e_3, e_1, e_2) lexicographical order such that there are three cubes at $(x_1 - 1, x_2, x_3)$, $(x_1, x_2 - 1, x_3)$, $(x_1, x_2, x_3 - 1)$. The cell (x_1, x_2, x_3) is the cell where a cube of c will be moved. In case $x_3 = j_3(c) - 1$ the cube–moving does nothing. Suppose $x_3 < j_3(c) - 1$. We now define the unit cube of c which is to be moved. We consider the extremal section $s_3^*(c)$ as a two dimensional polyomino. As such, it is a Ferrers diagram i.e. it belongs to the set $\mathcal{F}$ (see the two dimensional projections). The extreme bars $b_1^*(s_3^*(c))$ and $b_2^*(s_3^*(c))$ are well defined as well as the lengths $l_1(s_3^*(c))$, $l_2(s_3^*(c))$. We omit c and $s_3^*(c)$ in these notations and write b_1^*, b_2^*, l_1, l_2 in the following lines. Several cases arise according to the lengths of these bars.

- If $|b_1^*| < |b_2^*|$ we move $(|b_1^*| - 1, l_2 - 1, j_3 - 1)$.
- If $|b_1^*| > |b_2^*|$ we move $(l_1 - 1, |b_2^*| - 1, j_3 - 1)$.
- If $|b_1^*| = |b_2^*|$ and $l_1 \leq l_2$ we move $(|b_2^*| - 1, l_2 - 1, j_3 - 1)$.
- If $|b_1^*| = |b_2^*|$ and $l_1 > l_2$ we move $(l_1 - 1, |b_1^*| - 1, j_3 - 1)$.

We repeat this elementary operation until exhaustion of any possibility (that is, until the smallest empty cell belongs to the extreme slice s_3^*). The procedure necessarily ends since the number of available empty cells below the extreme slice s_3^* decreases.

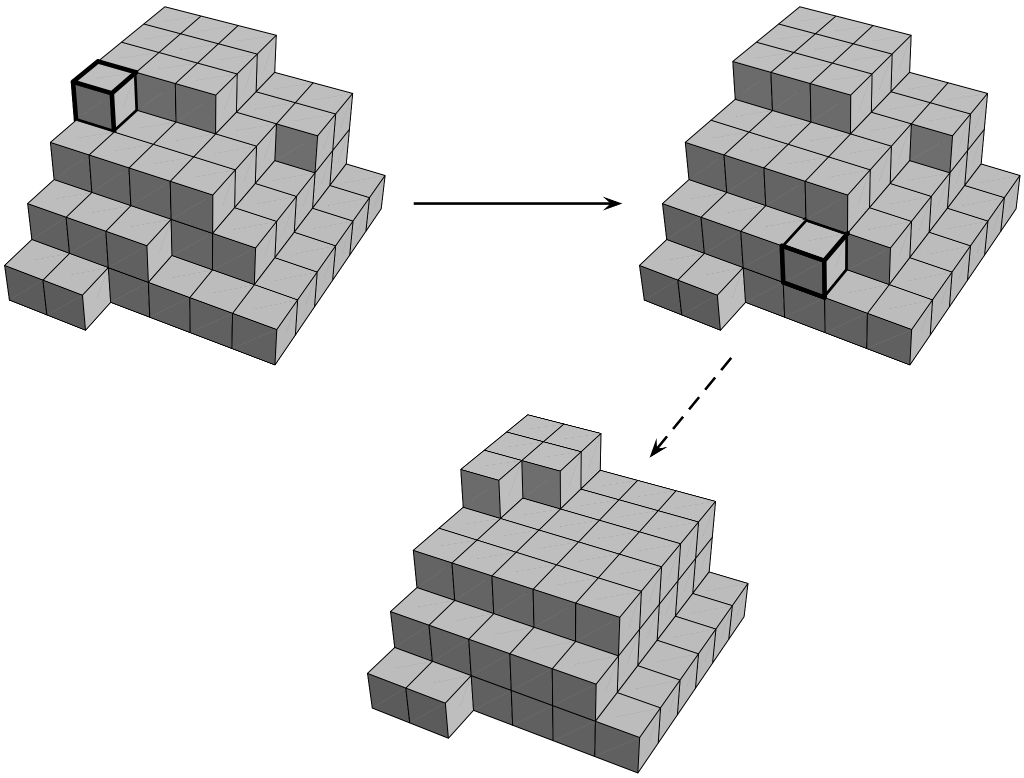

figure 3.7: cube–moving

The cube–moving does not change the volume nor increase the area.

Example. Suppose we apply the cube–moving to a polyomino c belonging to sli–rem($\mathcal{G}$): thus each two dimensional section $s_3(c, \gamma)$ of c is a canonical two dimensional polyomino. Putting $n(x_3) = |s_3(c, x_3)|$ we have $n(0) \geq \cdots \geq n(j_3(c) - 1)$. In this situation, the elementary cube–moving operation amounts to take the only cube belonging to $s_3^*(c)$ so that $s_3^*(c)$ becomes $m_{n(j_3(c))-1}$ and to put it at the smallest empty cell (x_1, x_2, x_3) such that there are three cubes at $(x_1 - 1, x_2, x_3)$, $(x_1, x_2 - 1, x_3)$, $(x_1, x_2, x_3 - 1)$. Notice that the section $s_3(c, x_3) = m_{n(x_3)}$ then becomes $m_{n(x_3)+1}$ so that the polyomino still belongs to sli–rem($\mathcal{G}$). When the cube–moving ends, we obtain a polyomino such that

$$\exists\, r \geq 0, \quad n(0) = \cdots = n(r), \quad \forall i,\, r < i < j_3 - 1, \quad n(i) \text{ is a quasisquare},$$

and all the slices along e_3 are two dimensional canonical polyominoes. Figure 3.8 shows the typical shape of the polyominoes we obtain after having completed the cube–moving.

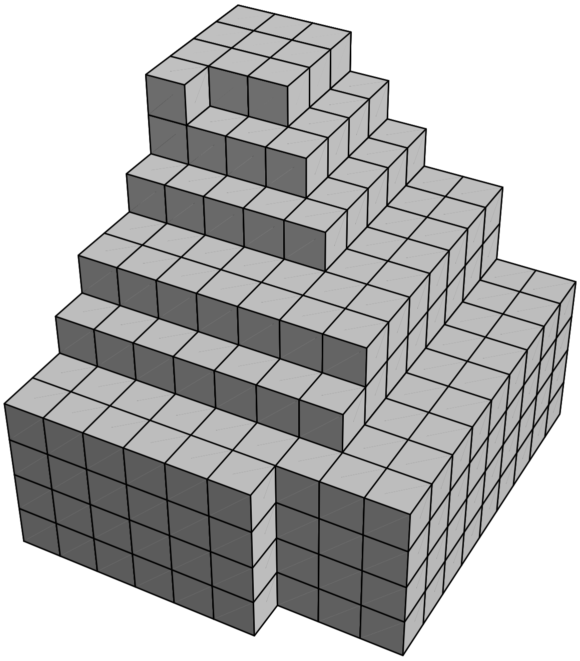

figure 3.8: after cube–moving

Bar–moving. The bar–moving transformation is defined on the set cube–moving($\mathcal{G}$). This transformation moves bars of the polyomino. Let c belong to cube–moving($\mathcal{G}$). The bar to be moved is one of the two extreme bars b_1^* or b_2^* of the extreme slice s_3^*. We first define the bar b which is to be moved.

- if $|b_1^*| < |b_2^*|$ we choose $b = b_1^*$.
- if $|b_1^*| > |b_2^*|$ we choose $b = b_2^*$.
- if $|b_1^*| = |b_2^*|$ and $l_1 \leq l_2$ we choose $b = b_2^*$.
- if $|b_1^*| = |b_2^*|$ and $l_1 > l_2$ we choose $b = b_1^*$.

Notice that the cube chosen to start the cube–moving operation would belong to this bar b. We next search for an appropriate place to move b. Let b_1' be the smallest empty bar $b_1(c, x_2, x_3)$ for the (e_3, e_2) lexicographical order such that the two bars $b_1(c, x_2 - 1, x_3)$ and $b_1(c, x_2, x_3 - 1)$ have a length greater or equal to the length of the bar b. We define b_2' similarly, using the vectors (e_3, e_1). We define b_3' using the vectors (e_1, e_2) (thus b_3' is a vertical bar) and we impose the additional condition that $|b'_3|$ is strictly less than $j_3(c)$. Notice that these bars might not exist. If none of these bars exist, the bar–moving does nothing. If only the bar b'_3 exists we move b to b'_3. If only one bar among b'_1, b'_2 exists, we move b to this bar. Suppose finally that both bars b'_1 and b'_2 do exist. Let $x_3(b'_1), x_3(b'_2)$ be the e_3–coordinate of the bars b'_1, b'_2 (the bars b'_1, b'_2 are one dimensional sections of c along the vectors e_1 and e_2). Several cases arise according to these coordinates and, in case of equality, according to the lengths of the s_3–section of c.

- If $x_3(b'_1) < x_3(b'_2)$ we move b to b'_1.
- If $x_3(b'_1) > x_3(b'_2)$ we move b to b'_2.
- If $x_3(b'_1) = x_3(b'_2)$ and $l_1(s_3(c, x_3)) < l_2(s_3(c, x_3))$ we move b to b'_1.
- If $x_3(b'_1) = x_3(b'_2)$ and $l_1(s_3(c, x_3)) \geq l_2(s_3(c, x_3))$ we move b to b'_2.

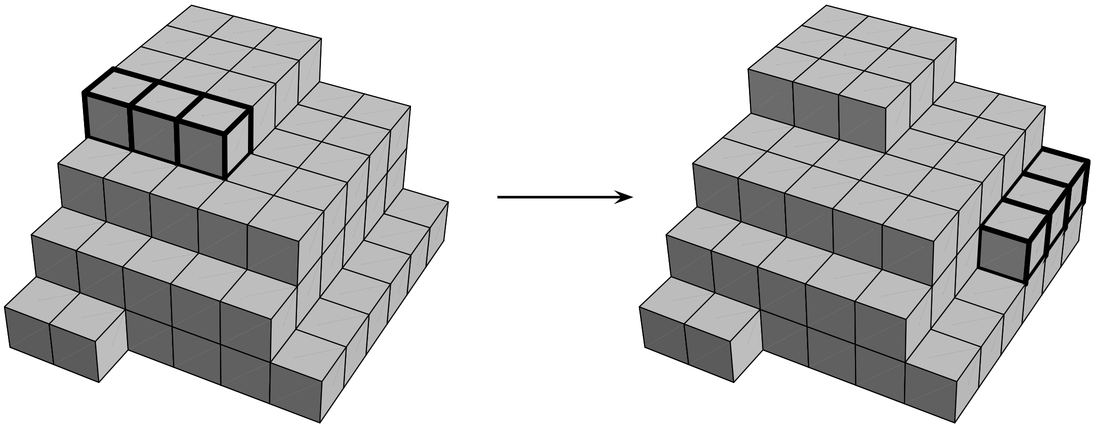

figure 3.9: bar–moving

The choices of the empty bar and of the bar which is moved are done in such a way that bar–moving$\circ$cube–moving$\circ$sli–rem($\mathcal{G}$) is included in sli–rem($\mathcal{G}$). As usual, the bar–moving does not alter the volume nor decrease the area.

Filling. The transformation filling(e_3) is the application, until exhaustion of any possibility, of the transformation bar–moving$\circ$cube–moving. Thus it is applied to polyominoes belonging to the set $\mathcal{G}$. The transformations filling(e_1) and filling(e_2) are defined in the same way: we just make a circular permutation of the axis so that e_1 or e_2 becomes the third vector of the basis and we apply the same scheme. When filling(e_3) is completed, we obtain a polyomino of the form $j_1 \times j_2 \times j_3 +_1 c +_2 d +_3 e$ where e is a two dimensional polyomino of $\mathcal{F}$ and c, d are either empty or they are rectangles such that no bar–moving operation can take place. In case the starting polyomino belongs to sli–rem($\mathcal{G}$), we obtain a polyomino of the form $(j + \epsilon) \times j \times j_3 +_2 d +_3 e$ where e is a two dimensional canonical polyomino, and d is a rectangle such that no bar–moving from e is possible.

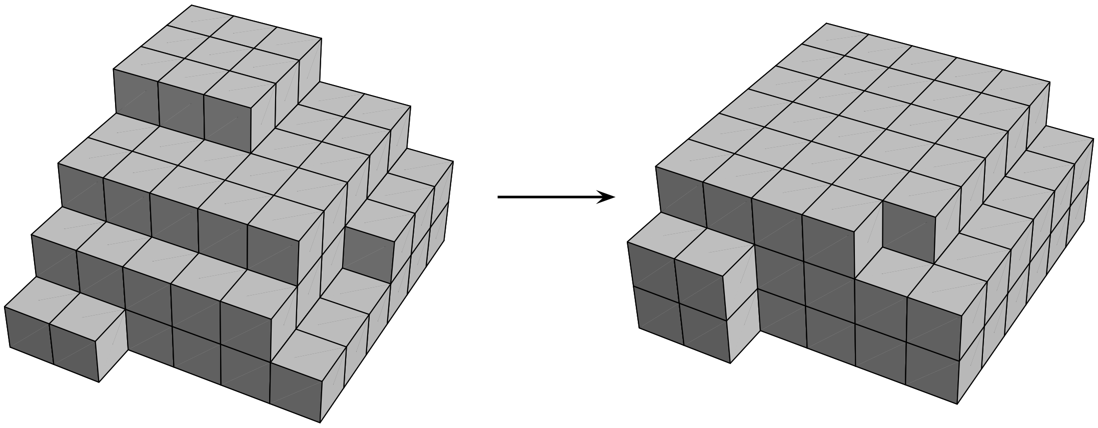

figure 3.10: filling(e_3)

Trividing. This transformation is defined for a polyomino c of the form $c = (j+\epsilon) \times j \times j_3 \oplus_3 d$ where $\epsilon \in \{0,1\}$, $j_3 > j+\epsilon$, and d is a non empty two dimensional polyomino which is included in the quasisquare $(j+\epsilon) \times j$ (d might be equal to $(j+\epsilon) \times j$). We decompose j_3 as $j_3 = j+\epsilon+3a+\delta+\theta$ where δ, θ are in $\{0,1\}$ and $\delta \leq \theta$. This decomposition is obviously unique.

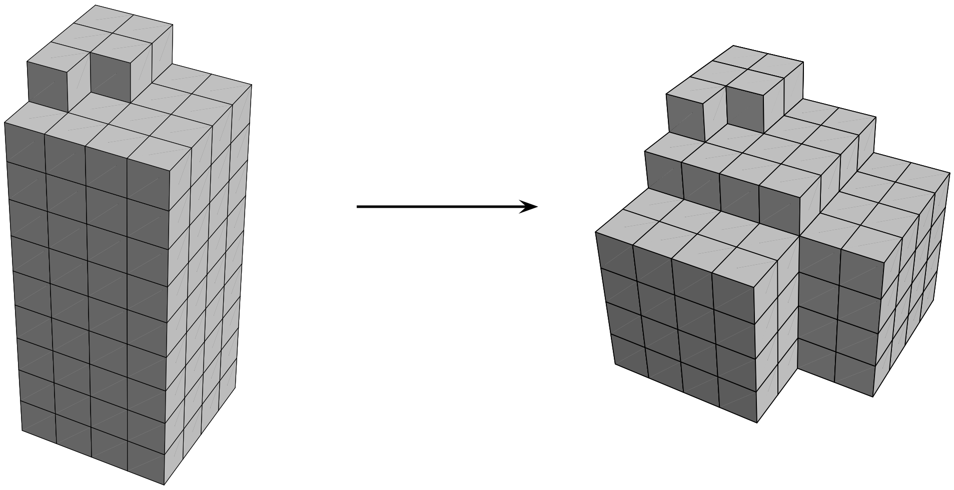

figure 3.11: trividing

The trivided polyomino trivided(c) is then

$$(j+\epsilon) \times j \times (j+\epsilon) +_1 (a+\epsilon') \times j \times (j+\epsilon) +_2 (j+\epsilon) \times (a+\epsilon'') \times j +_3 (j+\epsilon) \times j \times a +_3 d$$

where $(\epsilon', \epsilon'') = (\theta, \delta)$ if $\epsilon = 0$ and $(\epsilon', \epsilon'') = (\delta, \theta)$ if $\epsilon = 1$.
Considering separately each case, we have:

- If $\epsilon = 0$, trivided$(c) = j \times j \times j +_1 (a+\theta) \times j \times j \times +_2 j \times (a+\delta) \times j +_3 j \times j \times a +_3 d$.
- If $\epsilon = 1$, trivided$(c) = (j+1) \times j \times (j+1) +_1 (a+\delta) \times j \times (j+1) +_2 (j+1) \times (a+\theta) \times j +_3 (j+1) \times j \times a +_3 d$.

Once more, the trividing does not alter the volume and does not increase the area.

Swivelling. The swivelling is defined for a polyomino c of the form $c = j_1 \times j_2 \times j_3 +_i d$ where $1 \leq i \leq 3$ and d is a two dimensional polyomino which is included in the rectangle $j_1 \times j_2$. The aim of the swivelling is to move d on the side $j_1 \times j_2$ of the parallelepiped. Thus swivelling$(c) = j_1 \times j_2 \times j_3 +_3 d$.

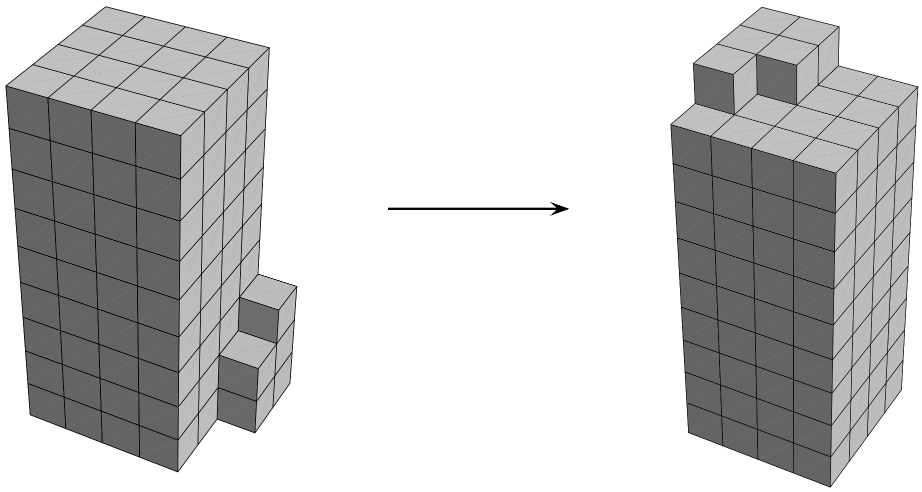

figure 3.12: swivelling

The sequence of transformations. We start with a polyomino c of $\mathcal{M}_n$. We first apply the three projections p_1, p_2, p_3. Let $c_1 = p_3 \circ p_2 \circ p_1(c)$. We next apply a rotation to c_1 to obtain a polyomino c_2 such that $j_3(c_2) \geq j_1(c_2)$ and $j_3(c_2) \geq j_2(c_2)$. We next apply the slice–remodelling and get a polyomino c_3. We then apply the filling filling(e_3) and obtain a polyomino c_4 of the form $c_4 = (j+\epsilon) \times j \times j_3 +_2 d +_3 e$. If d is empty, we set $c_5 = c_4$. If not, we apply the filling filling(e_2), obtaining a polyomino $c_4' = (j+\epsilon) \times j \times (j_3+1) +_2 d$ where d is included in the quasisquare $(j+\epsilon) \times j$ (if the result of the filling was different, the area would necessarily have decreased) and we then set $c_5 =$ swivelling(c_4'). In both cases, we have now a polyomino c_5 of the form $c_5 = (j+\epsilon) \times j \times j_3 +_3 e$. We replace e by the corresponding two dimensional canonical polyomino $m_{|e|}$, obtaining a polyomino c_6 (this amounts to apply the slice–remodelling). We apply the trividing to c_6 and obtain $c_7 =$ trividing(c_6). We end the sequence by a filling filling(e_3): $c_8 =$ filling$(e_3)(c_7)$. We claim that c_8 has the desired shape. $\square$

$$
\begin{tikzcd}
c \arrow[d, "p_3 \circ p_2 \circ p_1"] \\
c_1 \arrow[d, "\text{rotation}"] \\
c_2 \text{ with } j_3(c_2) \geq \max(j_1(c_2), j_2(c_2)) \arrow[d, "\text{slice–remodelling}"] \\
c_3 \arrow[d, "\text{filling}(e_3)"] \\
c_4 = (j + \epsilon) \times j \times j_3 +_2 d +_3 e \arrow[d, "d \neq \emptyset,\ \text{fill}(e_2)"] \arrow[dd, bend right=60, "d = \emptyset"', "\text{nothing}"' near end] \\
c_4' \arrow[d, "\text{swivelling}"] \\
c_5 = (j + \epsilon) \times j \times j_3 +_3 e \arrow[d, "\text{slice–remodelling}"] \\
c_6 \arrow[d, "\text{trividing}"] \\
c_7 \arrow[d, "\text{filling}(e_3)"] \\
c_8
\end{tikzcd}
$$

figure 3.13: the sequence of transformations

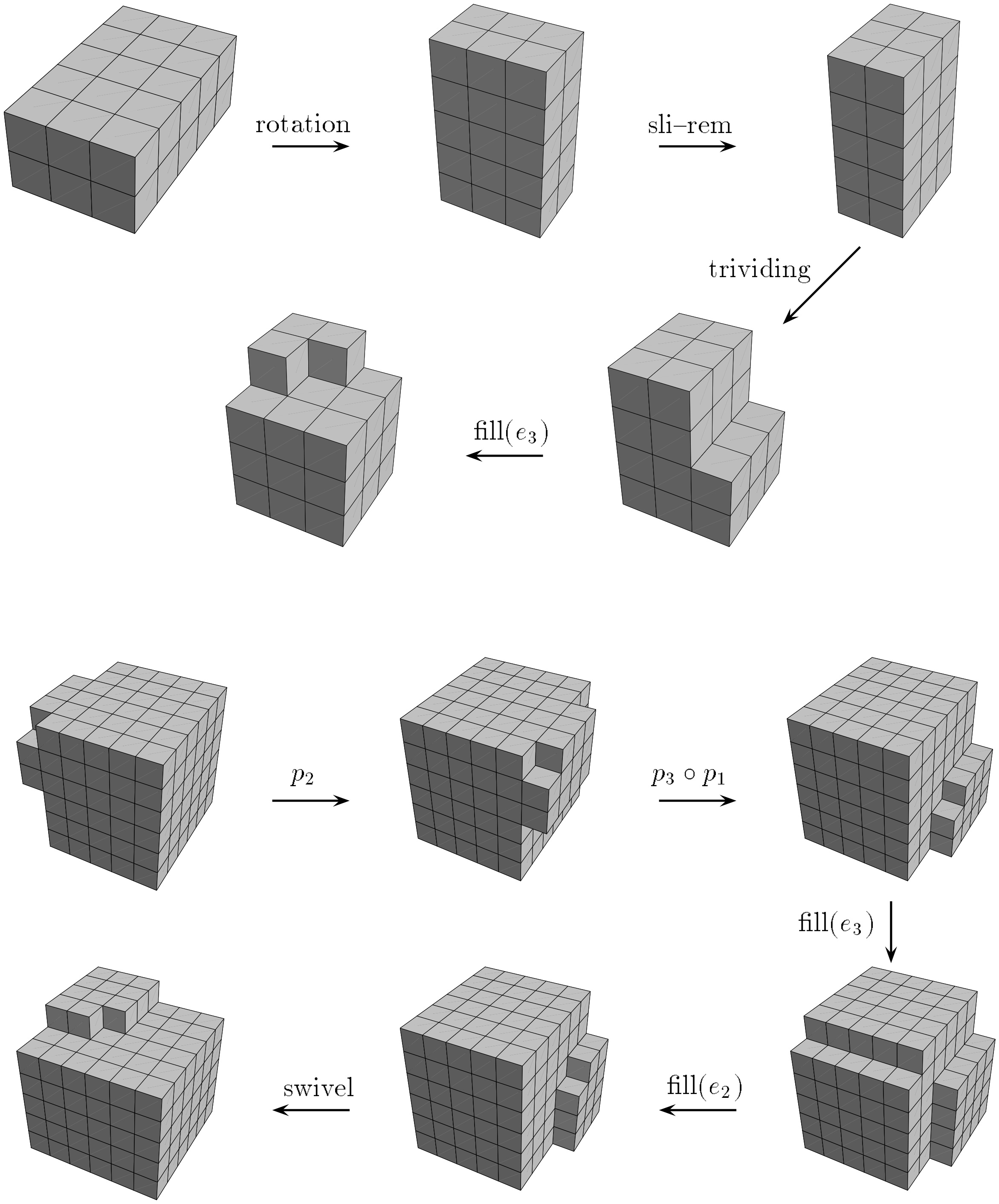

figure 3.14: two examples

Proposition 3.2. *For each integer n there exists a unique 6–uple $(j, l, k, \delta, \theta, \epsilon)$ such that $\delta, \theta, \epsilon \in \{0, 1\}$, $\delta \leq \theta$, $k < l+\epsilon, l(l+\epsilon)+k < (j+\delta)(j+\theta)$ and $n = j(j+\delta)(j+\theta)+l(l+\epsilon)+k$.*

Corollary 3.3. *The polyomino obtained at the end of the sequence of transformations does not depend on the polyomino initially chosen in the set $\mathcal{M}_n$.*

Throughout the sequel, we will refer to this decomposition as "the decomposition" of the integer n, without further detail. We thus have a method for computing the minimal area $A(n)$ of a polyomino of volume n.

Corollary 3.4. *The minimal area $A(n)$ of a polyomino of volume n is*

$$A(n) = 2(j(j+\delta) + j(j+\theta) + (j+\delta)(j+\theta)) + 2(2l+\epsilon) + 2\times 1_{\{k>0\}}$$

where $n = j(j+\delta)(j+\theta) + l(l+\epsilon) + k$ is the decomposition of n given by proposition 3.2.

Corollary 3.5. *(discrete isoperimetric inequality)*
For any polyomino c, we have $A(c) \geq A(V(c))$, where $A(c)$ is the area of c, $V(c)$ its volume and the function $n \mapsto A(n)$ is defined in corollary 3.4.

The canonical, standard and principal polyominoes. Proposition 3.2 and corollary 3.3 allow us to define a canonical polyomino $\mathfrak{m}_n$ belonging to $\mathcal{M}_n$.
Let $n = j(j+\delta)(j+\theta) + l(l+\epsilon) + k$ be the decomposition of n. We put $r = l(l+\epsilon) + k$. The canonical polyomino $\mathfrak{m}_n$ is obtained by adding the two dimensional canonical polyomino m_r to the right side of a right quasicube of volume $j(j+\delta)(j+\theta)$.

- If $\delta = \theta = 0$ we set $\mathfrak{m}_n = j \times j \times j +_1 m_r$.
- If $\delta = 0$, $\theta = 1$ we set $\mathfrak{m}_n = (j+1) \times j \times j +_2 m_r$.
- If $\delta = \theta = 1$ we set $\mathfrak{m}_n = (j+1) \times (j+1) \times j +_3 m_r$.

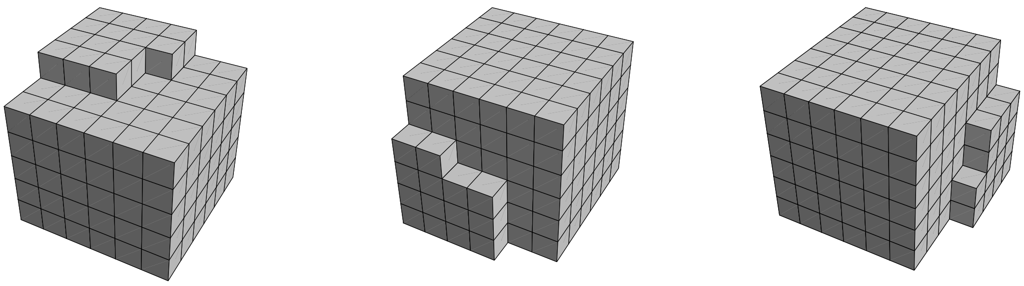

figure 3.15: the canonical polyominoes $\mathfrak{m}_{194}, \mathfrak{m}_{230}, \mathfrak{m}_{266}$

We may give a general formula:

$$\mathfrak{m}_n = (j+\theta) \times (j+\delta) \times j +_{1+\delta+\theta} \big((l+\epsilon) \times l +_{1+\epsilon} k\big).$$

The polyomino $\mathfrak{m}_n$ is called the canonical polyomino of volume n.
For c a polyomino, we denote by $\overline{c}$ its equivalence class modulo the spatial isometries which leave the integer lattice $\mathbb{Z}^3$ invariant. By $\overline{c}^{123}$ we denote the equivalence class modulo the three symmetries s_1, s_2, s_3 with respect to the planes $(e_2, e_3), (e_1, e_3), (e_1, e_2)$. If A is a subset of $\mathcal{C}$, we put

$$\overline{A} = \bigcup_{c \in A} \overline{c}, \qquad \overline{A}^{123} = \bigcup_{c \in A} \overline{c}^{123}.$$

The set $\mathcal{S}_n$ of the standard polyominoes is

$$\mathcal{S}_n = \overline{(j+\theta) \times (j+\delta) \times j \oplus_{1+\delta+\theta} \overline{(l+\epsilon) \times l \oplus_{1+\epsilon} k}}$$

and the set $\widetilde{\mathcal{M}}_n$ of the principal polyominoes is

$$\widetilde{\mathcal{M}}_n = \bigcup_{\substack{t=1,2,3 \\ u=1,2}} \overline{(j+\theta) \times (j+\delta) \times j \oplus_t \overline{(l+\epsilon) \times l \oplus_u k}}$$

The sets $\mathcal{S}_n$ and $\widetilde{M}_n$ coincide if $\delta = \theta = \epsilon = 0$. Moreover we have

$$\{\mathfrak{m}_n\} \subset \mathcal{S}_n \subset \widetilde{\mathcal{M}}_n \subset \mathcal{M}_n.$$

However, the inclusions might be strict, as shown by the examples of figure 3.16, 3.17.

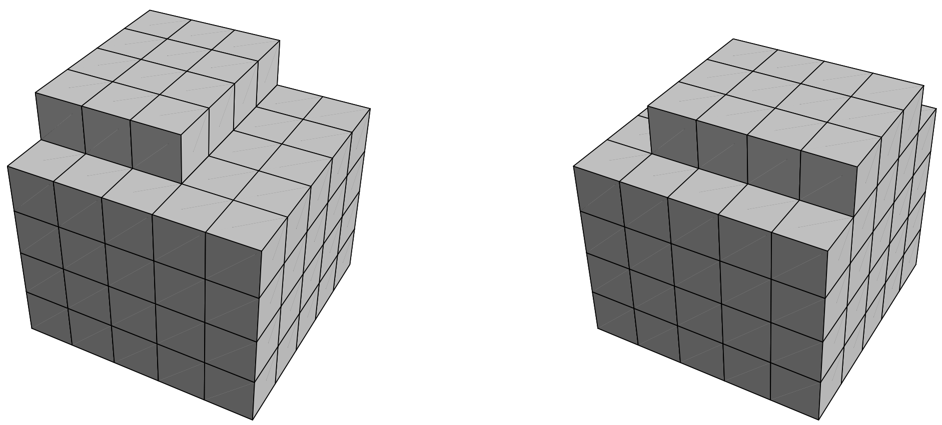

figure 3.16: polyominoes in $\mathfrak{m}_{112}$, $\mathcal{S}_{112} \setminus \mathfrak{m}_{112}$.

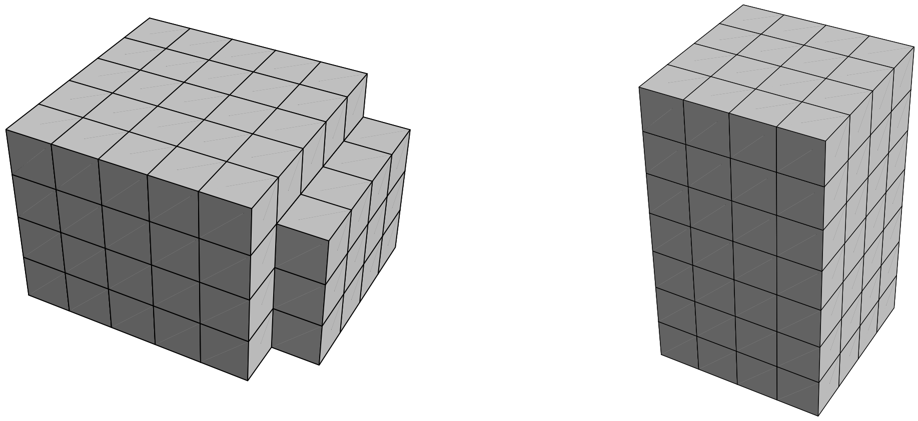

figure 3.17: polyominoes in $\widetilde{\mathcal{M}}_{112} \setminus \mathcal{S}_{112}$, $\mathcal{M}_{112} \setminus \widetilde{\mathcal{M}}_{112}$.

Theorem 3.6. *The set $\mathcal{M}_n$ of the minimal polyominoes of volume n coincides with the set $\widetilde{\mathcal{M}}_n$ of the principal polyominoes if and only if n is of the form quasicube+quasisquare*

$$\begin{array}{ccc} j^3 & j^3+l^2 & j^3+l(l+1) \\ j^2(j+1) & j^2(j+1)+l^2 & j^2(j+1)+l(l+1) \\ j(j+1)^2 & j(j+1)^2+l^2 & j(j+1)^2+l(l+1) \end{array}$$

(where $1 \le l < j$) or quasicube minus one $j^3-1, j^2(j+1)-1, j(j+1)^2-1$.

Proof. Suppose first that $\mathcal{M}_n$ is equal to $\widetilde{\mathcal{M}}_n$. This means that, modulo spatial isometries and moves of the two dimensional polyomino $l\times(l+\epsilon)+k$ on one side of the quasicube $j\times(j+\delta)\times(j+\theta)$, or moves of the bar $k\times 1$ along one side of the quasisquare $l\times(l+\epsilon)$ the set $\mathcal{M}_n$ is reduced to the six polyominoes

$$\begin{array}{ll} (j+\theta)\times(j+\delta)\times j +_1 (l+\epsilon)\times l +_1 k\,, & (j+\theta)\times(j+\delta)\times j +_1 (l+\epsilon)\times l +_2 k \\ (j+\theta)\times(j+\delta)\times j +_2 (l+\epsilon)\times l +_1 k\,, & (j+\theta)\times(j+\delta)\times j +_2 (l+\epsilon)\times l +_2 k \\ (j+\theta)\times(j+\delta)\times j +_3 (l+\epsilon)\times l +_1 k\,, & (j+\theta)\times(j+\delta)\times j +_3 (l+\epsilon)\times l +_2 k \end{array}$$

(which are isometric when $\delta=\theta=\epsilon=0$). The corresponding two dimensional result implies that $k=0$ or $k=l+\epsilon-1$. For $k=0$ we obtain the quasicubes plus the quasisquares. Suppose $k=l+\epsilon-1$. If $l+\epsilon$ is not equal to $j+\theta$, then the polyomino $((j+\theta-1)\times(j+\delta)+_1(j+\delta-1))+_3(j+\theta)\times(j+\delta)\times(j-1)+_3(l+\epsilon)\times(l+\epsilon)$ belongs to the set $\mathcal{M}_n$ and is not principal. Thus $l+\epsilon$ necessarily equals $j+\theta-1$ so that n is a quasicube minus one.

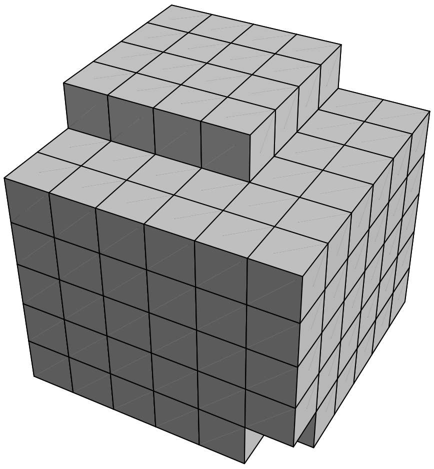

figure 3.18: an element of $\mathcal{M}_{195}$

Conversely, we examine for these particular values of n the possible actions of the sequence of transformations. That is, we seek the antecedents of the final polyomino obtained at the end of the sequence of transformations. The main idea is that we started the sequence of transformations with a polyomino belonging to $\mathcal{M}_n$ so that the area of the polyomino cannot change throughout the whole sequence. We first notice that for all these shapes the last filling fill(e_3) between c_7 and c_8 had necessarily no effect. When n is of the form $j^3, j^3+l^2, j^3+l(l+1)$, the transformation between c_6 and c_7 was also effectless; when n is of the form $j^2(j+1), j^2(j+1)+l^2, j^2(j+1)+l(l+1)$ or $j(j+1)^2, j(j+1)^2+l^2, j(j+1)^2+l(l+1)$, the trividing could have swivelled $l^2, l(l+1)$ from one face to another, yielding a principal polyomino of $\widetilde{\mathcal{M}}_n$. Moreover, for all these values of n, we have

$$\forall i \in \{1,2,3\} \qquad \text{fill}(e_i)^{-1}(\widetilde{\mathcal{M}}_n) \cap \mathcal{M}_n \;=\; \widetilde{\mathcal{M}}_n$$

i.e. the antecedents of a principal polyomino through a filling operation which are minimal are also principal. Finally we check that the same is true for the slice–remodelling: its only possible effect when the final shape is principal is to have swivelled or rotated $l\times l$ or $l\times(l+1)$ from one face to another, or to have applied an isometry to the cube j^3,$j^2(j+1)$,$j(j+1)^2$. The same kind of results hold concerning the first two transformations i.e. the projections between c and c_1 and the rotation between c_1 and c_2. Putting these facts together, we see that for the values of n listed in the theorem, the sets of the minimal antecedents through the sequence of transformations of the canonical polyomino coincides with the set of the principal polyominoes. $\square$

As a by–product, we obtain the following results.

Corollary 3.7. *The set $\mathcal{M}_n$ is reduced to $\{\mathfrak{m}_n\}$ if and only if n is of the form j^3. The set $\mathcal{M}_n$ is equal to $\mathcal{S}_n$ if and only if n is of the form $j^3, j^2(j+1), j(j+1)^2$ or $j^3-1, j^2(j+1)-1, j(j+1)^2-1$ (in which case $S_n = \overline{\mathfrak{m}}_n$), or $j^3+l^2, j^3+l(l+1)$.*

We were not able to prove or disprove results characterizing the three dimensional minimal polyominoes, comparable to theorems 2.1 and 2.11. We propose two conjectures instead.

Conjecture 3.8. *If a polyomino is minimal then its smallest surrounding parallelepiped is also minimal.*

Definition 3.9. A corner of a polyomino c is a unit cube of c having at least three sides belonging to the boundary of c.

Conjecture 3.10. *All the minimal polyominoes can be obtained by removing successively m corner cubes from a minimal parallelepiped $j_1 \times j_2 \times j_3$, in such a way that the number of bars removed from the parallelepiped is maximal, where $m < l(l+\epsilon)+k$, $j_1 j_2 j_3 = j(j+\delta)(j+\theta)+l(l+\epsilon)+k$.*

Moves through the minimal polyominoes. We are interested in moving through the polyominoes by adding or removing one unit cube at a time. How far is it possible to travel in this way through the minimal polyominoes?
Let us define three kernels q, q_-, q_+ on $\mathcal{C} \times \mathcal{C}$. First

$$\forall c, d \in \mathcal{C} \qquad q_-(c,d) = \begin{cases} 1 & \text{if } d \subset c \text{ and } |c \setminus d| = 1 \\ 0 & \text{otherwise} \end{cases}$$

that is, $q_-(c,d)=1$ if d may be obtained by removing a unit cube from c, and $q_-(c,d)=0$ otherwise. Next, we put $q_+(c,d)=q_-(d,c)$, that is $q_+(c,d)=1$ if d may be obtained by adding a unit cube to c and $q_+(c,d)=0$ otherwise. Finally we set $q(c,d)=q_-(c,d)+q_+(c,d)$ so that $q(c,d)=1$ if the polyominoes differ by a unit cube, and $q(c,d)=0$ otherwise. Two polyominoes c,d are said to communicate if $q(c,d)=1$. If Y is a subset of $\mathcal{C}$ and c is a polyomino, we set $q(c,Y)=1$ if c communicates with at least one element of Y and $q(c,Y)=0$ otherwise. The quantities $q_-(c,Y), q_+(c,Y)$ are defined similarly.
The next lemmas describe the way we can move starting from a canonical polyomino $\mathfrak{m}_n$.

Lemma 3.11. *Let ι be a spatial isometry. For any n which is not a quasicube plus a quasisquare, that is not of the form*

$$j^3+l^2, j^3+l(l+1), j^2(j+1)+l^2, j^2(j+1)+l(l+1), j(j+1)^2+l^2, j(j+1)^2+l(l+1)$$

(where $l<j$), $\iota(\mathfrak{m}_{n+1})$ is the unique polyomino of $\mathcal{M}_{n+1}$ which communicates with $\iota(\mathfrak{m}_n)$.

Lemma 3.12. *For n a quasicube i.e. of the form $j^3, j^2(j+1), j(j+1)^2$, we have*

$$\begin{aligned} \{\, c \in \mathcal{M}_{n-1} : q_-(\mathcal{M}_n, c) = 1 \,\} &= \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n+1} : q_+(\mathcal{M}_n, c) = 1 \,\} &= \widetilde{\mathcal{M}}_{n+1}. \end{aligned}$$

Lemma 3.13. *For n a quasicube plus a quasisquare i.e. of the form $j(j+\delta)(j+\theta)+l(l+\epsilon)$ (where $0 < l(l+\epsilon) < (j+\delta)(j+\theta)$) we have*

$$\begin{aligned} \{\, c \in \mathcal{M}_{n-1} : q_-(\widetilde{\mathcal{M}}_n \setminus S_n, c) = 1 \,\} &\supset \widetilde{\mathcal{M}}_{n-1} \setminus S_{n-1}, \\ \{\, c \in \mathcal{M}_{n-1} : q_-(\mathcal{S}_n, c) = 1 \,\} &\supset \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n+1} : q_+(\mathcal{M}_n, c) = 1 \,\} &= \widetilde{\mathcal{M}}_{n+1}, \end{aligned}$$

Lemma 3.14. *For the remaining integers i.e. the values of n which are not a quasicube or a quasicube plus a quasisquare we have*

$$\begin{aligned} \{\, c \in \mathcal{M}_{n-1} : q_-(\mathcal{S}_n, c) = 1 \,\} &\supset \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n+1} : q_+(\mathcal{S}_n, c) = 1 \,\} &= \mathcal{S}_{n+1}, \\ \{\, c \in \mathcal{M}_{n-1} : q_-(\widetilde{\mathcal{M}}_n \setminus \mathcal{S}_n, c) = 1 \,\} &\supset \widetilde{\mathcal{M}}_{n-1} \setminus \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n+1} : q_+(\widetilde{\mathcal{M}}_n \setminus S_n, c) = 1 \,\} &= \widetilde{\mathcal{M}}_{n+1} \setminus \mathcal{S}_{n+1}. \end{aligned}$$

Lemma 3.15. *Suppose that $j \times (j+\delta) \times (j+\theta) + l \times (l+r)$, $r \geq 2$ is minimal. It cannot be grown through the minimal polyominoes.*

Lemma 3.16. *The parallelepipeds $j \times j \times (j+2)$, $j \times (j+1) \times (j+2)$ cannot grow through the minimal polyominoes.*

Lemma 3.17. *Suppose n is not of the form $j^2(j+1), j(j+1)^2, j^3 + l(l+1), j^2(j+1) + l(l+1), j(j+1)^2 + l(l+1)$. Then*

$$\begin{aligned} \{\, c \in \mathcal{M}_{n+1} : q(\mathcal{S}_n, c) = 1 \,\} &= \mathcal{S}_{n+1}, \\ \{\, c \in \mathcal{M}_{n+1} : q_+(\widetilde{\mathcal{M}}_n \setminus \mathcal{S}_n, c) = 1 \,\} &= \widetilde{\mathcal{M}}_{n+1} \setminus \mathcal{S}_{n+1}. \end{aligned}$$

Lemma 3.18. *Suppose n is of the form $j^3 + l(l+1), j^2(j+1) + l(l+1), j(j+1)^2 + l(l+1)$ (where $0 < l < j$). Let c belong to $\mathcal{S}_n$ and suppose there is an increasing sequence of minimal polyominoes $c_n, \cdots, c_m$ such that $c_n = c$. Either c_{n+1} belongs to $\mathcal{S}_{n+1}$ or m is less than $n + l$; in this last case, none of the polyominoes $c_{n+1}, \cdots, c_m$ is standard, and they are all principal.*

Lemma 3.19. *Suppose n is a quasicube of the form $j^2(j+1)$ (respectively $j(j+1)^2$). Let c belong to $\mathcal{S}_n$ and suppose there is an increasing sequence of minimal polyominoes $c_n, \cdots, c_m$ such that $c_n = c$. Either c_{n+1} belongs to $\mathcal{S}_{n+1}$ or m is less than $n + j^2$ (resp. $n + j(j+1)$); in this last case, none of the polyominoes $c_{n+1}, \cdots, c_m$ is standard, and they are all principal.*

The next theorems sum up several results which are consequences of the preceding lemmas together with theorems 3.1 and corollary 3.7.

Theorem 3.20. *For any n and any principal polyomino c of $\widetilde{\mathcal{M}}_n$, there exists an increasing sequence of minimal polyominoes $c_0, \cdots, c_n$ starting at $c_0 = \emptyset$ and ending at $c_n = c$.*

Theorem 3.21. *Let n be an integer between the two cubes $j^3, (j+1)^3$ and let c be a principal polyomino belonging to $\widetilde{\mathcal{M}}_n$ which is not standard i.e. it does not belong to $\mathcal{S}_n$. Suppose there exists an increasing sequence of minimal polyominoes $c_n, \cdots, c_m$ such that $c_n = c$. Then necessarily $m < (j+1)^3$ and none of the polyominoes $c_n, \cdots, c_m$ is standard.*

Theorem 3.22. *The principal polyominoes can be completely shrinked through the minimal polyominoes: for any integer n and for any principal polyomino c in $\widetilde{\mathcal{M}}_n$, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$ and $c_n = c$.*

Theorem 3.23. *The standard polyominoes can be grown or shrinked arbitrarily far through minimal polyominoes: for any integers $m \leq n$ and for any standard polyomino c in $\mathcal{S}_m$, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$ and $c_m = c$.*

Theorem 3.24. *Let $c_0, \cdots, c_n$ be an increasing sequence of minimal polyominoes starting from the empty polyomino ($c_0 = \emptyset$). If c_n is a standard polyomino (i.e. belongs to $\mathcal{S}_n$) then all the polyominoes of the sequence are standard (i.e. $c_j \in \mathcal{S}_j$ for all $j \leq n$).*

Proposition 3.25. *The infinite sequence $\mathcal{S}_0, \cdots, \mathcal{S}_n, \cdots$ of the sets of standard polyominoes is the greatest infinite sequence of subsets of the sequence $\mathcal{M}_0, \cdots, \mathcal{M}_n, \cdots$ of the sets of minimal polyominoes enjoying the properties stated in theorems 3.23 and 3.24.*

Proof. Let $\mathcal{S}'_0, \cdots, \mathcal{S}'_n, \cdots$ be a sequence included in $\mathcal{M}_0, \cdots, \mathcal{M}_n, \cdots$ for which the theorems 3.23 and 3.24 hold. Suppose there exists n such that $\mathcal{S}'_n \not\subset \mathcal{S}_n$. Let n be the smallest such index and let c belong to $\mathcal{S}'_n \setminus \mathcal{S}_n$. There exists d in $\mathcal{S}'_{n-1} \subset \mathcal{S}_{n-1}$ such that $q_+(d, c) = 1$. Hence c is principal. Several cases arise: if $n = j(j+\delta)(j+\theta) + l(l+1) + k$, (with $(l, k) \neq (0, 0)$) the corresponding two dimensional proposition 2.22 yields the result; if $n = j(j+\delta)(j+\theta)$, we have necessarily $(\delta, \theta) \neq (0, 0)$ (otherwise c would be standard) and lemma 3.19 shows that c cannot be grown indefinitely through the minimal polyominoes. □

Shrinking or growing a parallelepiped. Let j_1, j_2, j_3, r be integers. We define

$$\begin{aligned}
\mathcal{M}(j_1 \times j_2 \times j_3, -r) &= \{\, c \in \mathcal{C}_{j_1 j_2 j_3 - r} : c \subset j_1 \times j_2 \times j_3,\ A(c) \text{ minimal}\,\},\\
\mathcal{M}(j_1 \times j_2 \times j_3, +r) &= \{\, c \in \mathcal{C}_{j_1 j_2 j_3 + r} : j_1 \times j_2 \times j_3 \subset c,\ A(c) \text{ minimal}\,\}.
\end{aligned}$$

The set $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$ is the set of the polyominoes of minimal area which can be obtained by removing r cubes from $j_1 \times j_2 \times j_3$. Analogously, $\mathcal{M}(j_1 \times j_2 \times j_3, +r)$ is the set of the polyominoes of minimal area which can be obtained by adding r cubes to $j_1 \times j_2 \times j_3$. A natural way to remove (add) r cubes (for $r < j_1 j_2, r < j_2 j_3, r < j_1 j_3$) is to remove (add) the cubes on only one side of the parallelepiped.

Theorem 3.26. *Let j_1, j_2, j_3, r be positive integers such that $r < j_1j_2$, $r < j_2j_3$, $r < j_3j_1$. The set $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$ is the set of the polyominoes obtained by removing from $j_1 \times j_2 \times j_3$ as many bars as possible, and then removing a succession of corner cubes (see definition 3.9) until reaching the volume $j_1j_2j_3 - r$.*

Proof. This result is straightforward. □

Corollary 3.27. *Let j_1, j_2, j_3, r be positive integers such that $r < j_1j_2$, $r < j_2j_3$, $r < j_3j_1$. A polyomino obtained from c by the successive removal of r cubes, in such a way that each cube removal takes place on a bar of minimal length of the polyomino is in the set $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$.*

Corollary 3.28. *Let j_1, j_2, j_3, r be positive integers such that $r < j_1j_2$, $r < j_2j_3$, $r < j_3j_1$. For each configuration c of the set $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$, there exists a configuration d in $\mathcal{M}(j_1 \times j_2 \times j_3, -r+1)$ such that $q^-(d,c) = 1$ i.e. c can be obtained by removing a (corner) cube from d.*

Proof. Let $N(r)$ be the number of bars one can remove from $j_1 \times j_2 \times j_3$ with a removal of r cubes. If $N(r-1) = N(r)$, then any polyomino obtained by removing a corner cube from a polyomino in $\mathcal{M}(j_1 \times j_2 \times j_3, -r+1)$ is in $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$. If $N(r) = N(r-1)+1$, then there exists a polyomino c in $\mathcal{M}(j_1 \times j_2 \times j_3, -r+1)$ having a cube with four sides in the boundary of c: the removal of this cube yields a polyomino of $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$. □

Corollary 3.29. *Suppose $j_1 \times j_2$ is the side of $j_1 \times j_2 \times j_3$ of smallest area.*
If $j_1j_2 - r \le j_1^2$ and $r \le j_1j_2$, then $j_1 \times j_2 \times (j_3 - 1) \oplus m_{j_1j_2-r} \subset \mathcal{M}(j_1 \times j_2 \times j_3, -r)$.
If $j_1j_2 - r > j_1^2$, then $j_1 \times j_2 \times (j_3 - 1) \oplus (j_1 \times q + s) \subset \mathcal{M}(j_1 \times j_2 \times j_3, -r)$, where $j_1j_2 - r = j_1q + s$, $0 \le s < j_1$.

Theorem 3.30. *Let j_1, j_2, j_3, r be positive integers such that $r \le \min(j_1^2, j_2^2, j_3^2)$. The best way to add r cubes to the parallelepiped $j_1 \times j_2 \times j_3$ is to add a minimal two dimensional polyomino of M_r on one side of the parallelepiped. Equivalently, we have*

$$\mathcal{M}(j_1 \times j_2 \times j_3, +r) \;=\; \overline{\{\, j_1 \times j_2 \times j_3 \oplus_i d,\ 1 \le i \le 3,\ d \in M_r \,\}}^{123}.$$

In particular, $j_1 \times j_2 \times j_3 \oplus m_r \subset \mathcal{M}(j_1 \times j_2 \times j_3, +r)$.

Proof. Let c belong to $\mathcal{M}(j_1 \times j_2 \times j_3, +r)$. Let d be the polyomino obtained by removing from c the slices $s_1(c,j), j \ge j_1(c)$ or $j < 0$. Let c_1 be a two dimensional polyomino formed by the union of all these slices laid out in $\mathbb{Z}^2$ in such a way that no two of them intersect. We have $A(c) \ge A(d) + P(c_1)$ (here P is the two dimensional perimeter). Let e be the polyomino obtained by removing from d all the slices $s_2(d,j), j \ge j_2(c)$ or $j < 0$. Let c_2 be a two dimensional polyomino formed by the union of all these slices laid out in $\mathbb{Z}^2$ in such a way that no two of them intersect. We have $A(d) \ge A(e) + P(c_2)$. Removing all the slices $s_3(e,j), j \ge j_3(c)$ or $j < 0$ from e, we obtain the parallelepiped $j_1 \times j_2 \times j_3$. Let c_3

be a two dimensional polyomino formed by the disjoint union of all these slices. Clearly, $A(e) \geq A(j_1 \times j_2 \times j_3) + P(c_3)$. We have $A(c) \geq A(j_1 \times j_2 \times j_3) + P(c_1) + P(c_2) + P(c_3)$. It follows that $A(c) \geq A(j_1 \times j_2 \times j_3) + P(c_4)$ where c_4 is a polyomino which is the disjoint union of c_1, c_2, c_3. Yet c_4 has area r, so that $P(c_4) \geq P(m_r)$; since $A(j_1 \times j_2 \times j_3 \oplus m_r) = A(j_1 \times j_2 \times j_3) + P(m_r)$ we see that the area of an element of $\mathcal{M}(j_1 \times j_2 \times j_3, +r)$ is exactly $A(j_1 \times j_2 \times j_3) + P(m_r)$; in particular c_4 must be a minimal two dimensional polyomino: thus c_4 belongs to M_r and c is necessarily in

$$\overline{\{\, j_1 \times j_2 \times j_3 \oplus_i d,\ 1 \leq i \leq 3,\ d \in M_r \,\}}^{123}.$$

Conversely, this set is clearly included in $\mathcal{M}(j_1 \times j_2 \times j_3, +r)$. $\square$

We finally state the results describing the best ways of growing and shrinking a parallelepiped plus a rectangle. Let $j_1, j_2, j_3, l_1, l_2, r$ be integers. We consider a polyomino c of the set $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2$. We suppose that the rectangle is added on a compatible side of the parallelepiped (i.e. $l_1 \times l_2$ is included in this side). We suppose also that $r < \min(l_1, l_2) \leq \min(j_1, j_2, j_3)$. As before, we let

$$\begin{aligned} \mathcal{M}(c, -r) &= \{\, d \in \mathcal{C}_{v(c)-r} : d \subset c,\ A(d) \text{ minimal} \,\}, \\ \mathcal{M}(c, +r) &= \{\, d \in \mathcal{C}_{v(c)+r} : c \subset d,\ A(d) \text{ minimal} \,\}. \end{aligned}$$

Theorem 3.31. *The set $\mathcal{M}(c, -r)$ is the set of the polyominoes obtained by removing successively r corner cubes from c. The set $\mathcal{M}(c, +r)$ is equal to the set of the polyominoes obtained by adding a bar of length r against a compatible side of the rectangle $l_1 \times l_2$ (in such a way that $l_1 \times l_2 \oplus r$ fits into the side of the parallelepiped).*

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