Rigid Body Motion, Interacting Billiards, and Billiards on Curved Manifolds

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Abstract

It is shown that the free motion of any three-dimensional rigid body colliding elastically between two parallel, flat walls is equivalent to a three-dimensional billiard system. Depending upon the inertial parameters of the problem, the billiard system may possess a potential energy field and a non-Euclidean configuration space. The corresponding curvilinear motion of the billiard ball does not necessarily lead to a decrease of the stable periodic orbits found in the analogous rectilinear system.

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I. INTRODUCTION

The traditional billiard problem of a point particle moving in a rectilinear manner and undergoing specular reflections at boundary walls has proved to be a lucid example of a Hamiltonian dynamical system exhibiting both integrable and chaotic motion [1–3]. Yet, billiards as dynamical systems first came to attention when it was shown by Hadamard that negative curvature billiards, *i.e.*, the study of geodesics on manifolds of strictly negative curvature (hyperbolic manifolds), provide a rich illustration of dynamical systems with dense stochastic trajectories. Clearly then, billiards with *non-rectilinear* motion are of some interest to the study of chaos. Such systems can be created by non-Euclidean configuration manifolds as above or by the introduction of some interacting potential field. Systems of this type have been considered before (*i.e.*, gravitational billiards, Aharonov-Bohm billiards) but their construction have been somewhat *ad hoc.*

In this paper, we present a systematic procedure for constructing a broad class of physically realizable curvilinear billiard systems. We begin with the demonstration that the motion of a freely moving three-dimensional rigid body making elastic collisions between two flat infinite parallel walls can, in general, be mapped to a three-dimensional non-rectilinear billiard system in which the corresponding point particle moves in a potential energy field and makes specular reflections at two suitable curved parallel walls (*i.e.*, two-dimensional manifolds). While the shape of these walls is determined solely by the geometric shape of the rigid body, the potential field and the geodesic nature of the configuration space are determined by the inertial properties (inertia tensor) of the rigid body. This strict separation between inertial and geometric properties will be used in constructing certain useful comparisons to two-dimensional billiards. The equivalence of rigid body motion with billiards was first demonstrated for two-dimensional motion in a previous paper [4]. There, all billiards were Euclidean in nature. It was noted, however, that since the motion of a rigid body in three (and higher) dimensions is associated with a non-commutative group, a new type of billiard motion was to be expected, which is certainly the case according to the statements above.

Let us now outline the contents of this paper. We begin in section II with a brief review of the elementary problem of a one-dimensional stick moving in two dimensions and then extend these results to three dimensions. As we shall see, this additional degree of freedom leads to an interacting billiard system on a flat manifold. Section III considers the motion of free billiards on certain curved manifolds constructed to describe the motion of cubical rigid bodies. After these two illustrative examples, we demonstrate in section IV that the general system of any freely moving rigid body leads to a billiard problem which combines the two phenomena found in the stick and cube problems (*i.e.*, interacting potential fields and curved manifolds). Most of these features will be illustrated in section V when we consider the motion of an ellipsoid with the mass distribution of a stick, a problem chosen because of its clear separation between geometric and inertial properties. This example is well-suited for a comparison of the resulting motion with its known two-dimensional analogue. Finally, a number of conclusions and directions for future study will be discussed in section VI.

II. STICKS AND INTERACTING BILLIARDS

We begin by considering the case of a one-dimensional stick of total mass M which is composed of two equal point masses separated by a rigid rod of length 2ℓ and which makes elastic collisions between two flat parallel walls separated by a distance h. Let us recall some results from [4] for the case when the stick moves in two dimensions. The coordinates of the stick will be z, the height of its center of mass above the lower wall, and the angle of rotation θ from the vertical. Scaling z with the radius of gyration, κ, as $\eta = z/\kappa$, the energy of the stick becomes

$$E = \frac{M}{2}(\dot{\theta}^2 + \dot{\eta}^2) \quad . \tag{1}$$

The distance of closest approach of the center of mass to the plane is $\eta_{\rm min} = (\ell/\kappa)|\cos(\theta)|$ so that this point moves between boundaries at the bottom, $b(\theta)$, and the top, $t(\theta)$, with

$$
\begin{aligned}
b(\theta) &= \frac{\ell}{\kappa}|\cos(\theta)|, \\
t(\theta) &= \frac{1}{\kappa}\left(h - \ell|\cos(\theta)|\right).
\end{aligned}
\tag{2}
$$

A collision between stick and wall is described by

$$
\begin{aligned}
(M\kappa)\Delta\dot{\eta} &= f_n\tau \ , \\
(M\kappa^2)\Delta\dot{\theta} &= \ell\sin(\theta) f_n\tau \ .
\end{aligned}
\tag{3}
$$

The impulse can be eliminated to obtain a relation between $\Delta\dot{\eta}$ and $\Delta\dot{\theta}$, and conservation of energy can be used to show that

$$
\Delta\dot{\eta} = \frac{-2(\dot{\eta} + (\ell/\kappa)\dot{\theta}\sin\theta)}{1 + (\ell/\kappa)^2\sin^2\theta}, \tag{4}
$$

Proof that the reflection is specular follows immediately and is given in [4]. Here, we merely note that, given the form of the energy (1), this system clearly describes a free particle.

Now, we want to allow the stick to move in three dimensions and make elastic collisions with two flat walls which are the planes $z = 0$ and $z = h$. We orient the stick using the usual polar angles, θ and ϕ. The corresponding energy can be written as

$$
E = \frac{1}{2}M\ell^2[\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2] + \frac{1}{2}M\dot{z}^2 \quad , \tag{5}
$$

and the angular momentum of the stick by

$$
\begin{aligned}
L_x &= M\ell^2[-\sin\phi\ \dot{\theta} - \sin\theta\ \cos\theta\ \cos\phi\ \dot{\phi}] \\
L_y &= M\ell^2[\cos\phi\ \dot{\theta} - \sin\theta\ \cos\theta\ \sin\phi\ \dot{\phi}] \\
L_z &= M\ell^2[\sin^2\theta\ \dot{\phi}] \quad .
\end{aligned}
\tag{6}
$$

This problem initially appears to be five-dimensional. However, it is clear that the x and y motion of the center of mass of the stick are trivial and can be ignored. It is also clear that the force-free motion of the stick does not result in θ and ϕ being linear functions of time except for geometrical accidents. Now consider a collision with the wall which imparts some impulse, $f\tau$, in the z-direction.

$$\Delta(M\dot{z}) = f\tau \quad . \tag{7}$$

There is a corresponding change in the angular momenta:

$$\begin{aligned}
\Delta L_x &= -\ell \sin\theta \ \sin\phi \ (f\tau) \\
\Delta L_y &= \ell \sin\theta \ \cos\phi \ (f\tau) \\
\Delta L_z &= 0 \quad .
\end{aligned} \tag{8}$$

As a consequence of the third of these equations, we see that

$$\Delta\dot{\phi} = 0 \quad . \tag{9}$$

Using this fact, the equations for L_x and L_y, and the equations for ΔL_x and ΔL_y, we find that

$$\Delta\dot{\theta} = \frac{1}{\ell} \sin\theta \ \Delta\dot{z} \quad . \tag{10}$$

Finally, we can use the fact that the collision is strictly elastic and equate kinetic energies before and after the collision. This leads us to a quadratic equation with a trivial solution $\Delta\dot{z} = 0$ and a non-trivial solution of

$$\Delta\dot{z} = \frac{-2(\dot{z} + \ell\dot{\theta}\sin\theta)}{1 + \sin^2\theta} \quad . \tag{11}$$

We are now in a position to draw all desired conclusions about this special problem. Since $\dot{\phi}$ does not change during the collisions, the coordinate ϕ is quite passive. It serves only to "complicate" the motion in θ. Equation (11) is *identical* to what was found in the above two-dimensional problem. There is no ϕ-dependence in this equation. Furthermore, there can be no ϕ-dependence in the wall function.[1] The wall function is also exactly what

[1]This result is general. A general three-dimensional body will be described by 3 angles. (See below.) Set the first two and bring the body in contact with the wall. Now rotate the body about an axis in the z-direction through the center of mass while maintaining contact with the wall. The height of the center of mass above the wall will not change.

we had in the two-dimensional case. Thus, with the scaling of variables described previously, we again find that we have specular reflection in the $(\ell\theta, z)$ plane for every collision. This apparently three-dimensional problem is really a two-dimensional problem in the $(\ell\theta, z)$ plane. The only difference is that, as a consequence of the more complicated equations of motion, the trajectories between consecutive wall hits are no longer straight lines. Although the time-dependence of θ is not linear, it is not complicated. We simply consider the free motion in a rotated coordinate system such that the angular momentum vector lies along the z'-axis. In this frame the angular velocity $\omega_{z'}$ will be a constant. It is then easy to transform back to the original θz-coordinates.

The energy (5) can be rewritten as

$$E = \frac{1}{2}M[(\ell\dot{\theta})^2 + \dot{z}^2] + \frac{L_z^2}{2M\ell^2\sin^2\theta} \quad . \tag{12}$$

Since all reference to ϕ has disappeared, this is the total energy of the billiard ball in the reduced $(\ell\theta, z)$ plane. The third term, $L_z^2/2M\ell^2\sin^2\theta$, can be interpreted as the potential energy for the two-dimensional billiard system. This explains the non-linear time dependence of the θ variable. Thus, a one-dimensional stick bouncing elastically between two flat walls is equivalent to an interacting billiard problem (with suitable walls) on a flat two-dimensional manifold (with a specific form of the interacting field).

III. CUBES AND CURVILINEAR BILLIARDS

Another specific example will be sufficient to point the way to the general problem. Consider a cube of total mass M and sides $2a$ composed of 8 identical point masses at the corners joined by rigid rods. (We have now proceeded from tossing coins to rolling dice.) Start by aligning the cube with its corners at $(\pm a,\ \pm a,\ \pm a)$.[2] Now consider the most general angular orientation of the cube by performing the rotations,

[2]For a general rigid body, we would start with the principal axes along the laboratory x, y and z-axes.

$$\mathcal{R}(\alpha\beta\gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha), \tag{13}$$

where all the rotations are about the laboratory-fixed axes, and the R's are the usual 3×3 orthogonal rotation matrices (see equation (67) in the Appendix for an explicit representation of $\mathcal{R}(\alpha\beta\gamma)$). It is straightforward to construct the rotational kinetic energy as

$$E_{\rm rot} = \frac{1}{2}\sum_\mu m_\mu \sum_{i=1}^{3}(\dot{\mathcal{R}}x_\mu)_i(\dot{\mathcal{R}}x_\mu)_i \quad . \tag{14}$$

The μ sum is over the point masses and allows us to recover the moment of inertia tensor in the general case. (We could have done this for the stick. If we had, the first rotation would have had no effect. We would have obtained the results above with some sign changes on the angles.) For the specific case of the cube we find

$$E = \frac{1}{2}Ma^2[2\dot\alpha^2 + 2\dot\beta^2 + 2\dot\gamma^2 + 4\dot\alpha\dot\gamma\cos\beta] + \frac{1}{2}M\dot z^2 \quad . \tag{15}$$

It is equally easy to generate the angular momentum components as

$$L_i = \sum_\mu m_\mu \varepsilon_{ijk}(\mathcal{R}x_\mu)_j(\dot{\mathcal{R}}x_\mu)_k \quad . \tag{16}$$

Again, we can recover the moment of inertia tensor in the general case. For the specific case of the cube, this produces

$$\begin{aligned}
L_x &= Ma^2[-2\sin\gamma\ \dot\beta + 2\cos\gamma\ \sin\beta\ \dot\alpha] \\
L_y &= Ma^2[-2\cos\gamma\ \dot\beta - 2\sin\gamma\ \sin\beta\ \dot\alpha] \\
L_z &= Ma^2[-2\dot\gamma - 2\cos\beta\ \dot\alpha] \quad .
\end{aligned} \tag{17}$$

Now, consider the situation when the (a, a, a) corner of the cube is in contact with the plane at $z = 0$. The coordinates of this point are

$$\begin{aligned}
x/a = &\cos\gamma\cos\beta(\cos\alpha + \sin\alpha) \\
&+ \sin\gamma(\cos\alpha - \sin\alpha) - \cos\gamma\sin\beta,
\end{aligned} \tag{18}$$

$$\begin{aligned}
y/a = &-\sin\gamma\cos\beta(\cos\alpha + \sin\alpha) \\
&+ \cos\gamma(\cos\alpha - \sin\alpha) + \sin\gamma\sin\beta,
\end{aligned} \tag{19}$$

and

$$z/a = \sin\beta(\cos\alpha + \sin\alpha) + \cos\beta \quad . \tag{20}$$

We can now repeat the manipulations from the example of the stick to study the nature of the present collision. Equation (7) is unaltered. An impulse in the z-direction will leave $\Delta L_z = 0$. This results in the relation

$$\Delta\dot{\gamma} = -\cos\beta\Delta\dot{\alpha} \quad . \tag{21}$$

The expressions for L_x and L_y can be used to determine $\Delta\dot{\alpha}$ and $\Delta\dot{\beta}$ in terms of $\Delta\dot{z}$. This leads to

$$\Delta\dot{\alpha} = \frac{\Delta\dot{z}}{2a\sin\beta}[\cos\alpha - \sin\alpha] \tag{22}$$

and

$$\Delta\dot{\beta} = \frac{\Delta\dot{z}}{2a}[-\sin\beta + \cos\beta(\cos\alpha + \sin\alpha)]. \tag{23}$$

We can impose the condition of conservation of energy under the collision and eliminate $\dot{\gamma}$ from the expression for L_z in (17). This gives

$$\begin{aligned} a^2 \ [\ &(2\dot{\alpha}\Delta\dot{\alpha} + \Delta\dot{\alpha}^2)\sin^2\beta \\ &+ (2\dot{\beta}\Delta\dot{\beta} + \Delta\dot{\beta}^2)] + \dot{z}\Delta\dot{z} + \Delta\dot{z}^2/2 = 0. \end{aligned} \tag{24}$$

Finally, using (22) and (23), we determine $\Delta\dot{z}$ to be

$$\begin{aligned} \Delta\dot{z} = 4(&\dot{z} + a[\dot{\beta}\cos\beta(\cos\alpha + \sin\alpha) \\ &+ \sin\beta(\dot{\alpha}(\cos\alpha - \sin\alpha) - \dot{\beta})]) \\ &/ \ [\sin^2\beta\sin 2\alpha + \sin 2\beta(\cos\alpha + \sin\alpha) - 4]. \end{aligned} \tag{25}$$

It is interesting to note that all reference to γ has disappeared. Neither does this angle play any role in the wall function. This represents a genuine reduction of the dimensionality of the problem. At this point, we have obtained all available information. What remains is to see if there is a natural way to interpret these results in order to recover specular reflection. With this in mind, it is useful to write the kinetic energy of equation (15) in a more suggestive way. Specifically,

$$E = \frac{1}{2}Ma^2[2\dot{\alpha}^2 \sin^2\beta + 2\dot{\beta}^2] + \frac{{L_z}^2}{4Ma^2} + \frac{1}{2}M\dot{z}^2 \quad . \tag{26}$$

Evidently, L_z is not changed by an elastic collisions with the wall in the xy-plane. It is apparent that it is useful to introduce a metric g_{ab} in the $\alpha\beta z$ subspace with non-zero elements $g_{\alpha\alpha} = \sin^2\beta$, $g_{\beta\beta} = 1$, and $g_{zz} = 1$. With the introduction of the velocity vector

$$v = (\dot{\alpha}, \dot{\beta}, \dot{z}/\sqrt{2}a) \ , \tag{27}$$

we can write the total energy of the billiard system,

$$E = Ma^2 v^a v^b g_{ab} + \frac{L_z^2}{2Ma^2} \ , \tag{28}$$

in a manner which emphasizes the non-Euclidean nature of the configuration space $(\alpha, \beta, z/\sqrt{2}a)$. This metric will be included in all subsequent scalar product operations. It is important to note that this metric is determined by the inertial tensor and is independent of the shape of the body. The fact that the $\alpha - \beta$ components of the metric g_{ab} correspond to the surface of a sphere in the present example is a consequence of the equality of the three principal moments of inertia of a cube. In general, we would expect an ellipsoid. Thus, our billiard space is curved.

In order to address the question of specular reflection, we define the wall function using equation (20).

$$\begin{aligned} b(\alpha, \beta) &= -z(\alpha, \beta)/\sqrt{2}a \\ &= -[\sin\beta(\cos\alpha + \sin\alpha) + \cos\beta]/\sqrt{2}. \end{aligned} \tag{29}$$

This leads to an associated wall surface,

$$S(\alpha, \beta) = (\alpha, \beta, b(\alpha, \beta)) \ \ , \tag{30}$$

and two tangents to this surface,

$$S_\alpha = \frac{\partial S}{\partial \alpha} = (1, 0, b_\alpha),$$
$$S_\beta = \frac{\partial S}{\partial \beta} = (0, 1, b_\beta) \quad . \tag{31}$$

The normal to the wall at each point, N, must be orthogonal to these two tangent vectors so that we have $N \cdot S_\alpha = 0$ and $N \cdot S_\beta = 0$ where the metric, g_{ab}, must be included in forming the scalar product. This leads to

$$N = (-b_\alpha / \sin^2 \beta, -b_\beta, 1) \quad . \tag{32}$$

The final ingredient required for the demonstration of specular reflection is the vector describing the change of the velocity due to the collision:

$$\Delta v = (\Delta\dot{\alpha}, \Delta\dot{\beta}, \Delta\dot{z}/\sqrt{2}a) \ , \tag{33}$$

where we know $\Delta\dot{\alpha}$, $\Delta\dot{\beta}$, and $\Delta\dot{z}$ are all known from equations (22), (23), and (25) above.

The conditions for specular reflection are now simply expressed. The two tangential components of the velocity must remain unchanged, and the normal component of the velocity must be reversed. Thus,

$$S_\alpha \cdot (v + \Delta v) = S_\alpha \cdot v$$
$$S_\beta \cdot (v + \Delta v) = S_\beta \cdot v$$
$$N \cdot (v + \Delta v) = -N \cdot v \quad , \tag{34}$$

or

$$S_\alpha \cdot \Delta v = 0$$
$$S_\beta \cdot \Delta v = 0$$
$$N \cdot (2v + \Delta v) = 0 \quad . \tag{35}$$

Equations (22), (23), and (25) reveal that the reflection is specular provided that the metric is included in the scalar product.

It is useful to look at the structure of this argument in a slightly different way. Conservation of energy leads to the condition

$$
\begin{aligned}
& a^2[2\dot\alpha\Delta\dot\alpha + (\Delta\dot\alpha)^2]\ g_{\alpha\alpha} \\
+ & a^2[2\dot\beta\Delta\dot\beta + (\Delta\dot\beta)^2]\ g_{\beta\beta} \\
+ & [2\dot z\Delta\dot z + (\Delta\dot z)^2]\ g_{zz}/2 = 0.
\end{aligned}
\tag{36}
$$

The third of equations (35) can be expressed as

$$
\begin{aligned}
& - ab_\alpha(2\dot\alpha + \Delta\dot\alpha)\frac{g_{\alpha\alpha}g_{zz}}{g_{\alpha\alpha}} \\
& - ab_\beta(2\dot\beta + \Delta\dot\beta)\frac{g_{\beta\beta}g_{zz}}{g_{\beta\beta}} \\
& + (2\dot z + \Delta\dot z)\frac{g_{zz}}{\sqrt{2}} = 0,
\end{aligned}
\tag{37}
$$

where we have used the fact that (32) can be written as

$$
N = (-b_\alpha\frac{g_{zz}}{g_{\alpha\alpha}}, -b_\beta\frac{g_{zz}}{g_{\beta\beta}}, 1). \tag{38}
$$

¿From the two tangential conditions,

$$
\begin{aligned}
& \Delta\dot\alpha\ g_{\alpha\alpha} + b_\alpha\Delta\dot z\ g_{zz}/\sqrt{2}a = 0 \\
& \Delta\dot\beta\ g_{\beta\beta} + b_\beta\Delta\dot z\ g_{zz}/\sqrt{2}a = 0\ ,
\end{aligned}
\tag{39}
$$

we find that

$$
\begin{aligned}
b_\alpha &= -\sqrt{2}a\frac{\Delta\dot\alpha}{\Delta\dot z}\frac{g_{\alpha\alpha}}{g_{zz}} \\
b_\beta &= -\sqrt{2}a\frac{\Delta\dot\beta}{\Delta\dot z}\frac{g_{\beta\beta}}{g_{zz}}\quad .
\end{aligned}
\tag{40}
$$

The substitution of (40) into the normal constraint (37) leads imediately to (36) which came from the requirement of conservation of energy. Thus, by imposing conservation of energy in the billiard space and verifying that the tangential conditions hold, the normal constraint is automatically satistfied. This is of practical value since it is no longer necessary to solve for $\Delta\dot z$ explicitly in terms of $\Delta\dot\alpha$ and $\Delta\dot\beta$. That the present billiard manifold is an orthogonal system (*i.e.*, the metric tensor is diagonal) and that it posseses a trivial potential (*i.e.*, free motion) is due to the fact that all principal moments of inertia are equal in this special case. As we will see in section IV, the most general metric will contain off-diagonal elements, and the potential will be more complicated.

However, much of the above does generalize to arbitrary shapes. The wall function will always be indpendent of γ. Collisions with xy-planes cannot change L_z, so that it is always possible to eliminate all reference to γ in the analogues of equations (25) and (26). Thus, γ will again be a neglectable coordinate. We will find a non-trivial, ellipsoidal metric in the coordinates α and β. As is the case here, the details of this metric will depend only on the inertial tensor of the system. Also, the wall function will be determined solely by the shape of the body, independent of the mass distribution and the generalized metric.

IV. GENERAL RIGID BODIES

The extension of the results of the preceding section to the case of arbitrary rigid bodies is straight-forward. Here, we shall describe the general approach. Details are given in the Appendix. It is elementary to determine the point of contact for any (α, β, γ). One starts with the body oriented so that the body-fixed x, y, and z-axes coincide with the laboratory axes. Label a point on the surface of the body by the usual polar angles θ and ϕ and specify the associated radius, $R(\theta, \phi)$. Apply $\mathcal{R}(\alpha, \beta, \gamma)$ to this vector. The condition that this point should be a point of contact is that the (inward) normal to the surface should point in the (laboratory) $+z$ direction. This defines the wall function in terms of $R(\theta, \phi)$, independent of the metric. As in the two-dimensional case, the various derivatives of R which enter into the proof of specular reflection through the analogue of (35) can be eliminated by the condition that the surface should be tangent to the plane at the point of contact.

The metric tensor, g_{ab}, can be read from the form of the energy once the angle γ has been eliminated using relations analogous to (17) and (21). This yields a billiard energy having the form

$$
\begin{aligned}
E = {} & \frac{\dot{\alpha}^2}{2D(\alpha,\beta)} \sin^2\beta[\mathcal{I}_x^2 \sin^2\alpha + \mathcal{I}_y^2 \cos^2\alpha \\
& + \mathcal{I}_x\mathcal{I}_y + \mathcal{I}_x\mathcal{I}_z + \mathcal{I}_y\mathcal{I}_z] \\
& + \frac{\dot{\beta}^2}{2D(\alpha,\beta)}[\mathcal{I}_x^2 \cos^2\alpha\cos^2\beta + \mathcal{I}_y^2 \sin^2\alpha\cos^2\beta \\
& + \mathcal{I}_z^2 \sin^2\beta + \mathcal{I}_x\mathcal{I}_y + \mathcal{I}_x\mathcal{I}_z + \mathcal{I}_y\mathcal{I}_z] \\
& + \frac{\dot{\alpha}\dot{\beta}}{D(\alpha,\beta)}(\mathcal{I}_y^2 - \mathcal{I}_x^2)\cos\beta\sin\beta\cos\alpha\sin\alpha \\
& + \frac{L_z^2}{2D(\alpha,\beta)} + \frac{1}{2}M\dot{z}^2 \ ,
\end{aligned}
\tag{41}
$$

where

$$
\begin{aligned}
D(\alpha,\beta) = {} & [\mathcal{I}_x(1-\cos^2\alpha\sin^2\beta) \\
& + \mathcal{I}_y(1-\sin^2\alpha\sin^2\beta) + \mathcal{I}_z\sin^2\beta] \quad .
\end{aligned}
\tag{42}
$$

Thus, we are led to define the metric tensor as

$$
\begin{aligned}
g_{\alpha\alpha} &= \frac{\sin^2\beta[\mathcal{I}_x^2\sin^2\alpha + \mathcal{I}_y^2\cos^2\alpha + \mathcal{I}_x\mathcal{I}_y + \mathcal{I}_x\mathcal{I}_z + \mathcal{I}_y\mathcal{I}_z]}{D(\alpha,\beta)} \\
g_{\beta\beta} &= [\mathcal{I}_x^2\cos^2\alpha\cos^2\beta + \mathcal{I}_y^2\sin^2\alpha\cos^2\beta + \mathcal{I}_z^2\sin^2\beta \\
&\quad + \mathcal{I}_x\mathcal{I}_y + \mathcal{I}_x\mathcal{I}_z + \mathcal{I}_y\mathcal{I}_z]/D(\alpha,\beta) \\
g_{\alpha\beta} &= \frac{(\mathcal{I}_y^2 - \mathcal{I}_x^2)(\cos\beta\sin\beta\cos\alpha\sin\alpha)}{D(\alpha,\beta)} \\
g_{zz} &= M \quad .
\end{aligned}
\tag{43}
$$

This allows us to rewrite (41) as

$$
E = \frac{v^a v^b g_{ab}}{2} + \frac{L_z^2}{2D(\alpha,\beta)}. \tag{44}
$$

(An implicit factor of 2 has been included in the definition of $g_{\alpha\beta}$ since $g_{\alpha\beta} = g_{\beta\alpha}$). When $\mathcal{I}_x = \mathcal{I}_y$, this tensor becomes diagonal (*i.e.*, the coordinate system becomes orthogonal). As expected, we are obtain a potential energy term, $(L_z^2/2D(\alpha,\beta))$, which depends only on the relative magnitude of the principal moments of inertia. In the completely symmetric case of $\mathcal{I}_x = \mathcal{I}_y = \mathcal{I}_z$, $D(\alpha,\beta)$ is a constant and the motion is free. The construction of the billiard geometry and the proof of specular reflection can be found in the Appendix.

V. ELLIPSOIDS

Since the results of the previous section and the Appendix are somewhat complicated, we consider the specific example of an ellipsoid of revolution whose surface is given by

$$(1+\epsilon)({x_1}^2+{x_2}^2)+{x_3}^3=1. \tag{45}$$

It is useful to parametrize the coordinates as

$$\begin{aligned} x_1 &= R(\theta,\phi)\sin\theta\cos\phi, \\ x_2 &= R(\theta,\phi)\sin\theta\sin\phi, \\ x_3 &= R(\theta,\phi)\cos\theta, \end{aligned} \tag{46}$$

where

$$R(\theta,\phi)=\frac{1}{[1+\epsilon\sin^2\theta]^{1/2}}. \tag{47}$$

We have deliberately chosen an ellipsoid (45) in order to have $R(\theta,\phi)$ independent of ϕ. It is our intention to consider a three-dimensional billiard system with strong similarities to the two-dimensional systems studied previously. The present choice will enable us to make a direct comparison with the results of ref. [4].

The billiard wall function is obtained from equation (70) of the Appendix.

$$\begin{aligned} b_{\rm wall} &= -z(\alpha,\beta,\pi-\theta_c,\pi-\phi_c) \\ &= \frac{[\sin\beta\sin\theta_c\cos(\alpha+\phi_c)+\cos\beta\cos\theta_c]}{[1+\epsilon\sin^2\theta_c]^{1/2}}, \end{aligned} \tag{48}$$

where θ_c and ϕ_c are the angles of contact when the ellipsoid hits the plane at $z=0$. That is, we must have

$$\frac{\partial z}{\partial\theta}(\theta=\pi-\theta_c,\phi=\pi-\phi_c)=0, \tag{49}$$

and,

$$\frac{\partial z}{\partial\phi}(\theta=\pi-\theta_c,\phi=\pi-\phi_c)=0. \tag{50}$$

This results in the relations

$$\sin\theta_c \cos\beta(1+\epsilon) = \cos\theta_c \sin\beta \cos(\alpha+\phi_c) \tag{51}$$

and

$$-\tan\alpha = \tan\phi_c, \tag{52}$$

whic indicates that $\phi_c = -\alpha$. Thus, b_{wall} is independent of ϕ_c. Finally, solving for $\sin\theta_c$ and $\cos\theta_c$ in (51) and substituting back into (48) produces

$$b_{\text{wall}}\ (\alpha,\beta) = \left[\frac{1+\epsilon\cos^2\beta}{1+\epsilon}\right]^{1/2}, \tag{53}$$

which is the exact result found for the (flat) ellipse in [4]. This is not surprising since the symmetrical rigid body under consideration is a surface of revolution. We also note that the corresponding billiard system has a *two*-dimensional configuration space (i.e. an appropriately scaled (β, z) subspace) because of this symmetry. As was mentioned before and demonstrated above, the shape of the walls for the equivalent billiard problem is determined solely by the geometrical properties of the rigid body.

In order to determine the trajectories of our billiard we must specify its inertial properties. We choose to endow this ellipsoid with the inertia tensor of the one-dimensional stick of section II. That is, the mass is concentrated along the symmetry axis of the ellipsoid. Thus, the energy of the billiard problem will be given by equation (12) with $l = 1$ and $\theta = \beta$ and the corresponding equations of motion (obtained from the Lagrangian (5)) are

$$\begin{gathered}\ddot{\beta} - \left(\frac{L_z^2}{M^2 l^4}\right)\frac{\cos\beta}{\sin^3\beta} = 0, \\ \ddot{z} = 0.\end{gathered} \tag{54}$$

This provides another illustration of the distinction between the nature of the billiard's trajectories and its wall function. The wall function of the ellipsoid (53) is not that of the stick given in (2). For this problem, the center of mass moves in a non-zero potential (if L_z is non-zero) but with a flat metric and, as always, with specular reflection at every collision.

An example is shown in Fig.1 for an ellipsoid with $\epsilon = 0.5$ and $h = 2$. In this case, the major axis of the ellipsoid is equal to the wall separation as indicated by the fact that the upper and lower walls touch at $\beta = 0$ and π. The ellipsoid does not have sufficient room to "turn around ". In spite of this fact, the motion of the analogous two-dimensional system is very rich displaying periodic orbits, resonance islands, and chaotic regions. A sample trajectory of the point particle is shown in Fig.1 for $(L_z^2/M^2l^4) = 1$. (Note that we have adopted different horizontal and vertical scales in the interest of visibility. This makes it difficult to recognize specular reflection.) For purposes of comparison, the same trajectory is followed in Fig.2 for $L_z = 0$. The results of this figure are *identical* to those of the analogous rectilinear problem of an ellipse moving in a plane studied in [4]. In spite of the strong physical similarities between these problems, there are two qualitative distinctions of interest. As indicated by (54), the motion is not rectilinear when L_z is non-zero. This is apparent in Fig.1. Further, we see from (12) that there must be turning points in the motion. (Between collisions, $\dot{z}$ is constant. Evidently, $\dot{\theta}$ must decrease in magnitude and ultimately change sign as θ approaches 0 or π.)

Figs.3 and 4 provide a more complete summary of the motion for these two systems. At each collision with the walls, we plot the angular orientation of the body and the angle of incidence (measured relative to the normal at the point of contact). The investigation here has been rather cursory with the inclusion of only 10^3 points. Again, plots with a content similar to Fig.4 have been considered in [4]. Although the figure is somewhat crude, there is clear evidence of the elliptic fixed point, periodic orbits, resonance islands, and chaotic regions. The results of Fig.3 are remarkably similar given the qualitative differences in the individual trajectories suggested in Fig.1. The only qualitative difference of note is the overall compression of the firgure as a consequence of the existence of turning points in the case of $L_z \neq 0$. The survey of Fig.3 does not reveal resonance islands; this is probably due to the roughness of the exploration.

VI. CONCLUSIONS

This paper has continued the demonstration, initiated in [4], of the equivalence of a class of problems in rigid body motion to billiard problems. The present three-dimensional considerations reveal a richness (in the metric and through the presence of a potential) not encountered in two dimensions. The utility of this equivalence can be recognized in either direction. On the one hand, it is extremely appealing to have physically motivated and even physically realizable examples of billiard problems in higher dimensions. On the other, familiarity with general billiard results can cut through some of the difficulties associated with rigid body motion. For example, the wall function for a cube making elastic collisions between parallel walls is everywhere convex and thus provides a strong suggestion that the resulting motion is chaotic. (Since, as noted, there is a non-flat metric in this problem, the issue cannot be regarded as completely settled.)

There are several extensions of the present three-dimensional results which could be made. In two and three dimensions we have seen that there is a single billiard problem equivalent to a given rigid body problem. In two dimensions we demonstrated that, for every periodic billiard problem, there exist infinitely many equivalent rigid body problems. (The multiplicity reflects freedom in choosing the separation, h, between the parallel walls which can have any value greater than some h_c determined by the shape of the billiard wall.) We expect that this proof can be extended to three dimensions without difficulty.

There is no reason to restrict attention to rigid bodies colliding with parallel walls. One can equally well imagine a rigid body rattling inside an infinite cylinder of arbitrary cross-sectional shape or confined within an arbitrary closed three-dimensional surface. It is expected that these problems can also be mapped uniquely onto billiard problems using arguments similar to those adopted here and in [4].

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VII. APPENDIX

In order to determine the general rotational contribution to the kinetic energy, we start from equation (14). All the time dependence is in $\mathcal{R}$. Initially, the principal axes of the body are aligned with the laboratory axes. The corresponding moments of inertia are now $\mathcal{I}_x$, $\mathcal{I}_y$, and $\mathcal{I}_z$, and thus, E becomes

$$\begin{aligned} E = & \frac{1}{2}\mathcal{I}_x \ [2\cos\beta \ \dot{\alpha}\dot{\gamma} + 2\sin\alpha\cos\alpha\sin\beta \ \dot{\beta}\dot{\gamma} \\ & + \cos^2\alpha\cos^2\beta \ \dot{\gamma}^2 + \dot{\alpha}^2 + \cos^2\alpha \ \dot{\beta}^2 + \sin^2\alpha \ \dot{\gamma}^2] \\ & + \frac{1}{2}\mathcal{I}_y \ [2\cos\beta \ \dot{\alpha}\dot{\gamma} - 2\sin\alpha\cos\alpha\sin\beta \ \dot{\beta}\dot{\gamma} \\ & + \sin^2\alpha\cos^2\beta \ \dot{\gamma}^2 + \dot{\alpha}^2 + \sin^2\alpha \ \dot{\beta}^2 + \cos^2\alpha \ \dot{\gamma}^2] \\ & + \frac{1}{2}\mathcal{I}_z \ [\sin^2\beta \ \dot{\gamma}^2 + \dot{\beta}^2] + \frac{1}{2}M\dot{z}^2. \end{aligned} \tag{55}$$

Similarly, we can work with equation (16) and find the general expression for the various components of the angular momentum.

$$\begin{aligned} L_x = & \mathcal{I}_x \ [\sin\beta\cos\gamma \ \dot{\alpha} - (\sin\alpha\cos\alpha\cos\beta\cos\gamma \\ & + \cos^2\alpha\sin\gamma)\dot{\beta} + (\cos^2\alpha\sin\beta\cos\beta\cos\gamma \\ & - \sin\alpha\cos\alpha\sin\beta\sin\gamma)\dot{\gamma}] \\ & + \mathcal{I}_y \ [\sin\beta\cos\gamma \ \dot{\alpha} + (\sin\alpha\cos\alpha\cos\beta\cos\gamma \\ & - \sin^2\alpha\sin\gamma)\dot{\beta} + (\sin^2\alpha\sin\beta\cos\beta\cos\gamma \end{aligned}$$

$$+\sin\alpha\cos\alpha\sin\beta\sin\gamma)\dot{\gamma}]$$

$$-\mathcal{I}_z\ [\sin\gamma\ \dot{\beta}+\sin\beta\cos\beta\cos\gamma\ \dot{\gamma}] \tag{56}$$

$$\begin{aligned}
L_y = \mathcal{I}_x\ &[-\sin\beta\sin\gamma\ \dot{\alpha}+(\sin\alpha\cos\alpha\cos\beta\sin\gamma \\
&-\cos^2\alpha\cos\gamma)\dot{\beta}+(-\cos^2\alpha\sin\beta\cos\beta\sin\gamma \\
&-\sin\alpha\cos\alpha\sin\beta\cos\gamma)\dot{\gamma}] \\
&+\mathcal{I}_y\ [-\sin\beta\sin\gamma\ \dot{\alpha}+(-\sin\alpha\cos\alpha\cos\beta\sin\gamma \\
&-\sin^2\alpha\cos\gamma)\dot{\beta}+(-\sin^2\alpha\sin\beta\cos\beta\sin\gamma \\
&+\sin\alpha\cos\alpha\sin\beta\cos\gamma)\dot{\gamma}] \\
&+\mathcal{I}_z\ [-\cos\gamma\ \dot{\beta}+\sin\beta\cos\beta\sin\gamma\ \dot{\gamma}]
\end{aligned} \tag{57}$$

and finally

$$\begin{aligned}
L_z = \mathcal{I}_x\ &[-\cos\beta\ \dot{\alpha}-\sin\alpha\cos\alpha\sin\beta\ \dot{\beta} \\
&+(-\sin^2\alpha-\cos^2\alpha\cos^2\beta)\ \dot{\gamma}] \\
&+\mathcal{I}_y\ [-\cos\beta\ \dot{\alpha}+\sin\alpha\cos\alpha\sin\beta\ \dot{\beta} \\
&+(-\cos^2\alpha-\sin^2\alpha\cos^2\beta)\ \dot{\gamma}] \\
&-\mathcal{I}_z\ [\sin^2\beta\ \dot{\gamma}].
\end{aligned} \tag{58}$$

For $\mathcal{I}_x = \mathcal{I}_y = \mathcal{I}_z$, the above equations reproduce the previous results for the cube, whereas setting $\mathcal{I}_x = \mathcal{I}_y = 0$ give the expressions for the stick.

We can eliminate reference to the angle γ from the expression for the total energy E by using the equation for L_z. This is useful for two reasons. First, at the point of contact, γ is a symmetry angle since it specifies a final rotation about the z-axis through the center of mass, and thus cannot alter the height of the center of mass above the wall. Second, L_z is a constant of the motion because the impact force is normal to the wall. Solving for $\dot{\gamma}$ in (58) gives

$$\dot{\gamma} = [(\mathcal{I}_y-\mathcal{I}_x)\sin\alpha\cos\alpha\sin\beta\dot{\beta}-L_z-(\mathcal{I}_x+\mathcal{I}_y)\cos\beta\dot{\alpha}]$$

$$/[\mathcal{I}_x \sin^2\alpha + \mathcal{I}_y \cos^2\alpha + \mathcal{I}_z \sin^2\beta$$
$$+ \cos^2\beta(\mathcal{I}_x \cos^2\alpha + \mathcal{I}_y \sin^2\alpha)]\ . \qquad (59)$$

By substituting this expression back into (55) and simplifying, one is left with the equations (41) and (42) of section IV.

Now, we wish to determine the changes in the various angular velocities due to the collision with th wall. We seek $\Delta\dot{\alpha}$, $\Delta\dot{\beta}$, and $\Delta\dot{\gamma}$ (but not $\Delta\dot{z}$!). First consider the following three quantities,

$$A_1 = [(L_x \cos\gamma - L_y \sin\gamma)\sin\beta - L_z \cos\beta]/(\mathcal{I}_x + \mathcal{I}_y),$$
$$A_2 = [((L_x \cos\gamma - L_y \sin\gamma)\cos\beta + L_z \sin\beta)\cos\alpha$$
$$-(L_x \sin\gamma + L_y \cos\gamma)\sin\alpha]/(\mathcal{I}_y + \mathcal{I}_z),$$
$$A_3 = -[((L_x \cos\gamma - L_y \sin\gamma)\cos\beta + L_z \sin\beta)\sin\alpha$$
$$+(L_x \sin\gamma + L_y \cos\gamma)\cos\alpha]/(\mathcal{I}_x + \mathcal{I}_z). \qquad (60)$$

It can be verified that

$$A_1 = \dot{\alpha} + \dot{\gamma}\cos\beta,$$
$$A_2 = \dot{\beta}\sin\alpha - \dot{\gamma}\cos\alpha\sin\beta,$$
$$A_3 = \dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta. \qquad (61)$$

Thus,

$$\Delta A_1 = \Delta\dot{\alpha} + \Delta\dot{\gamma}\cos\beta,$$
$$\Delta A_2 = \Delta\dot{\beta}\sin\alpha - \Delta\dot{\gamma}\cos\alpha\sin\beta,$$
$$\Delta A_3 = \Delta\dot{\beta}\cos\alpha + \Delta\dot{\gamma}\sin\alpha\sin\beta, \qquad (62)$$

and

$$\Delta\dot{\gamma} = [\Delta A_3 \sin\alpha - \Delta A_2 \cos\alpha]/\sin\beta,$$
$$\Delta\dot{\alpha} = \Delta A_1 - \cos\beta\Delta\dot{\gamma},$$
$$\Delta\dot{\beta} = \Delta A_2 \sin\alpha + \Delta A_3 \cos\alpha. \qquad (63)$$

Next, $\Delta A_1,\ \Delta A_2$, and ΔA_3 are obtained from the equations of motion

$$
\begin{aligned}
\Delta L_x &= y f_n \tau \\
\Delta L_y &= -x f_n \tau \\
\Delta L_z &= 0 \\
M\Delta\dot{z} &= f_n \tau.
\end{aligned}
\tag{64}
$$

Let us first define our x, y, and z coordinates. As explained earlier, we label a point on the surface of our rigid body by a radius vector and apply the rotation matrix $\mathcal{R}(\alpha,\beta,\gamma)$ given by

$$
\begin{aligned}
\mathcal{R}_{11} &= \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha \\
\mathcal{R}_{12} &= \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha \\
\mathcal{R}_{13} &= -\cos\gamma\sin\beta \\
\mathcal{R}_{21} &= -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha \\
\mathcal{R}_{22} &= -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha \\
\mathcal{R}_{23} &= \sin\gamma\sin\beta \\
\mathcal{R}_{31} &= \sin\beta\cos\alpha,\ \mathcal{R}_{32} = \sin\beta\sin\alpha,\ \mathcal{R}_{33} = \cos\beta,
\end{aligned}
\tag{65}
$$

to $\vec{R}(\theta,\phi)$ resulting in $x(\alpha,\beta,\gamma,\theta,\phi),\ y(\alpha,\beta,\gamma,\theta,\phi),$ and $z(\alpha,\beta,\gamma,\theta,\phi)$. Using these expressions for x and y in the equations of motion produces

$$
\begin{aligned}
\Delta A_1 &= \frac{R(\theta,\phi)\Delta\dot{z}}{M(\mathcal{I}_x+\mathcal{I}_y)}[-\sin\theta\cos\phi\sin\alpha\sin\beta \\
&\quad + \sin\theta\sin\phi\cos\alpha\sin\beta] \\
\Delta A_2 &= \frac{R(\theta,\phi)\Delta\dot{z}}{M(\mathcal{I}_y+\mathcal{I}_z)}[\sin\theta\sin\phi\cos\beta - \cos\theta\sin\alpha\sin\beta] \\
\Delta A_3 &= \frac{-R(\theta,\phi)\Delta\dot{z}}{M(\mathcal{I}_x+\mathcal{I}_z)}[-\sin\theta\cos\phi\cos\beta + \cos\theta\cos\alpha\sin\beta].
\end{aligned}
\tag{66}
$$

Finally, substituting these into the equations for $\Delta\dot{\gamma}$, $\Delta\dot{\alpha}$, and $\Delta\dot{\beta}$ (63) provides the desired expressions.

$$\Delta\dot{\gamma} = \frac{R(\theta,\phi)\Delta\dot{z}}{M\sin\beta}\left[\frac{\sin\alpha}{\mathcal{I}_x+\mathcal{I}_z}(\sin\theta\cos\phi\cos\beta\right.$$
$$-\cos\theta\cos\alpha\sin\beta) \tag{67}$$
$$\left.-\frac{\cos\alpha}{\mathcal{I}_y+\mathcal{I}_z}(\sin\theta\sin\phi\cos\beta-\cos\theta\sin\alpha\sin\beta)\right],$$

$$\Delta\dot{\alpha} = \frac{R(\theta,\phi)\Delta\dot{z}}{M}\left[\frac{1}{\mathcal{I}_x+\mathcal{I}_y}(-\sin\beta\sin\theta\sin\alpha\cos\phi\right.$$
$$+\sin\beta\sin\theta\cos\alpha\sin\phi) \tag{68}$$
$$-\cot\beta\left(\frac{\sin\alpha}{\mathcal{I}_x+\mathcal{I}_z}(\sin\theta\cos\phi\cos\beta-\cos\theta\cos\alpha\sin\beta)\right.$$
$$\left.\left.-\frac{\cos\alpha}{\mathcal{I}_y+\mathcal{I}_z}(\sin\theta\sin\phi\cos\beta-\cos\theta\sin\alpha\sin\beta)]\right)\right],$$

$$\Delta\dot{\beta} = \frac{R(\theta,\phi)\Delta\dot{z}}{M}\left[\frac{\sin\alpha}{\mathcal{I}_y+\mathcal{I}_z}(\sin\theta\sin\phi\cos\beta\right.$$
$$-\cos\theta\sin\alpha\sin\beta) \tag{69}$$
$$\left.+\frac{\cos\alpha}{\mathcal{I}_x+\mathcal{I}_z}(\sin\theta\cos\phi\cos\beta-\cos\theta\cos\alpha\sin\beta)\right].$$

One final piece of information is needed before we can proceed with the verification of specular reflection. From equation (65), it can be deduced that

$$z = R(\theta,\phi)(\sin\beta\cos\alpha\sin\theta\cos\phi$$
$$+\sin\beta\sin\alpha\sin\theta\sin\phi$$
$$+\cos\beta\cos\theta). \tag{70}$$

Using this, we can derive the curved billiard wall (a two-dimensional hypersurface) for our system. First, z must be extremized with respect to θ and ϕ i.e. $\nabla z(\theta_c,\phi_c)=0$. This results in two conditions,

$$\frac{\partial z}{\partial\theta}(\theta=\theta_c,\phi=\phi_c) = \left(\frac{\partial R}{\partial\theta_c}\right)(\sin\beta\cos\alpha\sin\theta_c\cos\phi_c$$
$$+\sin\beta\sin\alpha\sin\theta_c\sin\phi_c+\cos\beta\cos\theta_c)$$
$$+R(\sin\beta\cos\alpha\cos\theta_c\cos\phi_c$$
$$+\sin\beta\sin\alpha\cos\theta_c\sin\phi_c-\cos\beta\sin\theta_c)=0 \tag{71}$$

and

$$\frac{\partial z}{\partial \phi}(\theta = \theta_c, \phi = \phi_c) = \left(\frac{\partial R}{\partial \phi_c}\right)(\sin\beta\cos\alpha\sin\theta_c\cos\phi_c$$
$$+\sin\beta\sin\alpha\sin\theta_c\sin\phi_c + \cos\beta\cos\theta_c)$$
$$+R(-\sin\beta\cos\alpha\sin\theta_c\sin\phi_c$$
$$+\sin\beta\sin\alpha\sin\theta_c\cos\phi_c) = 0. \quad (72)$$

As a consequence of these relations, all reference to derivatives of R disappear in the evaluation of $\partial z/\partial\alpha$ and $\partial z/\partial\beta$, which will be needed below to construct the tangents and normals to the curved billiard wall.

Our billiard problem exists in a three-dimensional space given by the coordinates (α, β, z) which specify the height of the center of mass and the orientation of the body. The center of mass moves between wall surfaces which depend on the orientation of the body. We denote the lower wall as $S(\alpha, \beta, -z)$. (The upper wall is displaced by h, reflected, and subject to an evident phase shift). Of course, the θ and ϕ dependence of z must be eliminated *via* equations (71) and (72), i.e. $\theta_c = \theta_c(\alpha, \beta)$ and $\phi_c = \phi_c(\alpha, \beta)$. The two principle tangent vectors to the lower wall are $S_\alpha = (1, 0, -\partial z/\partial\alpha)$ and $S_\beta = (0, 1, -\partial z/\partial\beta)$. Using equation (70) and the two extremum conditions (71) and (72), one finds that

$$\frac{\partial z}{\partial \alpha} = R(\theta, \phi)[-\sin\beta\sin\alpha\sin\theta\cos\phi$$
$$+\sin\beta\cos\alpha\sin\theta\sin\phi] \quad (73)$$

and,

$$\frac{\partial z}{\partial \beta} = R(\theta, \phi)[\cos\beta\cos\alpha\sin\theta\cos\phi$$
$$+\cos\beta\sin\alpha\sin\theta\sin\phi - \sin\beta\cos\theta]. \quad (74)$$

As in the specific example of the cube, there are two conditions for specular reflection with respect to the tangent vectors S_α and S_β. Since the velocity of the billiard ball is $v = (\dot\alpha, \dot\beta, \dot z)$, these tangent conditions read

$$S_\alpha \cdot \Delta v = \Delta\dot\alpha \; g_{\alpha\alpha} + \Delta\dot\beta \; g_{\alpha\beta} - \Delta\dot z \frac{\partial z}{\partial \alpha} \; g_{zz} = 0 \tag{75}$$

and,

$$S_\beta \cdot \Delta v = \Delta\dot\alpha \; g_{\alpha\beta} + \Delta\dot\beta \; g_{\beta\beta} - \Delta\dot z \frac{\partial z}{\partial \beta} \; g_{zz} = 0. \tag{76}$$

It can be shown following considerable tedious algebra that these two equations are satisfied.

Finally, we prove that the last condition for specular reflection, *viz*, $N \cdot (2v + \Delta v) = 0$ holds automatically if the tangent conditions hold and energy conservation is valid in the billiard space. Denoting the normal as $N = (N_1, N_2, 1)$, the equations $N \cdot S_\alpha = 0, N \cdot S_\beta = 0$ are explicitly given by

$$N_1 \; g_{\alpha\alpha} \; + N_2 \; g_{\alpha\beta} \; = \frac{\partial z}{\partial \alpha} \; g_{zz} \tag{77}$$

and

$$N_1 \; g_{\alpha\beta} \; + N_2 \; g_{\beta\beta} \; = \frac{\partial z}{\partial \beta} \; g_{zz}. \tag{78}$$

Solving for N_1 and N_2, one finds

$$N_1 = \frac{(\partial z/\partial\alpha \; g_{\beta\beta} - \partial z/\partial\beta \; g_{\alpha\beta}) \; g_{zz}}{(g_{\alpha\alpha} \; g_{\beta\beta} - \; g_{\alpha\beta} \; g_{\alpha\beta})} \tag{79}$$

and,

$$N_2 = \frac{(\partial z/\partial\alpha \; g_{\alpha\beta} - \partial z/\partial\beta \; g_{\beta\beta}) \; g_{zz}}{(g_{\alpha\beta} \; g_{\alpha\beta} - \; g_{\alpha\alpha} \; g_{\beta\beta})}. \tag{80}$$

Using the above expressions, $N \cdot (2v + \Delta v) = 0$ becomes

$$\begin{aligned} (2\dot\alpha + \Delta\dot\alpha)(N_1 \; g_{\alpha\alpha} + N_2 \; g_{\alpha\beta}) \\ +(2\dot\beta + \Delta\dot\beta)(N_1 \; g_{\alpha\beta} + N_2 \; g_{\beta\beta}) \\ +(2\dot z + \Delta\dot z) \; g_{zz} = 0, \end{aligned} \tag{81}$$

which simplifies to

$$
\begin{aligned}
(2\dot{\alpha} + \Delta\dot{\alpha})\ (\partial z/\partial\alpha)\ g_{zz} \\
+(2\dot{\beta} + \Delta\dot{\beta})\ (\partial z/\partial\beta)\ g_{zz} \\
+(2\dot{z} + \Delta\dot{z})\ g_{zz} = 0.
\end{aligned} \tag{82}
$$

¿From the tangent equations (75) and (76) , we find that

$$
\frac{\partial z}{\partial\alpha} = \frac{\Delta\dot{\alpha}}{\Delta\dot{z}}\frac{g_{\alpha\alpha}}{g_{zz}} + \frac{\Delta\dot{\beta}}{\Delta\dot{z}}\frac{g_{\alpha\beta}}{g_{zz}} \tag{83}
$$

and

$$
\frac{\partial z}{\partial\beta} = \frac{\Delta\dot{\alpha}}{\Delta\dot{z}}\frac{g_{\alpha\beta}}{g_{zz}} + \frac{\Delta\dot{\beta}}{\Delta\dot{z}}\frac{g_{\beta\beta}}{g_{zz}}. \tag{84}
$$

Substituting (83) and (84) into (81) results in

$$
\begin{aligned}
&(2\dot{\alpha} + \Delta\dot{\alpha})\ \frac{\Delta\dot{\alpha}}{\Delta\dot{z}}\ g_{\alpha\alpha} \\
&+ (2\dot{\beta} + \Delta\dot{\beta})\ \frac{\Delta\dot{\beta}}{\Delta\dot{z}}\ g_{\beta\beta} \\
&+ (2\dot{z} + \Delta\dot{z})\ g_{zz} \\
&+ 2(\dot{\alpha}\Delta\dot{\beta} + \dot{\beta}\Delta\dot{\alpha} + \Delta\dot{\alpha}\Delta\dot{\beta})\ \frac{g_{\alpha\beta}}{\Delta\dot{z}} = 0
\end{aligned} \tag{85}
$$

which is simply the conservation of energy condition. Therefore, the normal condition for specular reflection also holds and thus, we have proven that any three-dimensional rigid body colliding elastically between two parallel, flat walls is equivalent to a billiard problem.

FIGURES

FIG. 1. A trajectory in configuration space for the ellipsoidal system $\epsilon = 0.5$, $h = 2$, and $L_z \neq 0$ as indicated in the text.

FIG. 2. A trajectory in configuration space for the ellipsoidal system $\epsilon = 0.5$, $h = 2$, and $L_z = 0$ (noninteracting case).

FIG. 3. A phase space surface of section for the ellipsoidal system of Fig.1.

FIG. 4. A phase space surface of section for the ellipsoidal system of Fig.2.