Localization of Classical Waves I: Acoustic Waves.

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Abstract

We consider classical acoustic waves in a medium described by a position dependent mass density $\varrho(x)$. We assume that $\varrho(x)$ is a random perturbation of a periodic function $\varrho_0(x)$ and that the periodic acoustic operator $A_0 = -\nabla \cdot \frac{1}{\varrho_0(x)} \nabla$ has a gap in the spectrum. We prove the existence of localized waves, i.e., finite energy solutions of the acoustic equations with the property that almost all of the wave's energy remains in a fixed bounded region of space at all times, with probability one. Localization of acoustic waves is a consequence of Anderson localization for the self-adjoint operators $A = -\nabla \cdot \frac{1}{\varrho(x)} \nabla$ on $L^2(\mathbb{R}^d)$. We prove that, in the random medium described by $\varrho(x)$, the random operator A exhibits Anderson localization inside the gap in the spectrum of A_0 . This is shown even in situations when the gap is totally filled by the spectrum of the random operator; we can prescribe random environments that ensure localization in almost the whole gap.

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Contents

1 Introduction

Localization of classical waves, acoustic and electromagnetic, have received much attention in recent years (e.g., [An2, J1, J2, DE, Sc, VP, M] and references therein). This phenomenon arises from coherent multiple scattering and interference and occurs when the scale of the coherent multiple scattering reduces to the wavelength itself. Numerous potential applications (e.g., [DE, J2, VP]), for instance, the optical transistor, and the fundamental significance of localization of classical waves motivate the interest in this phenomenon.

In spite of the clear similarities between localization of quantum-mechanical electrons (studied by Anderson [An1] for the tight binding model), and localization of classical waves, there are some important differences. In particular, classical waves are harder to localize [J2]. A substantial difference is apparent when we multiply the coefficient describing the medium (the position dependent density for acoustic media, the dielectric constant for the dielectric media, the potential for electrons) by a constant: for classical waves the spectrum of the relevant operators will be just rescaled, in contrast to the case of Schrödinger operators. Another significant difference is that a local change in a homogeneous medium cannot create localized eigenfunctions for classical waves operators, but it can certainly create localized states for Schrödinger operators. For classical waves the bottom of the spectrum is always at 0 and clearly does not depend on the medium; for Schrödinger operators the bottom of the spectrum is movable so we may expect the rise of localized states for appropriate potentials.

Thus, in order for localization of classical waves to occur we have to play with the space distribution of inhomogeneities. S. John [J2] writes: *Photon localization arises here not as the by-product of high degree of uncontrolled disorder, but rather as a result of a subtle interplay between order and disorder. The true criterion for localization, in fact, depends strongly on the underlying static structure factor of the medium.* This suggests that the localization of classical waves can be achieved only in appropriately prepared random environments.

To create an environment which would favor localization one considers first a perfectly periodic medium. It is well known that the spectrum associated with a periodic medium has band gap structure and that the most significant manifestation of coherent multiple scattering is the rise of a gap in the spectrum. If such a periodic medium with a gap in the spectrum is slightly randomized, eigenvalues with exponentially localized eigenfunctions should arise in the gap. If the disorder is increased further within some limits the localized states can fill the gap completely. *This is exactly the medium in which we study acoustic and electromagnetic waves.* Acoustic waves are treated in this paper; electromagnetic waves will be discussed in a sequel [FK3]. *We assume that the underlying periodic medium has a gap in the spectrum.* The existence of periodic media exhibiting gaps in the spectrum is proved for acoustic and $2D$-periodic dielectric structures [FKu1, FKu2]. We will slightly randomize such periodic media with a gap in the spectrum and show that, under pretty reasonable hypotheses, Anderson localization occurs in a vicinity of the edges of the gap.

We previously considered these questions and media in a lattice approximation, both for classical waves [FK2] and for Schrödinger operators [FK1]. The strategy of this paper is the same one we used in [FK2], the main differences are of technical nature and due to working on the continuum instead of the lattice.

1.1 Acoustic waves and localization

In an inhomogeneous medium the basic linear acoustic equations are (e.g., [MI])

$$\kappa \frac{\partial p}{\partial t} = -\nabla \cdot \mathbf{u}, \quad \varrho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p \tag{1}$$

where $p = p(x,t)$ is the pressure, $\mathbf{u} = \mathbf{u}(x,t)$ is the velocity, $\kappa = \kappa(x)$ is the compressibility, and $\varrho = \varrho(x)$ is the mass density, so the pressure satisfies the second order partial diferential equation

$$\kappa \frac{\partial^2 p}{\partial t^2} = \nabla \cdot \frac{1}{\varrho} \nabla p. \tag{2}$$

The energy density $\mathcal{E}(x,t)$ and the (conserved) energy $\mathcal{E}$ are given by

$$\mathcal{E}(x,t) = \frac{1}{2}\left[\varrho(x)|u(x,t)|^2 + \kappa(x)|p(x,t)|^2\right], \quad \mathcal{E} = \int \mathcal{E}(x,t)\, dx. \tag{3}$$

It is convenient to introduce the momentum potential $\Psi = \Psi(x,t)$ by $\varrho u = -\nabla \Psi$, and rewrite (1) as

$$\kappa \frac{\partial p}{\partial t} = \nabla \cdot \frac{1}{\varrho} \nabla \Psi, \quad \frac{\partial \Psi}{\partial t} = \nabla p. \tag{4}$$

It follows that Ψ obeys the same second order partial diferential equation (2) as the pressure p, i.e.,

$$\kappa \frac{\partial^2 \Psi}{\partial t^2} = \nabla \cdot \frac{1}{\varrho} \nabla \Psi, \tag{5}$$

and the energy density can be rewritten as

$$\mathcal{E}(x,t) = \frac{1}{2}\left[\varrho(x)|\nabla\Psi(x,t)|^2 + \kappa(x)\left|\frac{\partial \Psi}{\partial t}(x,t)\right|^2\right]. \tag{6}$$

Motivated by (2), (4) and (5), we set $A = -\nabla \cdot \frac{1}{\varrho}\nabla$. It follows from (4) that $\Psi = -A^{-1}\kappa\frac{\partial p}{\partial t}$, so we can recover the velocity $\mathbf{u}$ from the pressure p by $\mathbf{u} = \frac{1}{\varrho}\nabla A^{-1}\kappa\frac{\partial p}{\partial t}$, so the energy density can also be written in terms of the pressure as

$$\mathcal{E}(x,t) = \frac{1}{2}\left[\frac{1}{\varrho(x)}\left|\nabla\left(A^{-1}\kappa\frac{\partial p}{\partial t}\right)(x,t)\right|^2 + \kappa(x)|p(x,t)|^2\right]. \tag{7}$$

In this article we work with a position dependent mass density $\varrho(x)$ and constant compressibility κ (we set $\kappa = 1$). We will define A as a nonnegative self-adjoint operator on $L^2(\mathbb{R}^d)$. Finite energy solutions for equations (2) and (5) are given by (e.g., [RS3, Section IX.10])

$$\varphi(t) = \cos(tA^{\frac{1}{2}})\varphi_0 + A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})\dot{\varphi}_0, \tag{8}$$

where the Cauchy data $(\varphi_0, \dot{\varphi}_0)$ can be taken in $L^2(\mathbb{R}^d) \times \mathcal{D}(A^{-\frac{1}{2}})$ for the pressure p, and in $\mathcal{D}(A^{\frac{1}{2}}) \times L^2(\mathbb{R}^d)$ for the potential Ψ.

A localized acoustic wave should be a finite energy solution of (1) with the property that almost all of the wave's energy remains in a fixed bounded region of space at all times, e.g.,

$$\lim_{R\to\infty} \inf_t \frac{1}{\mathcal{E}} \int_{|x|\le R} \mathcal{E}(x,t)\, dx = 1. \tag{9}$$

Our strategy for proving the existence localized waves is the following: we first prove that the operator A has pure point spectrum in some closed interval $I \subset (0, \infty)$, with all the corresponding eigenfunctions being exponentially decaying. For this operator it will follow that the gradient of an exponentially decaying eigenfunction has exponentially decaying local L^2-norms, so the corresponding energy densities (given by either (6) or (7)) also have exponentially decaying local L^2-norms, uniformly in the time t. If $\chi_I(A)$ is the corresponding spectral projection, then any solution of either (2) or (5) given by (8), with Cauchy data in the range of $\chi_I(A)$, satisfies (9).

Localization of acoustic waves is thus a consequence of Anderson localization for operators $A = -\nabla \cdot \frac{1}{\varrho}\nabla$ on $L^2(\mathbb{R}^d)$, i.e., the existence of closed intervals where these operators have pure point spectrum with exponentially decaying eigenfunctions.

Similarly, the localization of electromagnetic waves is a consequence of Anderson localization for operators $M = \nabla \times \frac{1}{\varepsilon}\nabla\times$ on $L^2(\mathbb{R}^3;\, \mathbb{C}^3)$, where $\varepsilon = \varepsilon(x)$ is the position dependent dielectric constant. Localization of electromagnetic waves will be studied in a sequel to this article [FK3].

1.2 Statement of results

We study acoustic waves which are described by the following formally self-adjoint operator on $L^2(\mathbb{R}^d)$:

$$A \;=\; A(\varrho) \;=\; -\nabla \cdot \frac{1}{\varrho(x)}\nabla \;=\; -\sum_{j=1}^{d} \partial_j \frac{1}{\varrho(x)} \partial_j, \tag{10}$$

where we always assume that $\varrho(x)$ is a measurable real valued function satisfying

$$0 < \varrho_- \leq \varrho(x) \leq \varrho_+ < \infty \quad \text{a.e. for some constants } \ \varrho_- \ \text{ and } \ \varrho_+. \tag{11}$$

A is rigorously defined as the unique nonnegative self-adjoint operator on $L^2(\mathbb{R}^d)$ uniquely defined by the quadratic form given as the closure of the nonnegative densely defined quadratic form

$$\mathcal{A}(\psi) \;=\; \langle \nabla\psi, \frac{\infty}{\varrho(\S)}\nabla\psi\rangle \equiv \;\sum_{|=\infty}^{\lceil} \langle \partial_| \psi, \frac{\infty}{\varrho(\S)}\partial_| \psi\rangle \text{ , with } \ \psi \in \mathcal{C}_\prime^\infty(\mathbb{R}^\lceil)\ . \tag{12}$$

In this article we consider acoustic waves in random media obtained by random perturbations of a periodic medium. The properties of the medium are described by the position dependent quantity $\varrho(x)$, which we will take to always satisfy the following assumptions.

Assumption 1 (The Random Media) *$\varrho_g(x) = \varrho_{g,\omega}(x)$ is a random function of the form*

$$\varrho_{g,\omega}(x) = \varrho_0(x)\gamma_{g,\omega}(x) \ \text{ , with } \ \gamma_{g,\omega}(x) = 1 + g \sum_{i\in\mathbb{Z}^d} \omega_i u_i(x), \tag{13}$$

where

(i) $\varrho_0(x)$ *is a measurable real valued function which is q-periodic for some* $q \in \mathbb{N}$, *i.e.,* $\varrho_0(x) = \varrho_0(x + qi)$ *for all* $x \in \mathbb{R}$ *and* $i \in \mathbb{Z}^d$, *with*

$$0 < \varrho_{0,-} \leq \varrho_0(x) \leq \varrho_{0,+} < \infty \ \textit{for a.e.}\ x \in \mathbb{R}^d \tag{14}$$

for some constants $\varrho_{0,-}$ *and* $\varrho_{0,+}$.

(ii) $u_i(x) = u(x - i)$ *for each* $i \in \mathbb{Z}^d$, u *being a nonnegative measurable real valued function with compact support, say* $u(x) = 0$ *if* $\|x\|_\infty \leq r_u$ *for some* $r_u < \infty$, *such that*

$$0 < U_- \leq U(x) \equiv \sum_{i \in \mathbb{Z}^d} u_i(x) \leq U_+ < \infty \ \textit{for a.e.}\ x \in \mathbb{R}^d, \tag{15}$$

for some constants U_- *and* U_+.

(iii) $\omega = \{\omega_i;\, i \in \mathbb{Z}^d\}$ *is a family of independent, identically distributed random variables taking values in the interval* $[-1, 1]$, *whose common probability distribution* μ *has a bounded density* $\rho > 0$ *a.e. in* $[-1, 1]$.

(iv) g, *satisfying* $0 \leq g < \frac{1}{U_+}$, *is the disorder parameter.*

For acoustic waves $\varrho_{g,\omega}(x)$ is the random position dependent mass density of the medium. Notice that Assumption 1 implies that each $\varrho_{g,\omega}$ satisfies (11) with

$$\varrho_\pm = \varrho_{g,\pm} = \varrho_{0,\pm}(1 \pm gU_+). \tag{16}$$

For later use we set

$$\delta_\pm(g) = \frac{U_\pm}{1 \mp gU_+}, \ \text{ with } \ 0 \leq g < \frac{1}{U_+}. \tag{17}$$

The periodic operator associated with the coefficient $\varrho_0(x)$ is denoted by A_0, i.e., $A_0 = A(\varrho_0)$. We will study the random acoustic operators (see appendix A)

$$A_g = A_{g,\omega} = A(\varrho_{g,\omega}). \tag{18}$$

It follows from ergodicity (measurability follows from Theorem 38 below) that there exists a nonrandom set Σ_g, such that $\sigma(A_{g,\omega}) = \Sigma_g$ with probability one, where $\sigma(A)$ denotes the spectrum of the operator A. In addition, the decomposition of $\sigma(A_{g,\omega})$ into pure point spectrum, absolutely continuous spectrum and singular continuous spectrum is also independent of the choice of ω with probability one (e.g., [PF]).

In this article we are interested in the phenomenon of localization. According to the philosophy of Anderson localization we will assume that the operator A_0 has at least one gap in the spectrum.

Assumption 2 (The gap in the spectrum) *There is a gap in the spectrum of the operator* A_0. *More precisely, there exist* $0 \leq \hat{a} < a < b < \hat{b}$ *such that*

$$\sigma(A_0) \bigcap [\hat{a}, \hat{b}] = [\hat{a}, a] \bigcup [b, \hat{b}],$$

so the interval (a, b) *is a gap in* $\sigma(A_0)$.

The following theorem gives information on the location of Σ_g, the (nonrandom) spectrum of the random acoustic operator A_g.

Theorem 3 (Location of the Spectrum) *Let the random operator A_g defined by (18) satisfy Assumptions 1 and 2. There exists g_0, with*

$$\frac{1}{U_+}\left(1-\left(\frac{a}{b}\right)^{\frac{1}{2}}\right) \leq g_0 \leq \frac{1}{U_+}\min\left\{1, \left(\left(\frac{b}{a}\right)^{\frac{U_+}{2U_-}}-1\right)\right\}, \tag{19}$$

and strictly increasing, Lipschitz continuous real valued functions $a(g)$ and $-b(g)$ on the interval $[0, \frac{1}{U_+})$, with $a(0) = a$, $b(0) = b$ and $a(g) \leq b(g)$, such that:

(i)

$$\Sigma_g \bigcap [\hat{a}, \hat{b}] = [\hat{a}, a(g)] \bigcup [b(g), \hat{b}] \ . \tag{20}$$

(ii) *For $g < g_0$, we have $a(g) < b(g)$ and $(a(g), b(g))$ is a gap in the spectrum of the random operator A_g, located inside the gap (a, b) of the unperturbed periodic operator A_0. Moreover, we have*

$$a \leq a(1+gU_+)^{\frac{U_-}{U_+}} \leq a(g) \leq \frac{a}{1-gU_+} \tag{21}$$

and

$$b(1-gU_+) \leq b(g) \leq \frac{b}{(1+gU_+)^{\frac{U_-}{U_+}}} \leq b. \tag{22}$$

(iii) *If $g_0 < \frac{1}{U_+}$, we have $a(g) = b(g)$ for all $g \in [g_0, \frac{1}{U_+})$, and the random operator A_g has no gap inside the gap (a, b) of the unperturbed periodic operator A_0, i.e., $[\hat{a}, \hat{b}] \subset \Sigma_g$.*

Definition 4 (Exponential localization) *We say that the random operator A_g exhibits* localization *in an interval $I \subset \Sigma_g$, if A_g has* only *pure point spectrum in I with probability one. We have* exponential localization *in I if we have localization and, with probability one, all the eigenfunctions corresponding to eigenvalues in I have exponential decay.*

Remark 5 *The gradients of exponentially decaying eigenfunctions of A_g always have exponentially decaying local L^2-norms (Corollary 40). Thus the corresponding energy densities (given by either (6) or (7)) also have exponentially decaying local L^2-norms, uniformly in the time t.*

Our main results show that random perturbations create exponentially localized eigenfunctions near the edges of the gap. Our method requires low probability of extremal values for the random variables; the following two theorems achieve this in different ways. The results are formulated for the left edge of the gap; with similar results holding at the right edge.

Theorem 6 (Localization at the edge) *Let the random operator A_g defined by (18) satisfy Assumptions 1 and 2, with*

$$\mu\{(1-\gamma, 1]\} \leq K\gamma^{\eta} \text{ for } 0 \leq \gamma \leq 1, \tag{23}$$

where $K < \infty$ and $\eta > d$. For any $g < g_0$ there exists $\delta(g) > 0$, depending only on the constants d, g, q, $\varrho_{0,\pm}$, $U_{\pm}$, r_u, a, b, K, η, $\|\rho\|_\infty$, such that the random operator A_g exhibits exponential localization in the interval $[a(g)-\delta(g), a(g)]$.

Theorem 7 (Localization in a specified interval) *Let the random operator A_g defined by (18) satisfy Assumptions 1 and 2. For any $g < g_0$, given $a < a_1 < a_2 < a(g)$, with $a(g) - a_1 \leq b(g) - a(g)$, there exists $p_1 > 0$, depending only on the constants d, g, q, $\varrho_{0,\pm}$, $U_{\pm}$, r_u, a, an upper bound on $\|\rho\|_\infty$ and on the given a_1, a_2, such that if*

$$\mu\left(\left(\frac{g_1}{g}, 1\right]\right) < p_1, \tag{24}$$

where g_1 is defined by $a(g_1) = a_1$, the random operator A_g exhibits exponential localization in the interval $[a_2, a(g)]$.

Theorems 6 and 7 can be extended to the situation when the gap is totally filled by the spectrum of the random operator, we then establish the existence of an interval (inside the original gap) where the random acoustic operator exhibits exponential localization. Notice that the extension of Theorem 7 tells us that we can arrange for localization in as much of the gap as we want.

Theorem 8 (Localization at the meeting of the edges) *Let the random operator A_g defined by (18) satisfy Assumptions 1 and 2, with*

$$\mu\{(1-\gamma, 1]\}, \;\; \mu\{[-1, -1+\gamma)\} \leq K\gamma^{\eta} \text{ for } 0 \leq \gamma \leq 1, \tag{25}$$

where $K < \infty$ and $\eta > d$. Suppose $g_0 < \frac{1}{U_+}$ (e.g., if $\left(\frac{b}{a}\right)^{\frac{U_+}{2U_-}} < 2$), so the random operator A_g has no gap inside (a,b) for $g \in [g_0, \frac{1}{U_+})$. Then there exist $0 < \epsilon < \frac{1}{U_+} - g_0$ and $\delta > 0$, depending only on the constants d, q, $\varrho_{0,\pm}$, $U_{\pm}$, r_u, a, b, K, η, $\|\rho\|_\infty$, such that the random operator A_g exhibits exponential localization in the interval $[a(g_0)-\delta, a(g_0)+\delta]$ for all $g_0 \leq g < g_0 + \epsilon$.

Theorem 9 (Localization in a specified interval in the closed gap) *Let the random operator A_g defined by (18) satisfy Assumptions 1 and 2. Suppose $g_0 < \frac{1}{U_+}$ (e.g., if $\left(\frac{b}{a}\right)^{\frac{U_+}{2U_-}} < 2$), so the random operator A_g has no gap inside (a,b) for $g \in [g_0, \frac{1}{U_+})$. Let $a < a_1 < a_2 < a(g_0) = b(g_0) < b_2 < b_1 < b$ be given. For any $g \in [g_0, \frac{1}{U_+})$ there exist $p_1, p_2 > 0$, depending* *only on the constants d, g, q, $\varrho_{0,\pm}$, $U_\pm$, r_u, a, b, an upper bound on $\|\rho\|_\infty$ and on the given a_1, a_2, b_1, b_2, such that if*

$$\mu\left(\left(\frac{g_1}{g}, 1\right]\right) < p_1, \quad \mu\left(\left[-1, -\frac{g_2}{g}\right)\right) < p_2, \tag{26}$$

where g_1 and g_2 are defined by $a(g_1) = a_1$ and $b(g_2) = b_1$ (notice $0 < g_1,\ g_2 < g_0 \le g$), the random operator A_g exhibits exponential localization in the interval $[a_2, b_2]$.

Theorems 8 and 9 are proved exactly as Theorems 6 and 7, respectively, taking into account both edges of the gap.

Remark 10 *The methods of this paper also apply to random Schrödinger operators on $\mathbb{R}^d$. They give a new proof of localization at the bottom of the spectrum, as in [HM, CH, Kp]. For random perturbations of a periodic Schrödinger operator with a gap in the spectrum, we obtain the analogues of Theorems 3, 6, 7, 8 and 9, extending our results on the lattice [FK1] to the continuum.*

Remark 11 *Theorems 6 and 8 should be true without the extra hypotheses (23) and (25). They are used in conjunction with a Combes-Thomas argument to obtain the starting hypothesis for the multiscale analysis, in the proof of localization. One may expect estimates similar to Lifshitz tails (e.g., [PF]) for the density of states inside the gap, which would replace (23) and (25) in the proofs. This is how the starting hypothesis is obtained for random Schrödinger operators at the bottom of the spectrum [HM].*

Combes and Hislop have announced an improved Combes-Thomas argument inside a gap; they obtain a decay rate proportional to the square root of the product of the distances to the edges of the gap. With this result we would only need $\eta > \frac{d}{2}$ in Theorem 6, but we would still need to require $\eta > d$ in Theorem 8.

Theorem 3 is proved in Section 4; the proof requires periodic operators and periodic boundary conditions, studied in Section 3. A Combes-Thomas argument for acoustic operators is given in Section 2. Theorems 6 and 7 are proved in Section 6 by multiscale analyses. The required Wegner-type estimate is in Section 5. The starting hypotheses are proved first for finite volume acoustic operators with periodic boundary condition, using a Combes-Thomas argument for operators with periodic boundary condition (Section 3) and Theorem 3. Appendix A contains a result on measurability of random operators, from which follows the desired measurability for the acoustic operators we study. We collect some results on elliptic operators in Appendix B.

We adopt the following definitions and notations:

- For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we let $|x|_p = (x_1^p + \ldots + x_d^p)^{1/p}$ for $1 \le p < \infty$, and $|x|_\infty = \max_{1 \le j \le d} |x_j|$. We set $|x| = |x|_2$ and $\|x\| = |x|_\infty$.

- $\Lambda_L(x) = \{y \in \mathbb{R}^d;\ \|y - x\| < \frac{L}{2}\}$ is the (open) cube of side L centered at $x \in \mathbb{R}^d$; $\bar{\Lambda}_L(x)$ is the closed cube, and $\check{\Lambda}_L(x) = \{y \in \mathbb{R}^d;\ -\frac{L}{2} \le y_i - x_i < \frac{L}{2},\ i = 1, \dots, d\}$ the half-open/half-closed cube.
- χ_Λ is the characteristic function of the set Λ; we write $\chi_{x,L} = \chi_{\Lambda_L(x)}$.
- A function f on $\mathbb{R}^d$ is called q-periodic for some $q > 0$ if $f(x+qi) = f(x)$ for all $x \in \mathbb{R}^d$ and $i \in \mathbb{Z}^d$.
- A domain Ω is an open connected subset of $\mathbb{R}^d$; its boundary is denoted by $\partial\Omega$.
- $L^p(\Omega)$ is the space of measurable complex-valued functions $u(x)$, $x \in \Omega$ with the norm $\|u\|_p = \|u\|_{p,\Omega} = [\int_\Omega |u(x)|^p\ dx]^{1/p}$. We will often use the space $L^2(\Omega)$ and in this case we will write $\|u\|_\Omega$ for $\|u\|_{2,\Omega}$. If $\Omega = \mathbb{R}^d$ we may omit it from the notation.
- $C^1(\Omega)$ is the linear space of continuously differentiable functions on the domain Ω, $C_0^1(\Omega)$ is the subspace of functions with compact support.
- $W^{1,p}(\Omega)$ is the Sobolev space of complex-valued functions $u(x)$, $x \in \Omega$ with the norm $\|u\|_{1,p,\Omega} = \left[\|u\|_{p,\Omega}^p + \||\nabla u|\|_{p,\Omega}^p\right]^{1/p}$.
- $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the Banach space of bounded operators from the normed space $\mathcal{X}$ to the normed space $\mathcal{Y}$; $\mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X})$.
- The domain, spectrum and adjoint of a linear operator A are denoted by $\mathcal{D}(\mathcal{A})$, $\sigma(A)$ and A^*, respectively .
- The domain of a quadratic form $\mathcal{A}$ is denoted by $\mathcal{D}(\mathcal{A})$.
- For a complex number z its conjugate is denoted by z^*.

2 A Combes-Thomas argument

Let the operator A be given by (10). If $z \notin \sigma(A)$, we write $R(z) = (A - z)^{-1}$.

Lemma 12 *Let the operator A be given by (10) with (11). Then for any $z \notin \sigma(A)$, $n \in \mathbb{N}$ and $\ell > 0$ we have*

$$\|\chi_{x,\ell} R(z)^n \chi_{y,\ell}\| \ \le \left(\frac{9}{\eta}\right)^n e^{(\sqrt{d}\ell/4)} e^{-m_z|x-y|} \quad \textit{for all } \ x, y \in \mathbb{R}^d, \tag{27}$$

with

$$m_z = \frac{\eta}{4\left[\varrho_+^{-1} + |z| + \eta\right]}, \tag{28}$$

where $\eta = \operatorname{dist}(z, \sigma(A))$.

Proof. We start by defining the operators formally given by

$$A_a = e^{a\cdot x} A e^{-a\cdot x}, \;\; a \in \mathbb{R}^d \tag{29}$$

as the closed densely defined operators uniquely defined by the corresponding quadratic forms. More precisely, for each $a \in \mathbb{R}^d$ we define quadratic forms on $C_0^1(\mathbb{R}^d)$ by

$$\mathcal{A}_\dashv[\psi] = \langle \nabla \rceil^{\dashv\cdot\S}\psi, \frac{\infty}{\varrho(\S)} \nabla \rceil^{-\dashv\cdot\S}\psi\rangle = \langle (\nabla + \dashv)\psi, \frac{\infty}{\varrho(\S)}(\nabla - \dashv)\psi\rangle \tag{30}$$

and

$$\begin{aligned} \mathcal{Q}_\dashv[\psi] &= \mathcal{A}_\dashv[\psi] - \mathcal{A}[\psi] && (31)\\ &= \langle \psi, \frac{1}{\varrho(x)} a\cdot\nabla\psi\rangle - \langle a\cdot\nabla\psi, \frac{1}{\varrho(x)}\psi\rangle - |a|^2\langle\psi, \frac{1}{\varrho(x)}\psi\rangle && (32) \end{aligned}$$

Notice that

$$|\langle \psi, \frac{1}{\varrho(x)} a\cdot\nabla\psi\rangle| \le \frac{1}{2}\{\langle\psi, \frac{1}{\varrho(x)}\psi\rangle + \langle a\cdot\nabla\psi, \frac{1}{\varrho(x)} a\cdot\nabla\psi\rangle\} \tag{33}$$

and

$$\langle a\cdot\nabla\psi, \frac{1}{\varrho(x)} a\cdot\nabla\psi\rangle \le |a|^2\mathcal{A}[\psi]\,. \tag{34}$$

Thus, if $|a| \le 1$ we have

$$|\mathcal{Q}_\dashv[\psi]| \le |\dashv|\mathcal{A}[\psi] + |\dashv|(\infty - |\dashv|)\varrho_+^{-\infty}\|\psi\|^\in \quad \text{for all } \psi \in \mathcal{C}_\prime^\infty(\mathbb{R}^\lceil)\,. \tag{35}$$

We now require $|a| < 1$ and use [Kato; Theorem VI.3.9] to conclude that $\mathcal{A}_\dashv$ is a closable sectorial form and define A_a as the unique m-sectorial operator associated with it. If in addition $z \notin \sigma(A)$ and

$$\Omega \equiv 2\|(|a|(1-|a|)\varrho_+^{-1} + |a|A)R(z)\| < 1 \tag{36}$$

we can conclude that $z \notin \sigma(A_a)$ and

$$\|R(z) - R_a(z)\| \le \frac{4\Omega}{(1-\Omega)^2}\|R(z)\|, \tag{37}$$

where $R_a(z) = (A_a - z)^{-1}$.

Since

$$\begin{aligned} &\Omega = 2\|((|a|(1-|a|)\varrho_+^{-1} + |a|z)R(z) + |a|\| && (38)\\ \le\;& 2|a|((1-|a|)\varrho_+^{-1} + |z|)\eta^{-1} + 1) \le 2|a|((\varrho_+^{-1} + |z|)\eta^{-1} + 1), && (39) \end{aligned}$$

it suffices to take

$$|a| < \frac{\eta}{2(\varrho_+^{-1} + |z| + \eta)} \tag{40}$$

to ensure $\Omega < 1$. In fact, we get $\Omega \le \frac{1}{2}$ for $|a| \le m_z$ (given by (28)), so

$$\|R_a(z)\| \le (1 + \frac{4\Omega}{(1-\Omega)^2})\|R(z)\| \le \frac{9}{\eta}. \tag{41}$$

Now let $x_0, y_0 \in \mathbb{R}^d$, $n \in \mathbb{N}$, $\ell > 0$, and take $a = \frac{m_z}{|x_0 - y_0|}(x_0 - y_0)$. We have

$$\begin{aligned} \chi_{x_0,\ell} R(z)^n \chi_{y_0,\ell} &= \chi_{x_0,\ell} e^{-a\cdot x} R_a(z)^n e^{a\cdot x} \chi_{y_0,\ell} && (42)\\ &= e^{-m_z|x_0-y_0|} \chi_{x_0,\ell} e^{-a\cdot(x-x_0)} R_a(z)^n e^{a\cdot(x-y_0)} \chi_{y_0,\ell}, && (43) \end{aligned}$$

so

$$\|\chi_{x_0,\ell} R(z)^n \chi_{y_0,\ell}\| \le \left(\frac{9}{\eta}\right)^n \|\chi_{x_0,\ell} e^{-a\cdot(x-x_0)}\|_\infty \|\chi_{y_0,\ell} e^{a\cdot(x-y_0)}\|_\infty e^{-m_z|x_0-y_0|}. \tag{44}$$

Since

$$\|\chi_{x_0,\ell} e^{\pm a\cdot(x-x_0)}\|_\infty \le e^{\frac{\sqrt{d}\ell}{2}|a|} = e^{\frac{\sqrt{d}\ell}{2} m_z}, \tag{45}$$

and clearly $m_z \le \frac{1}{4}$, the theorem is proved. □

The next lemma gives an exponential estimate for the gradient of the resolvent.

Lemma 13 *Let the operator A be given by (10) with (11), and let $z \notin \sigma(A)$ with η, m_z as in Lemma 12. Then $\nabla R(z):\ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d, \mathbb{C}^d)$ is a bounded operator with*

$$\|\nabla R(z)\| \le \Theta_1 \left(\frac{(1+|z|)}{\eta} + 1 \right), \tag{46}$$

where $\Theta_1 = \Theta(d, \varrho_\pm, 1)$ is given in (237). Furthermore, for each $\ell > 0$ we have

$$\|\chi_{x,\ell} \nabla R(z) \chi_{y,\ell}\| \le \Theta_1 (1+|z|) \frac{9}{\eta} e^{(3\sqrt{d}\ell/4)} e^{-m_z|x-y|} \tag{47}$$

for all $x,\ y \in \mathbb{R}^d$ with $|x-y| \ge 2\ell$.

Proof. For any $\psi \in L^2(\mathbb{R}^d)$ we have $R(z)\psi \in W^{1,2}\left(\mathbb{R}^d\right)$, since $R(z)\psi \in \mathcal{D}(\mathcal{A})$ and we have (11). The bound (46) is thus an immediate consequence of Proposition 41 and of the evident identity $AR(z)\psi = (I + zR(z))$. The proof of (47) is based upon Lemma 12 and Proposition 39. Let $\ell > 0,\ \ x,\ y \in \mathbb{R}^d$ with $|x - y| \ge 2\ell$. For any $\psi \in L^2(\mathbb{R}^d)$ we can use Proposition 39 to obtain

$$\begin{aligned} \|\chi_{x,\ell} \nabla R(z) \chi_{y,\ell}\psi\| &\le \Theta_1 \{\|\chi_{x,3\ell} R(z) \chi_{y,\ell}\psi\| + \|\chi_{x,3\ell} AR(z) \chi_{y,\ell}\psi\|\} && (48)\\ &\le \Theta_1 \{(1+|z|)\,\|\chi_{x,3\ell} R(z)\chi_{y,\ell}\psi\| + \|\chi_{x,3\ell}\chi_{y,\ell}\psi\|\} && (49)\\ &= \Theta_1 (1+|z|)\,\|\chi_{x,3\ell} R(z) \chi_{y,\ell}\psi\| && (50) \end{aligned}$$

Thus

$$\begin{aligned} &\|\chi_{x,\ell} \nabla R(z) \chi_{y,\ell}\| \le \Theta_1 (1+|z|)\,\|\chi_{x,3\ell} R(z) \chi_{y,\ell}\| && (51)\\ &\quad \le \Theta_1(1+|z|)\,\|\chi_{x,3\ell} R(z) \chi_{y,3\ell}\| \le \Theta_1 (1+|z|) \frac{9}{\eta} e^{(3\sqrt{d}\ell/4)} e^{-m_z|x-y|}, \end{aligned}$$

where we used Lemma 12. □

3 Periodic operators and periodic boundary condition

As in the matrix case [FK1, FK2], the (non-random) spectrum of random acoustic operators can be represented as the union of the spectra of finite volume acoustic operators with periodic boundary condition.

In this section we study acoustic operators in periodic media. We say that the acoustic operator A given by (10) with (11) is q-periodic for some $q > 0$ if $\varrho(x)$ is a q-periodic function. In this section A will always denote such an operator.

3.1 Periodic boundary condition

We start by defining the restriction of such A to a cube with periodic boundary condition. Given a cube $\Lambda = \Lambda_\ell(x)$, where $x \in \mathbb{R}^d$ and $\ell > 0$; we will denote by $\mathring{\Lambda}$ the torus we obtain by identifying the edges of the closed cube $\bar{\Lambda}$ in the usual way. We introduce the usual distance in the torus:

$$\mathring{d}\,(x, y) \equiv \min_{m \in \ell\mathbb{Z}^d} |x - y + m| \leq \frac{\sqrt{d}\ell}{2} \text{ for all } x, y \in \bar{\Lambda}. \tag{52}$$

We will identify functions on $\mathring{\Lambda}$ with their ℓ-periodic extensions to $\mathbb{R}^d$; for example, $C^1(\mathring{\Lambda})$ will be identified with the space of continuously differentiable ℓ-periodic functions on $\mathbb{R}^d$. We define $W^{1,2}(\mathring{\Lambda})$ as the closure of $C^1(\mathring{\Lambda})$ in $W^{1,2}(\Lambda)$.

We will always take $\ell \in q\mathbb{N}$ and define $\mathring{A}_\Lambda$, the restriction of A to Λ with periodic boundary condition, as the unique nonnegative self-adjoint operator on $L^2(\mathring{\Lambda}) \cong L^2(\Lambda)$, defined by the nonnegative densely defined closed quadratic form

$$\mathring{\mathcal{A}}_\Lambda\,(\psi) \;=\; \langle \nabla\psi, \frac{1}{\varrho(x)} \nabla\psi \rangle \equiv \sum_{j=1}^{d} \langle \partial_j \psi, \frac{1}{\varrho(x)} \partial_j \psi \rangle \text{ , with } \psi \in W^{1,2}(\mathring{\Lambda}) \,, \tag{53}$$

the inner product being in $L^2(\Lambda)$. It follows from (11) that $\mathring{A}_\Lambda \geq -\frac{1}{\varrho_+}\, \mathring{\Delta}_\Lambda$, where $\mathring{\Delta}_\Lambda$ is the Laplacian with periodic boundary condition on Λ. Since $-\,\mathring{\Delta}_\Lambda$ has compact resolvent, using the min-max principle [see Reed-Simon IV, Theorem XIII.2] we conclude that $\mathring{A}_\Lambda$ has compact resolvent.

The shift operators U^y; $y \in \mathbb{R}^d$ are defined by

$$U^y \varphi(x) = \varphi(x - y), \text{ for all } x \in \mathbb{R}^d. \tag{54}$$

They are unitary operators in $L^2(\mathbb{R}^d)$, and if $m \in q\mathbb{Z}^d$ we have $U^m A U^{-m} = A$ (as unbounded operators), since $\varrho(x)$ is a q-periodic function. Using the identification of functions on the torus with periodic functions, it is easy to see that for each $y \in \mathbb{R}^d$ we have that U^y is a unitary operator from $L^2(\mathring{\Lambda}_\ell\,(x))$ to $L^2(\mathring{\Lambda}_\ell\,(x + y))$, for any $x \in \mathbb{R}^d$, and

$$U^y\, \mathring{A}_{\Lambda_\ell(x)}\, U^{-y} = \mathring{A}_{\Lambda_\ell(x+y)}, \text{ so } \sigma(\mathring{A}_{\Lambda_\ell(x)}) = \sigma(\mathring{A}_{\Lambda_\ell(x+y)}). \tag{55}$$

3.2 Spectrum of periodic operators

If $k, n \in \mathbb{N}$, we say that $k \preceq n$ if $n \in k\mathbb{N}$ and that $k \prec n$ if $k \preceq n$ and $k \neq n$.

Theorem 14 *Suppose the operator A given by (10) with (11) is q-periodic. Let $\{\ell_n;\ n = 0, 1, 2, \ldots\}$ be a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0, 1, 2, \ldots$. Then*

$$\sigma\left(\overset{\circ}{A}_{\Lambda_{\ell_n}(0)}\right) \subset \sigma\left(\overset{\circ}{A}_{\Lambda_{\ell_{n+1}}(0)}\right) \subset \sigma(A) \text{ for all } n = 0, 1, 2, \ldots, \tag{56}$$

and

$$\sigma(A) = \overline{\bigcup_{n \geq 1} \sigma\left(\overset{\circ}{A}_{\Lambda_{\ell_n}(0)}\right)}. \tag{57}$$

Notice that it follows from (56) that a gap in the spectrum of A is inside a gap in the spectrum of $\overset{\circ}{A}_{\Lambda_\ell(x)}$ for any $x \in \mathbb{R}^d$ and $\ell \in q\mathbb{N}$.

A proof of (57) for periodic Schrödinger operators can be found in [Ea], based on Floquet theory. This proof can be adapted to operators as in the theorem in the case of smooth coefficients. For the nonsmooth coefficients we are interested in some aspects of the Floquet theory have to be revised. We give an alternative proof which does not use Floquet theory.

To prove the theorem we will relate the resolvent $\overset{\circ}{R}_\Lambda(z) = \left(\overset{\circ}{A}_\Lambda - zI\right)^{-1}$, a bounded operator on $L^2(\Lambda)$, to the resolvent $R(z) = (A - zI)^{-1}$, a bounded operator on $L^2(\mathbb{R}^d)$. Let us fix a cube $\Lambda = \Lambda_\ell(x)$ for some $x \in \mathbb{R}^d$ and $\ell \in q\mathbb{N}$, the map

$$\varphi \in L^2(\mathbb{R}^d) \mapsto \tilde{\varphi}_\Lambda = \left\{\tilde{\varphi}_{\Lambda,m};\ m \in \ell\mathbb{Z}^d\right\} \in \ell^2\left(\ell\mathbb{Z}^d, L^2(\Lambda)\right), \tag{58}$$

where

$$\tilde{\varphi}_{\Lambda,m} = \chi_\Lambda(U^{-m}\varphi) = U^{-m}(\chi_{\Lambda+m}\varphi) \text{ for each } m \in \ell\mathbb{Z}^d, \tag{59}$$

establishes a unitary isomorphism between the Hilbert spaces $L^2(\mathbb{R}^d)$ and $\ell^2\left(\ell\mathbb{Z}^d, L^2(\Lambda)\right)$. Under this unitary map, bounded operators T on $L^2(\mathbb{R}^d)$ have the following matrix representation on $\ell^2\left(\ell\mathbb{Z}^d, L^2(\Lambda)\right)$:

$$\widetilde{T}_\Lambda = \left\{\widetilde{T}_{\Lambda,m,n} = \chi_\Lambda U^{-m} T U^n \chi_\Lambda;\ m, n \in \ell\mathbb{Z}^d\right\}, \tag{60}$$

where each $\widetilde{T}_{\Lambda,m,n}$ is a bounded operator on $L^2(\Lambda)$. If R is a bounded ℓ-periodic operator, i.e., $U^{-m} R U^m = T$ for all $m \in \ell\mathbb{Z}^d$, we have

$$\widetilde{R}_{\Lambda,m,n} = \widehat{R}_{\Lambda,n-m} \equiv \chi_\Lambda R U^{n-m} \chi_\Lambda, \tag{61}$$

so $\widetilde{R}_\Lambda$ is a Toeplitz matrix with operator valued entries $\widehat{R}_{\Lambda,n-m}$. Notice that R is uniquely determined by

$$\widehat{R}_\Lambda = \left\{\widehat{R}_{\Lambda,m};\ m \in \ell\mathbb{Z}^d\right\}. \tag{62}$$

We now introduce the Banach spaces

$$\mathcal{L}_\Lambda = \left\{\varphi \in L^2(\mathbb{R}^d);\ \widetilde{\varphi}_\Lambda \in \ell^1\left(\ell\mathbb{Z}^d, L^2(\Lambda)\right)\right\}, \text{ with } \|\varphi\|_{\mathcal{L}_\Lambda} = \|\widetilde{\varphi}_\Lambda\|_{\ell^1\left(\ell\mathbb{Z}^d, L^2(\Lambda)\right)}, \tag{63}$$

and

$$\mathcal{W}_\Lambda = \left\{\varphi \in W^{1,2}(\mathbb{R}^d);\ \varphi, |\nabla\varphi| \in \mathcal{L}_\Lambda\right\}, \text{ with } \|\varphi\|^2_{\mathcal{W}_\Lambda} = \|\varphi\|^2_{\mathcal{L}_\Lambda} + \||\nabla\varphi|\|^2_{\mathcal{L}_\Lambda}, \tag{64}$$

and the normed space

$$\mathcal{B}_\Lambda = \left\{R \in \mathcal{B}\left(L^2(\mathbb{R}^d)\right) \ \ell\text{-periodic};\ \widehat{R}_\Lambda \in \ell^1\left(\ell\mathbb{Z}^d, \mathcal{B}\left(L^2(\Lambda)\right)\right)\right\}, \tag{65}$$

$$\text{with } \|R\|_{\mathcal{B}_\Lambda} = \|\widehat{R}_\Lambda\|_{\ell^1\left(\ell\mathbb{Z}^d, \mathcal{B}(L^2(\Lambda))\right)}.$$

We define the maps

$$\varphi \in \mathcal{L}_\Lambda \mapsto P_\Lambda\varphi = \sum_{m\in\ell\mathbb{Z}^d} \widetilde{\varphi}_{\Lambda,m} = \chi_\Lambda \sum_{m\in\ell\mathbb{Z}^d} U^{-m}\varphi \in L^2(\Lambda) \tag{66}$$

and

$$R \in \mathcal{B}_\Lambda \mapsto \mathcal{P}_\Lambda(R) = \sum_{m\in\ell\mathbb{Z}^d} \widehat{R}_{\Lambda,m} = \chi_\Lambda R \sum_{m\in\ell\mathbb{Z}^d} U^m\chi_\Lambda \in \mathcal{B}\left(L^2(\Lambda)\right). \tag{67}$$

Lemma 15 *The maps $P_\Lambda : \mathcal{L}_\Lambda \to L^2(\Lambda)$ and $\mathcal{P}_\Lambda : \mathcal{B}_\Lambda \to \mathcal{B}(L^2(\Lambda))$ are linear contractions, with P_Λ being onto. Moreover:*

(i)

$$\mathcal{B}_\Lambda \subset \mathcal{B}(\mathcal{L}_\Lambda) \text{ with } \|R\|_{\mathcal{B}(\mathcal{L}_\Lambda)} \le \|R\|_{\mathcal{B}_\Lambda} \text{ for all } R \in \mathcal{B}_\Lambda, \tag{68}$$

and

$$P_\Lambda R\varphi = \mathcal{P}_\Lambda(R)P_\Lambda\varphi \text{ for all } R \in \mathcal{B}_\Lambda, \varphi \in \mathcal{L}_\Lambda. \tag{69}$$

(ii)

$$RS \in \mathcal{B}_\Lambda \text{ and } \mathcal{P}_\Lambda(RS) = \mathcal{P}_\Lambda(R)\mathcal{P}_\Lambda(S) \text{ for all } R, S \in \mathcal{B}_\Lambda. \tag{70}$$

(iii)

$$P_\Lambda(\mathcal{W}_\Lambda) = W^{1,2}(\mathring{\Lambda}) \tag{71}$$

and

$$\nabla P_\Lambda\varphi = P_\Lambda\nabla\varphi \text{ for all } \varphi \in \mathcal{W}_\Lambda. \tag{72}$$

Proof. We will prove (iii), the other statements in the lemma being straightforward. We start by showing that

$$P_\Lambda\left(C^1_0(\mathbb{R}^d)\right) = C^1(\mathring{\Lambda}), \tag{73}$$

and

$$\nabla P_\Lambda\varphi = P_\Lambda\nabla\varphi \text{ for all } \varphi \in C^1_0(\mathbb{R}^d). \tag{74}$$

Let $\varphi \in C_0^1(\mathbb{R}^d)$, since it has compact support we have that $\sum_{m\in\ell\mathbb{Z}^d} U^{-m}\varphi$ is an ℓ-periodic function in $C^1(\mathbb{R}^d)$ (the sum is locally finite), so $P_\Lambda\varphi \in C^1(\mathring{\Lambda})$ and (74) holds by the definition of P_Λ. To see that we have equality in (73), it now suffices to notice that

$$C_c^1(\mathring{\Lambda}) \equiv \left\{\varphi \in C^1(\mathring{\Lambda});\ \sup\{\mathring{d}\,(x,y);\ x,y \in \text{supp}\ \varphi\} \le \frac{\ell}{2}\right\} \subset P_\Lambda\left(C_0^1(\mathbb{R}^d)\right), \tag{75}$$

and any $\varphi \in C^1(\mathring{\Lambda})$ can be written as a finite linear combination of functions in $C_c^1(\mathring{\Lambda})$.

Since $P_\Lambda : \mathcal{L}_\Lambda \to L^2(\Lambda)$ is a contraction, it follows from (73) and (74) that P_Λ is also a contraction from $C_0^1(\mathbb{R}^d)$, equipped with the norm of $\mathcal{W}_\Lambda$, to $C^1(\mathring{\Lambda})$, with the norm of $W^{1,2}(\mathring{\Lambda})$. As $C_0^1(\mathbb{R}^d)$ and $C^1(\mathring{\Lambda})$ are dense in $\mathcal{W}_\Lambda$ and $W^{1,2}(\mathring{\Lambda})$, respectively, we can conclude that P_Λ is a contraction from $\mathcal{W}_\Lambda$ to $W^{1,2}(\mathring{\Lambda})$, with dense range (because of (73), and (72) holds. To show the equality in (71), define $W_c^{1,2}(\mathring{\Lambda})$ as the closure of $C_c^1(\mathring{\Lambda})$ (they are not linear subspaces), notice that $W_c^{1,2}(\mathring{\Lambda}) \subset P_\Lambda\,(\mathcal{W}_\Lambda)$, and that any $\varphi \in W^{1,2}(\mathring{\Lambda})$ can be written as a finite linear combination of functions in $W_c^{1,2}(\mathring{\Lambda})$. □

For each $N \in \mathbb{N}$ we define the operator

$$W_N = \sum_{m\in\ell\mathbb{Z}^d;\ |m|\le N} U^{-m}. \tag{76}$$

Lemma 16 **(i)** *$W_N \in \mathcal{B}(\mathcal{L}_\Lambda)$ for each $N \in \mathbb{N}$ and for all $\psi, \varphi \in \mathcal{L}_\Lambda$ we have*

$$\lim_{N\to\infty} \langle \psi, W_N\varphi\rangle_{L^2(\mathbb{R}^d)} = \langle P_\Lambda\psi, P_\Lambda\varphi\rangle_{L^2(\Lambda)}. \tag{77}$$

(ii) *$W_N \in \mathcal{B}(\mathcal{W}_\Lambda)$ for each $N \in \mathbb{N}$ and for all $\psi, \varphi \in \mathcal{W}_\Lambda$ we have*

$$\lim_{N\to\infty} \mathcal{A}(\psi, W_N\varphi) = \mathring{\mathcal{A}}_\Lambda\,(P_\Lambda\psi, P_\Lambda\varphi). \tag{78}$$

Proof. (i) and the fact that $W_N \in \mathcal{B}(\mathcal{W}_\Lambda)$ follow easily from the definitions. To prove (78), let $\psi, \varphi \in \mathcal{W}_\Lambda$; we have

$$\mathcal{A}(\psi, W_N\varphi) = \langle \nabla\psi, \frac{1}{\varrho(x)}\nabla W_N\varphi\rangle_{L^2(\mathbb{R}^d)} = \langle \nabla\psi, W_N\frac{1}{\varrho(x)}\nabla\varphi\rangle_{L^2(\mathbb{R}^d)}, \tag{79}$$

where we used the ℓ-periodicity of $\varrho(x)$. Since $\nabla\psi, \frac{1}{\varrho(x)}\nabla\varphi \in \mathcal{L}_\Lambda$, we can use first (77) and then Lemma 15(iii) to get

$$\begin{aligned}\lim_{N\to\infty} \mathcal{A}(\psi, W_N\varphi) &= \langle P_\Lambda\nabla\psi, P_\Lambda\frac{1}{\varrho(x)}\nabla\varphi\rangle_{L^2(\Lambda)} \\ &= \langle \nabla P_\Lambda\psi, \frac{1}{\varrho(x)}\nabla P_\Lambda\varphi\rangle_{L^2(\Lambda)} = \mathring{\mathcal{A}}_\Lambda\,(P_\Lambda\psi, P_\Lambda\varphi).\end{aligned} \tag{80}$$

□

Lemma 17 *Suppose $z \notin \sigma(A)$, then:*

(i) $R(z) \in \mathcal{B}_\Lambda \cap \mathcal{B}(\mathcal{L}_\Lambda, \mathcal{W}_\Lambda)$.

(ii) $z \notin \sigma(\overset{\circ}{A}_\Lambda)$ *and*

$$\overset{\circ}{R}_\Lambda(z) = \mathcal{P}_\Lambda(R(z)). \tag{81}$$

Proof. Let us fix $z \notin \sigma(A)$; $R(z)$ is then a bounded ℓ-periodic operator since A is q-periodic and $\ell \in q\mathbb{N}$. Using Lemmas 12 and 13 we get $R(z) \in \mathcal{B}_\Lambda$ and $R(z) \in \mathcal{B}(\mathcal{L}_\Lambda, \mathcal{W}_\Lambda)$.

Now let $\varphi \in \mathcal{L}_\Lambda$, $\psi \in \mathcal{W}_\Lambda$, then $R(z)\varphi \in \mathcal{W}_\Lambda$ and, by Lemma 15, $P_\Lambda\psi$, $P_\Lambda R(z)\varphi = \mathcal{P}_\Lambda(R(z))P_\Lambda\varphi \in W^{1,2}(\overset{\circ}{\Lambda})$. Thus we can use Lemma 16 and the identity $AR(z) = zR(z) + I$ to obtain

$$\begin{aligned}\overset{\circ}{\mathcal{A}}_\Lambda(P_\Lambda\psi, \mathcal{P}_\Lambda(R(z))P_\Lambda\varphi) &= \overset{\circ}{\mathcal{A}}_\Lambda(P_\Lambda\psi, P_\Lambda R(z)\varphi) \\ &= \lim_{N\to\infty} \mathcal{A}(\psi, W_N R(z)\varphi) = \lim_{N\to\infty} \mathcal{A}(\psi, R(z)W_N\varphi) \\ &= \langle \psi, (zR(z)+I)W_N\varphi\rangle_{L^2(\mathbb{R}^d)} = \langle \psi, W_N(zR(z)+I)\varphi\rangle_{L^2(\mathbb{R}^d)} \\ &= \langle P_\Lambda\psi, P_\Lambda(zR(z)+I)\varphi\rangle_{L^2(\Lambda)} = \langle P_\Lambda\psi, (z\mathcal{P}_\Lambda(R(z))+I)P_\Lambda\varphi\rangle_{L^2(\Lambda)}.\end{aligned} \tag{82}$$

Recalling that $P_\Lambda(\mathcal{L}_\Lambda) = L^2(\Lambda)$ and $P_\Lambda(\mathcal{W}_\Lambda) = W^{1,2}(\overset{\circ}{\Lambda})$ (see Lemma 15), we can rewrite (82) as

$$\overset{\circ}{\mathcal{A}}_\Lambda(\psi, \mathcal{P}_\Lambda(R(z))\varphi) = \langle \psi, (z\mathcal{P}_\Lambda(R(z))+I)\varphi\rangle_{L^2(\Lambda)} \text{ for all } \varphi \in L^2(\Lambda),\ \psi \in W^{1,2}(\overset{\circ}{\Lambda}). \tag{83}$$

We can now conclude that $\mathcal{P}_\Lambda(R(z))\varphi \in \mathcal{D}(\overset{\circ}{A}_\Lambda)$ for all $\varphi \in L^2(\Lambda)$, and

$$\left(\overset{\circ}{A}_\Lambda - zI\right)\mathcal{P}_\Lambda(R(z))\varphi = \varphi \text{ for all } \varphi \in L^2(\Lambda). \tag{84}$$

If $z \notin \sigma(\overset{\circ}{A}_\Lambda)$ we can immediately conclude that $\overset{\circ}{R}_\Lambda(z) = \mathcal{P}_\Lambda(R(z))$. Thus if $\operatorname{Im} z \neq 0$, we are done. If $z \notin \sigma(A)$ and z is real, notice that it follows from (77), (69) and the ℓ- periodicity of $R(z)$ that $\mathcal{P}_\Lambda(R(z))$ is a bounded self-adjoint operator. Since $\overset{\circ}{A}_\Lambda$ is self-adjoint, it follows from (84) by taking adjoints that we also have

$$\mathcal{P}_\Lambda(R(z))\left(\overset{\circ}{A}_\Lambda - zI\right)\varphi = \varphi \text{ for all } \varphi \in \mathcal{D}(\overset{\circ}{A}_\Lambda), \tag{85}$$

hence $z \notin \sigma(\overset{\circ}{A}_\Lambda)$ and $\overset{\circ}{R}_\Lambda(z) = \mathcal{P}_\Lambda(R(z))$. $\square$

Proof of Theorem 14. We already proved that $\sigma\left(\overset{\circ}{A}_{\Lambda_\ell(x)}\right) \subset \sigma(A)$ in Lemma 17, for all $x \in \mathbb{R}^d$ and $\ell \succ q$. To complete the proof of (56), it suffices to notice that if $q \prec \ell \prec L$, the theory developed in this section, up to and including Lemma 17, can be applied with $\Lambda = \Lambda_\ell(0)$ and with the torus $\overset{\circ}{\Lambda}_L(0)$ substituted for $\mathbb{R}^d$, with the obvious modifications.

Thus it only remains to show that

$$\sigma(A) \subset \Sigma \equiv \overline{\bigcup_{n\geq 1} \sigma\left(\mathring{A}_{\Lambda_{\ell_n}(0)}\right)}. \tag{86}$$

We only have to worry about real E. So let E be real, with $E \notin \Sigma$, and let $\delta \equiv d(E,\Sigma) > 0$. We have

$$\| \mathring{R}_{\Lambda_{\ell_n}(0)}\,(E+i\eta)\| \leq \frac{1}{\delta} \text{ for all } n = 1,2,\ldots \text{ and } \eta \in \mathbb{R}^d. \tag{87}$$

Now let φ be an arbitrary function in $L^2(\mathbb{R}^d)$ with compact support. It follows that for all sufficiently large n we have $\chi_{\Lambda_{\ell_n}(0)}\varphi = \varphi$, so $\widetilde{\varphi}_{\Lambda_{\ell_n}(0),m} = 0$ for all $m \in \ell\mathbb{Z}^d$, $m \neq 0$ and $\widetilde{\varphi}_{\Lambda_{\ell_n}(0),0} = \chi_{\Lambda_{\ell_n}(0)}\varphi = \varphi$. Thus it follows from (81) that

$$\mathring{R}_{\Lambda_{\ell_n}(0)}\,(z)\varphi = \chi_{\Lambda_{\ell_n}(0)}R(z)\varphi + \chi_{\Lambda_{\ell_n}(0)}R(z) \sum_{m\in \ell_n\mathbb{Z}^d;\, m\neq 0} U^m \chi_{\Lambda_{\ell_n}(0)}\varphi, \tag{88}$$

for all $z = E + i\eta$, $\eta \neq 0$ and sufficiently large n. Using Lemma 12 we can verify that

$$\lim_{n\to\infty} \left\| \chi_{\Lambda_{\ell_n}(0)}R(z) \sum_{m\in \ell_n\mathbb{Z}^d;\, m\neq 0} U^m \chi_{\Lambda_{\ell_n}(0)} \right\| = 0, \tag{89}$$

so it follows from (87) and (88) that

$$\|R(z)\varphi\| = \lim_{n\to\infty} \|\chi_{\Lambda_{\ell_n}(0)}R(z)\varphi\| \leq \frac{1}{\delta}\|\varphi\| \text{ for all } z = E+i\eta,\ \eta\neq 0. \tag{90}$$

Since functions with compact support are dense in $L^2(\mathbb{R}^d)$, we conclude that

$$\|R(z)\| \leq \frac{1}{\delta} \text{ for all } z = E+i\eta,\ \eta \neq 0. \tag{91}$$

Since A is self-adjoint, we have $E \notin \sigma(A)$. □

3.3 A Combes-Thomas argument for the torus

Lemma 18 *Let the operator A given by (10) with (11) be q-periodic, and let $\Lambda = \Lambda_\ell(x_0)$ for some $x_0 \in \mathbb{R}^d$ and $\ell \in q\mathbb{N}$, $\ell > 2r+8$, where $r > 0$. Then for any $z \notin \sigma(\mathring{A}_\Lambda)$ and $n \in \mathbb{N}$ we have*

$$\|\chi_{x,r}\,\mathring{R}_\Lambda\,(z)^n \chi_{y,r}\| \ \leq \left(\frac{9}{\eta}\right)^n e^{\frac{\sqrt{d}r\mathring{m}_{z,r,\ell}}{2}} e^{-\mathring{m}_{z,r,\ell}\mathring{d}(x,y)} \quad \text{for all } x,y \in \mathring{\Lambda}, \tag{92}$$

with

$$\mathring{m}_{z,r,\ell} = \frac{\eta}{4\left(\frac{2\sqrt{d}}{1-\frac{2r+8}{\ell}}+1\right)\left[\varrho_+^{-1}+|z|+\eta\right]}, \tag{93}$$

where $\eta = \operatorname{dist}(z,\sigma(\mathring{A}_\Lambda))$.

Proof. Let us fix $x_1, y_1 \in \bar{\Lambda}$; changing the representation of the torus $\overset{\circ}{\Lambda}$ by a shift (see (55) and the discussion preceding it), we can assume $x_0 = \frac{1}{2}(x_1 + y_1)$ and $x_1, y_1 \in \overline{\Lambda_{\ell/2}(x_0)}$. In particular, $\overset{\circ}{d}(x_1, y_1) = |x_1 - y_1|$. Let $\ell > 2r + 8$, we pick a real valued function $\xi \in C_0^1(\mathbb{R})$ with $0 \le \xi(t) \le 1$ for all $t \in \mathbb{R}$, such that $\xi(t) = 1$ for $|t| \le \frac{\ell}{4} + \frac{r}{2}$, $\xi(t) = 0$ for $|t| \ge \frac{\ell}{2} - 1$, and $|\xi'(t)| \le \left(\frac{\ell}{4} - \frac{r}{2} - 2\right)^{-1}$ for all $t \in \mathbb{R}$. We set $\Xi(x) = \prod_{i=1}^{d} \xi(x_i)$ for $x \in \mathbb{R}^d$. Notice $\operatorname{supp} \Xi(x - x_0) \subset \Lambda$.

We now proceed as in the proof of Lemma 12 with Λ substituted for $\mathbb{R}^d$ and definition (29) replaced by

$$\left(\overset{\circ}{A}_\Lambda\right)_a = e^{\Xi(x-x_0)a\cdot(x-x_0)} \ \overset{\circ}{A}_\Lambda \ e^{-\Xi(x-x_0)a\cdot(x-x_0)}, \ a \in \mathbb{R}^d. \tag{94}$$

Notice that

$$\begin{aligned} |\nabla\left[\Xi(x - x_0)a \cdot (x - x_0)\right]| &\le \left(\frac{\left(\frac{\ell}{2} - 1\right)\sqrt{d}}{\frac{\ell}{4} - \frac{r}{2} - 2} + 1\right)|a| \\ &\le \left(\frac{2\sqrt{d}}{1 - \frac{2r+8}{\ell}} + 1\right)|a| \ \text{ for all } x \in \Lambda. \end{aligned} \tag{95}$$

We can thus repeat the proof of Lemma 12, except that we must now require that $\left(\frac{2\sqrt{d}}{1-\frac{2r+8}{\ell}} + 1\right)|a| < 1$, and we must substitute $\left(\frac{2\sqrt{d}}{1-\frac{2r+8}{\ell}} + 1\right)|a|$ for $|a|$ in (36) and in the rest of the proof. Thus if $\left(\frac{2\sqrt{d}}{1-\frac{2r+8}{\ell}} + 1\right)|a| \le m_z$, i.e., $|a| \le \overset{\circ}{m}_{z,r,\ell}$, we have the equivalent of (41). We thus choose

$$a = \frac{\overset{\circ}{m}_{z,r,\ell}}{|x_1 - y_1|}(x_1 - y_1), \tag{96}$$

and complete the proof of as before (with x_1, y_1 substituted for x, y in (27)), as $\chi_{x_1} e^{-\Xi(x-x_0)a\cdot(x-x_0)} = \chi_{x_1} e^{-a\cdot(x-x_0)}$ and $\chi_{y_1} e^{\Xi(x-x_0)a\cdot(x-x_0)} = \chi_{y_1} e^{a\cdot(x-x_0)}$. $\square$

4 Location of the spectrum of random operators

In this section we prove Theorem 3.

4.1 Approximation by periodic operators

Let us fix a disorder parameter g. In order to investigate the samples of the random quantity $\varrho_{g,\omega}(x)$ we set

$$\mathcal{T}_{\}} = \{\tau : \tau = \{\tau_{\rangle}, \rangle \in \mathbb{Z}^{\lceil}\}, -\} \le \tau_{\rangle} \le \}\}, \tag{97}$$

$$\mathcal{T}_{\}}^{(\backslash)} = \{\tau \in \mathcal{T} : \tau_{\rangle+\backslash|} = \tau_{\rangle} \ \text{ for all } \ \rangle, | \in \mathbb{Z}^{\lceil}\}, \ \ \backslash \in \mathbb{N}, \tag{98}$$

and

$$\mathcal{T}_{\}}^{(\infty)} = \bigcup_{\backslash \succeq \mathrm{II}} \mathcal{T}_{\}}^{(\backslash)}. \tag{99}$$

For $\tau \in \mathcal{T}_{\}}$ we let

$$\varrho_\tau(x) = \varrho_0(x) \left[1 + \sum_{i \in \mathbb{Z}^d} \tau_i u(x-i) \right] \tag{100}$$

and

$$A(\tau) = A(\varrho_\tau). \tag{101}$$

In addition, we set

$$\mathcal{E}_{\}}^{(\backslash)} = \left\{ \varrho_\tau : \tau \in \mathcal{T}_{\}}^{(\backslash)} \right\} \ \text{ and } \ \mathcal{E}_{\}}^{(\infty)} = \bigcup_{\backslash \succeq \mathrm{II}} \mathcal{E}_{\}}^{(\backslash)}. \tag{102}$$

To approximate acoustic operators by periodic operators, given $\tau \in \mathcal{T}_{\}}$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we specify $\tau_{\Lambda_n(x)} \in \mathcal{T}_{\}}^{(\backslash)}$ by requiring $\left(\tau_{\Lambda_n(x)}\right)_i = \tau_i$ for all $i \in \breve{\Lambda}_n(x) \cap \mathbb{Z}^d$, and define

$$A_{\Lambda_n(x)}(\tau) = A(\tau_{\Lambda_n(x)}). \tag{103}$$

Lemma 19 *Let the random operator A_g defined by (18) satisfy Assumption 1, and let*

$$\Sigma_g = \overline{\bigcup_{\tau \in \mathcal{T}_{\}}^{(\infty)}} \sigma\left(A(\tau)\right)}. \tag{104}$$

Then $\sigma(A_g) = \Sigma_g$ with probability one.

Proof. We start by showing that

$$\sigma(A(\tau)) \subset \Sigma_g \ \text{ for all } \ \tau \in \mathcal{T}_{\}}. \tag{105}$$

Let $\Lambda_n = \Lambda_{\ell_n}(0)$, where $\{\ell_n;\ n = 0, 1, 2, \ldots\}$ is a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0, 1, 2, \ldots$. Let us pick $\tau \in \mathcal{T}_{\}}$, and consider the associated sequence of operators $A_n(\tau) = A_{\Lambda_n}(\tau)$. We define bounded nonnegative measurable functions

$$\theta_{\tau,n}^{\pm}(x) = \max\left\{ \pm \left(\frac{1}{\varrho_{\tau_{\Lambda_n}}(x)} - \frac{1}{\varrho_\tau(x)} \right), 0 \right\}, \tag{106}$$

and corresponding self-adjoint operators

$$\Theta_n^{\pm}(\tau) = -\nabla \cdot \theta_{\tau,n}^{\pm}(x) \nabla, \tag{107}$$

defined by the corresponding quadratic forms as in (12). It is not hard to see that

$$0 \leq \Theta_n^{\pm}(\tau) \leq \alpha^{\pm} A(\tau), \tag{108}$$

with

$$0 \leq \alpha^- = 1 - \frac{\varrho_{g,-}}{\varrho_{g,+}} < 1; \quad 0 \leq \alpha^+ = \frac{\varrho_{g,+}}{\varrho_{g,-}} - 1 < \infty, \tag{109}$$

and

$$A(\tau) - \Theta_n^-(\tau) \leq A_n(\tau) \leq A(\tau) + \Theta_n^+(\tau), \tag{110}$$

as quadratic forms. $\mathcal{D} = \mathcal{C}_{\prime}^\infty(\mathbf{R^d})$ is a core for $\sqrt{A(\tau)}$, and for any $\psi \in \mathcal{D}$ we have $\langle \psi, \Theta_n^\pm(\tau)\psi \rangle = 0$ for sufficiently large n, since $(\varrho_{\tau_{\Lambda_n}}(x) - \varrho_\tau(x))\chi_{0,(\ell_n - 4r_u)}(x) = 0$. The last observation and (108)-(110) allow us to apply Lemma 45 and conclude that (105) is true.

To prove the opposite inclusion to (105), with probability one, we introduce the countable set

$$\mathcal{T}_{\}}^{(\infty)}(\mathbb{Q}) = \mathcal{T}_{\}}^{(\infty)} \cap \mathbb{Q}^{\mathbb{Z}^{\lceil}} \tag{111}$$

Approximating any $\tau \in \mathcal{T}_{\}}^{(\infty)}$ uniformly by a sequence $\tau_n \in \mathcal{T}_{\}}^{(\infty)}(\mathbb{Q})$, and using Lemma 45 together with appropriate inequalities analogous to (108)-(110), one can show that

$$\sigma(A(\tau)) \subset \overline{\bigcup_{\tau' \in \mathcal{T}_{\}}^{(\infty)}(\mathbb{Q})} \sigma\left(A(\tau')\right)} \tag{112}$$

which readily implies that

$$\Sigma_g = \overline{\bigcup_{\tau \in \mathcal{T}_{\}}^{(\infty)}(\mathbb{Q})} \sigma\left(A(\tau)\right)}. \tag{113}$$

Thus, to finish the proof of Lemma (19) it suffices to show that, with probability one, we have

$$\sigma(A(\tau)) \subset \sigma(A_g) \ \text{ for any } \ \tau \in \mathcal{T}_{\}}^{(\infty)}(\mathbb{Q}). \tag{114}$$

Notice that $A_g = A_{g,\omega} = A(g\omega)$, with $g\omega = \{g\omega_i, i \in \mathbb{Z}^d\} \in \mathcal{T}_{\}}$, since $\omega \in \Omega \equiv [-1,1]^{\mathbb{Z}^d}$. Let $\{\ell_n;\ n = 0,1,2,\ldots\}$ be a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0,1,2,\ldots$. For each n and $\tau \in \mathcal{T}_{\}}^{(\mathrm{II}')}(\mathbb{Q})$ for some $q' \succeq q$, we consider the event

$$\Omega_{\tau,n} = \{\omega \in \Omega; \max_{i \in \Lambda_{\ell_n}(m_{\omega,\tau,n}) \cap \mathbb{Z}^d} |g\omega_i - \tau_i| \leq (\ell_n + 1)^{-(d+1)} \text{ for some } m_{\omega,\tau,n} \in q'\mathbb{Z}^d\}; \tag{115}$$

notice $\mathbb{P}(\Omega_{\tau,n}) = 1$. We now take the countable intersection

$$\widehat{\Omega} = \bigcap_{n=0}^{\infty} \bigcap_{q' \succeq q} \bigcap_{\tau \in \mathcal{T}_{\}}^{(\mathrm{II}')}(\mathbb{Q})} \Omega_{\tau,n}, \tag{116}$$

so we have $\mathbb{P}(\widehat{\Omega}) = 1$. We will show that (114) is true for any $\omega \in \widehat{\Omega}$.

So let $\omega \in \widehat{\Omega}$, $\tau \in \mathcal{T}_{\}}^{(\mathrm{II}')}(\mathbb{Q})$ for some $q' \succeq q$, $n \in \mathbb{N}$, and let $m_{\omega,\tau,n}$ be as in (115). We set $\omega^{(n)} = \{\omega_i^{(n)} = \omega_{i-m_{\omega,\tau,n}},\ i \in \mathbb{Z}^d\}$, and notice that $\sigma(A_{g,\omega^{(n)}}) = \sigma(A_{g,\omega})$. Similarly to (106), we define bounded nonnegative measurable functions

$$\theta_{\omega,\tau,n}^{\pm}(x) = \max\left\{\pm\left(\frac{1}{\varrho_{g,\omega^{(n)}}(x)} - \frac{1}{\varrho_\tau(x)}\right), 0\right\}, \tag{117}$$

and corresponding self-adjoint operators $\Theta_n^{\pm}(\omega,\tau)$ defined as in (107), so we get analogous inequalities to (108)-(110). We have $\lim_{n\to\infty}\langle\psi, \Theta_n^{\pm}(\omega,\tau)\psi\rangle = 0$ for any $\psi \in \mathcal{D}$, since

$$|\varrho_{g,\omega^{(n)}}(x) - \varrho_\tau(x)|\chi_{0,(\ell_n-4r_u)}(x) \leq (\ell_n+1)^{-(d+1)}\varrho_{0,+}U_+\chi_{0,(\ell_n-4r_u)}(x).$$

As before, this last observation allows us to apply Lemma 45 and conclude that

$$\sigma(A(\tau)) \subset \overline{\bigcap_{n=0}^{\infty}\sigma(A_{g,\omega^{(n)}})} = \sigma(A_{g,\omega}), \tag{118}$$

which implies the validity of (114). □

Given a real number h; $|h| < \frac{1}{U_+}$, let

$$A(h) = A(\varrho_h) \ \text{ with } \ \varrho_h(x) = \varrho_0(x)\left[1 + hU(x)\right] \tag{119}$$

If $|h| \leq g$, and we define $\tau(h) \in \mathcal{T}_{\}}$ by $\tau(h)_i = h$ for all $i \in \mathbb{Z}^d$, we have $\varrho_h = \varrho_{\tau(h)}$ and $A(h) = A(\tau(h))$.

Lemma 20 *Let $A(h)$, $|h| < \frac{1}{U_+}$, be given by (119), with ϱ_0 and U given in Assumption 1. Let $\Lambda = \Lambda_\ell(x_0)$ for some $x_0 \in \mathbb{R}^d$ and $\ell \succeq q$. The positive self-adjoint operator $\overset{\circ}{A}(h)_\Lambda$ has compact resolvent, so let $0 < \mu_1(h) \leq \mu_2(h) \leq \ldots$ be its eigenvalues, repeated according to their (finite) multiplicity. Then each $\mu_j(h)$, $j = 1, 2, \ldots$, is a Lipschitz continuous, strictly decreasing function of h, with*

$$\delta_-(g)\max_{l=1,2}\{\mu_j(h_l)\} \leq \frac{\mu_j(h_1) - \mu_j(h_2)}{h_2 - h_1} \leq \delta_+(g)\min_{l=1,2}\{\mu_j(h_l)\} \tag{120}$$

for any $h_1, h_2 \in (-g, g)$, $0 < g < \frac{1}{U_+}$, where $\delta_\pm(g)$ are given in (17).

Proof. Let $h_1, h_2 \in (-g, g)$, $0 < g < \frac{1}{U_+}$, it follows that

$$\frac{1}{\varrho_{h_1}(x)} - \frac{1}{\varrho_{h_2}(x)} = \frac{(h_2 - h_1)\varrho_0(x)U(x)}{\varrho_{h_1}(x)\varrho_{h_2}(x)}, \tag{121}$$

so

$$\frac{1 + \delta_-(g)(h_2 - h_1)}{\varrho_{h_2}(x)} \leq \frac{1}{\varrho_{h_1}(x)} \leq \frac{1 + \delta_+(g)(h_2 - h_1)}{\varrho_{h_2}(x)}, \tag{122}$$

and

$$\frac{1-\delta_+(g)(h_2-h_1)}{\varrho_{h_1}(x)} \leq \frac{1}{\varrho_{h_2}(x)} \leq \frac{1-\delta_-(g)(h_2-h_1)}{\varrho_{h_1}(x)}. \tag{123}$$

¿From (122) we get

$$(1+\delta_-(g)(h_2-h_1))\ \mathring{A}(h_2)_\Lambda \leq \mathring{A}(h_1)_\Lambda \leq (1+\delta_+(g)(h_2-h_1))\ \mathring{A}(h_2)_\Lambda, \tag{124}$$

so it follows from the min-max principle that, for any $j = 1, 2, \ldots$,

$$(1+\delta_-(g)(h_2-h_1))\mu_j(h_2) \leq \mu_j(h_1) \leq (1+\delta_+(g)(h_2-h_1))\mu_j(h_2), \tag{125}$$

i.e.,

$$\delta_-(g)(h_2-h_1) \leq \frac{\mu_j(h_1)-\mu_j(h_2)}{\mu_j(h_2)} \leq \delta_+(g)(h_2-h_1). \tag{126}$$

Similarly, using (123) we get

$$\delta_-(g)(h_2-h_1) \leq \frac{\mu_j(h_1)-\mu_j(h_2)}{\mu_j(h_1)} \leq \delta_+(g)(h_2-h_1). \tag{127}$$

Equation (120) follows from (126) and (127). The properties of the functions $\mu_j(h)$ follows. □

The following corollary follows immediately from Theorem 14, Lemmas 19 and 20, and the min-max principle.

Corollary 21 *Let the random operator A_g defined by (18) satisfy Assumption 1, and let $\{\ell_n;\ n = 0, 1, 2, \ldots\}$ be a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0, 1, 2, \ldots$. Then*

$$\Sigma_g = \overline{\bigcup_{h\in[-g,g]} \sigma\left(A(h)\right)} = \overline{\bigcup_{h\in[-g,g]} \bigcup_{n\geq 1} \sigma\left(\mathring{A}(h)_{\Lambda_{\ell_n}(0)}\right)}. \tag{128}$$

In particular, Σ_g is increasing in g.

4.2 Inside the gap

We now prove Theorem 3. As Σ_g is increasing in g, we expect the gap to shrink as we increase g until it either disappears at some g_0, or it remains open for all allowed g. Thus we define

$$g_0 = \sup\left\{g \in \left[0, \frac{1}{U_+}\right);\ \ \Sigma_g \cap (a,b) \neq (a,b)\right\}. \tag{129}$$

Let $\{\ell_n;\ n = 0, 1, 2, \ldots\}$ be as in Corollary 21, $h \in [-g, g]$, and let $0 < \mu_1^{(n)}(h) \leq \mu_2^{(n)}(h) \leq \ldots$ be the eigenvalues of $\mathring{A}(h)_{\Lambda_n}$, $\Lambda_n = \Lambda_{\ell_n}(0)$, repeated according to their (finite) multiplicity; notice $\lim_{j\to\infty}\mu_j^{(n)}(h)=\infty$. By Lemma 20 each $\mu_j^{(n)}(h)$ is a strictly decreasing continuous function of h, hence it follows from Corollary 21 that

$$\Sigma_g=\overline{\bigcup_{n\geq 1}\bigcup_{h\in[-g,g]}\sigma\left(\overset{\circ}{A}(h)_{\Lambda_n}\right)}=\overline{\bigcup_{n\geq 1}\bigcup_{j=0}^{\infty}[\mu_j^{(n)}(g),\mu_j^{(n)}(-g)]}. \tag{130}$$

In particular, Σ_g is a countable union of disjoint closed intervals, none contained in the original gap (a,b), so for $g<g_0$ we can define $a(g)$ and $b(g)$ by (20). Since Σ_g is increasing in $g\in[0,\frac{1}{U_+})$ by Corollary 21, it follows that $a(g)$ and $-b(g)$ are increasing functions in $[0,g_0)$.

For each n let

$$j_n=\max\{j;\ \mu_j^{(n)}(0)\leq a\}, \tag{131}$$

so using Assumption 2 and (56) in Theorem 14, we have

$$j_n+1=\min\{j;\ \mu_j^{(n)}(0)\geq b\}. \tag{132}$$

If $g<g_0$, it follows from the definition of j_n, Assumption 2 and (56)-(57) in Theorem 14, that $\mu_{j_n}(-g)$ and $-\mu_{j_n+1}(g)$ are both increasing in n, and

$$a(g) = \lim_{n\to\infty}\mu_{j_n}(-g), \tag{133}$$

$$b(g) = \lim_{n\to\infty}\mu_{j_n+1}(g). \tag{134}$$

Thus, given $0\leq g_1<g_2<g_0$, we can conclude from (120) that

$$\delta_-(g_2)a(g_2)\leq\frac{a(g_2)-a(g_1)}{g_2-g_1}\leq\delta_+(g_2)a(g_1), \tag{135}$$

$$\delta_-(g_2)b(g_1)\leq\frac{b(g_1)-b(g_2)}{g_2-g_1}\leq\delta_+(g_2)b(g_2). \tag{136}$$

The Lipschitz continuity of $a(g)$ and $b(g)$ follows, as well as

$$\int_{g_1}^{g_2}\delta_-(h)dh\leq\log\frac{a(g_2)}{a(g_1)},\log\frac{b(g_1)}{b(g_2)}\leq\int_{g_1}^{g_2}\delta_+(h)dh, \tag{137}$$

so we obtain (21) and (22), from which we get (19).

If $g_0<\frac{1}{U_+}$, we must have $\lim_{g\uparrow g_0}a(g)=\lim_{g\uparrow g_0}b(g)$. This follows from (130), (133) and (134), since by (120) each $\mu_j^{(n)}(h)$ is a Lipschitz continuous functions of $h\in(-\frac{1}{U_+},\frac{1}{U_+})$, uniformly in n. Thus, if $g\in[g_0,\frac{1}{U_+})$ it follows that $[\hat{a},\hat{b}]\subset\Sigma_g$; we set $a(g)=b(g)=\lim_{g\uparrow g_0}a(g)$.

Theorem 3 is proven.

5 A Wegner-type estimate

Given an open cube Λ in $\mathbb{R}^d$, we will denote by $A_{g,\Lambda} = A_{g,\omega,\Lambda}$ the restriction of $A_{g,\omega}$ to Λ with Dirichlet boundary condition (e.g., [RS4]). Notice that each $A_{g,\omega,\Lambda}$ is a nonnegative self-adjoint operator on $L^2(\Lambda)$ with compact resolvent, and measurability follows from Theorem 38. We can thus define $n_{g,\Lambda}(E) = n_{g,\omega,\Lambda}(E)$ as the number of eigenvalues of $A_{g,\omega,\Lambda}$ less than E; clearly $n_{g,\omega,\Lambda}(E) = 0$ for $E \leq 0$. Notice that $n_{g,\omega,\Lambda}(E)$ is the distribution function of the measure $n_{g,\omega,\Lambda}(dE)$ given by

$$\int h(E) n_{g,\omega,\Lambda}(dE) = \mathrm{Tr}(h(A_{g,\omega,\Lambda})) \tag{138}$$

for positive continuous functions h of a real variable.

We will say that the random operator A_g defined by (18) satisfies Assumption 1′, if it satisfies all of Assumption 1 with the exception of the requirement that $\varrho_0(x)$ be a q-periodic function.

We have the following "a priori" estimate:

Lemma 22 *Let the random operator A_g defined by (18) satisfy Assumption 1′. There exists a constant $C_1 = C_1(d, \varrho_{0,+}) < \infty$ such that we have*

$$n_{g,\omega,\Lambda}(E) \leq C_1 |\Lambda| E^{\frac{d}{2}} \tag{139}$$

for all $\omega \in [-1,1]^{\mathbb{Z}^d}$, for all $E \geq 0$ and all cubes Λ in $\mathbb{Z}^d$.

Proof. Let Δ_Λ be the Laplacian on Λ with Dirichlet boundary condition. We have

$$A_{g,\omega,\Lambda} \geq -\frac{1}{\varrho_{0,+}(1+gU_+)} \Delta_\Lambda \geq -\frac{1}{2\varrho_{0,+}} \Delta_\Lambda \; . \tag{140}$$

Thus (139) follows from [Reed-Simon, vol. IV, p. 267, (118)]. □

Theorem 23 (Wegner-type estimate) *Let the random operator A_g defined by (18) satisfy Assumption 1′. There exists a constant $Q < \infty$, depending only on the dimension d and the constants r_u and $\varrho_{0,+}$, such that*

$$\mathbb{P}\{\,\mathrm{dist}(\sigma(A_{g,\omega,\Lambda}), E) \leq \eta\} \leq Q \frac{U_- + 2U_+}{gU_+(1-gU_+)U_-} \|\rho\|_\infty |E|^{\frac{d}{2}-1} \eta |\Lambda|^2 \tag{141}$$

for all $E > 0$, cubes Λ in $\mathbb{R}^d$, and all $\eta \in [0, E]$.

Proof. Let us pick $\kappa \in (1, \frac{1}{gU_+})$, say $\kappa = \frac{1+gU_+}{2gU_+}$, we write

$$\gamma_{g,\omega} = \hat{\gamma} + g \sum_{i \in \mathbb{Z}^d} s_i u_i \; , \tag{142}$$

where

$$\hat{\gamma} = 1 - \kappa g \sum_{i \in \mathbb{Z}^d} u_i > \frac{1 - gU_+}{2} > 0 \, , \tag{143}$$

and $s_i = \omega_i + \kappa \in [\kappa - 1, \kappa + 1]$ for each $i \in \mathbb{Z}^d$.

Now let f be an arbitrary continuous function on the real line with compact support. As in [FK, Lemma 7], we have

$$\begin{aligned} -\frac{\partial}{\partial s_i} \int n_{g,\omega,\Lambda}(E) f(E) dE &= \operatorname{Tr}\left\{ \left(-\frac{\partial}{\partial s_i} A_{g,\omega,\Lambda} \right) f(A_{g,\omega,\Lambda}) \right\} \\ &= \operatorname{Tr}\left\{ A_{g,\omega,\Lambda} \left(\frac{g u_i}{\varrho_{g,\omega} \gamma_{g,\omega}} \right) f(A_{g,\omega,\Lambda}) \right\}, \end{aligned}$$

where $A_{g,\omega,\Lambda}(\xi)$ is defined in the same way as $A_{g,\omega,\Lambda}$ but with $\frac{1}{\varrho_{g,\omega}}$ replaced by the function ξ.

Thus,

$$\begin{aligned} -\sum_{i \in \mathbb{Z}^d} s_i \frac{\partial}{\partial s_i} \int n_{g,\omega,\Lambda}(E) f(E) dE &= \operatorname{Tr}\left\{ A_{g,\omega,\Lambda} \left(\frac{1}{\varrho_{g,\omega}} \frac{\gamma_{g,\omega} - \widehat{\gamma}}{\gamma_{g,\omega}} \right) f(A_{g,\omega,\Lambda}) \right\} \\ = \operatorname{Tr}\left\{ A_{g,\omega,\Lambda} f(A_{g,\omega,\Lambda}) \right\} &- \operatorname{Tr}\left\{ A_{g,\omega,\Lambda} \left(\frac{1}{\varrho_{g,\omega}} \frac{\widehat{\gamma}}{\gamma_{g,\omega}} \right) f(A_{g,\omega,\Lambda}) \right\}. \end{aligned}$$

But

$$\frac{\gamma_{g,\omega}}{\hat{\gamma}} = \frac{\hat{\gamma} + g \sum_{i \in \mathbb{Z}^d} s_i u_i}{\hat{\gamma}} \geq 1 + \frac{(\kappa - 1) g U_-}{1 - \kappa g U_-} \geq 1 + \frac{(1 - gU_+) U_-}{2U_+},$$

and $f \geq 0$, so

$$\operatorname{Tr}\left\{ A_{g,\omega,\Lambda} \left(\frac{1}{\varrho_{g,\omega}} \frac{\widehat{\gamma}}{\gamma_{g,\omega}} \right) f(A_{g,\omega,\Lambda}) \right\} \leq \left(1 + \frac{(1 - gU_+) U_-}{2U_+} \right)^{-1} \operatorname{Tr}\left\{ A_{g,\omega,\Lambda} f(A_{g,\omega,\Lambda}) \right\}.$$

We conclude that

$$\begin{aligned} &\operatorname{Tr}\left\{ A_{g,\omega,\Lambda} f(A_{g,\omega,\Lambda}) \right\} \\ &\quad \leq \left(1 + \frac{2U_+}{(1 - gU_+) U_-} \right) \left(-\sum_{i \in \mathbb{Z}^d} s_i \frac{\partial}{\partial s_i} \int n_{g,\omega,\Lambda}(E) f(E) dE \right). \end{aligned} \tag{144}$$

For given $j \in \mathbb{Z}^d$ let $\omega^{(j)} = \{\omega_i ;\, i \in \mathbb{Z}^d \backslash \{j\}\}$, and denote the corresponding expectation by $\mathbb{E}^{(j)}$. We have

$$\mathbb{E}\left(-\frac{\partial}{\partial s_j} \int n_{g,\omega,\Lambda}(E) f(E) dE \right) \tag{145}$$

$$
\begin{aligned}
&= \mathbb{E}^{(j)}\left(\int_{\kappa-1}^{\kappa+1}\left[-\frac{\partial}{\partial s_j}\int n_{g,\omega,\Lambda}(E)f(E)dE\right]\rho(s_j-\kappa)ds_j\right)\\
&\le \|\rho\|_\infty \mathbb{E}^{(j)}\left(\int\left|n_{g,(\omega^{(j)},\omega_j=-1),\Lambda}(E)-n_{g,(\omega^{(j)},\omega_j=1),\Lambda}(E)\right|f(E)dE\right)\\
&\le \delta_{j,\Lambda}\|\rho\|_\infty 2C_1|\Lambda|\int E^{\frac{d}{2}}f(E)dE,
\end{aligned}
$$

where we used (139); here $\delta_{j,\Lambda}=0$ if $u_j\equiv 0$ in Λ and $\delta_{j,\Lambda}=1$ otherwise. Since the function u has support in a cube of side $2r_u$, there exists a constant $C_2<\infty$, depending only on r_u and the dimension d, such that

$$
\sum_{i\in\mathbb{Z}^d}\delta_{i,\Lambda}\le C_2|\Lambda| \tag{146}
$$

for all cubes Λ in $\mathbb{Z}^d$.

Let $\hat{n}_{g,\Lambda}(dE)=\mathbb{E}(n_{g,\omega,\Lambda}(dE))$. For functions f as above, it now follows from (144), (145) and (146) that

$$
\int Ef(E)\hat{n}_{g,\Lambda}(dE)\le 2C_1C_2(\kappa+1)\left(1+\frac{2U_+}{(1-gU_+)U_-}\right)\|\rho\|_\infty|\Lambda|^2\int E^{\frac{d}{2}}f(E)dE. \tag{147}
$$

It follows that $\hat{n}_{g,\Lambda}(dE)$ is absolutely continuous with

$$
\frac{\hat{n}_{g,\Lambda}(dE)}{dE}\le C_3|\Lambda|^2E^{\frac{d}{2}-1}\ \text{ for }\ E\ge 0, \tag{148}
$$

where

$$
C_3=4C_1C_2\frac{U_-+2U_+}{gU_+(1-gU_+)U_-}\|\rho\|_\infty. \tag{149}
$$

The estimate (141) now follows by a standard argument:

$$
\begin{aligned}
\mathbb{P}\{\operatorname{dist}(\sigma(A_{g,\omega,\Lambda}),E)<\eta\}&\le\mathbb{P}\left\{\int_{[E-\eta,E+\eta]}n_{g,\omega,\Lambda}(dE)\ge 1\right\}\\
&\le\int_{[E-\eta,E+\eta]}\hat{n}_{g,\Lambda}(dE)\le 2^{\frac{d}{2}}C_3E^{\frac{d}{2}-1}\eta|\Lambda|^2,
\end{aligned} \tag{150}
$$

for all $E>0$ and $0\le\eta\le E$. □

6 Localization

In this section we prove Theorems 6 and 7. To do so we develop a multiscale analysis appropriate for random perturbations of periodic operators on $\mathbb{R}^d$, based on the von Dreifus-Klein [DK] multiscale analysis. We will work with random operators A_g of the form given in (18), but our method works also for random Schrodinger operators.

Let the operator A be as in (10) with (11). Given an open cube Λ in $\mathbb{R}^d$, we will denote by A_Λ the restriction of A to Λ with Dirichlet boundary condition (e.g., [RS4]). Each A_Λ is a nonnegative self-adjoint operator on $L^2(\Lambda)$ with compact resolvent $R_\Lambda(z) = (A_\Lambda - z)^{-1}$. If $\Lambda = \Lambda_L(x)$, we will write $A_{x,L} = A_{\Lambda_L(x)}$ and $R_{x,L}(z) = R_{\Lambda_L(x)}(z)$. The norm in $L^2(\Lambda_L(x))$ and also the corresponding operator norm will both be denoted by $\| \|_{x,L}$. If $\Lambda_1 \subset \Lambda_2$ are open cubes, $J_{\Lambda_1}^{\Lambda_2} : L^2(\Lambda_1) \to L^2(\Lambda_2)$ is the canonical injection. If $\Lambda_i = \Lambda_{L_i}(x_i)$, $i = 1, 2$, we write $\| \|_{x_1,L_1}^{x_2,L_2}$ for the (operator) norm in $\mathcal{B}\left(\mathcal{L}^{\in}(*_{\mathcal{L}_\infty}(\S_\infty)), \mathcal{L}^{\in}(*_{\mathcal{L}_\in}(\S_\in))\right)$ and $J_{x_1,L_1}^{x_2,L_2} = J_{\Lambda_{L_1}(x_1)}^{\Lambda_{L_2}(x_2)}$. If $\varphi \in L^\infty(\Lambda)$, we also denote the operator multiplication by φ on $L^2(\Lambda)$ by φ.

6.1 The basic technical tools

The basic tool to relate resolvents in different scales is a *local resolvent identity* (LRI), which was used in a similar context by Combes and Hislop [CH].

Lemma 24 (LRI) *Let $\Lambda_1 \subset \Lambda_2$ be open cubes in $\mathbb{R}^d$, let H_i be a self-adjoint operator on $L^2(\Lambda_i)$, $i = 1, 2$, with $R_i(z) = (H_i - z)^{-1}$, and let $\varphi_1 \in L^\infty(\Lambda_1)$. Then, for any $z \notin \sigma(H_1) \cup \sigma(H_2)$ we have*

$$R_2(z) J_{\Lambda_1}^{\Lambda_2} \varphi_1 = J_{\Lambda_1}^{\Lambda_2} \varphi_1 R_1(z) + R_2(z) \left(J_{\Lambda_1}^{\Lambda_2} \varphi_1 H_1 - H_2 J_{\Lambda_1}^{\Lambda_2} \varphi_1 \right) R_1(z) \tag{151}$$

as quadratic forms on $L^2(\Lambda_2) \times L^2(\Lambda_1)$.

Proof. We clearly have

$$R_2(z) J_{\Lambda_1}^{\Lambda_2} \varphi_1 \psi_1 = R_2(z) J_{\Lambda_1}^{\Lambda_2} \varphi_1 (H_1 - z) R_1(z) \psi_1 \text{ for all } \psi_1 \in L^2(\Lambda_1), \tag{152}$$

and

$$\langle \psi_2, J_{\Lambda_1}^{\Lambda_2} \varphi_1 R_1(z) \psi_1 \rangle_{L^2(\Lambda_2)} = \langle (H_2 - z^*) R_2(z^*) \psi_2, J_{\Lambda_1}^{\Lambda_2} \varphi_1 R_1(z) \psi_1 \rangle_{L^2(\Lambda_2)} \tag{153}$$

for all $\psi_1 \in L^2(\Lambda_1), \psi_2 \in L^2(\Lambda_2)$, so (151) follows. □

For operators of the form given be given by (10), the LRI yields the *smooth resolvent identity* (SRI).

Lemma 25 (SRI) *Let the operator A be given by (10) with (11), let $\Lambda_1 \subset \Lambda_2$ be open cubes in $\mathbb{R}^d$, and let $\varphi_1 \in C_0^1(\Lambda_1)$. Then, for any $z \notin \sigma(A_{\Lambda_1}) \cup \sigma(A_{\Lambda_2})$ we have*

$$\begin{aligned} & R_{\Lambda_2}(z) J_{\Lambda_1}^{\Lambda_2} \varphi_1 = \\ & \quad J_{\Lambda_1}^{\Lambda_2} \varphi_1 R_{\Lambda_1}(z) + R_{\Lambda_2}(z) \left(- J_{\Lambda_1}^{\Lambda_2} \frac{1}{\varrho} (\nabla \varphi_1) \cdot \nabla + \nabla \cdot J_{\Lambda_1}^{\Lambda_2} (\nabla \varphi_1) \frac{1}{\varrho} \right) R_{\Lambda_1}(z) \end{aligned} \tag{154}$$

as quadratic forms on $L^2(\Lambda_2) \times L^2(\Lambda_1)$.

Proof. Using (151), (10), and the definition of Dirichlet boundary condition we get (154). □

The other basic tool for the multiscale analysis is a *Simon-Lieb-type inequality* (SLI), which we need to adapt to the continuum. It is used to obtain decay in a larger scale from decay in a given scale.

Since we are working in a background medium of period $q \in \mathbb{N}$ (see Assumption 1), we will work with boxes $\Lambda_L(x)$ with $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$, so the background will be the same in all boxes in a given scale L. For such boxes (with $L \geq 4q$) we set

$$\Upsilon_L(x) = \{y \in q\mathbb{Z}^d; \ \ \|y - x\| = \frac{L}{2} - q\} \tag{155}$$

and

$$\tilde{\Upsilon}_L(x) = \Lambda_{L-q}(x) \setminus \overline{\Lambda_{L-3q}(x)}, \ \ \hat{\Upsilon}_L(x) = \Lambda_{L-\frac{3q}{2}}(x) \setminus \overline{\Lambda_{L-\frac{5q}{2}}(x)}. \tag{156}$$

We also set

$$\chi_x = \chi_{x,q} \ \text{ and } \ \Gamma_{x,L} = \chi_{\tilde{\Upsilon}_L(x)}, \ \ \hat{\Gamma}_{x,L} = \chi_{\hat{\Upsilon}_L(x)}. \tag{157}$$

Notice

$$\Gamma_{x,L} = \sum_{y \in \Upsilon_L(x)} \chi_y \ \text{ a.e.} \tag{158}$$

and

$$|\Upsilon_L(x)| \leq d(L - 2q + 1)^{d-1}. \tag{159}$$

In addition each $\Lambda_L(x)$ will be equipped with a function $\Phi_{x,L}$ constructed in the following way: we fix an even function $\xi \in C_0^1(\mathbb{R})$ with $0 \leq \xi(t) \leq 1$ for all $t \in \mathbb{R}$, such that $\xi(t) = 1$ for $|t| \leq \frac{q}{4}$, $\xi(t) = 0$ for $|t| \geq \frac{3q}{4}$, and $|\xi'(t)| \leq \frac{3}{q}$ for all $t \in \mathbb{R}$. (Such a function always exists.) We define

$$\xi_L(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{L}{2} - \frac{5q}{4} \\ \xi\left(|t| - \left(\frac{L}{2} - \frac{3q}{2}\right)\right), & \text{if } |t| \geq \left(\frac{L}{2} - \frac{3q}{2}\right) \end{cases} \tag{160}$$

and set

$$\Phi_{x,L}(y) = \Phi_L(y - x) \ \text{ for } \ y \in \mathbb{R}^d, \ \text{ with } \ \Phi_L(y) = \prod_{i=1}^{d} \xi_L(y_i). \tag{161}$$

We have $\Phi_{x,L} \in C_0^1(\Lambda_L(x))$, $0 \leq \Phi_{x,L} \leq 1$,

$$\chi_{x,\frac{L}{2}-\frac{5q}{4}} \Phi_{x,L} = \chi_{x,\frac{L}{2}-\frac{5q}{4}}, \quad \chi_{x,\frac{L}{2}-\frac{3q}{4}} \Phi_{x,L} = \Phi_{x,L}, \tag{162}$$

and

$$\hat{\Gamma}_{x,L} \nabla \Phi_{x,L} = \nabla \Phi_{x,L}, \quad |\nabla \Phi_{x,L}| \leq \frac{3\sqrt{d}}{q}. \tag{163}$$

Lemma 26 (SLI) *Let the operator A be given by (10) with (11). Then for any $\ell, L \in 2q\mathbb{N}$ with $4q \leq \ell < L - 3q$, $x, y \in q\mathbb{Z}^d$ with $2\|y - x\| \leq L - \ell - 3q$ (so $\Lambda_\ell(y) \subset \Lambda_{L-3q}(x)$), and $z \notin \sigma(A_{x,L}) \cup \sigma(A_{y,\ell})$, we have*

$$\|\Gamma_{x,L} R_{x,L}(z) \chi_y\|_{x,L} \leq \gamma_z \ell^{d-1} \|\Gamma_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} \|\Gamma_{x,L} R_{x,L}(z) \chi_{y'}\|_{x,L} \tag{164}$$

for some $y' \in \Upsilon_{y,\ell}$, with

$$\gamma_z = \frac{6d^{\frac{3}{2}}}{q\varrho_-} \Theta_{\frac{q}{4}}(1 + |z|), \tag{165}$$

where $\Theta_{\frac{q}{4}} = \Theta(d, \varrho_\pm, \frac{q}{4})$ is the constant given in Proposition 39.

Proof. Using (162) , (154) and $\Gamma_{x,L} J_{y,\ell}^{x,L} \Phi_{y,\ell} = 0$ we obtain

$$\Gamma_{x,L} R_{x,L}(z) J_{y,\ell}^{x,L} \chi_y = \Gamma_{x,L} R_{x,L}(z) J_{y,\ell}^{x,L} \Phi_{y,\ell} \chi_y = \tag{166}$$
$$\Gamma_{x,L} R_{x,L}(z) \left(-J_{y,\ell}^{x,L} \frac{1}{\varrho} (\nabla \Phi_{y,\ell}) \cdot \nabla + \nabla \cdot J_{y,\ell}^{x,L} (\nabla \Phi_{y,\ell}) \frac{1}{\varrho} \right) R_{y,\ell}(z) \chi_y$$

as quadratic forms on $L^2(\Lambda_L(x)) \times L^2(\Lambda_\ell(y))$.

We now use (163) and (11) to get

$$\begin{aligned}
&\|\Gamma_{x,L} R_{x,L}(z) J_{y,\ell}^{x,L} \frac{1}{\varrho} (\nabla \Phi_{y,\ell}) \cdot \nabla R_{y,\ell}(z) \chi_y \|_{y,\ell}^{x,L} \\
&\quad = \|\Gamma_{x,L} R_{x,L}(z) \Gamma_{y,\ell} J_{y,\ell}^{x,L} \frac{1}{\varrho} \hat{\Gamma}_{y,\ell} (\nabla \Phi_{y,\ell}) \cdot \nabla R_{y,\ell}(z) \chi_y \|_{y,\ell}^{x,L} &(167)\\
&\quad \le \frac{3\sqrt{d}}{q\varrho_-} \|\hat{\Gamma}_{y,\ell} \nabla R_{y,\ell}(z) \chi_y\|_{y,\ell} \|\Gamma_{x,L} R_{x,L}(z) \Gamma_{y,\ell}\|_{x,L} &(168)
\end{aligned}$$

and

$$\begin{aligned}
&\|\Gamma_{x,L} R_{x,L}(z) \nabla \cdot J_{y,\ell}^{x,L} (\nabla \Phi_{y,\ell}) \frac{1}{\varrho} R_{y,\ell}(z) \chi_y \|_{y,\ell}^{x,L} \\
&\quad = \|\Gamma_{x,L} R_{x,L}(z) \nabla \cdot (\nabla \Phi_{y,\ell}) \hat{\Gamma}_{y,\ell} J_{y,\ell}^{x,L} \Gamma_{y,\ell} \frac{1}{\varrho} R_{y,\ell}(z) \chi_y \|_{y,\ell}^{x,L} &(169)\\
&\quad \le \frac{1}{\varrho_-} \|\Gamma_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} \|\Gamma_{x,L} R_{x,L}(z) \nabla \cdot (\nabla \Phi_{y,\ell}) \hat{\Gamma}_{y,\ell}\|_{x,L} &(170)\\
&\quad \le \frac{3\sqrt{d}}{q\varrho_-} \|\Gamma_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} \|\hat{\Gamma}_{y,\ell} \nabla R_{x,L}(z^*) \Gamma_{x,L}\|_{x,L} &(171)
\end{aligned}$$

Appealing to Proposition 39 we obtain

$$\begin{aligned}
\|\hat{\Gamma}_{y,\ell} \nabla R_{y,\ell}(z) \chi_y\|_{y,\ell} &\le \Theta_{\frac{q}{4}} \left[\|\Gamma_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} + \|\Gamma_{y,\ell} A_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} \right] \\
&\le \Theta_{\frac{q}{4}} (1+|z|) \|\Gamma_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} &(172)
\end{aligned}$$

and

$$\begin{aligned}
\|\hat{\Gamma}_{y,\ell} \nabla R_{x,L}(z^*) \Gamma_{x,L}\|_{x,L} &\le \Theta_{\frac{q}{4}} \|\Gamma_{y,\ell} R_{x,L}(z^*) \Gamma_{x,L}\|_{x,L} + \|\Gamma_{y,\ell} A_{x,L} R_{x,L}(z^*) \Gamma_{x,L}\|_{x,L}] \\
&\le \Theta_{\frac{q}{4}} (1+|z|) \|\Gamma_{y,\ell} R_{x,L}(z^*) \Gamma_{x,L}\|_{x,L} &(173)\\
&= \Theta_{\frac{q}{4}} (1+|z|) \|\Gamma_{x,L} R_{x,L}(z) \Gamma_{y,\ell}\|_{x,L}. &(174)
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Gamma_{x,L} R_{x,L}(z) \chi_y\|_{x,L} &= \|\Gamma_{x,L} R_{x,L}(z) J_{y,\ell}^{x,L} \chi_y\|_{y,\ell}^{x,L} &(175)\\
&\le 2\Theta_{\frac{q}{4}} (1+|z|) \|\Gamma_{y,\ell} R_{y,\ell}(z) \chi_y\|_{y,\ell} \|\Gamma_{x,L} R_{x,L}(z) \Gamma_{y,\ell}\|_{x,L}.
\end{aligned}$$

Hence (164) follows from (175), (158) and (159). □

The last tool in this subsection is the *eigenfunction decay inequality* (EDI), an inequality that gives decay for generalized eigenfunctions (see Appendix B.2) from decay of local resolvents.

Lemma 27 (EDI) *Let the operator A be given by (10) with (11), and let ψ be a generalized eigenfunction for a given $z \in \mathbb{C}$. For any $x \in q\mathbb{Z}^d$ and $\ell \in 2q\mathbb{N}$ with $\ell \geq 4q$, such that $z \notin \sigma(A_{x,\ell})$, we have*

$$\|\chi_x \psi\| \leq \gamma_z \ell^{d-1} \|\Gamma_{x,\ell} R_{x,\ell}(z^*) \chi_x\|_{x,\ell} \|\chi_y \psi\| \tag{176}$$

for some $y \in \Upsilon_{y,\ell}$, with γ_z as in (165).

Proof. Since $(A - zI)\psi = 0$ weakly (see (239)), we also have $\Phi_{x,\ell}(A - zI)\psi = 0$ weakly, so $J^{x,\ell}\Phi_{x,\ell}(A - zI)\psi = 0$ weakly in $L^2(\Lambda_\ell(x))$, where $J^{x,\ell} = J_{\mathbb{R}^d}^{\Lambda_\ell(x)}$. Thus

$$\begin{aligned}
J^{x,\ell}\chi_x\psi &= \chi_x R_{x,\ell}(z)(A_{x,\ell} - zI) J^{x,\ell}\Phi_{x,\ell}\psi & (177)\\
&= \chi_x R_{x,\ell}(z)(A_{x,\ell} J^{x,\ell}\Phi_{x,\ell} - J^{x,\ell}\Phi_{x,\ell} A)\psi & (178)\\
&= \chi_x R_{x,\ell}(z)\left(-\nabla \cdot J^{x,\ell}\frac{1}{\varrho}(\nabla\Phi_{x,\ell}) + J^{x,\ell}(\nabla\Phi_{x,\ell}) \cdot \frac{1}{\varrho}\nabla\right)\psi & (179)
\end{aligned}$$

weakly in $L^2(\Lambda_\ell(x))$.

Proceeding as in the proof of Lemma 26, we have

$$\begin{aligned}
&\|\chi_x R_{x,\ell}(z)\nabla \cdot J^{x,\ell}\frac{1}{\varrho}(\nabla\Phi_{x,\ell})\psi\|_{x,\ell} & (180)\\
&\quad = \|\chi_x R_{x,\ell}(z)\nabla \cdot J^{x,\ell}\frac{1}{\varrho}(\nabla\Phi_{x,\ell})\hat{\Gamma}_{x,\ell}\Gamma_{x,\ell}\psi\|_{x,\ell} & (181)\\
&\quad \leq \frac{1}{\varrho_-}\|\hat{\Gamma}_{x,\ell}(\nabla\Phi_{x,\ell}) \cdot \nabla R_{x,\ell}(z^*)\chi_x\|_{x,\ell}\|\Gamma_{x,\ell}\psi\| & (182)\\
&\quad \leq \frac{3\sqrt{d}}{q\varrho_-}\Theta_{\frac{q}{4}}(1 + |z|)\|\Gamma_{x,\ell}R_{x,\ell}(z^*)\chi_x\|_{x,\ell}\|\Gamma_{x,\ell}\psi\|, & (183)
\end{aligned}$$

and

$$\begin{aligned}
&\|\chi_x R_{x,\ell}(z) J^{x,\ell}(\nabla\Phi_{x,\ell}) \cdot \frac{1}{\varrho}\nabla\psi\|_{x,\ell} & \\
&\quad = \|\chi_x R_{x,\ell}(z)\Gamma_{x,\ell} J^{x,\ell}\hat{\Gamma}_{x,\ell}(\nabla\Phi_{x,\ell}) \cdot \frac{1}{\varrho}\nabla\psi\|_{x,\ell} & (184)\\
&\quad \leq \frac{3\sqrt{d}}{q\varrho_-}\Theta_{\frac{q}{4}}(1 + |z|)\|\Gamma_{x,\ell}R_{x,\ell}(z^*)\chi_x\|_{x,\ell}\|\Gamma_{x,\ell}\psi\|. & (185)
\end{aligned}$$

Thus (176) follows. □

6.2 The multiscale analysis

Motivated by the SLI (164) and the EDI (176) we make the following *deterministic* definition (see [CH] for a similar definition).

Definition 28 *Let the operator A be as in (10) with (11). Given $m > 0$, $E > 0$, $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$, $L \geq 4q$, we say that the cube $\Lambda_L(x)$ is (m, E)-regular, if $E \notin \sigma(A_{x,L})$ and*

$$\|\Gamma_{x,L} R_{x,L}(E) \chi_x\|_{x,L} \leq \mathrm{e}^{-m\frac{L}{2}}. \tag{186}$$

We say that $\psi \in L^2(\mathbb{R}^d)$ decays exponentially fast with mass $m > 0$ if

$$\limsup_{\|x\| \to \infty} \frac{\log |\psi(x)|}{\|x\|} \leq -m, \tag{187}$$

Motivated by the EDI, which is formulated for $\|\chi_x \psi\|$, we say that ψ decays q-exponentially fast with mass $m > 0$ if

$$\limsup_{\|x\| \to \infty} \frac{\log \|\chi_x \psi\|}{\|x\|} \leq -m. \tag{188}$$

Notice that if ψ is an eigenfunction for an operator A is as in (10) with (11), then ψ decays exponentially fast with mass $m > 0$ if it decays q-exponentially fast with mass $m > 0$ [Ag, Theorem 5.1]. *Moreover, $\nabla\psi$ also decays q-exponentially fast with mass $m > 0$* (see Corollary 40).

We now adapt Theorem 2.1 in [DK] to our setting.

Theorem 29 *Let A be a random operator as in (18) satisfying Assumption 1. Given $E_0 > 0$, $p > d$ and $m_0 > 0$, suppose:*

(P1) *There exists $L_0 \in 2q\mathbb{N}$ such that*

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (m_0, E_0)\text{-regular}\} \geq 1 - \frac{1}{L_0^p}. \tag{189}$$

(P2) *There exist $s > 0$, $r > 4p + 6d$ and $C < \infty$ such that*

$$\mathbb{P}\left\{\operatorname{dist}(\sigma(A_{0,L}, E)) \leq \frac{1}{L^s}\right\} \leq \frac{C|E|^{\frac{d}{2}-1}}{L^r} \tag{190}$$

for all $E > 0$ and $L \in 2q\mathbb{N}$ with $\frac{1}{L^s} < E$.

Then, given m, $0 < m < m_0$, there exists

$$\mathcal{B} = \mathcal{B}(d, q, \varrho_\pm, r_u, E_0, p, s, r, C, m_0, m) < \infty,$$

depending only on the indicated constants, nondecreasing in E_0, such that, if we have $L_0 > \mathcal{B}$, we can find $\delta = \delta(E_0, L_0, m_0, m, s, C) > 0$, so, with probability one, A has only pure point spectum in $(E_0 - \delta, E_0 + \delta)$, and the corresponding eigenfunctions decay q-exponentially fast with mass $m > 0$.

Remark 30 *A random operator A_g as in (18) satisfying Assumption 1 always satisfies hypothesis* (P2) *for any $s > 4p + 8d$ and $r = s - 2d$, with*

$$C = Q\frac{U_- + 2U_+}{gU_+(1 - gU_+)U_-}\|\rho\|_\infty, \quad Q = Q(d, \varrho_{0,\pm}, r_u) < \infty. \tag{191}$$

This is just a special case of Theorem 23. Theorem 29 as stated is true with weaker hypotheses on A_g; the proof requires only (P2), *the* SLI, *the* EDI, *and Theorem 43.*

Remark 31 *Hypothesis* (P1) *says that we have localization in a large, but finite, volume, with high enough probability. It is the* starting hypothesis *for the multiscale analysis. The proof of Theorem 7 from Theorem 29 will consist of verifying* (P1), *using Assumption 2*

Proof. Theorem 29 is proven as Theorem 2.1 in [DK], using the SLI and the EDI, with the following modifications:

1. We always take boxes $\Lambda_L(x)$ with $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$.

2. For two cubes $\Lambda_{L_1}(x_1)$ and $\Lambda_{L_2}(x_2)$ to be non-overlapping we require that $\|x_1 - x_2\| > \frac{1}{2}(L_1 + L_2 + r_u)$. In this case the random operators A_{x_1,L_1} and A_{x_2,L_2} are independent.

3. The probabilistic statement in [DK, Theorems 2.2 and 2.3] (called $R(L, m)$ in [DK, p. 290]),

$$\mathbb{P}\left\{\text{for any } E \in I \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (m, E)\text{-regular}\right\} \geq 1 - \frac{1}{L^{2p}}, \tag{192}$$

   is now stated for any $x, y \in q\mathbb{Z}^d$ with $\|x - y\| > L + \frac{r_u}{2}$.

4. The length scales L_k^α of [DK, Theorems 2.2 and 2.3] are now defined by $L_{k+1} = [L_k^\alpha]_{2q}$, $k = 0, 1, \ldots$, where $[t]_{2q} = \sup\{n \in 2q\mathbb{N};\ n \leq t\}$. We also take $L = [\ell^\alpha]_{2q}$ in [DK, Lemmas 4.1 and 4.2].

5. The basic tool for the proof of [DK, Lemma 4.2] in our setting is the SLI (164); it replaces inequality [DK, (4.1)].

6. We prove q-exponential decay of eigenfunctions in the analogues of [DK, Theorem 2.3 and Lemma 3.1]. The basic tool in the proof of [DK, Lemma 3.1] is now the EDI (176). The basic facts about generalized eigenfunctions are given in Theorem 43, we use (242) with $\ell = q$.

7. The fact that $\mathcal{B}$ can be chosen nondecreasing in E_0 follows from the fact that E_0 only appears in the proof when we use either (190), (164) or (176).

□

Theorem 29 suffices to prove Theorem 7, but Theorem 6 requires a somewhat different starting hypothesis, in which m_0 and L_0 are related.

Theorem 32 *Let A be a random operator as in (18) satisfying Assumption 1. Given $E_0 > 0$ and $p > d$, suppose we have* (P2) *as in Theorem 29 and*

(Q1) *There exists $L_0 \in 2q\mathbb{N}$ such that*

$$\mathbb{P}\left\{\Lambda_{L_0}(0) \;\; is \;\left(\tfrac{\lambda \log L_0}{L_0}, E_0\right)\text{-}regular\right\} \geq 1 - \frac{1}{L_0^p} \tag{193}$$

for some $\lambda > s + d - 1$.

Then, given ξ, $0 < \xi < 1$, there exists

$$\mathcal{C} = \mathcal{C}(d, q, \varrho_\pm, r_u, E_0, p, s, r, C, \lambda, \xi) < \infty,$$

depending only on the indicated constants, nondecreasing in E_0, such that, if we have $L_0 > \mathcal{C}$, we can find $\delta = \delta(E_0, L_0, s, C, \lambda, \xi) > 0$, so, with probability one, A has only pure point spectum in $(E_0 - \delta, E_0 + \delta)$, and the corresponding eigenfunctions decay q-exponentially fast with mass $\frac{\xi \lambda \log L_0}{L_0}$.

Proof. This theorem is just Theorem 29 with $m_0 = \frac{\lambda \log L_0}{L_0}$; the same proof applies. We need $\lambda > s + d - 1$ in (Q1) to control the contribution of a singular region by the decay of a regular cube, using the SLI (164) and (P2) (see [DK, Lemma 4.2]). □

Remark 33 *Notice that (193) is the same as*

$$\mathbb{P}\left\{ \|\Gamma_{0,L_0} R_{0,L_0}(E_0)\chi_0\|_{0,L_0} \leq \frac{1}{L_0^{\frac{\lambda}{2}}} \right\} \geq 1 - \frac{1}{L_0^p}. \tag{194}$$

Theorem 32 suffices to prove Theorem 6 with the stronger hypothesis $\eta > 2d$ in (23). To deal with the weaker hypothesis $\eta > d$ we adapt an argument of Spencer [Sp] to obtain the starting hypothesis (Q1) from a weaker (and easier to verify) hypothesis.

Definition 34 *Let the operator A be as in (10) with (11). Given $\nu > 0$, $E > 0$, $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$, $L \geq 4q$, we say that the cube $\Lambda_L(x)$ is (ν, E)-suitable, if $E \notin \sigma(A_{x,L})$ and*

$$\sup_{y \in \bar{\Lambda}_{\frac{L}{2}}(x) \cap q\mathbb{Z}^d} \|\Gamma_{x,L} R_{x,L}(E)\chi_y\|_{x,L} \leq \frac{1}{L^\nu}. \tag{195}$$

Theorem 35 *Let A be a random operator as in (18) satisfying Assumption 1. Given $E_0 > 0$ and $\nu > 2(d-1)$, suppose:*

(H1)

$$\limsup_{L \to \infty} \mathbb{P}\{\Lambda_L(0) \;\; is \; (\nu, E_0)\text{-}suitable\} = 1. \tag{196}$$

(H2) *There exist* $s \in (0, \nu - 2(d-1))$, $r > 10d$ *and* $C < \infty$ *such that*

$$\mathbb{P}\left\{\operatorname{dist}(\sigma(A_{0,L}), E_0) \leq \frac{1}{L^s}\right\} \leq \frac{C|E_0|^{\frac{d}{2}-1}}{L^r} \tag{197}$$

for all $L \in 2q\mathbb{N}$ *with* $\frac{1}{L^s} < E_0$.

Then there exists $\delta = \delta(d, q, \varrho_\pm, r_u, E_0, \nu, s, r, C) > 0$, *so, with probability one,* A *has only pure point spectum in* $(E_0 - \delta, E_0 + \delta)$, *and the corresponding eigenfunctions decay* q*-exponentially fast.*

Theorem 35 follows from Theorem 32 and from the following lemma.

Lemma 36 *Let* A *be a random operator as in (18) satisfying Assumption 1. Given* $E_0 > 0$, $\nu > d - 1$ *and* $p > d$, *suppose:*

(X1) *There exists* $L_0 \in 2q\mathbb{N}$ *and* $\delta_0 \in (0,1)$ *such that*

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\nu, E_0)\text{-suitable}\} \geq 1 - \delta_0. \tag{198}$$

(X2) *There exist* $s \in (0, \nu - 2(d-1))$, $r > p$ *and* $C < \infty$ *such that*

$$\mathbb{P}\left\{\operatorname{dist}(\sigma(A_{0,L}, E_0)) \leq \frac{1}{L^s}\right\} \leq \frac{C|E_0|^{\frac{d}{2}-1}}{L^r} \tag{199}$$

for all $L \in 2q\mathbb{N}$ *with* $\frac{1}{L^s} < E_0$.

Then there exist $\mathcal{Y} = \mathcal{Y}(d, q, \varrho_\pm, r_u, E_0, \nu, s) < \infty$, *depending only on the indicated constants, nondecreasing in* E_0, *such that given* $\alpha > \mathcal{Y}$, $\alpha \in \mathbb{N}$, *we can find* $\mathcal{Z} = \mathcal{Z}(d, q, \varrho_\pm, r_u, E_0, \nu, p, r, s, \alpha) < \infty$, *depending only on the indicated constants, nondecreasing in* E_0 *and in* α, *so if we have* $L_0 > \mathcal{Z}$ *and* $\delta_0 < \left[2\,(2\alpha)^{2d}\right]^{-1}$ *in* (X1), *and we set* $L_{k+1} = \alpha L_k$, $k = 1, 2, \ldots$, *we have*

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\nu, E_0)\text{-suitable}\} \geq 1 - \frac{1}{L^p} \tag{200}$$

for all $k \geq \mathcal{K}$, *where* $\mathcal{K} = \mathcal{K}(\alpha, p) < \infty$.

Proof. The proof is by a multiscale analysis which combines ideas from [Sp, Theorem 1] and [DK]. We set

$$p_L = \mathbb{P}\{\Lambda_L(0) \text{ is not } (\nu, E_0)\text{-suitable}\}. \tag{201}$$

The multiscale analysis is based on an induction argument; let $\ell \in 2q\mathbb{N}$, $\alpha \in \mathbb{N}$ and $L = \alpha\ell$. We set $\Xi_{L,\ell} = \Lambda_L(0) \cap \frac{\ell}{2}\mathbb{Z}^d \subset q\mathbb{Z}^d$, notice $|\Xi_{L,\ell}| \leq (2\alpha)^d$. The induction step is similar to [DK, Lemma 4.1], it is based on the the SLI, but only cubes in $\mathcal{C}_{L,\ell} = \{\Lambda_\ell(x);\ \ x \in \Xi_{L,\ell}\}$ are allowed; notice that $\Lambda_L(0) \subset \bigcup_{x \in \Xi_{L,\ell}} \bar{\Lambda}_{\frac{\ell}{2}}(x)$.

Let $\mathcal{F}_{L,\ell}$ denote the event that either there are at least two non-overlapping cubes in $\mathcal{C}_{L,\ell}$ which are not (ν, E_0)-suitable, or $\operatorname{dist}(\sigma(A_{x,3\ell}, E_0)) \leq \frac{1}{(3\ell)^s}$ for some $x \in \Xi_{L,\ell}$, or $\operatorname{dist}(\sigma(A_{0,L}, E_0)) \leq \frac{1}{L^s}$. As in [DK, Lemma 4.2], we will show that

$$\{\Lambda_L(0) \text{ is not } (\nu, E_0)\text{-suitable}\} \subset \mathcal{F}_{L,\ell}, \tag{202}$$

so

$$p_L \leq \mathbb{P}(\mathcal{F}_{L,\ell}) \leq (2\alpha)^{2d} p_\ell^2 + (2\alpha)^d \frac{C|E_0|^{\frac{d}{2}-1}}{(3\ell)^r} + \frac{C|E_0|^{\frac{d}{2}-1}}{L^r} \tag{203}$$

$$\leq (2\alpha)^{2d} p_\ell^2 + \frac{C|E_0|^{\frac{d}{2}-1}}{L^r}\left(1 + \frac{2^d \alpha^{d+r}}{3^d}\right), \tag{204}$$

where we used (X2) to obtain the last two terms in (203).

To prove (202), we take $\ell > 2r_u + 2q$, so if $\omega \notin \mathcal{F}_{L,\ell}$ there exist $x_1 \in \Xi_{L,\ell}$, so that $\Lambda_\ell(x)$ is (ν, E_0)-suitable for any $x \in \Xi_{L,\ell} \backslash \Lambda_{3\ell}(x_1)$. We control the "bad region" $\Lambda_{3\ell}(x_1)$ as in [DK, Lemma 4.2], using the SLI (164) twice, we must require

$$[\gamma_{E_0}(3\ell)^{d-1}(3\ell)^s][\gamma_{E_0}\ell^{d-1}] < \ell^\nu, \tag{205}$$

which is true for $\nu > s + 2(d-1)$ and ℓ large (γ_{E_0} is given in (165)). We then have

$$\sup_{y \in \bar{\Lambda}_{\frac{L}{2}}(0) \cap q\mathbb{Z}^d} \|\Gamma_{0,L} R_{0,L}(E)\chi_y\|_{0,L} \leq \left[\frac{\gamma_{E_0}\ell^{d-1}}{\ell^\nu}\right]^N L^s, \tag{206}$$

where N is the number of times we are allowed to use the SLI on (ν, E_0)-suitable boxes (without using the result for the control of a "bad box"). We have

$$N \geq \frac{\frac{L}{2} - q - 6\ell}{\frac{3\ell}{4} - q} \geq \frac{2}{3}(\alpha - 14), \tag{207}$$

so for $\Lambda_L(0)$ to be (ν, E_0)-suitable we need

$$\left[\frac{\gamma_{E_0}\ell^{d-1}}{\ell^\nu}\right]^{\frac{2}{3}(\alpha-14)} L^s < \frac{1}{L^\nu}, \tag{208}$$

which is true for α sufficiently large since $\nu > d - 1$. Thus (202) is proven.

Let $\mathcal{Z}$ be such that

$$\frac{C|E_0|^{\frac{d}{2}-1}}{\mathcal{Z}^{r-p}}\left(1 + \frac{2^d \alpha^{d+r}}{3^d}\right) \leq \frac{1}{2}. \tag{209}$$

If we pick $L_0 > \mathcal{Z}$, and set $L_{k+1} = \alpha L_k$, $k = 1, 2, \ldots$, $p_k = p_{L_k}$, and $\gamma = (2\alpha)^{2d}$, it follows from (204) that

$$p_{k+1} \leq \gamma p_k^2 + \frac{1}{2L_{k+1}^p} \quad \text{for } k = 1, 2, \ldots. \tag{210}$$

Notice first that if $p_k < \frac{1}{L_k^p}$, then

$$p_{k+1} \le \gamma \frac{1}{L_k^{2p}} + \frac{1}{2L_{k+1}^p} \le \frac{2\gamma\alpha^p}{L_0^p}\frac{1}{2L_{k+1}^p} + \frac{1}{2L_{k+1}^p} \le \frac{1}{L_{k+1}^p} \tag{211}$$

if we require $\mathcal{Z}$ also to satisfy $\frac{2\gamma\alpha^p}{\mathcal{Z}^p} \le 1$. Now, suppose $p_{k+1} \ge \frac{1}{L_{k+1}^p}$ for $k = 1, 2, \ldots, n$, it follows from (210) that $\gamma p_k^2 \ge \frac{1}{2L_{k+1}^p}$ for $k = 0, 1, 2, \ldots, n-1$, so $p_{k+1} \le \gamma p_k^2$ for $k = 1, 2, \ldots, n$, so we have

$$\frac{1}{\alpha_{k+1}^p L_0} \le p_{n+1} \le \frac{1}{2\gamma}(2\gamma p_0)^{2^{n+1}} \tag{212}$$

Thus, if $2\gamma p_0 < 1$, there must be $\mathcal{K}$ so $p_k < \frac{1}{L_k^p}$ for all $k \le \mathcal{K}$. Thus (200) is proven. □

6.3 Boundary condition: from periodic to Dirichlet

The starting hypothesis (P1) for the multiscale analysis is formulated for operators with Dirichlet boundary condition. But under the hypotheses of Theorems 6 and 7 the natural starting hypothesis is the analogue of (P1) for *periodic* boundary condition. The following lemma will enable us to go from periodic boundary condition to Dirichlet boundary condition.

Let A_g be as in (18) satisfying Assumption 1. Given $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$, we set (with the notation of (103))

$$\mathring{A}_{g,\omega,x,L} = (\mathring{A}\,((g\omega)_{\Lambda_L(x)}))_{\Lambda_L(x)}, \tag{213}$$

which is a random operator by Theorem 38. We write $\mathring{R}_{g,\omega,x,L}\,(z)$ for its resolvent.

Lemma 37 *Let A_g be as in (18) satisfying Assumption 1. Let $E > 0$, $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$, $L \ge 4q$; set $\hat{L} = L + [2r_u]_{2q} + 2q$. If ω is such that $E \notin \sigma(A_{g,\omega,x,L} \cup \sigma(\mathring{A}_{g,\omega,x,\hat{L}})$, then*

$$\begin{aligned} &\|\Gamma_{x,L} R_{g,\omega,x,L}(E)\chi_x\|_{x,L} \le \\ &\quad \left(1 + \frac{3\sqrt{d}}{q\varrho_-}(1 + 2(1+E)\|R_{g,\omega,x,L}(E)\|_{x,L})\right) \|\Gamma_{x,L}\,\mathring{R}_{g,\omega,x,\hat{L}}\,(E)\chi_x\|_{x,\hat{L}}. \end{aligned} \tag{214}$$

Proof. ¿From Lemma 24 and the definition of periodic and Dirichlet boundary condition, plus the choice of $\hat{L}$ which ensures that

$$\varrho_{(g\omega)_{\Lambda_{\hat{L}}(x)}}(y) = \varrho_{g,\omega}(y) \text{ for all } y \in \Lambda_L(x), \tag{215}$$

we get the following analogue to the SRI (154):

$$\begin{aligned} &\mathring{R}_{g,\omega,x,\hat{L}}\,(E) J_{x,L}^{x,\hat{L}} \Phi_{x,L} = J_{x,L}^{x,\hat{L}} \Phi_{x,L} R_{g,\omega,x,L}(E) + \\ &\quad \mathring{R}_{g,\omega,x,\hat{L}}\,(E)\left(-J_{x,L}^{x,\hat{L}} \frac{1}{\varrho_{g,\omega}}(\nabla\Phi_{x,L}) \cdot \nabla + \nabla \cdot J_{x,L}^{x,\hat{L}}(\nabla\Phi_{x,L})\frac{1}{\varrho_{g,\omega}}\right) R_{g,\omega,x,L}(E) \end{aligned} \tag{216}$$

as quadratic forms on $L^2(\Lambda_{x,\hat{L}}) \times L^2(\Lambda_{x,L})$.

Proceeding as in the proof of Lemma 26, we get (we omit g and ω from the notation)

$$\begin{aligned}
&\|\chi_x \overset{\circ}{R}_{x,\hat{L}}(E) J_{x,L}^{x,\hat{L}} \frac{1}{\varrho} (\nabla \Phi_{x,L}) \cdot \nabla R_{x,L}(E) \Gamma_{x,L}\|_{x,L}^{x,\hat{L}} \\
&\quad \le \frac{3\sqrt{d}}{q\varrho_-} \|\hat{\Gamma}_{x,L} \nabla R_{x,L}(E) \Gamma_{x,L}\|_{x,L} \|\chi_x \overset{\circ}{R}_{x,\hat{L}}(E) \Gamma_{x,L}\|_{x,\hat{L}} \\
&\quad \le \frac{3\sqrt{d}}{q\varrho_-} \Theta_{\frac{q}{4}} (1 + (1+E)\|R_{x,L}(E)\|_{x,L}) \|\Gamma_{x,L} \overset{\circ}{R}_{g,\omega,x,\hat{L}}(E) \chi_x\|_{x,\hat{L}}
\end{aligned} \tag{217}$$

and

$$\begin{aligned}
&\|\chi_x \overset{\circ}{R}_{x,\hat{L}}(E) \nabla \cdot J_{x,L}^{x,\hat{L}} (\nabla \Phi_{x,L}) \frac{1}{\varrho} R_{x,L}(E) \Gamma_{x,L}\|_{x,L}^{x,\hat{L}} \\
&\quad \le \frac{3\sqrt{d}}{q\varrho_-} \|\hat{\Gamma}_{x,L} R_{x,L}(E) \Gamma_{x,L}\|_{x,L} \|\hat{\Gamma}_{x,L} \nabla \overset{\circ}{R}_{x,\hat{L}}(E) \chi_x\|_{x,\hat{L}} \\
&\quad \le \frac{3\sqrt{d}}{q\varrho_-} \Theta_{\frac{q}{4}} (1+E) \|R_{x,L}(E)\|_{x,L} \|\Gamma_{x,L} \overset{\circ}{R}_{g,\omega,x,\hat{L}}(E) \chi_x\|_{x,\hat{L}}.
\end{aligned} \tag{218}$$

(214) now follows from (216), (217) and (218). $\square$

6.4 Localization in a specified interval

We now prove Theorem 7 from Theorem 29. Let the random operator A_g and a_1, a_2, g_1 be as in Theorem 7, let us fix an upper bound $\tilde{\rho}$ for $\|\rho\|_\infty$, and set

$$p_1' = \mu\left(\left(\frac{g_1}{g}, 1\right]\right). \tag{219}$$

(P2) follows from Theorem 23 (see remark 30). To prove Theorem 7, since $\mathcal{B}$ in Theorem 29 is nondecreasing in E_0, we need only to verify (P1) uniformly on $E \in [a_2, a(g)]$ for some sufficiently large L_0 and all sufficiently small p_1'; this will be done using Assumption 2. It thus suffices to show that we can find $m_0 > 0$ and $p > d$ such that

$$\limsup_{p_1' \to 0} \inf_{E \in [a_2, a(g)]} \mathbb{P}\{\Lambda_L(0) \text{ is } (m_0, E)\text{-regular}\} > 1 - \frac{1}{L^p} \tag{220}$$

for some sufficiently large $L \in 2q\mathbb{N}$.

We will prove more. Let us fix $p > d$. For a given $L \in 2q\mathbb{N}$, $L \ge 4q$, we set $\hat{L}$ as in Lemma 37 and define the event

$$\mathcal{E}_L = \left\{\omega_i \le \frac{g_1}{g} \text{ for all } i \in \mathbb{Z}^d \cap \check{\Lambda}_{\hat{L}}(0)\right\} \tag{221}$$

Since $a(g_1) = a_1$, we can conclude from Theorem 3, Lemma 19 and Theorem 14 that

$$(a_1, b(g)) \subset \mathbb{R} \backslash \sigma(\overset{\circ}{A}_{g,\omega,0,\hat{L}}) \text{ for all } \omega \in \mathcal{E}_L. \tag{222}$$

It now follows from Lemma 18 (with $r = q$), (158) and (159) , that for $\omega \in \mathcal{E}_L$ and $E \in (a_1, a(g)]$ we have

$$\begin{aligned} \|\Gamma_{0,L} \overset{\circ}{R}_{g,\omega,0,\hat{L}} (E)\chi_0\|_{0,\hat{L}} &\leq \left(\frac{9dL^{d-1}}{\eta}\right) e^{\frac{\sqrt{d}q\overset{\circ}{m}_{E,q,\hat{L}}}{2}} e^{-\overset{\circ}{m}_{E,q,\hat{L}}|\frac{L}{2}-q|} \\ &\leq \left(\frac{9dL^{d-1}}{\eta}\right) e^{(\frac{\sqrt{d}}{2}+1)q\overset{\circ}{m}_{E,q,\hat{L}}} e^{-\overset{\circ}{m}_{E,q,\hat{L}}\frac{L}{2}}, \end{aligned} \tag{223}$$

where $\overset{\circ}{m}_{E,q,\hat{L}}$ is given in (93) and $\eta = E - a_1$ since $a(g) - a_1 \leq b(g) - a(g)$; we assume $\hat{L} > 2(2q + 8)$.

If we now take $E \in [a_2, a(g)]$, so $a_2 - a_1 \leq \eta \leq a(g) - a \leq a\delta_+(g)g$ (we used (21)), we have

$$M \equiv \frac{a_2 - a_1}{4(4\sqrt{d}+1)\left[\varrho_+^{-1} + a + a\delta_+(g)g\right]} \leq \overset{\circ}{m}_{E,q,\hat{L}} \leq \frac{1}{4}, \tag{224}$$

so

$$\|\Gamma_{0,L} \overset{\circ}{R}_{g,\omega,0,\hat{L}} (E)\chi_0\|_{0,\hat{L}} \leq \left(\frac{9dL^{d-1}}{a_2 - a_1}\right) e^{\frac{(\sqrt{d}+2)q}{8}} e^{-M\frac{L}{2}}. \tag{225}$$

At this point we introduce the events

$$\mathcal{W}_{E,L} = \left\{\operatorname{dist}(\sigma(A_{0,L}, E)) > \frac{1}{L^{2p+2d}}\right\}, \tag{226}$$

and set $\mathcal{G}_{E,L} = \mathcal{E}_L \cap \mathcal{W}_{E,L}$. If $E \in [a_2, a(g)]$ and $\omega \in \mathcal{G}_{E,L}$, we can apply Lemma 37, (21), (225) and (226) to obtain

$$\begin{aligned} &\|\Gamma_{0,L} R_{g,\omega,0,L}(E)\chi_0\|_{0,L} \leq \\ &\quad \left(\frac{9dL^{d-1}}{a_2 - a_1}\right) e^{\frac{(\sqrt{d}+2)q}{8}} \left(1 + \frac{3\sqrt{d}}{q\varrho_-}\left(1 + 2\left(1 + \frac{a}{1 - gU_+}\right) L^{2p+2d}\right)\right) e^{-M\frac{L}{2}}. \end{aligned} \tag{227}$$

Thus, if we take $m_0 = \frac{M}{2}$, we can find

$$T_1 = T_1(d, q, g, \varrho_\pm, r_u, a, a_1, a_2, p) < \infty,$$

such that if $\omega \in \mathcal{G}_{E,L}$, with $E \in [a_2, a(g)]$, we have that the cube $\Lambda_L(0)$ is (m_0, E)-regular for any $L > T_1$. To conclude the proof of Theorem 7, notice that from (221), (219), (226) and (191) we get, for $\frac{1}{L^{2p+2d}} < E \leq a(g)$,

$$\begin{aligned} \mathbb{P}(\mathcal{G}_{E,L}) &\geq 1 - p_1' \hat{L}^d - \frac{C}{L^{2p}}\left(\frac{a}{1 - gU_+}\right)^{\frac{d}{2}-1} &(228) \\ &\geq 1 - p_1' \hat{L}^d - \frac{1}{2L^p} &(229) \end{aligned}$$

with (229) valid for $L > T_2$, where

$$T_2 = T_2(d, q, g, \varrho_{0,\pm}, U_\pm, r_u, a, \tilde{\rho}, p) < \infty. \tag{230}$$

We can conclude that

$$\inf_{E\in[a_1,a(g)]} \mathbb{P}\{\Lambda_L(0) \text{ is } (m_0, E)\text{-regular}\} \geq \inf_{E\in[a_1,a(g)]} \mathbb{P}(\mathcal{G}_{E,L}) > 1 - \frac{1}{L^p} \tag{231}$$

for all $L > \max\{T_1, T_2\}$ and $p_1' < \frac{1}{2L^{p+d}}$, so (220) follows.

Theorem 7 is proven.

6.5 Localization at the edge

We now prove Theorem 6 from either Theorem 32 or Theorem 35. Let the random operator A_g be as in Theorem 6. Since we have (P2) and (H2) in the form given in Remark 30, to prove Theorem 6 it suffices to establish either (Q1) or (H1) for $E_0 = a(g)$.

So let us fix $p > d$, $\lambda > 4p+9d-1$, and let $a_1, g_1, p_1', \hat{L}, \mathcal{E}_L, \mathcal{W}_{E,L}, \mathcal{G}_{E,L}$ be as in Subsection 6.4. Notice that (23) can be written as

$$p_1' \leq K\left(\frac{g-g_1}{g}\right)^\eta. \tag{232}$$

For $\omega \in \mathcal{G}_{a(g),L}$ we have (227) with $a_2 = a(g)$ and M as in (224). If we choose $a_1 = a(g) - \kappa\frac{\log L}{L}$, with $\kappa \geq 1$ and L such that $a < a_1$ and $a(g) - a_1 \leq b(g) - a(g)$, we get

$$\|\Gamma_{0,L} R_{g,\omega,0,L}(a(g))\chi_0\|_{0,L} \leq C_1 L^{-\frac{c\kappa}{2}+2p+3d} \leq L^{-\frac{\lambda}{2}} \tag{233}$$

where the constants C_1 and c can be read from (227) and (224); the last inequality in (233) is valid for $\kappa > c^{-1}(\lambda + 4p + 6d)$ and L sufficiently large.

If $L > T_2$ (with T_2 given in (230)), it follows from (229) and (232) that

$$\begin{aligned} \mathbb{P}(\mathcal{G}_{a(g),L}) &\geq 1 - K\left(\frac{g-g_1}{g}\right)^\eta \hat{L}^d - \frac{1}{2L^p} &(234)\\ &\geq 1 - K\left(\frac{\kappa}{g\delta_-(g)a}\right)^\eta \left(\frac{\log L}{L}\right)^\eta \hat{L}^d - \frac{1}{2L^p}, &(235) \end{aligned}$$

where we used (135) to get (235).

If $\eta > 2d$, it follows immediately from (233) and (235) that the starting hypothesis (Q1) holds for all L sufficiently large, so Theorem 6 follows from Theorem 32.

If we only have $\eta > d$ we prove (H1). Let $r > 10d$, $s = r + 2d$ and $\nu > s + 2(d-1)$. For $\omega \in \mathcal{G}_{a(g),L}$ (we take $p = r$ in (226)), we have

$$\sup_{y\in\bar{\Lambda}_{\frac{L}{2}}(0)\cap q\mathbb{Z}^d} \|\Gamma_{0,L} R_{g,\omega 0,L}(a(g))\chi_y\|_{0,L} \leq C_1 L^{-\frac{c\kappa}{4}+2p+3d} \leq L^{-\nu}, \tag{236}$$

where (236) is derived in the same way as (233), with the same constants C_1 and c, with the last inequality valid for $\kappa > 4c^{-1}(\nu + 2p + 3d)$ and L sufficiently large. Since we have (235) and $\eta > d$, the starting hypothesis (H1) follows, so Theorem 6 follows from Theorem 35.

Theorem 6 is proven.

A Measurability of operator valued functions

In this appendix we will prove the desired measurability properties for our random operators. We say that a mapping $\omega \mapsto A_\omega$ from a probability space Ω to nonnegative self-adjoint operators on a separable Hilbert space $\mathcal{H}$ is measurable if the mappings $\omega \in \Omega \mapsto f(A_\omega) \in \mathcal{B}(\mathcal{H})$ are weakly (and hence strongly) measurable for all bounded measurable functions f on $\mathbb{R}$. In this case A_ω is called a *random operator*.

Theorem 38 *Let $\omega \mapsto A_\omega$ be a mapping from a probability space Ω to nonnegative self-adjoint operators on a separable Hilbert space $\mathcal{H}$, such that $c_- S \leq A_\omega \leq c_+ S$ with probability one, for some nonnegative self-adjoint operator S and constants $0 < c_- \leq c_+ < \infty$. Let $\mathcal{A}_\omega$ and $\mathcal{S}$ be the corresponding quadratic forms, $\mathcal{D} = \mathcal{D}(\mathcal{S})$ being the common domain. Suppose $\omega \mapsto \mathcal{A}_\omega(\psi)$ is measurable for all $\psi \in \mathcal{D}$, then A_ω is a random operator.*

Proof. Let $R_\omega = (A_\omega + I)^{-1}$; it suffices to prove that the mapping $\omega \mapsto R_\omega$ is weakly measurable (e.g., the argument in [PF, p. 40]).

Since $\mathcal{H}$ is separable, we can find a countable set $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{D}$, which is dense in $\mathcal{D}$ with the inner product $\langle \psi, \varphi \rangle + \mathcal{S}(\psi, \varphi)$. For each ω we apply the Gram-Schmidt orthogonalization process to $\{\varphi_n\}_{n=1}^\infty$, with respect to the inner product $\langle \psi, \varphi \rangle + \mathcal{A}_\omega(\psi, \varphi)$, obtaining the orthonormal basis $\{\psi_{\omega,n}\}_{n=1}^\infty \subset \mathcal{D}$. Each $\psi_{\omega,m}$ is measurable, since it is a linear combination of the $\{\varphi_n\}_{n=1}^m$ with measurable coefficients. We let $P_{\omega,n}$ be the orthogonal projection onto the linear span of $\{\psi_{\omega,k}\}_{k=1}^n$ and define the quadratic forms $\mathcal{A}_{\omega,n}(\psi) = \mathcal{A}_\omega(P_{\omega,n}\psi)$ for $\psi \in \mathcal{D}$; notice $\lim_{n\to\infty} \mathcal{A}_{\omega,n}(\psi) = \mathcal{A}_\omega(\psi)$ for all $\psi \in \mathcal{D}$.

Let $A_{\omega,n}$ be the nonnegative self-adjoint operator corresponding to $\mathcal{A}_{\omega,n}$. By construction, the operators $A_{\omega,n} + I$ form a monotone increasing sequence which converges weakly in $\mathcal{D}$ to $A_\omega + I$. Setting $R_{\omega,n} = (A_{\omega,n} + I)^{-1}$, we have

$$I \geq R_{\omega,1} \geq \ldots \geq R_{\omega,n} \geq R_{\omega,n+1} \geq \ldots \geq R_\omega > 0.$$

So let $\tilde{R}_\omega$ be the weak limit of the $R_{\omega n}$, it must be a bounded self-adjoint operator such that

$$I \geq R_{\omega,1} \geq \ldots \geq R_{\omega,n} \geq R_{\omega,n+1} \geq \ldots \geq \tilde{R}_\omega \geq R_\omega > 0,$$

hence $\tilde{R}_\omega = (\tilde{A}_\omega + I)^{-1}$ for some nonnegative self-adjoint operator $\tilde{A}_\omega$, with, by the monotonicity of the inverse,

$$0 \leq A_{\omega,1} \leq \ldots \leq A_{\omega,n} \leq A_{\omega,n+1} \leq \ldots \leq \tilde{A}_\omega \leq A_\omega.$$

Since $A_{\omega,n} \to A_\omega$ weakly on $\mathcal{D}$, we must have $\tilde{A}_\omega = A_\omega$ so $\tilde{R}_\omega = R_\omega$. Thus the theorem is proved if we can show that the map $\omega \mapsto R_{\omega,n}$ is weakly measurable for each n. But since each $P_{\omega,n}$ is finite dimensional, this follows from the measurability of the functions $\mathcal{A}_\omega(\psi_{\omega,n})$, a consequence from the measurability of the scalar functions $\mathcal{A}_\omega(\psi)$ ($\psi \in \mathcal{D}$) and of the vector functions $\psi_{\omega,n}$. $\square$

B Some results on elliptic operators

Divergence operators of the form (18) can have nonsmooth coefficient $\varrho(x)$ that implies, in particular, that the standard functional space C_0^∞ cannot be used even as a subset of the domains of such operators. In this appendix we provide some tools to treat these operators.

B.1 An interior estimate

Proposition 39 ([GT, Problem 8.2]) *Let A be an operator of the form (10) satisfying the bounds (11). For any $\delta > 0$ there exists a constant $\Theta_\delta = \Theta(d, \varrho_\pm, \delta) < \infty$, depending only on the indicated parameters, such that for any open subset Ω of $\mathbb{R}^d$, if $u \in W^{1,2}(\Omega)$ is a weak solution for the equation $Au = f$ in Ω, with $f \in L^2(\Omega)$, we have*

$$\|\nabla u\|_{2,\Omega'} \leq \Theta_\delta \left[\|u\|_{2,\Omega} + \|f\|_{2,\Omega}\right] \tag{237}$$

for any $\Omega' \subset \Omega$ with $\operatorname{dist}(\Omega', \partial\Omega) \geq \delta$.

We have the following immediate corollary.

Corollary 40 *Let A be an operator of the form (10) satisfying the bounds (11), and let φ be an eigenfunction for A. Suppose φ has exponentially decaying local L^2-norms, i.e., $\|\chi_{x,\ell}\varphi\|_2$ decays exponentially as $\|x\| \to \infty$ for some $\ell > 0$. Then $\nabla\varphi$ also has exponentially decaying local L^2-norms.*

Proposition 39 has the following obvious extension to $\mathbb{R}^d$.

Proposition 41 *Let A be an operator of the form (10) satisfying the bounds (11). If $u \in W^{1,2}(\mathbb{R}^d)$ is a weak solution for the equation $Au = f$ in $\mathbb{R}^d$, with $f \in L^2(\mathbb{R}^d)$, we have*

$$\|\nabla u\|_2 \leq \Theta_\infty \left[\|u\|_2 + \|f\|_2\right], \tag{238}$$

with $\Theta_\infty = \inf_{\delta>0} \Theta(d, \varrho_\pm, \delta)$.

B.2 Generalized eigenfunctions

Let A be an operator of the form (10) satisfying the bounds (11). Given $z \in \mathbb{C}$, a measurable function φ on $\mathbb{R}^d$ will be called a *generalized eigenfunction* for z if $\chi_\Omega\varphi \in W^{1,2}(\Omega)$ for all bounded domains Ω in $\mathbb{R}^d$ and φ is a weak solution for the equation $A\psi = z\psi$ on $\mathbb{R}^d$, i.e.,

$$\langle \nabla\psi, \frac{1}{\varrho}\nabla\varphi \rangle = z\langle \psi, \varphi \rangle \ \text{ for all } \ \psi \in C_0^1(\mathbb{R}^d). \tag{239}$$

To obtain properties of generalized eigenfunctions we use the following estimates on the Green's functions of divergence operators with nonsmooth uniformly bounded coefficients [D, Corollary 3.2.8 and Lemma 3.4.2].

Proposition 42 *Let A be an operator of the form (10) satisfying the bounds (11), let $\ell > d/2$ and $\mu > 0$. Then the operator $R^{\ell} = (\mu I + A)^{-\ell}$ has a kernel $R^{\ell}(x, y)$ satisfying the following a priori estimate:*

$$0 \leq R^{\ell}(x, y) \leq ce^{-\beta|x-y|} \text{ for all } x, y \in \mathbb{R}^d, \tag{240}$$

where the constant c depends only on d, $\varrho_{\pm}$, ℓ and μ, and $\beta = \sqrt{\frac{\mu\varrho_-}{2d}}$.

Theorem 43 *Let A be an operator of the form (10) satisfying the bounds (11), $\rho\,(d\lambda)$ its the spectral measure, and let $\ell > d/2$ be a positive integer. Let $w(x) = \left(|x|^{d'} + 1\right)^{-1}$, $d' > d$. Then for $\rho\,(d\lambda)$-almost all λ A has a generalized eigenfunction φ_λ satisfying*

$$\int_{\mathbb{R}^d} |\varphi_\lambda(x)|^2 w(x)\, dx < \infty, \tag{241}$$

so for any $\ell \in \mathbb{N}$ we have

$$\|\chi_{x,\ell}\varphi_\lambda\| \leq C_\ell \left(|x|^{d'} + 1\right) \text{ for all } x \in \ell\mathbb{Z}^d, \tag{242}$$

for some constant $C_\ell < \infty$ depending only on ℓ, d, $\varrho_{\pm}$ and the LHS of (241). Moreover, for such φ_λ we also have

$$|\varphi_\lambda(x)|^2 \leq C\left(|x|^{d'} + 1\right) \text{ for all } x \in \mathbb{R}^d, \tag{243}$$

for some constant $C < \infty$ depending only on d, $\varrho_{\pm}$ and the LHS of (241).

Proof. Notice that in view of Proposition 42 $R^{\ell} = (A + \mu I)^{-\ell}$ for $\ell > d/2$ is a Carleman operator such that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |R^{\ell}(x, y)|^2 w(x)\, dxdy < \infty. \tag{244}$$

This and [B, Theorem V.4.1] imply (241), from which (242) follows. The estimate (243) follows from (242) and [Ag, Theorem 5.1]. □

C Lemmas on convergence of operators

Lemma 44 *Let $S_n^+, S_n^-, K_n, n \geq 1$ be sequences of self-adjoint bounded operators in a Hilbert space $\mathcal{H}$ such that for some constant $C < \infty$ we have*

$$0 \leq S_n^{\pm} \leq CI \text{ for all } n \geq 1, \tag{245}$$

and

$$\lim_{n\to\infty} \langle \varphi, S_n^{\pm}\varphi\rangle = 0 \text{ for all } \varphi \in \mathcal{H}. \tag{246}$$

Then

$$\lim_{n\to\infty} S_n^{\pm}\varphi = 0 \text{ for all } \varphi \in \mathcal{H}. \tag{247}$$

If in addition we have

$$-S_n^- \leq K_n \leq S_n^+, \tag{248}$$

then

$$\lim_{n\to\infty} K_n\varphi = 0 \text{ for all } \varphi \in \mathcal{H}. \tag{249}$$

Proof. Notice that if S_n stands for either S_n^- or S_n^+ then it follows from (245) that

$$S_n^2 = \sqrt{S_n}S_n\sqrt{S_n} \leq \sqrt{S_n}C\sqrt{S_n} = CS_n.$$

This together with (246) implies that $\|S_n\varphi\|^2 \leq C\langle\varphi, S_n\varphi\rangle \to 0$ as $n \to \infty$, so (247) is true.

Observe now that in view of (245)and (246) the operators $G_n = K_n + S_n^-$ satisfy

$$0 \leq G_n \leq S_n^- + S_n^+ \leq 2CI$$

and

$$\lim_{n\to\infty}\langle\varphi, G_n\varphi\rangle = 0, \ \varphi \in \mathcal{H}.$$

Hence, using (247), we obtain

$$\lim_{n\to\infty} G_n\varphi = 0, \ \varphi \in \mathcal{H}. \tag{250}$$

Recalling now that $K_n = G_n - S_n$ and using (247) and (250) we obtain (249). □

Given a nonnegative operator B the associated quadratic form will be denoted by

$$B[\varphi,\psi] \equiv \langle\sqrt{B}\varphi, \sqrt{B}\psi\rangle, \ , \ \varphi, \ \psi \in \mathcal{D}\left(\sqrt{\mathcal{B}}\right).$$

We will write $B[\varphi] \equiv B[\varphi,\varphi]$.

Lemma 45 *Let S and $S_n, s_n^{\pm}(n \geq 1)$ be self-adjoint operators in a Hilbert space $\mathcal{H}$ such that*

$$S - s_n^- \leq S_n \leq S + s_n^+ \tag{251}$$

(the sums of possibly unbounded operators being understood in the quadratic form sense),

$$0 \leq s_n^{\pm} \leq \alpha^{\pm}S \text{ for some } 0 \leq \alpha^- < 1, \ \ 0 \leq \alpha^+ < \infty, \tag{252}$$

and

$$\lim_{n\to\infty} s_n^{\pm}[\varphi] = 0 \text{ for all } \varphi \in \mathcal{D}, \tag{253}$$

where $\mathcal{D} \subset \mathcal{D}\left(\sqrt{S}\right)$ is a core for the operator $\sqrt{S}$. Then

$$\lim_{n\to\infty}(I + S_n)^{-1}\psi = (I + S)^{-1}\psi \text{ for all } \psi \in \mathcal{H}, \tag{254}$$

so S_n converge to S in the strong resolvent sense and we have

$$\overline{\bigcup_{n\geq 1}\sigma(S_n)} \supset \sigma(S). \tag{255}$$

Proof. We set $R_n = (I+S_n)^{-1}$; $R = (I+S)^{-1}$, and notice that Notice that $-r_n^- \leq R_n - R \leq r_n^+$, where

$$\begin{aligned} r_n^+ &= \rho_n^+ - (I+S)^{-1} \geq 0,\ \rho_n^+ = (I+S-s_n^-)^{-1} \\ r_n^- &= (I+S)^{-1} - \rho_n^- \geq 0,\ \rho_n^- = (I+S+s_n^+)^{-1} \end{aligned} \tag{256}$$

Clearly

$$\|R_n\|, \|R\|, \|r_n^\pm\|, \|\rho_n^\pm\| \leq 1, \tag{257}$$

and, in view of (251) and (252),

$$\begin{aligned} \mathcal{D}\left(\sqrt{\mathcal{S}_\backslash}\right) &= \mathcal{D}\left(\sqrt{\mathcal{S}\pm\int_\backslash^\pm}\right) = \mathcal{D}\left(\sqrt{\mathcal{S}}\right), \\ R\varphi, \rho_n^\pm\varphi &\in \mathcal{D}\left(\sqrt{\mathcal{S}}\right), \varphi \in \mathcal{H}. \end{aligned} \tag{258}$$

In addition, (253) together with $s_n^\pm \leq \alpha^\pm S$ imply

$$\lim_{n\to\infty} s_n^\pm[\varphi] = 0 \text{ for all } \varphi \in \mathcal{D}\left(\sqrt{\mathcal{S}}\right). \tag{259}$$

In view of Lemma 44, to prove (254) it is sufficient to verify that

$$\lim_{n\to\infty} \langle r_n^\pm\psi, \psi\rangle = 0 \text{ for all } \psi \in \mathcal{H}. \tag{260}$$

Let us consider r_n^+. Observe that from (256), (257) and (258) we have

$$\begin{aligned} \langle r_n^+\psi, \psi\rangle &= \left\langle \rho_n^+\psi, (I+S)R\psi \right\rangle - \left\langle (I+S-s_n^-)\rho_n^+\psi, R\psi \right\rangle = \\ &= (I+S)\left[\rho_n^+\psi, R\psi\right] - (I+S-s_n^-)\left[\rho_n^+\psi, R\psi\right] = \\ &= s_n^-\left[\rho_n^+\psi, R\psi\right] = \left\langle \sqrt{s_n^-}\rho_n^+\psi, \sqrt{s_n^-}R\psi \right\rangle,\ \psi \in \mathcal{H}. \end{aligned} \tag{261}$$

The norms of the vectors $\sqrt{s_n^-}\rho_n^+\psi$ and $\sqrt{s_n^-}R\psi$ can be estimated as follows. Since $s_n^- \leq \alpha^- S$, we have

$$s_n^- \leq \frac{\alpha^-}{1-\alpha^-}(S - s_n^-). \tag{262}$$

Notice that for any nonnegative operator K we have

$$K \leq (I+K)^2.$$

Combining (262) with the last inequality for $K = S - s_n^-$, we obtain

$$s_n^- \leq \frac{\alpha^-}{1-\alpha^-}(I+S-s_n^-)^2 = \frac{\alpha^-}{1-\alpha^-}(\rho_n^+)^{-2},$$

which readily implies

$$\rho_n^+ s_n^- \rho_n^+ \leq \frac{\alpha^-}{1-\alpha^-} I \text{ and } \|\sqrt{s_n^-}\rho_n^+\|^2 \leq \frac{\alpha^-}{1-\alpha^-}. \tag{263}$$

It now follows from (261) and (263) that for $\psi \in \mathcal{H}$ we have

$$\langle r_n^+\psi, \psi\rangle^2 \leq \|\sqrt{s_n^-}R\psi\|^2\|\sqrt{s_n^-}\rho_n^+\|^2\|\psi\|^2 \leq \frac{\alpha^-}{1-\alpha^-}s_n^-\left[R\psi\right]\|\psi\|^2.$$

The last inequality together with (258) and (259) imply (260) for r_n^+. The proof of this statement for r_n^- is analogous, using $s_n^+ \leq \frac{\alpha^+}{1+\alpha^+}(S+s_n^+)$.

To prove (255) it is sufficient to prove the similar inclusion for the resolvents R_n and R, namely

$$\overline{\bigcup_{n\geq 1}\sigma(R_n)} \supset \sigma(R) \tag{264}$$

Suppose that $\lambda \in \sigma(R)$. Then for any positive γ there exists a vector $\psi \in \mathcal{H}$ such that $\|R\psi - \lambda\psi\| \leq \gamma$, $\|\psi\| = 1$. In view of the strong convergence of R_n to R this implies that for sufficiently large n we have $\|R_n\psi - \lambda\psi\| \leq 2\gamma$. Since γ is an arbitrary number the last inequality implies (264) and, hence, the desired inclusion (255). □

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