On the 2D Stochastic Ising Model in the Phase Coexistence Region Near the Critical Point

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Abstract. We consider the two dimensional stochastic Ising model in finite square Λ with free boundary conditions, at inverse temperature $\beta > \beta_c$ and zero external field. Using duality and recent results of Ioffe on the Wulff construction close to the critical temperature, we extend some of the results obtained by one of us (F.M.) in the low temperature regime, to any temperature below the critical one. In particular we show that the gap in the spectrum of the generator of the dynamics goes to zero in the thermodynamic limit as an exponential of the sidelength of Λ, with a rate constant determined by the surface tension along one of the coordinate axes. We also extend to the same range of temperatures the result due to Shlosman on the equilibrium large deviations of the magnetization with free boundary conditions.

Key Words: Stochastic Ising model, phase coexistence, relaxation time, spectral gap, surface tension, large deviations.

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[3] Work partially supported by grant SC1-CT91-0695 of the Commission of European Communities

[4] Work partially supported by the American NSF under grants DMS 9100725 and DMS 9400644, and by the Istituto Nazionale di Fisica della Materia, Italy

Mathematics Subject Classification: 60K35, 82A05

0. Introduction

In the last two years there has been considerable progress in the detailed description of the two dimensional kinetic Ising model inside and outside the phase coexistence region.

When the inverse temperature β and the external field h are such that there is no phase transition, it has been finally established that the relaxation time (that in our discussion we take as the inverse of the gap in the spectrum of the generator of the dynamics) of any Glauber–type dynamics in a finite rectangle is bounded from above uniformly in the size of the rectangle and in the boundary conditions. Such a bound has been first proved for $\beta < \beta_c$ and arbitrary h or $h \neq 0$ and $\beta \gg \beta_c$ (see [MOSc], [MO] and [MOS]) and only recently it has been extended to any $\beta > \beta_c$ and arbitrary $h \neq 0$ (see [SS1]). Here β_c denotes the critical point.

On the other hand, when $h = 0$ and $\beta > \beta_c$, the infinite volume Glauber dynamics is no longer ergodic due to the existence of two extremal Gibbs states and one easily realizes that the relaxation time in a $L \times L$ square with free boundary conditions diverges when $L \to \infty$. In [M] (see Theorem 4.2) the precise logarithmic asymptotics of the gap was computed for $\beta \gg \beta_c$ and found to be

$$\text{gap}(L,\ \text{free boundary conditions}\ ) \sim \exp(-\beta\tau_b(0)L)$$

where $\tau_\beta(0)$ stands for the surface tension in the direction of one of the coordinate axes. A similar behaviour has recently been proved in [HY] for other boundary conditions that, roughly speaking, do not favor either one of the two phases, without however a precise estimate of the rate constant.

The picture of the relaxation to the Gibbs equilibrium measure that comes out of the results of [M], and subsequently substantially improved in [MaM], is the following: the system first relaxes rather rapidly to one of the two phases and then it creates, via a large fluctuation, a thin layer of the opposite phase along one of the sides of Λ. Such a process requires already a time of the order of $e^{\beta\tau_\beta(0)L}$. After that, the opposite phase invades the whole system by moving, in a much shorter time scale, the interface to the side opposite to the initial one and equilibrium is finally reached. Once this picture is established it is not difficult to show that, under a suitable stretching of the time by a factor $a(L) \approx e^{\beta\tau_\beta(0)L}$, the magnetization in the square Λ behaves in time as a continuous Markov chain with state space $\{-m^*, +m^*\}$ and unitary jump rates , where m^* is the spontaneous magnetization (see Theorem 6.1 of [M]).

The key ingredients of the results in [M] were

- (i) a geometric bound on the gap in the spectrum of the generator of the dynamics in a rectangle
- (ii) some detailed equilibrium estimate on the fluctuations of an horizontal interface of length L
- (iii) a precise estimate of the equilibrium probability of having anomalous magnetization $m_\Lambda(\sigma) \in (-m^*, m^*)$.

The first one (i) was borrowed from a clever technique to bound from below the spectral gap of symmetric Markov chains on complicated graphs introduced in [JS] in the framework of hard computational problems. It is important for us that its validity is independent of β. The second and the third one were available at low enough temperature thanks to the results of [DKS] and [Sh] on the rigorous Wulff construction. Here the requirement of large β was quite essential since most of the results in [DKS] have been obtained through cluster expansions methods.

Recently however there have been some remarkable progress in the rigorous theory of the Wulff construction for *any* $\beta > \beta_c$, using the powerful methods of duality (see [I1], [I2]). Duality had alrealdy played a relevant role in the approach to the Wulff construction and phase separation developed in [P] for the 2D Ising model with + boundary conditions at low temperature and the main contribution of the above two papers was to combine it with the Fortuin-Kasteleyn representation of the Ising model to push some of the most relevant results of [P] up to the critical point excluded. The ideas and techniques developed in these two papers were also fundamental to the analysis of metastability for the 2D Ising model very close to the line of first order phase transition (see [SS3]). Relevant simplifications of the techniques of [I1] and [I2] can be found in [SS1] and [SS2].

In this paper we use duality techniques to extend up to β_c the basic estimates (ii) and (iii) above. As a consequence we compute for any $\beta > \beta_c$ the logarithmic asymptotics of the relaxation time in the thermodynamic limit and show that it has the same form as the one computed in [M]. Our result is however stated in a slightly weaker form than the corresponding one in [M] since we do not give any explicit bounds on the finite–volume correction. This is due to the fact that, contrary to what happens in the low temperature regime where the powerful methods of cluster expansion are available, duality methods seems, so far, able to provide good estimates on the fluctuations of an interface only when these are of the same order of magnitude as the length of the interface itself.

Once we have (ii) for any $\beta > \beta_c$, it follows from the same technique described in Section 3 of [M] that, if we replace on one side of the square Λ the free boundary conditions with *plus* b.c, then the relaxation time of the new system becomes much smaller than it was before and in particular it can be bounded from above by an exponential in L with a rate that vanishes in the thermodynamic limit. Using this result one could extend up to β_c the results of Section 6 of [M] on the Markov chain description of the magnetization mentioned above. We decided however to skip this part in order to avoid too much repetition.

We conclude this short introduction with a brief summary of the various sections.

- In Section 1 we fix the notation and state the major result.
- In Section 2 we recall the main steps of the argument given in [M] and prove the main theorem assuming three key propositions.
- In Section 3, the core of the paper, we prove the main technical estimates.
- In Section 4 we use the results of Section 3 to give the proof of two of the three propositions used in Section 2.
- In Section 5 we prove the last proposition of Section 2 by extending to any $\beta > \beta_c$ the result of [Sh] on the large deviations of the magnetization with free boundary conditions.

Acknowledgements. RHS thanks the warm hospitality provided to him at Rome by his coauthors in this papers as well as by E. Olivieri and E. Scoppola.

1. Notation and Results

1.1. General definitions

We consider the two dimensional lattice $\mathbb{Z}^2$ whose elements are called *sites* and its dual $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. For $x, y \in \mathbb{R}^2$ we define the distances

$$d(x,y) = |x-y| = \sum_{i=1}^{2} |x_i - y_i| \qquad d_2(x,y) = |x-y|_2 = \Big(\sum_{i=1}^{2} |x_i - y_i|^2\Big)^{1/2}$$

$[x,y]$ is the *closed segment* with x, y as its endpoints. The *edges* of $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$) are those $e = [x,y]$ with x, y nearest neighbors in $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$). Given an edge e of $\mathbb{Z}^2$, e^* is the unique edge in $\mathbb{Z}^2_*$ that intersects e. The *boundary of an edge* $e = [x,y]$ is $\delta e = \{x, y\}$ The *boundary of a subset of edges* α is the set of sites $\delta\alpha$ that belong to an odd number of edges of α. A set of edges is called *closed* if its boundary is empty. We denote by $\mathcal{E}_\Lambda$ the set of all edges such that both endpoints are in Λ and by $\bar{\mathcal{E}}_\Lambda$ the set of all edges with at least one endpoint in Λ. Viceversa, for a set of edges X, $\mathcal{V}(X)$ stands for the set of all sites which are endpoints of at least one edge in X.

Given $\Lambda \subset \mathbb{Z}^2$ we let $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$ and define Λ^* as the set of all $x \in \mathbb{Z}^2_*$ such that $d(x, \Lambda) = 1$. The set of the dual edges is defined as

$$\mathcal{E}^*_\Lambda = \{e^* : e \in \bar{\mathcal{E}}_\Lambda\}\,.$$

Notice that, in general, $\mathcal{E}^*_\Lambda \subset \mathcal{E}_{\Lambda^*}$ (the equality holds, for instance, in the case of rectangles).

We will often consider our model on a $(2L+1) \times (2M+1)$ rectangle

$$Q_{L,M} = \{\, (x_1, x_2) \in \mathbb{Z}^2 : -L \le x_1 \le L\,,\ -M \le x_2 \le M \,\}$$

Q_L stands for $Q_{L,L}$. If Λ is finite we write $\Lambda \subset\subset \mathbb{Z}^2$. The cardinality of Λ is denoted by $|\Lambda|$. We define the *boundaries*

$$\partial\Lambda = \{x \in \Lambda : d(x, \Lambda^c) = 1\} \quad \partial^+\Lambda = \{x \in \Lambda^c : d(x, \Lambda) = 1\} \quad \delta\Lambda = \{e^* : e \in \bar{\mathcal{E}}_\Lambda \setminus \mathcal{E}_\Lambda\}$$

$(x_1, \ldots, x_n)$ is called a *path* from x_1 to x_n if $|x_{i+1} - x_i| = 1$ for $i = 1, \ldots, n-1$. A $*$–path is the same as a path with $|x_{i+1} - x_i| = 1$ replaced by $|x_i - x_{i+1}|_2 \in 1, \sqrt{2}$. A ($*$–) path is called *self-avoiding* if $x_i \neq x_j$ for all $\{i, j\}$ such that $i \neq j$ and $\{i, j\} \neq \{1, n\}$. If $x_1 = x_n$ the ($*$–) path is called *closed.*

$\Lambda \subset \mathbb{Z}^2$ is said to be *connected* ($*$–connected) if for all x, y in Λ there exists a path ($*$–path) from x to y which is entirely contained in Λ. $\Lambda \subset\subset \mathbb{Z}^2$ is called *simply connected* if Λ^c is connected. A set of edges α is *connected* if the union of all its edges is connected in $\mathbb{R}^2$.

If $\alpha \subset \mathcal{E}^*_{\mathbb{Z}^2}$ is a finite, closed set of dual edges, then we define the *interior* of α as the set of all sites $x = (x_1, x_2) \in \mathbb{Z}^2$ such that the half line

$$\{x_1\} \times [x_2, +\infty)$$

intersects α in an odd number of points (see Fig. 1).

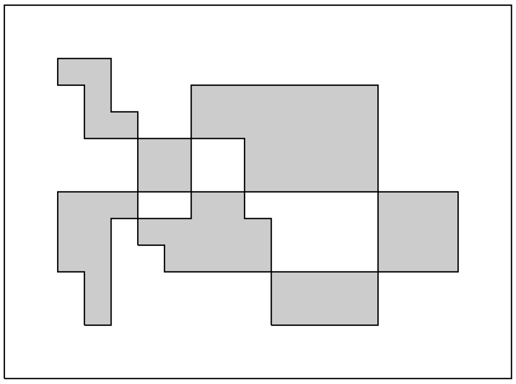

Fig. 1

The interior of α is denoted by $\operatorname{int}\alpha$.

1.2. The Ising model

We consider the standard 2D Ising model with *configuration space* $\Omega = \{-1,+1\}^{\mathbb{Z}^2}$, or $\Omega_V = \{-1,+1\}^V$ for some $V \subset \mathbb{Z}^2$. An element of Ω_V will usually be denoted by $\sigma = \{\sigma(x),\, x \in V\}$. If $U \subset V \subset \mathbb{Z}^2$, and $\sigma \in \Omega_V$ we denote by σ_U the restriction of σ to the set U. If U, V are disjoint, $\sigma_U\eta_V$ is the configuration on $U \cup V$ which is equal to (…guess what) σ on U and η on V.

Given $V \subset\subset \mathbb{Z}^2$ and some *boundary condition* (b.c.) $\eta \in \Omega$, we consider the hamiltonian

$$-H_V^{J,\eta}(\sigma) = \frac{1}{2}\sum_{\substack{x,y\in V\\ |x-y|=1}} J(x,y)\big(\sigma(x)\sigma(y)-1\big) + \sum_{\substack{x\in V,\, y\in V^c\\ |x-y|=1}} J(x,y)\big(\sigma(x)\eta(y)-1\big) \tag{1.1}$$

The coupling J has been introduced for technical reasons, but our main result is for $J = 1$. We always assume $0 \le J(x,y)$ for all x, y.

For further convenience we define the hamiltonian with *free boundary conditions* using a different normalization

$$-H_V^{J,\emptyset}(\sigma) = \frac{1}{2}\sum_{\substack{x,y\in V\\ |x-y|=1}} J(x,y)\,\sigma(x)\sigma(y) \tag{1.2}$$

The partition function is given by

$$Z^{\beta,J,\eta}(V) = \sum_{\sigma\in\Omega_V} \exp\Big[-\beta H_V^{J,\eta}(\sigma)\Big]\,. \tag{1.3}$$

When $J(x,y) = 1$ for all x, y, we drop the superscript J. We sometimes consider also the hamiltonian with free b.c. on a restricted set of bonds: If $X \subset \mathcal{E}_V$ we write

$$-H_{V,X}^{J,\emptyset}(\sigma) = \sum_{[x,y]\in X} J(x,y)\,\sigma(x)\sigma(y) \tag{1.4}$$

Of course this is equivalent to having $J(x,y) = 0$ unless $[x,y] \in X$. The partition function corresponding to (1.4) is denoted by $Z^{\beta,J,\emptyset}(V,X)$.

When V is a rectangle $V = Q_{L,M}$, we define the $[k]$ boundary condition by (let $x = (x_1,x_2)$)

$$[k](x) = \begin{cases} -1 & \text{if } x_2 \ge M-k+1 \\ +1 & \text{if } x_2 \le M-k \end{cases} \tag{1.5}$$

so, in particular, [0] b.c. means -1 on the top side of the rectangle and $+1$ on the remaining three sides. A rectangle V has a δ–boundary δV consisting of a top, bottom, left and right side, that we denote respectively with $\delta_t V$, $\delta_b V$, $\delta_l V$ and $\delta_r V$. So, for instance

$$\delta_t Q_{L,M} = \{ e = [x,y]^* : [x,y] = [(j,M),(j,M+1)] \; j = -L,\dots,L \}$$

In the following we will often choose $J = 1$ everywhere with the exception of one or more sides of a certain rectangle where we take $J = \varepsilon$. So, we introduce the notation

$$\begin{aligned} J_\varepsilon^{\square}(V;x,y) &= \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta\Lambda \\ 1 & \text{otherwise} \end{cases} & J_\varepsilon^{\sqcup}(V;x,y) &= \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta\Lambda \setminus \delta_t\Lambda \\ 1 & \text{otherwise} \end{cases} \\ J_\varepsilon^{\|}(V;x,y) &= \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta_l\Lambda \cup \delta_r\Lambda \\ 1 & \text{otherwise} \end{cases} & J_\varepsilon^{\sqcap}(V;x,y) &= \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta\Lambda \setminus \delta_b\Lambda \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{1.6}$$

The (finite volume) conditional Gibbs measure associated with the Hamiltonian (1.1) is defined as

$$\mu_V^{\beta,J,\eta}(\sigma) = \begin{cases} \left(Z^{\beta,J,\eta}(V)\right)^{-1} e^{-\beta H_V^{J,\eta}(\sigma)} & \text{if } \sigma(x) = \eta(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

The expectation, with respect to the measure (1.7) is denoted by $\mathbf{E}_V^{\beta,J,\tau}(\cdot)$. The set of measures (1.7) satisfies the DLR compatibility conditions

$$\mu_\Lambda^{\beta,J,\tau}(\sigma) = \sum_{\sigma' \in \Omega} \mu_\Lambda^{\beta,J,\tau}(\sigma')\, \mu_V^{\beta,J,\sigma'}(\sigma) \qquad \text{for all } V \subset \Lambda \subset\subset \mathbb{Z}^2 \tag{1.8}$$

One introduces a partial order on Ω_V by saying that $\sigma \le \sigma'$ if $\sigma(x) \le \sigma'(x)$ for all $x \in V$. A function $f : \Omega_V \mapsto \mathbb{R}$ is called *monotone increasing (decreasing)* if $\sigma \le \sigma'$ implies $f(\sigma) \le f(\sigma')$ ($f(\sigma) \ge f(\sigma')$). An event is called *positive* (*negative*) if its characteristic function is increasing (decreasing). Given two probability measures μ, μ' on Ω_V we write $\mu \le \mu'$ if $\mu(f) \le \mu'(f)$ for all increasing funcions f (with $\mu(f)$ we denote the expectation with respect μ).

In the following we will take advantage of the FKG inequalities [FKG] which state that

(1) if $\eta \le \eta'$, then $\mu_V^{\beta,J,\eta} \le \mu_V^{\beta,J,\eta'}$
(2) if f and g are increasing then $\mathbf{E}_V^{\beta,J,\tau}(fg) \ge \mathbf{E}_V^{\beta,J,\tau}(f)\, \mathbf{E}_V^{\beta,J,\tau}(g)$

The surface tension. We denote by $\tau_\beta(\vartheta)$ the surface tension at angle ϑ ($0 \le \vartheta \le \pi/2$) which measures the free energy of an interface in direction orthogonal to the vector $n_\vartheta = (\cos\vartheta, \sin\vartheta)$. We refer the reader to [P] for a precise definition. In the standard Ising model we have, by symmetry,

$$\tau_\beta(\vartheta) = \tau_\beta(\vartheta + \pi/2) = \tau_b(\pi/2 - \vartheta) \tag{1.9}$$

We extend τ_β to a function on $\mathbb{R}^2$ by setting

$$\tau_\beta(x) = |x|_2\, \tau_\beta\left(\frac{x}{|x|_2}\right)$$

We also let for simplicity

$$\tau_\beta = \tau_\beta(\vartheta = 0) = \tau_\beta(\vartheta = \pi/2)$$

For the reader's convenience we recall (see Lemma 6.3 in [P]) that

$$\mathbf{E}_\Lambda^{\beta,\emptyset}\, \sigma(x)\sigma(y) \le \exp(-\beta^* \tau_{\beta^*}(x-y)) \tag{1.10}$$

where β^* is the dual value for β (see (1.19)). We are also going to make use of the so called sharp triangle inequality (Lemma 2.1 of [I1]) which, in particular, implies that there exists $D(\beta) > 0$ such that

$$\tau_\beta(\vartheta) \geq \tau_\beta(0)\cos\vartheta + D(\beta)\sin^2(\vartheta/2) \tag{1.11}$$

As a consequence of (1.11) and (1.9) we have

$$\tau_b(\vartheta) \geq \frac{\tau_\beta}{\sqrt{2}} \tag{1.12}$$

For further relevant properties of the surface tension the reader is again referred to [P].

1.3. Contours

We use the contour representation with the so called *splitting rules* (see for instance [DKS]). Given (σ, η, Λ) we define $\mathcal{B}^\eta_\Lambda(\sigma)$ as the set of all unsatisfied edges in $\mathcal{E}^*_\Lambda$ of the configuration σ subject to b.c. η, *i.e.* , if $\rho = \sigma_\Lambda \eta_{\Lambda^c}$, then

$$\mathcal{B}^\eta_\Lambda(\sigma) = \{\, e^* = [x,y]^* \in \mathcal{E}^*_\Lambda : \rho(x) \neq \rho(y) \,\}$$

If the b.c. η is $+$ (or $-$), then $\mathcal{B}^\eta_\Lambda(\sigma)$ is closed, while in general it has a nonempty boundary. We have

Proposition 1.1. *If $\Lambda \subset\subset \mathbb{Z}^2$, $\eta \in \Omega$, then*

(i) The boundary of $\mathcal{B}^\eta_\Lambda(\sigma)$ does not depend on σ, so, in particular $\delta\mathcal{B}^\eta_\Lambda(\sigma) = \delta\mathcal{B}^\eta_\Lambda(+1)$
(ii) $\delta\mathcal{B}^\eta_\Lambda(\sigma) \subset \Lambda^*$
(iii) If Λ is simply connected $\mathcal{B}^\eta_\Lambda$ is a one–to–one mapping from Ω_Λ onto the set

$$\{\, X \subset \mathcal{E}^*_\Lambda : \delta X = \delta\mathcal{B}^\eta_\Lambda(+1) \,\}$$

(i) and (ii) are straightforward. For (iii) see for instance Lemma 6.1 in [P] which deals with the case $\eta = +1$ (the generalization is easy).

It is useful to decompose $\mathcal{B}^\eta_\Lambda(\sigma)$ and, in general, an arbitrary set X of dual edges as a collection of *contours* γ_i

$$X = \gamma_1 \cup \cdots \cup \gamma_n \tag{1.13}$$

which have the advantage that they can be associated with simple self–avoiding (open or closed) curves in $\mathbb{R}^2$. Decomposition (1.13) is intuitively obtained by cutting all three and four edges meetings, along the south–west to north–east direction. More precisely we first consider the subset of $\mathbb{R}^2$

$$\hat{X} = \bigcup_{e \in X} e$$

then, at each dual site where three or four edges of X meet, we operate the *rounding the corner* operation which is shown in Fig. 2.

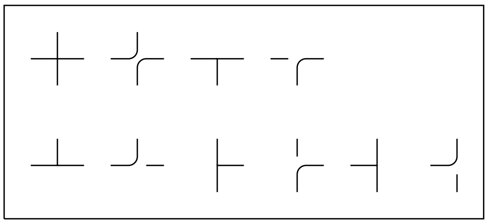

Fig. 2

In this way we obtain another subset $\tilde{X}$ of $\mathbb{R}^2$ whose Hausdorff distance from $\hat{X}$ is less than, say, $1/3$. $\tilde{X}$ can now be decomposed as a union of its connected components $\tilde{X} = Y_1 \cup \cdots \cup Y_n$. Finally we define γ_i as the set of all edges which "belong" to Y_i, *i.e.*

$$\gamma_i = \{e \in X : \text{the Hausdorff distance of } e \text{ from } Y_i \text{ is less than } 1/3\}$$

Fig. 3 shows a set of edges on the left and the corresponding collection of contours on the right.

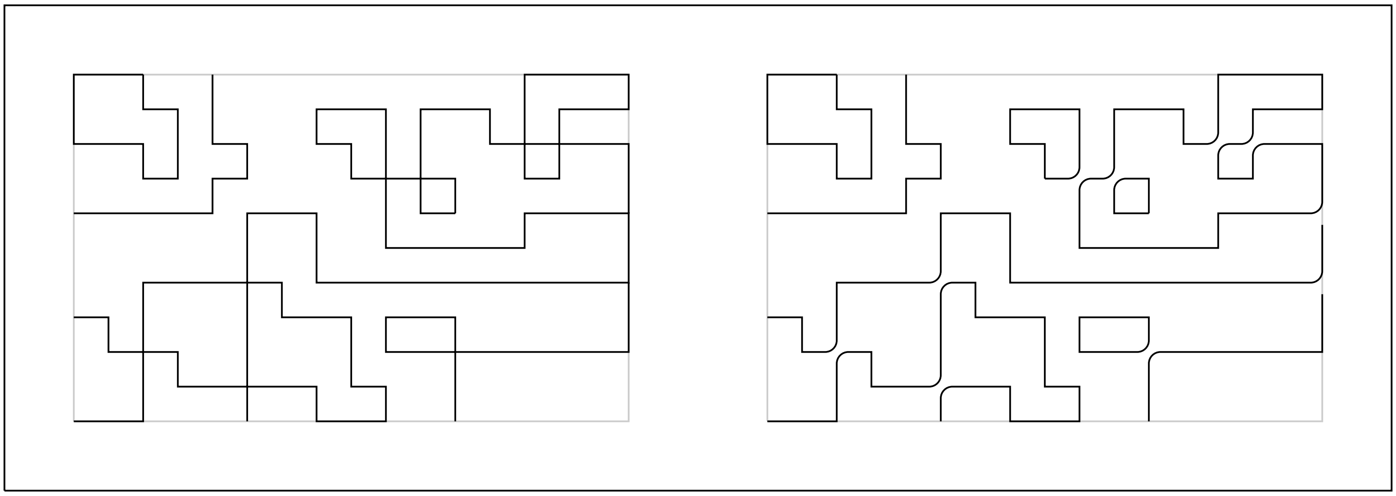

Fig. 3

We now give an independent definition of contours and we introduce the notion of *compatibility* between contours in such a way to have a one–to–one correspondence between set of edges and compatible collections of contours.

Let's denote by $\hat{e}_n$, $\hat{e}_s$, $\hat{e}_e$, $\hat{e}_w$ the unit vectors in $\mathbb{R}^2$ pointing in direction respectively north, south, east and west. Let also $\hat{e}_n(x)$ be the edge whose endpoints are x and $x + \hat{e}_n$ (and similarly for the other directions). To each edge e we associate a set of three edges $\Delta(e)$ given by

$$\Delta(\hat{e}_e(x)) = \hat{e}_e(x) \cup \hat{e}_s(x) \cup \hat{e}_n(x + \hat{e}_e)$$
$$\Delta(\hat{e}_n(x)) = \hat{e}_n(x) \cup \hat{e}_w(x) \cup \hat{e}_e(x + \hat{e}_n)$$

The elements in $\Delta(e)$ are said to be *forbidden* by e (for reasons which will become clear when the "compatibility" will be introduced). We extend the notion of forbidden edges to contours by setting

$$\Delta(\gamma) = \bigcup_{e \in \gamma} \Delta(e)$$

The definition of $\Delta(e)$ is such that, for any two edges e, e', we have

$$\Delta(e) \ni e' \qquad \text{if and only if} \qquad e \in \Delta(e')$$

Given a set E of edges (or dual edges), a *contour* in E is a finite sequence of elements of E, $\gamma = (e_1, \ldots, e_n)$ such that, it is possible to write $e_i = [x_i, y_i]$ and

(1) all edges e_i are distinct

(2) $y_i = x_{i+1}$
(3) for any two non–consecutive edges e, e' we have $e \notin \Delta(e')$

If $x_1 = y_n$ the contour is called *closed*, otherwise it is *open*. We do not distinguish between two contours with the same set of edges. The representation $(e_1, \ldots, e_n)$ of γ is thus always to be interpreted modulo a reversal of the ordering of the edges and, for closed contours, modulo a cyclic permutation. Two edges e and e' are said to be *consecutive* in a contour γ if they are consecutive in at least one representation.

We will sometime use the same symbol γ to denote the set $\{e_1, \ldots, e_n\}$ and we will improperly write things like $e \in \gamma$ and $\gamma \subset X$.

For further convenience we introduce a modified version of $\Delta(e)$. If $x \in \delta e$ we let

$$\Delta'(e, x) = \{e' \in \Delta(e) : x \notin \delta e'\} \cup e$$

and, for all open contours $\gamma = (e_1, \ldots, e_n)$ with $e_i = [x_{i-1}, x_i]$

$$\begin{aligned} &\Delta'(\gamma, x_0) = \bigcup_{i=2}^{n} \Delta(e_i) \cup \Delta(e_1, x_0) \qquad \Delta'(\gamma, x_n) = \bigcup_{i=1}^{n-1} \Delta(e_i) \cup \Delta(e_n, x_n) \\ &\Delta''(\gamma) = \bigcup_{i=2}^{n-1} \Delta(e_i) \cup \Delta(e_1, x_0) \cup \Delta(e_n, x_n) \end{aligned} \tag{1.14}$$

Sometimes it is more convenient to consider the forbidden sites, so we let

$$\Delta^s(\gamma) = \{x \in \mathbb{Z}^2 : d_2(x, \Delta(\gamma)) = 1/2\} \tag{1.15}$$

The *boundary* $\delta\gamma$ of a contour is given by the usual boundary of γ when γ is thought of as a set of edges. Thus $\delta\gamma$ can either be empty or consist of pair of sites. Given a collection of contours $\underline{\gamma} = \{\gamma\}$, its *boundary* $\delta\underline{\gamma}$ is defined as

$$\delta\underline{\gamma} = \bigcup_{\gamma \in \underline{\gamma}} \delta\gamma$$

(notice that the boundary of a collection of contours is not equal to the boundary of the set of all edges in some $\gamma \in \underline{\gamma}$). Given a set of edges X, we let

$$\begin{aligned} &C(X) = \{\gamma : \ \gamma \text{ is a closed contour in } X\ \} \\ &C(X, \{x, y\}) = \{\gamma : \ \gamma \text{ is a contour in } X \text{ with boundary } \delta\gamma = \{x, y\}\ \} \end{aligned}$$

Two contours γ_1 and γ_2 are said to be *compatible* if their boundaries are disjoint and

$$\Delta(\gamma_1) \cap \gamma_2 = \emptyset \quad \text{or, equivalently} \quad \gamma_1 \cap \Delta(\gamma_2) = \emptyset$$

A collection $\underline{\gamma}$ of contours in X is called *compatible* if each pair in $\underline{\gamma}$ is compatible. We define

$$C^*(X, U) = \text{the set of all compatible collections } \underline{\gamma} \text{ of contours in } X \text{ such that } \delta\underline{\gamma} = U$$

For simplicity $C^*(X) = C^*(X, \emptyset)$ denotes compatible collections of closed contours.

The *interior* of a compatible collection of closed contours $\underline{\gamma}$ is defined as

$$\operatorname{int} \underline{\gamma} = \operatorname{int} \Big[\bigcup_{\gamma \in \underline{\gamma}} \gamma \Big] \tag{1.16}$$

(the interior of a closed set of edges was defined in Section 1.1).

Is is straightforward to check that there is a one–to–one correspondence between $C^*(X,U)$ and the set of all subsets $Y \subset X$ such that $\delta Y = U$. The collection of contours corresponding to the set of edges $\mathcal{B}^\eta_\Lambda(\sigma)$ will be denoted by $\mathcal{G}^\eta_\Lambda(\sigma)$.

We introduce now the *contour partition function* which is a basic object in what follows. Let then $X \subset\subset \mathcal{E}^*_{\mathbb{Z}^2}$, $U \subset\subset \mathbb{Z}^2_*$ and let $\underline{\lambda}$ be a compatible collection of contours in X whose boundary is contained in U, *i.e.* $\underline{\lambda} \in C^*(X,V)$ for some $V \subset U$. We define

$$\widetilde{Z}^{\beta,J}(X,U;\underline{\lambda}) = \sum_{\underline{\gamma}\,:\,\underline{\gamma}\cup\underline{\lambda}\in C^*(X,U)} w_{\beta,J}(\underline{\gamma})$$

where the *weight* of a collection of contours is given by

$$w_{\beta,J}(\underline{\gamma}) = \prod_{\gamma\in\underline{\gamma}} w_{\beta,J}(\gamma) = \prod_{\gamma\in\underline{\gamma}}\prod_{e\in\gamma} \exp(-2\beta J(e))$$

If the arguments U or $\underline{\lambda}$ are missing in $\widetilde{Z}(X,U;\underline{\lambda})$ they must be interpreted as the empty set, so in particular $\widetilde{Z}(X)$ is the sum of the weight of all compatible collections of closed contours in X. The notion of compatibility between contours is such that we have

$$\widetilde{Z}^{\beta,J}(X,U;\underline{\lambda}) = \widetilde{Z}^{\beta,J}(X\setminus\Delta(\underline{\lambda}), U\setminus\delta\underline{\lambda})$$

If $\Lambda \subset\subset \mathbb{Z}^2$ is simply connected, we can express the usual partition function as a contour partition function, by

$$Z^{\beta,J,\eta}(\Lambda) = \widetilde{Z}^{\beta,J}(\mathcal{E}^*_\Lambda, \delta\mathcal{B}^\eta_\Lambda(+1))$$

In the special case that Λ is a rectangle $\Lambda = Q_{L,M}$ with $[k]$ b.c. (see (1.5)), then an open contour will appear going from a_k to b_k where a_k and b_k are the two extreme sites in the $(k+1)^{th}$ row (from the top) of Λ^*, *i.e.*

$$a_k = (-L-\tfrac{1}{2}, M+\tfrac{1}{2}-k) \qquad b_k = (+L+\tfrac{1}{2}, M+\tfrac{1}{2}-k) \tag{1.17}$$

and the partition function is given by

$$Z^{\beta,J,[k]}(\Lambda) = \widetilde{Z}^{\beta,J}(\mathcal{E}^*_\Lambda,\{a_k,b_k\}) = \sum_{\gamma\in C(\mathcal{E}^*_\Lambda,\{a_k,b_k\})} w_{\beta,J}(\gamma)\,\widetilde{Z}^{\beta,J}(\mathcal{E}^*_\Lambda;\gamma)$$

Thanks to the duality properties of the 2D Ising model (see [P]) one can also write the partition function with free b.c. as

$$Z^{\beta,J,\emptyset}(\Lambda,X) = 2^{|\Lambda|}\prod_{e\in X}\cosh(\beta J(e))\,\widetilde{Z}^{\beta^*,J^*}(X) \qquad \text{for any } X\subset\mathcal{E}_\Lambda \tag{1.18}$$

where β^* and J^* are determined by the duality relationships

$$e^{-2\beta} = \tanh\beta^* \qquad e^{-2\beta J(e)} = \tanh(\beta^* J(e)^*) \tag{1.19}$$

(1.18) implies

$$Z^{\beta,J,+}(\Lambda) = 2^{-|\Lambda^*|}\prod_{e\in\mathcal{E}^*_\Lambda}(\cosh(\beta^*J(e)^*))^{-1}\,Z^{\beta^*,J^*,\emptyset}(\Lambda^*,\mathcal{E}^*_\Lambda) \tag{1.20}$$

The critical value β_c is determined by the equality $\beta_c = \beta_c^*$. Expectations of products of spin variables can be expressed as quotients of contour partition functions [P]

$$\mathbf{E}^{\beta,J,\emptyset}_{\Lambda,X}\Big(\prod_{x\in U}\sigma(x)\Big) = \frac{\widetilde{Z}^{\beta^*,J^*}(X,U)}{\widetilde{Z}^{\beta^*,J^*}(X)} \tag{1.21}$$

where $\mathbf{E}^{\beta,J,\emptyset}_{\Lambda,X}$ is the expectation with respect to the Gibbs measure associated to the hamiltonian (1.4).

We anticipate a result which will be useful in the following

Proposition 1.2. *Let X, Y be two finite set of dual bonds such that $X \subset Y$ and let*

$$F(J) = \frac{\widetilde{Z}^{\beta,J}(X)}{\widetilde{Z}^{\beta,J}(Y)}$$

Then $F(J)$ is nondecreasing in each variable $J(x,y)$ such that $[x,y]^* \notin Y \setminus X$.

Proof. Let Λ be any finite set of sites such that $Y \subset \mathcal{E}^*_\Lambda$. Then, by (1.18)

$$F(J) = \frac{Z^{\beta^*,J^*,\emptyset}(\Lambda, X)}{Z^{\beta^*,J^*,\emptyset}(\Lambda, Y)} \prod_{e=[x,y]^* \in Y \setminus X} \cosh(\beta^* J^*(x,y))$$

Thus, if $e \notin Y \setminus X$, we have

$$\frac{1}{\beta^*} \frac{d}{dJ^*(x,y)} \log F(J) = \mathbf{E}^{\beta^*,J^*,\emptyset}_{\Lambda,X}\, \sigma(x)\sigma(y) - \mathbf{E}^{\beta^*,J^*,\emptyset}_{\Lambda,Y}\, \sigma(x)\sigma(y)$$

which is non positive by the second Griffiths' inequality. $\square$

Contours with free boundary conditions. A configuration σ in Λ with free boundary conditions produces a collection of contours $\underline{\gamma} = \mathcal{G}^{\emptyset}_\Lambda(\sigma)$ such that

(*i*) $\underline{\gamma} \subset \mathcal{E}^*_\Lambda \setminus \delta\Lambda$
(*ii*) each contour $\gamma \in \underline{\gamma}$ is either closed or open with its boundary $\delta\gamma \subset \mathcal{V}(\delta\Lambda)$

So we let, $\hat{\mathcal{E}}_\Lambda = \mathcal{E}^*_\Lambda \setminus \delta\Lambda$ and, for an arbitrary set of dual edges X and any $U \subset \mathbb{Z}^2$ we define

$$C(X, \subset U) = \bigcup_{V \subset U} C(X, V) \qquad C^*(X, \subset U) = \bigcup_{V \subset U} C^*(X, V)$$

There is a two to one mapping $\underline{\gamma} \mapsto \underline{\gamma}'$ from $C^*(\mathcal{E}^*_\Lambda)$ to $C^*(\hat{\mathcal{E}}_\Lambda, \subset \mathcal{V}(\delta\Lambda))$ given by

$$\underline{\gamma}' = \Big(\bigcup_{\gamma \in \underline{\gamma}} \gamma\Big) \setminus \delta\Lambda \tag{1.22}$$

where it is understood that the LHS of (1.22) must be partitioned in the proper way according to the splitting rules. In this way we can write, for each simply connected Λ,

$$Z^{\beta,\emptyset}(\Lambda) = 2 \sum_{\underline{\gamma} \in C^*(\hat{\mathcal{E}}_\Lambda, \subset \mathcal{V}(\delta\Lambda))} w(\underline{\gamma})$$

The following proposition will be used later in the paper.

Proposition 1.3. *Let Λ be a rectangle, $\sigma \in \Omega_\Lambda$ and let γ be a contour in $\mathcal{G}^{\emptyset}_\Lambda(\sigma)$. Then*

(*i*) *if γ is open there exist $\lambda' \in C(\mathcal{E}^*_\Lambda \setminus \Delta''(\gamma), \delta\gamma)$ and $\lambda \in C(\mathcal{E}^*_\Lambda)$ such that $\lambda = \gamma \cup \lambda'$ and $\lambda \in \mathcal{G}^{+}_\Lambda(\sigma)$*
(*ii*) *if γ is closed then $\gamma \in \mathcal{G}^{+}_\Lambda(\sigma)$*

The proof is more or less straightforward and we omit it. We just observe that the result is false for arbitrary shapes of Λ.

1.4. The dynamics and our result

The stochastic dynamics we want to study is defined by the Markov generator

$$(L_V^\eta f)(\sigma) = \sum_{x\in V} c(x,\sigma)\,[f(\sigma^x) - f(\sigma)] \tag{1.23}$$

(in this subsection we consider β and J fixed and we mostly omit them) acting on $L^2(\Omega, d\mu_V^\eta)$, where

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x \end{cases}$$

In (1.23) σ denotes a configuration on the whole lattice $\mathbb{Z}^2$ which, in view of (1.7), agrees with the b.c. η on V^c. In general we identify $L^2(\Omega, d\mu_V^\eta)$ with $L^2(\Omega_V, d\mu_V^\eta)$. The nonnegative real quantities

$$c(x,\sigma) \quad x\in\mathbb{Z}^2, \quad \sigma\in\Omega\,,$$

are the *transition rates* for the process.

The assumptions on the transition rates are

(H_1) *Nearest neighbor interactions.* If $\sigma(y) = \sigma'(y)$ for all y such that $d(x,y)\le 1$, then $c(x,\sigma) = c(x,\sigma')$

(H_2) *Attractivity.* If $\sigma\le\sigma'$ and $\sigma(x) = \sigma'(x)$, then

$$\sigma(x)\,c(x,\sigma) \le \sigma'(x)\,c(x,\sigma') \tag{1.24}$$

(H_3) *Detailed balance.*

$$\exp\left[-\beta H^\sigma_{\{x\}}(\sigma(x))\right] c(x,\sigma) = \exp\left[-\beta H^\sigma_{\{x\}}(-\sigma(x))\right] c(x,\sigma^x) \tag{1.25}$$

(H_4) *Positivity and boundedness.* There exist $c_m(\beta)$ and $c_M(\beta)$ such that

$$0 < c_m(\beta) \le \inf_{a,\sigma} c(x,\sigma) \le \sup_{x,\sigma} c(x,\sigma) \le c_M(\beta) < \infty \tag{1.26}$$

Two cases one may want to keep in mind are

$$c(x,\sigma) = \min\left\{e^{-\beta\Delta_x H(\sigma)}\,,\,1\right\}$$

and

$$c(x,\sigma) = \left[1+\exp\left(\beta\Delta_x H(\sigma)\right)\right]^{-1}$$

where

$$\Delta_x H(\sigma) = H^\sigma_{\{x\}}(-\sigma(x)) - H^\sigma_{\{x\}}(\sigma(x))\,.$$

$(H_1)-(H_4)$ garantee that there exists a unique Markov process with semigroup $T_V^\eta(t)$ and generator L_V^η. L_V^η is a bounded operator on $L^2(\Omega, d\mu_V^\eta)$. The process has a unique invariant measure given by μ_V^η. Moreover μ_V^η is *reversible* with respect to the process, *i.e.* L_V^η is self–adjoint on $L^2(\Omega, d\mu_V^\eta)$. Given $\sigma\in\Omega$ we denote by σ_t the random configuration at time t evolving according to the process, so that

$$\mathbb{E}_V^\sigma\, f(\sigma_t) = \int f(\sigma_t)\,d\,\mathbb{P}_V^\sigma = (T_V^\eta(t))\,f(\sigma)\,, \quad \forall\sigma\in\Omega \text{ such that } \sigma_{V^c} = \eta_{V^c}\,.$$

$\mathbb{E}^\sigma$ and $\mathbb{P}^\sigma$ stand respectively for the expectation and the probability measure associated with the process starting from σ_V at time zero and subject to b.c. σ_{V^c}.

Attractivity assumption implies (see for instance [L])

(1) If f is an increasing function on Ω_V then $T_V^\eta(t)f$ is also increasing for all $t \geq 0$
(2) If ρ_1, ρ_2 are two probability measures on Ω_V such that $\rho_1 \leq \rho_2$ then
$\rho_1 T_V^\eta(t) \leq \rho_2 T_V^\eta(t)$ for all $t \geq 0$
(3) For any σ, σ' in Ω such that $\sigma \leq \sigma'$, the *standard coupling* [L] $\mathbb{P}_V^{\sigma,\sigma'}$ of σ_t, σ'_t is such that
$\mathbb{P}_V^{\sigma,\sigma'}\{\sigma_t \leq \sigma'_t\} = 1$, for all $t \geq 0$.

This last property allows us to define a *standard* coupling of two Gibbs measures which preserve the order of the b.c. Take in fact $\hat\nu_V^{\eta,\eta'}$ as the unique invariant measure of the (standard) coupled process (σ_t, σ'_t). Then we have

(1) $\hat\nu_V^{\eta,\eta'}\{(\sigma,\sigma') : \sigma = \sigma_0\} = \mu_V^\eta(\sigma_0)$ for all $\sigma_0 \in \Omega_V$
(2) $\hat\nu_V^{\eta,\eta'}\{(\sigma,\sigma') : \sigma' = \sigma_0\} = \mu_V^{\eta'}(\sigma_0)$ for all $\sigma_0 \in \Omega_V$
(3) If $\eta \leq \eta'$, then $\hat\nu_V^{\eta,\eta'}\{(\sigma,\sigma') : \sigma \leq \sigma'\} = 1$

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\operatorname{gap}(V,\eta) = \operatorname{gap}(L_V^\eta) = \inf \operatorname{spec}\left(-L_V^\eta \restriction \mathbb{1}^\perp\right)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega, d\mu_V^\eta)$ orthogonal to the constant functions. The gap can be also characterized as

$$\operatorname{gap}(V,\eta) = \inf_{f \in L^2(\Omega, d\mu_V^\eta)} \frac{\mathcal{E}_V^\eta(f,f)}{\operatorname{Var}_V^\eta(f)} \tag{1.27}$$

where $\mathcal{E}$ is the Dirichlet form associated with the generator L,

$$\mathcal{E}_V^\eta(f,f) = \frac{1}{2} \sum_{\sigma \in \Omega} \sum_{x \in V} \mu_V^\eta(\sigma)\, c(x,\sigma)\, [f(\sigma^x) - f(\sigma)]^2 \tag{1.28}$$

and Var_V^h is the variance relative to the probability measure μ_V^η.

The main result in this paper is then

Theorem 1.4. *Assume $(H_1) - (H_4)$. If $\beta > \beta_c$ then*

$$\lim_{L\to\infty} \left[-\frac{1}{\beta(2L+1)} \log \operatorname{gap}(Q_L, \beta, J = 1, \emptyset)\right] = \tau_\beta$$

2. Strategy of the proof

2.1. Lower bound

We proceed as in Section 4 of [M], *i.e.* we consider the system on a square Q_L and replace the free boundary conditions with very weak $+\varepsilon$ b.c. ($\varepsilon \ll 1$). More precisely we take $J_\varepsilon \equiv J_\varepsilon^{\square}(Q_L)$ (see (1.6)).

Then, using the variational characterization of the gap in term of the Dirichlet form (1.27), one gets

$$\mathrm{gap}(Q_L, \beta, 1, \emptyset) \geq \frac{c_m}{c_M}\, e^{-16\beta\varepsilon(2L+1)}\, \mathrm{gap}(Q_L, \beta, J_\varepsilon, +)$$

where c_m and c_M where defined in (1.26). In order to prove the lower bound on the gap, it is therefore sufficient to show that

$$\lim_{\varepsilon\to 0} \limsup_{L\to\infty} -\frac{1}{\beta(2L+1)} \log\left[\, \mathrm{gap}(\beta, Q_L, J_\varepsilon, +)\right] \leq \tau_\beta \tag{2.1}$$

To establish (2.1) we take $M = \lfloor \varepsilon L \rfloor$ and we introduce a covering $\mathcal{R} = \{R_i\}_{i=1}^K$ of Q_L by means of rectangles

$$R_i = Q_{L,M} + (0, h_i)$$

where the h_i's are chosen in such a way that the height of the overlap between any two consecutive rectangles $R_i \cap R_{i+1}$ is at least $M/2$ and not greater than M (remember that the height of each R_i is $2M+1$). So we assume that

$$\begin{aligned} &h_1 = -L + M \qquad h_K = L - M \\ &\frac{M}{2} \leq h_i - h_{i+1} + (2M+1) = \text{ height of the overlap} \leq M \end{aligned} \tag{2.2}$$

Let $L_{\mathcal{R}}^{\beta, J_\varepsilon, +}$ be the new generator

$$(L_{\mathcal{R}}^{\beta, J_\varepsilon, +} f)(\sigma) = \sum_{i=1}^{K} \sum_{\eta \in \Omega_{R_i}} \mu_{R_i}^{\beta, J_\varepsilon, \sigma}(\eta)\left[\, f(\sigma^\eta) - f(\sigma)\right] \tag{2.3}$$

Here, for any $\eta \in \Omega_{R_i}$, σ^η denotes the configuration in Ω_{Q_L} which is equal to η in R_i and to σ in $Q_L \setminus R_i$. The generator $L_{\mathcal{R}}^{\beta, J_\varepsilon, +}$, in the sequel refered to as the block-dynamics generator, is symmetric in $L^2(\Omega_{Q_L}, d\mu_{Q_L}^{\beta, J_\varepsilon, +})$ and we will denote by $\mathrm{gap}(\mathcal{R}, \beta, J_\varepsilon, +)$ the absolute value of its first non zero eigenvalue.

Theorem 2.1 in [M] relates the gap in the spectrum of the generator of the single site Glauber dynamics to the gap of generator $L_{\mathcal{R}}^{\beta, J_\varepsilon, +}$ and states that

$$\mathrm{gap}(Q_L, \beta, J_\varepsilon, +) \geq \frac{1}{2\, M\, (2L+1)}\, c_m\, e^{-4\beta(2M+1)}\, \mathrm{gap}(\mathcal{R}, \beta, J_\varepsilon, +) \tag{2.4}$$

where $c_m = \inf_{\sigma, x} c(x, \sigma)$.

If we now combine (2.4) with the definition of M, we conclude that (2.1) follows once we can prove that

$$\lim_{\varepsilon\to 0} \limsup_{L\to\infty} -\frac{1}{\beta(2L+1)} \log\left[\, \mathrm{gap}(\mathcal{R}, \beta, J_\varepsilon, +)\right] \leq \tau_\beta \tag{2.5}$$

Remark. It is important to remark at this point that our choice of the parameters M and ε is different from that of [M]. There in fact the only varying parameter was the size L of the square Q_L and the numbers ε, M were directly related to L through the choice

$$\varepsilon = L^{-\frac{1}{2}} \quad ; \quad M = L^{\frac{1}{2}+\delta}$$

where $\delta \ll 1$ was a fixed number. Here, unfortunately, we cannot do that, and we are able to take the limit $\varepsilon \to 0$ only *after* the thermodynamic limit $L \to \infty$. The reason for this difference is that, for β close to the critical value β_c, we are still unable to control the moderate fluctuations of an interface of length L between the plus and minus phase, i.e. fluctuations that occur on a scale L^α with $\frac{1}{2} < \alpha < 1$ (see e.g. the proof of Proposition 3.1). At very low temperature the powerful methods of the cluster expansion are available and they give a detailed control of all fluctuations, from the normal ones occurring on scale $\sqrt{L}$ to the large ones on scale L (see [DKS]).

The advantage of studying the block–dynamics, instead of the single site dynamics, is that the relaxation time can be estimated from above via natural probabilistic methods in a rather precise way. Such a probabilistic approach was developed in Section 4 of [M] and its implementation requires only two key equilibrium estimates described in the propositions below. Once these estimates are at hand then (2.5) follows.

Proposition 2.1. *Let $\beta > \beta_c$ and let $\varepsilon, \alpha \in (0,1]$. Given a positive integer L we set $M = \lfloor \varepsilon L \rfloor$, $k = \lfloor \varepsilon L/10 \rfloor$. Let $\Lambda = Q_{L,M} + (0,h)$ be a vertical translate of $Q_{L,M}$ contained in Q_L and let $\bar\Lambda$ be a vertical translate of $Q_{L,N}$ (with $M \le N < L$) such that the bottom sides of Λ and $\bar\Lambda$ coincide. We also let*

$$\Lambda_{\rm bot} = \{x = (x_1, x_2) \in \Lambda : x_2 \le M - 3k + h\}$$

Take $J_\alpha = J^\square_\alpha(Q_L)$. Then there exists $L_0 = L_0(\beta, \alpha, \varepsilon)$ and $m = m(\beta, \alpha, \varepsilon)$ such that if $L \ge L_0$

(*i*) *If the horizontal sides of Λ do not touch the horizontal sides of Q_L (and so the boundary coupling is identically equal to one on these sides) then*

$$\mu_\Lambda^{\beta, J_\alpha, +}\{\sigma(x) = +1\} - \mu_\Lambda^{\beta, J_\alpha, [0]}\{\sigma(x) = +1\} \le e^{-mL} \qquad \forall x \in \Lambda_{\rm bot} \tag{2.6}$$

(*ii*)

$$\mu_\Lambda^{\beta, J_\alpha, +}\{\sigma(x) = +1\} - \mu_{\bar\Lambda}^{\beta, J_\alpha, +}\{\sigma(x) = +1\} \le e^{-mL} \qquad \forall x \in \Lambda_{\rm bot} \tag{2.7}$$

Proposition 2.2. *Under the same assumptions as in Proposition 2.1, let $\Lambda_{\rm top} = \Lambda \setminus \Lambda_{\rm bot}$ and let $\Gamma_\Lambda^{[0]}(\sigma)$ be the unique open contour produced by the configuration σ with boundary condition $[0]$. Consider the event*

$$F(L, \varepsilon) = \{\sigma \in \Omega_\Lambda : \Gamma_\Lambda^{[0]}(\sigma) \subset \mathcal{E}^*_{\Lambda_{\rm top}}\}\,.$$

Then there exists $L_0 = L_0(\beta, \varepsilon)$ such that if $L \ge L_0$ then

$$\mu_\Lambda^{\beta, J_\varepsilon, [0]}(F(L, \varepsilon)) \ge \exp(-\beta(\tau_\beta + 14\varepsilon)(2L + 1)) \tag{2.8}$$

The proof of the propositions is postponed to Section 4.

2.2. Upper bound

Following [M], let $f(\sigma)$ be the trial function

$$f_\Lambda(\sigma) = \chi\{m_\Lambda(\sigma) > 0\} - \chi\{m_\Lambda(\sigma) < 0\} \tag{2.9}$$

where

$$m_\Lambda(\sigma) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \sigma(x)$$

If we plug f_Λ in the variational characterization of the gap (1.27) we get

$$\mathrm{gap}(Q_L,\beta,\emptyset) \le 4\,c_M\,(2L+1)^2\,\frac{\mu^{\beta,\emptyset}_{Q_L}\{\,|m_{Q_L}(\sigma)| \le \frac{2}{|Q_L|}\,\}}{1-\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)=0\,\}} \tag{2.10}$$

The upper bound then follows once we show that, for any $\beta > \beta_c$

$$\liminf_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\Bigl[\mu^{\beta,\emptyset}_{Q_L}\{\,|m_{Q_L}(\sigma)| \le \frac{2}{|Q_L|}\,\}\Bigr] \ge \tau_\beta \tag{2.11}$$

In the framework of the rigorous Wulff construction, at very low temperature, Shlosman (see [Sh]) computed the logarithmic asymptotics of the probability of rare events like $\{m_{Q_L}(\sigma)=m\}$ when $m\in(-m^*,m^*)$ (m^* denotes the spontaneous magnetization) and, in particular, he proved (2.11). We are going to show that (2.11) holds at any $\beta \ge \beta_c$, *i.e.* that

Proposition 2.3. *Let $\beta > \beta_c$ and $m \in (-m^*,m^*)$. Then*

$$\liminf_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\Bigl[\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)=m\,\}\Bigr] \ge \varphi(m)$$

where

$$\varphi(m) = \frac{1}{2}w\sqrt{\frac{m^*(\beta)-(|m|\vee m_1)}{2m^*(\beta)}}$$

where the constant w is the value of the Wulff functional W_τ on the Wulff curve (see e.g. [Sh] or [DKS]) and the singularity point m_1 satisfies the equation

$$\frac{1}{2}w\sqrt{\frac{m^*(\beta)-m_1}{2m^*(\beta)}} = \tau_\beta$$

The proof of the proposition is postponed to Section 5.

Remark. One can actually strengthen the above proposition by proving that

$$\lim_{\varepsilon\to 0}\lim_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\Bigl[\mu^{\beta,\emptyset}_{Q_L}\{\,|m_{Q_L}(\sigma)-m| \le \varepsilon\,\}\Bigr] = \varphi(m)$$

We decided however to omit the proof of the necessary lower bound since it follows without any significant change the proof given in Section 5 of [I1] (see also [SS2]). It is interesting to remark however that, at least for all values of m in the interval $[0,m_1]$, the stronger version of the proposition follows at once from the result of the first part of the section. In fact, if in the variational characterization of the gap we replace the trial function $f(\sigma)$ defined in (2.9) with the new function

$$g_{\Lambda,m}(\sigma) = \chi\{m_\Lambda(\sigma)>m\} - \chi\{m_\Lambda(\sigma)<m\} \qquad m\in[0,m^*)$$

we obtain

$$\frac{\mathrm{Var}^{\beta,\emptyset}_{Q_L}(g_{Q_L,m})}{4\,c_M\,(2L+1)^2}\,\mathrm{gap}(Q_L,\beta,\emptyset) \le \mu^{\beta,\emptyset}_{Q_L}\{\,|m_{Q_L}(\sigma)-m| \le \frac{2}{|Q_L|}\,\}$$

Since

$$\limsup_{L\to\infty}\mu^{\beta,\emptyset}_{Q_L}\{\,m_{Q_L}(\sigma)\in[-m^*+\delta,\ m^*-\delta]\,\} = 0$$

for any $\delta>0$ and any $\beta>\beta_c$ (see [CCS]), one gets that $\lim_{L\to\infty}\mathrm{Var}(g_m)=1$. Thus, using the correct lower bound on the gap proven in the first part of the section, we conclude that

$$\limsup_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\Bigl[\mu^{\beta,\emptyset}_{Q_L}\{\,|m_{Q_L}(\sigma)-m| \le \frac{2}{|Q_L|}\,\}\Bigr] \le \tau_\beta$$

which is the same as the lower bound proven in the proposition above if $m\in[0,m_1]$.

3. First basic results

We consider the system on a rectangle $Q_{L,M}$ with $[k]$ boundary conditions (see (1.5)). We let

$$\Gamma^{[k]}_{\Lambda}(\sigma) = \text{the unique open contour produced by a configuration } \sigma \tag{3.1}$$

The aim of this section is to show that

Proposition 3.1. *Let $\beta > \beta_c$ and let $\varepsilon, \alpha \in (0,1]$. Given a positive integer L we set $M = \lfloor \varepsilon L \rfloor$, $k = \lfloor \varepsilon L/10 \rfloor$, $\Lambda(L) = Q_{L,M}$ and $J = J^{\|}_{\alpha}(\Lambda)$ (see (1.6)). We also let*

$$V(L) = \{x = (x_1, x_2) \in \Lambda(L) : x_2 \geq M - 2k + 1\}$$

and consider the event

$$F(L, \varepsilon) = \{\sigma \in \Omega_\Lambda : \Gamma^{[0]}_{\Lambda}(\sigma) \subset \mathcal{E}^*_{V(L)}\}\,.$$

Then

$$\liminf_{L\to\infty} \Big[-\frac{1}{L} \log \mu^{\beta,J,[0]}_{\Lambda(L)} \left(F(L,\varepsilon)^c \right) \Big] \equiv m(\beta, \alpha, \varepsilon) > 0\,.$$

Proof. Since $F(L, \varepsilon)$ is a positive event, we can use the FKG inequality and write

$$\mu^{\beta,J,[0]}_{\Lambda(L)} \left(F(L,\varepsilon)^c \right) \leq \mu^{\beta,J,[k]}_{\Lambda(L)} \left(F(L,\varepsilon)^c \right)$$

Then we notice that, letting $R = \{x = (x_1, x_2) \in \Lambda^* : x_2 = M - 2k - 1/2\}$, we have

$$F(L,\varepsilon)^c \subset \{\sigma \in \Omega_\Lambda : \Gamma^{[k]}_{\Lambda}(\sigma) \subset \mathcal{E}^*_{V(L)}\} = \bigcup_{x \in R} \{\sigma \in \Omega_\Lambda : \Gamma^{[k]}_{\Lambda}(\sigma) \ni x\}$$

where, by $\Gamma \ni x$ we improperly mean that one edge in Γ has x as one of its endpoints, *i.e.* that $\mathcal{V}(\Gamma) \ni x$. Using now statements 3.2, 3.6 and 3.7 given below, one can easily finish the proof. □

We start by rewriting the quantity we want to estimate in terms of two point functions in the dual lattice. Letting, for simplicity,

$$\langle \cdot \rangle^* = \mathbf{E}^{\beta^*, J^*, \emptyset}_{\Lambda^*, \mathcal{E}^*_\Lambda}(\,\cdot\,)$$

we have

Lemma 3.2. *Let $\beta > 0$, $J \geq 0$, let L, M be two positive integers and let $\Lambda = Q_{L,M}$. Choose now an integer k with $0 \leq k \leq 2M+1$. Let a and b be the two extreme sites in the $(k+1)^{th}$ row (from the top) in Λ^*, i.e.*

$$a = (-L - \tfrac{1}{2}, M + \tfrac{1}{2} - k) \qquad b = (+L + \tfrac{1}{2}, M + \tfrac{1}{2} - k) \tag{3.2}$$

Then for each $x \in \Lambda^*$ we have

$$\mu^{\beta,J,[k]}_{\Lambda}\{\sigma \in \Omega_\Lambda : \Gamma^{[k]}_{\Lambda}(\sigma) \ni x\} \leq \frac{\langle\, \sigma(a)\sigma(x) \,\rangle^* \ \langle\, \sigma(x)\sigma(b) \,\rangle^*}{\langle\, \sigma(a)\sigma(b) \,\rangle^*} \tag{3.3}$$

Proof. If x coincides with a or b the statement is trivial, so we assume $x \neq a, b$. For all $\sigma \in \Omega_\Lambda$, we let $\Gamma = \Gamma^{[k]}_{\Lambda}(\sigma)$. If $\Gamma \ni x$ then it is possible to represent Γ as

$$\Gamma = (e_1, \ldots, e_l, e_{l+1}, \ldots, e_m)$$

in such a way that $a \in \delta e_1$, $b \in \delta e_m$ and e_l is the first edge whose boundary contains x. We then let

$$\Gamma_1 = (e_1, \ldots, e_l) \quad \text{and} \quad \Gamma_2 = (e_{l+1}, \ldots, e_m)$$

Γ_1 is thus an open contour in $\mathcal{E}^*_\Lambda$ with boundary $\{a,x\}$, but it has the additional property that there is exactly one edge e in Γ_1 such that $\delta e \ni x$, *i.e.* Γ_1 cannot come back to x after a first visit. We denote by $C(\mathcal{E}^*_\Lambda, \{a, \underline{x}\})$ the set of all open contours which satisfy this further requirement.

As far as Γ_2 is concerned, we observe that, since Γ is a contour, then none of the edges of Γ_2 with the exception of, perhaps, the first, lies in $\Delta(\Gamma_1)$. Let then $\bar{e}$ be the unique forbidden edge of e_l (not equal to e_l) which originates from x. Then it is possible that $e_{l+1} = \bar{e}$ unless $\bar{e}$ is a forbidden by another edge in Γ_1. For this reason we claim that the set of all possible pairs (Γ_1, Γ_2) is equal to the set (remember (1.14))

$$\{\, (\Gamma', \Gamma'') : \Gamma' \in C(\mathcal{E}^*_\Lambda, \{a, \underline{x}\}),\ \Gamma'' \in C(\mathcal{E}^*_\Lambda \setminus \Delta'(\Gamma', x), \{x, b\})\, \}$$

Then we can write the "numerator" of the LHS of (3.3) as

$$\begin{aligned} Z^{\beta,J,[k]}(\Lambda)\, \mu_\Lambda^{\beta,J,[k]}\{\, \Gamma \ni x\,\} &= \sum_{\substack{\Gamma \in C(\mathcal{E}^*_\Lambda, \{a,b\}) \\ \Gamma \ni x}} w(\Gamma)\, \widetilde{Z}(\mathcal{E}^*_\Lambda; \Gamma) = \\ &= \sum_{\Gamma_1 \in C(\mathcal{E}^*_\Lambda, \{a, \underline{x}\})} w(\Gamma_1) \sum_{\Gamma_2 \in C(\mathcal{E}^*_\Lambda \setminus \Delta'(\Gamma_1, x), \{x,b\})} w(\Gamma_2)\, \widetilde{Z}(\mathcal{E}^*_\Lambda \setminus \Delta(\Gamma)) \end{aligned} \tag{3.4}$$

In general $\Delta'(\Gamma_1, x) \cup \Delta(\Gamma_2) \neq \Delta(\Gamma_1) \cup \Delta(\Gamma_2) = \Delta(\Gamma)$, nevertheless if γ is a *closed* contour which does not intersects $\Delta'(\Gamma_1, x) \cup \Delta(\Gamma_2)$, then one can check, by considering all cases, that γ does not contain the possible extra edge $\bar{e}$ forbidden by Γ. So we have

$$\widetilde{Z}(\mathcal{E}^*_\Lambda \setminus \Delta(\Gamma)) = \widetilde{Z}(\mathcal{E}^*_\Lambda \setminus [\Delta'(\Gamma_1, x) \cup \Delta(\Gamma_2)])$$

By consequence the product of the last two terms in (3.4) can be written as

$$\widetilde{Z}^{\beta,J}(\mathcal{E}^*_\Lambda \setminus \Delta'(\Gamma_1, x), \{x, b\})$$

which, by (1.21), becomes

$$\mathbf{E}^{\beta^*, J^*, \emptyset}_{\Lambda^*, \mathcal{E}^*_\Lambda \setminus \Delta'(\Gamma_1, x)}(\sigma(x)\sigma(b))\, \widetilde{Z}(\mathcal{E}^*_\Lambda \setminus \Delta'(\Gamma_1, x))$$

Using the second Griffiths' inequality we obtain

$$\text{LHS of (3.4)} \le \langle \sigma(x)\sigma(b)\rangle^* \sum_{\Gamma_1 \in C(\mathcal{E}^*_\Lambda, \{a, \underline{x}\})} w(\Gamma_1)\, \widetilde{Z}(\mathcal{E}^*_\Lambda \setminus \Delta'(\Gamma_1, x)) \tag{3.5}$$

A straightforward check shows now that the simultaneous substitution

$$\begin{array}{ccc} C(\mathcal{E}^*_\Lambda, \{a, \underline{x}\}) & \to & C(\mathcal{E}^*_\Lambda, \{a, x\}) \\ \Delta'(\Gamma_1, x) & \to & \Delta(\Gamma_1) \end{array}$$

has no effect on the RHS of (3.5). Thus, using (1.21) again, we get

$$\mu_\Lambda^{\beta,J,[k]}\{\, \Gamma \ni x\,\} \le \frac{\widetilde{Z}(\mathcal{E}^*_\Lambda)}{Z^{\beta,J,[k]}(\Lambda)} \langle \sigma(a)\sigma(x)\rangle^* \,\langle \sigma(x)\sigma(b)\rangle^*$$

which is equivalent to (3.3). $\square$

The next proposition is a fairly general inequality which will be used later. We use ideas contained in [MOS]

Proposition 3.3. *Let $\Lambda \subset\subset \mathbb{Z}^2$, $x \in \Lambda$ and $y \in \partial^+\Lambda$. Assume there is a family of nested subsets*

$$\Lambda = \Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_n \ni x\,.$$

For each $i = 1, \ldots, n$, let $Y_i = \partial^+\Lambda_i \cap (\Lambda \cup \{y\})$ and let U_i be an arbitrary subset of Y_i. Let $|J|_\infty = \sup_{u,v\in\mathbb{Z}^2} J(u,v)$. For each $\tau \in \Omega$ we define the boundary conditions

$$(\tau_\pm)(u) = \begin{cases} \tau(u) & \text{if } u \in (\Lambda \cup \{y\})^c \\ \pm 1 & \text{if } u \in \Lambda \cup \{y\} \end{cases} \tag{3.6}$$

Then we have

$$\left[\, \mu_\Lambda^{\beta,J,\tau_+}\{\sigma(x)=1\} - \mu_\Lambda^{\beta,J,\tau_-}\{\sigma(x)=1\}\,\right] \le \prod_{i=0}^{n-1} d_i \tag{3.7}$$

where

$$d_i = 1 - \Big(\frac{1}{2} e^{-8\beta|J|_\infty}\Big)^{2|U_{i+1}|} \left[1 - \sum_{u\in Y_{i+1}\setminus U_{i+1}} \left[\, \mu_{\Lambda_i}^{\beta,J,\tau_+}\{\sigma(u)=1\} - \mu_{\Lambda_i}^{\beta,J,\tau_-}\{\sigma(u)=1\}\,\right] \right].$$

Proof. In this proof β and J are fixed and we usually omit them. Let

$$a_i = \mu_{\Lambda_i}^{\tau_+}\{\sigma(x)=1\} - \mu_{\Lambda_i}^{\tau_-}\{\sigma(x)=1\}$$

(notice that in particular a_0 is the LHS of (3.7)). Then we claim that

$$a_i \le b_i\, a_{i+1}\,, \tag{3.8}$$

where

$$b_i = \inf_{\nu\in K(\mu_{\Lambda_i}^{\tau_+},\mu_{\Lambda_i}^{\tau_-})} b_i(\nu)$$

$$b_i(\nu) = \nu\{\,\eta \neq \eta' \text{ on } Y_{i+1}\,\} \equiv \nu\{\,\exists z \in Y_{i+1} \text{ such that } \eta(z) \neq \eta'(z)\,\}$$

and $K(\mu_1,\mu_2)$ denotes the set of all couplings (joint representations) of μ_1 and μ_2. In fact, by the DLR property, for each coupling $\nu \in K(\mu_{\Lambda_i}^{\tau_+},\mu_{\Lambda_i}^{\tau_-})$,

$$a_i = \sum_{\eta,\eta'\in\Omega} \nu(\eta,\eta')\left[\, \mu_{\Lambda_{i+1}}^{\eta}\{\sigma(x)=1\} - \mu_{\Lambda_{i+1}}^{\eta'}\{\sigma(x)=1\}\,\right].$$

Since the sum is actually restricted over those η,η' which agree with τ outside Λ, the quantity inside the brackets vanishes unless $\eta \neq \eta'$ on Y_{i+1} and (3.8) follows. So, what is left is to show that there is a coupling $\bar\nu$ such that $b_i(\bar\nu) \le d_i$. To do that we apply the so called "surgery technique" [DS] to the standard coupling $\hat\nu$ as in [MOS], *i.e.* we define for each $\tau,\tau' \in \Omega$

$$\bar\nu_{\Lambda_i}^{\tau,\tau'}(\sigma,\sigma') = \sum_{\eta,\eta'} \hat\nu_{\Lambda_i}^{\tau,\tau'}(\eta,\eta')\, \mu_{U_{i+1}}^{\eta}(\sigma)\, \mu_{U_{i+1}}^{\eta'}(\sigma')\,.$$

It can be shown (see for instance [MOS]) that $\bar\nu_{\Lambda_i}^{\tau,\tau'} \in K(\mu_{\Lambda_i}^{\tau},\mu_{\Lambda_i}^{\tau'})$. Then, letting $W_i = Y_i \setminus U_i$, we get

$$\begin{aligned} &b_i(\bar\nu_{\Lambda_i}^{\tau_+,\tau_-}) = 1 - \bar\nu_{\Lambda_i}^{\tau_+,\tau_-}\{\sigma=\sigma' \text{ on } Y_{i+1}\} \le \\ &1 - \min_{\eta\in\Omega}\big(\mu_{U_{i+1}}^{\eta}\{\sigma(x)=1\ \forall x \in U_{i+1}\}\big)^2\, \hat\nu_{\Lambda_i}^{\tau_+,\tau_-}\{\sigma=\sigma' \text{ on } W_{i+1}\} \end{aligned}$$

Hence, by FKG, we have

$$\mu^{\eta}_{U_{i+1}}\{\sigma(x)=1\ \forall x\in U_{i+1}\}\geq\prod_{x\in U_{i+1}}\mu^{\eta}_{U_{i+1}}\{\sigma(x)=1\}\geq\Big[\frac{1}{2}e^{-8\beta|J|_\infty}\Big]^{|U_{i+1}|}. \tag{3.9}$$

On the other side

$$\hat\nu^{\tau+,\tau-}_{\Lambda_i}\{\sigma=\sigma' \text{ on } W_{i+1}\}\geq 1-\sum_{x\in W_{i+1}}\hat\nu^{\tau+,\tau-}_{\Lambda_i}\{\sigma(x)\neq\sigma'(x)\}=$$

$$=1-\sum_{x\in W_{i+1}}\big[\,\mu^{\tau+}_{\Lambda_i}\{\sigma(x)=1\}-\mu^{\tau-}_{\Lambda_i}\{\sigma(x)=1\}\big]$$

which, together with (3.9), implies $b_i(\bar\nu^{\tau+,\tau-}_{\Lambda_i})\leq d_i$. $\square$

We now use previuos proposition to show that

Proposition 3.4. *For each $\beta<\beta_c$ and $\alpha>0$ there exist $m_2(\beta,\alpha)>0$ and $q_2(\beta,\alpha)$ such that, if $\Lambda=Q_{L,M}$, and J is a set of couplings satisfying*

(i) $\sup_{x,y}J(x,y)\equiv|J|_\infty\leq\alpha$
(ii) $J(x,y)=1$ *if* $\{x,y\}\cap(\Lambda\setminus\ \partial\Lambda)\neq\emptyset$

then for all $\tau\in\Omega$ and for all $x,y\in\Lambda$ such that $|x-y|\geq q_2(\beta,\alpha)$ we have

$$\mathrm{Cov}^{\beta,J,\tau}_{\Lambda}(\sigma(x),\sigma(y))\leq e^{-m_2\sqrt{|x-y|}} \tag{3.10}$$

$$\mathbf{E}^{\beta,J,\emptyset}_{\Lambda}\ \sigma(x)\sigma(y)\leq e^{-m_2\sqrt{|x-y|}} \tag{3.11}$$

Proof. Choose x and y in Λ and $\tau\in\Omega$. Let then

$$\tau_\pm(u)=\begin{cases}\tau(u) & \text{if } u\in\Lambda^c\\ \pm1 & \text{if } u\in\Lambda\end{cases}$$

Let also $\Lambda_y=\Lambda\setminus\{y\}$. A straightforward computation shows that

$$\mathrm{Cov}^{\beta,J,\tau}_{\Lambda}(\sigma(x),\sigma(y))=4\,D\,\mu^{\beta,J,\tau}_{\Lambda}\{\sigma(y)=+1\}\,\mu^{\beta,J,\tau}_{\Lambda}\{\sigma(y)=-1\}\,,$$

where

$$D=\mu^{\beta,J,\tau+}_{\Lambda_y}\{\sigma(x)=+1\}-\mu^{\beta,J,\tau-}_{\Lambda_y}\{\sigma(x)=+1\}\,. \tag{3.12}$$

Let now $s=|x-y|$ and $k=(\lfloor\sqrt{s}\rfloor-10)\vee 0$. It is possible to find k positive integers $l_1,\dots,l_k$ such that

(a) $l_i\geq l_{i+1}+k$ and $s\geq l_1+k$
(b) if we set, $\Lambda_0=\Lambda_y$ and, for each $i=1,\dots,k$, $\Lambda_i\equiv\Lambda\cap Q_{l_i}(x)$ and $Y_i=\partial^+\Lambda_i\cap\Lambda$ then we have

$$\#\{u\in Y_{i+1}:d(u,\partial\Lambda_i)=j\}\leq 2\quad\forall j\leq k-2 \tag{3.13}$$

For each $i=1,\dots,k$ and $u\in\Lambda_i$ we define $B_i(u)$ as the largest square centered on u, contained in Λ_i and such that $J(v,z)=1$ if either v or z are in $B_i(u)$. We have then

$$B_i(u)=Q_{\alpha_i(u)}(u)$$

where

$$\alpha_i(u)=d(u,\partial\Lambda_i)-1\,.$$

Because of the definition of the $B_i(u)$ we have

$$\mu_{\Lambda_i}^{\beta,J,\tau+}\{\sigma(u)=1\}-\mu_{\Lambda_i}^{\beta,J,\tau-}\{\sigma(u)=1\}\le\mu_{B_i(u)}^{\beta,J=1,+}\{\sigma(u)=1\}-\mu_{B_i(u)}^{\beta,J=1,-}\{\sigma(u)=1\}$$

Thanks to Theorem 2 in [H] we know that there exist $C(\beta)$ and $m_1(\beta)>0$ such that, for each $u\in\Lambda_i$,

$$\mu_{B_i(u)}^{\beta,+}\{\sigma(u)=1\}-\mu_{B_i(u)}^{\beta,-}\{\sigma(u)=1\}\le C(\beta)e^{-m_1(\beta)\alpha_i(u)} \tag{3.14}$$

We take an integer $r(\beta)$ such that

$$C(\beta)\sum_{j=r(\beta)}^{\infty}e^{-m_1(\beta)j}\le\frac{1}{8} \tag{3.15}$$

and let

$$U_i=\{u\in Y_i:d(u,\partial\Lambda)\le r(\beta)\}\,.$$

If we take L large enough, we can assume $k\ge r+2$ and, thanks to (3.13),

$$|U_i|\le 2(r+1)\,. \tag{3.16}$$

In this way, letting $W_i=Y_i\setminus U_i$, and using (3.14), we obtain

$$\begin{aligned}A_i\equiv&\sum_{u\in W_{i+1}}\left[\mu_{\Lambda_i}^{\beta,J,\tau+}\{\sigma(u)=1\}-\mu_{\Lambda_i}^{\beta,J,\tau-}\{\sigma(u)=1\}\right]\le\\ \le&\sum_{j=0}^{\infty}C(\beta)e^{-m_1(\beta)j}\#\{u\in W_{i+1}:\alpha_i(u)=j\}\end{aligned}$$

By (3.13) and the trivial bound $|W_i|\le 4(2l_i+1)$, we get

$$A_i\le\sum_{j=r}^{k-2}2\,C(\beta)\,e^{-m_1(\beta)j}+4\,(2l_{i+1}+1)\,C(\beta)\,e^{-m_1(\beta)(k-1)}\le 2\frac{1}{8}+9sC(\beta)e^{-m_1(\beta)(k-1)}\,,$$

which implies

$$A_i\le\frac{1}{2}\,, \tag{3.17}$$

if $s=|x-y|\ge q_2(\beta,\alpha)$ with $q_2(\beta,\alpha)$ large enough.

By (3.12), Proposition 3.3, (3.16) and (3.17), we find

$$D\le\left[1-\frac{1}{2}\Big(\frac{1}{2}e^{-8\beta|J|_\infty}\Big)^{2r+1}\right]^k$$

which gives (3.10). (3.11) is obtained by taking $J(u,v)=0$ if $[u,v]^*\in\delta\Lambda$. $\square$

The previous result can be improved to get an exponential decay, thanks to Simon's inequality.

Corollary 3.5. *For each $\beta < \beta_c$ and $\alpha > 0$ there exist $m_3(\beta,\alpha) > 0$ and $C_3(\beta,\alpha)$ such that, if Λ and J are as in Proposition 3.4, then for all $x, y \in \Lambda$ we have*

$$\mathbf{E}_\Lambda^{\beta,J,\emptyset}\, \sigma(x)\sigma(y) \le C_3\, e^{-m_3|x-y|} \tag{3.18}$$

Proof. Let $l_0 = l_0(\beta,\alpha) > q_2(\beta,\alpha)$ be such that

$$4(2l_0+1)e^{-m_2\sqrt{l_0}} \equiv b < 1\,,$$

and let

$$f(u,v) = \mathbf{E}_\Lambda^{\beta,J,\emptyset}\, \sigma(u)\sigma(v) \qquad u, v \in \Lambda$$

Theorem 2.1 in [Si] says that, if $|x-y| > l_0$, then

$$f(x,y) \le \sum_{z\in\Lambda:|z-x|=l_0} f(x,z)f(z,y)\,.$$

Iterating this inequality, we get

$$f(x,y) \le b^{\lfloor l_0^{-1}|x-y|\rfloor}$$

which proves the corollary. □

Proposition 3.6. *For each $\beta < \beta_c$, $\alpha > 0$, $t > 0$ there exist $C_4(\beta,\alpha,t)$ and $m_4(\beta,\alpha,t) > 0$ such that, if $\Lambda = Q_{L,M}$, and J is a set of couplings satisfying*

(i) $\sup_{x,y} J(x,y) \equiv |J|_\infty \le \alpha$

(ii) $J(x,y) = 1$ *unless (let $x = (x_1,x_2)$, $y = (y_1,y_2)$) $x_1 = y_1 = -L$ or $x_1 = y_1 = +L$.*

then for all $x, y \in \Lambda$ such that $|x_2 - y_2| \ge t|x_1 - y_1|$ we have

$$\mathbf{E}_\Lambda^{\beta,J,\emptyset}\, \sigma(x)\sigma(y) \le C_4\, e^{-\beta^*\tau_{\beta^*}|x_1-y_1| - m_4|x_2-y_2|} \tag{3.19}$$

where τ_β is the surface tension at zero degrees.

Proof. The idea is to use the Lieb's improvement of Simon's inequality in order to get rid of those sites where $J \ne 1$ and obtain the surface tension in this way.

Define

$$f_\Lambda(u,v) = \mathbf{E}_\Lambda^{\beta,J,\emptyset}\, \sigma(u)\sigma(v) \qquad u, v \in \Lambda$$

If $|x_1 - y_1| < 3$, (3.19) is a consequence of (3.18). Assume then $y_1 \ge x_1 + 3$ and let

$$B_1 = \{z = (z_1,z_2) \in \Lambda : z_1 = x_1 + 1\} \qquad B_2 = \{z = (z_1,z_2) \in \Lambda : z_1 = y_1 - 1\}\,.$$

Define also

$$\Lambda' = \{z = (z_1,z_2) \in \Lambda : z_1 \ge x_1 + 1\} \qquad \Lambda'' = \{z = (z_1,z_2) \in \Lambda : x_1 + 1 \le z_1 \le y_1 - 1\}\,.$$

By Lieb's improvement [L] of Simon's inequality (as given in Theorem 2.2 in [Si]), and by Griffiths' second inequality we get

$$f_\Lambda(x,y) \le \sum_{z\in B_1} f_\Lambda(x,z)f_{\Lambda'}(z,y) \le \sum_{z\in B_1}\sum_{u\in B_2} f_\Lambda(x,z)f_{\Lambda''}(z,u)f_\Lambda(u,y)\,.$$

Since $J = 1$ on Λ'' we can write (using (1.10) and (3.18))

$$f_\Lambda(x,y) \le C_3(\beta,\alpha)^2 \sum_{z\in B_1}\sum_{u\in B_2} e^{-\beta^*\tau_{\beta^*}(z-u)-m_3(\beta,\alpha)(|x-z|+|u-y|)} \tag{3.20}$$

Let $0 \le \vartheta \le \pi/2$ be such that

$$|z_2 - u_2| = |z_1 - u_1| \tan\vartheta$$

The RHS of (3.20) can be written as $A_1 + A_2$, where A_1 is the contribution of those terms with

$$\sin(\vartheta) \ge s \equiv \frac{t\wedge 1}{100}$$

and A_2 is the sum of the other terms. Using (1.11), we get, for each term in A_1

$$\begin{aligned}
&\beta^*\tau_{\beta^*}(z-u) + m_3(\beta,\alpha)(|x-z|+|u-y|) \ge \\
&\ge \beta^*\tau_{\beta^*}(0)|z_1-u_1| + \beta^* D s^2 |z-u|_2 + \beta m_3(\beta,\alpha)(|x-z|+|u-y|) \ge \\
&\ge \beta^*\tau_{\beta^*}|x_1-y_1-2| + (\beta^* D s^2 \wedge \beta\frac{m_3}{2})|x_2-y_2| + \beta\frac{m_3}{2}(|x-z|+|u-y|)
\end{aligned}$$

while for the terms in A_2 (remember that $|x_2 - y_2| \ge t|x_1 - y_1|$) we write

$$\begin{aligned}
&\beta^*\tau_{\beta^*}(z-u) + m_3(\beta,\alpha)(|x-z|+|u-y|) \ge \\
&\ge\beta^*\tau_{\beta^*}(0)|z_1-u_1| + \beta\frac{m_3}{2}(|x_2-y_2|-|u_2-z_2|) + \beta\frac{m_3}{2}(|x-z|+|u-y|) \ge \\
&\ge \beta^*\tau_{\beta^*}|x_1-y_1-2| + \beta\frac{m_3}{4}|x_2-y_2| + \beta\frac{m_3}{2}(|x-z|+|u-y|)
\end{aligned}$$

In this way we obtain

$$A_1 + A_2 \le C_4 \exp\left[-\beta^*\tau_{\beta^*}(0)|x_1-y_1-2| - \beta m_4|x_2-y_2|\right] \quad \square$$

Proposition 3.7. *Under the same hypotheses as in Proposition 3.1, we have*

$$\liminf_{L\to\infty}\Big[-\frac{1}{(2L+1)}\log \mathbf{E}_{\Lambda^*}^{\beta^*,J^*,\emptyset}\,\sigma(a)\sigma(b)\Big] \le \beta\tau_\beta$$

Proof. We just need to observe that, thanks to the second Griffiths' inequality, since $J^* \ge 1$,

$$\mathbf{E}_{\Lambda^*}^{\beta^*,J^*,\emptyset}\,\sigma(a)\sigma(b) \ge \mathbf{E}_{V(L)^*}^{\beta^*,\emptyset}\,\sigma(a)\sigma(b)$$

which, by duality is equal to

$$\frac{Z_V^{\beta,[k]}(L)}{Z_V^{\beta,+}(L)}\,.$$

The proposition then follows from the definition of surface tension. $\square$

4. Proof of Propositions 2.1, 2.2

4.1. Proof of Proposition 2.1

To prove part (i) let V be the top portion of Λ of height $2k$, *i.e.*

$$V = \{x = (x_1, x_2) \in \Lambda : x_2 \geq h + M - 2k + 1\}$$

and let $F(L,\varepsilon)$ be the event

$$F(L,\varepsilon) = \{\sigma \in \Omega_\Lambda : \Gamma_\Lambda^{[0]}(\sigma) \subset \mathcal{E}_V^*\}\,.$$

Then, using FKG, one shows (see the appendix in [M]) that for any $x \in \Lambda_{\mathrm{bot}}$

$$\mu_\Lambda^{\beta,J,+}\{\sigma(x) = +1\} - \mu_\Lambda^{\beta,J,[0]}\{\sigma(x) = +1\} \leq \mu_\Lambda^{\beta,J,[0]}(F(L,\varepsilon)^c) \tag{4.1}$$

which, combined with Proposition 3.1, yields (i) of 2.1.

To prove the second inequality we observe that, using the DLR equation and a standard coupling argument, it is sufficient to prove the required bound only for $x = (x_1, x_2)$ with $x_2 = h + M - 3k$. We set,

$$\begin{bmatrix} \Lambda & J & \eta \\ \Lambda' & J' & \eta' \end{bmatrix} = \mu_\Lambda^{\beta,J,\eta}\{\sigma(x) = +1\} - \mu_{\Lambda'}^{\beta,J',\eta'}\{\sigma(x) = +1\}$$

and, for simplicity, we denote by $\Box$ the coupling $J_a^{\Box}$ and so on (see (1.6)). There are three cases to consider.

Case 1. $\delta_b\Lambda = \delta_b\bar\Lambda = \delta_b Q_L$

The LHS of (2.7) can be written as

$$\begin{bmatrix} \Lambda & \sqcup & + \\ \bar\Lambda & \sqcup & + \end{bmatrix} \leq \begin{bmatrix} \Lambda & \| & + \\ \bar\Lambda & \sqcup & + \end{bmatrix} = \begin{bmatrix} \Lambda & \| & + \\ \Lambda & \| & [0] \end{bmatrix} + \begin{bmatrix} \Lambda & \| & [0] \\ \bar\Lambda & \| & + \end{bmatrix} + \begin{bmatrix} \bar\Lambda & \| & + \\ \bar\Lambda & \sqcup & + \end{bmatrix} \equiv A + B + C$$

A and C can be estimated with (2.6) (for C one has to flip $\bar\Lambda$ upside down), while B is nonpositive.

Case 2. $\delta_b\Lambda = \delta_b\bar\Lambda \neq \delta_b Q_L$ and $\delta_t\bar\Lambda \neq \delta_t Q_L$

In this case we write

$$\text{LHS of (2.7)} = \begin{bmatrix} \Lambda & \| & + \\ \bar\Lambda & \| & + \end{bmatrix} = A + B$$

Case 3. $\delta_b\Lambda = \delta_b\bar\Lambda \neq \delta_b Q_L$ and $\delta_t = \delta_t Q_L$

We have

$$\text{LHS of (2.7)} = \begin{bmatrix} \Lambda & \| & + \\ \bar\Lambda & \sqcap & + \end{bmatrix} = \begin{bmatrix} \Lambda & \| & + \\ \Lambda & \| & [0] \end{bmatrix} + \begin{bmatrix} \Lambda & \| & [0] \\ \bar\Lambda & \sqcap & + \end{bmatrix} \leq \begin{bmatrix} \Lambda & \| & + \\ \Lambda & \| & [0] \end{bmatrix} = A \quad \Box$$

4.2. Proof of Proposition 2.2

Case 1. $\delta_b\Lambda \neq \delta_b Q_L$

In this case, by monotonicity,

$$\text{LHS of (2.8)} \geq \mu_\Lambda(F(L,\varepsilon); \beta, J_\varepsilon^{\|}(\Lambda), [0]) \tag{4.2}$$

(the inequality is strict only if $\delta_t\Lambda = \delta_t Q_L$) The RHS of (4.2) tends to one as $L \to \infty$ by Proposition 3.1.

Case 2. $\delta_b\Lambda = \delta_b Q_L$

We first observe that

$$\mu_\Lambda^{\beta,J_\varepsilon,[0]}(F(L,\varepsilon)) \geq e^{-8\beta\varepsilon(2L+1)}\mu_\Lambda^{\beta,\bar{J}_\varepsilon,[0]}(F(L,\varepsilon))$$

where

$$\bar{J}_\varepsilon(x,y) = \begin{cases} \varepsilon & \text{on the bottom side of } \Lambda \\ 1 & \text{elsewhere} \end{cases}$$

Define now a, b as in (3.2) with $k = 0$. Let also denote with A and B the two subsets of $\Lambda \setminus \Delta^s(\Gamma)$ that are respectively above and below Γ. Then we can write

$$\mu_\Lambda^{\beta,\bar{J}_\varepsilon,[0]}(F(L,\varepsilon)) = \sum_{\Gamma \in C(\mathcal{E}^*_{\Lambda_{\text{top}}},\{a,b\})} w_{\bar{J}_\varepsilon}(\Gamma)\, \frac{Z^{\beta,-}(A)\, Z^{\beta,\bar{J}_\varepsilon,+}(B)}{Z^{\beta,\bar{J}_\varepsilon,[0]}(\Lambda)} \tag{4.3}$$

Notice that for any $\Gamma \subset \mathcal{E}^*_{\Lambda_{\text{top}}}$, we have $w_{\bar{J}_\varepsilon}(\Gamma) = w(\Gamma)$. Using now the spin flip symmetry we obtain

$$Z^{\beta,\bar{J}_\varepsilon,[0]}(\Lambda) \leq \exp\big[\,2\beta(|\delta_e\Lambda| + |\delta_w\Lambda|) + 2\beta\varepsilon|\delta_b\Lambda|\,\big]\, Z^{\beta,\bar{J}_\varepsilon,-}(\Lambda) \leq e^{9\beta\varepsilon L}\, Z^{\beta,\bar{J}_\varepsilon,+}(\Lambda) = \tag{4.4}$$

Next we consider the ratio

$$G(\varepsilon) \equiv \frac{Z^{\beta,\bar{J}_\varepsilon,+}(B)}{Z^{\beta,\bar{J}_\varepsilon,+}(\Lambda)}$$

and we claim that there exists $m = m(\beta, e) > 0$ such that

$$\frac{Z^{\beta,\bar{J}_\varepsilon,+}(B)}{Z^{\beta,\bar{J}_\varepsilon,+}(\Lambda)} \geq \frac{Z^{\beta,+}(B)}{Z^{\beta,+}(\Lambda)}\, e^{-\beta e^{-mL}} \tag{4.5}$$

To prove it we compute the logarithmic derivative of G obtaining

$$\begin{aligned} \frac{d}{da}\log G(a) &= \beta \sum_{\substack{x=(x_1,x_2)\in\partial\Lambda \\ x_2=-M}} \big[\,\mathbf{E}_B^{\beta,\bar{J}_a,+}(\sigma(x)) - \mathbf{E}_\Lambda^{\beta,\bar{J}_a,+}(\sigma(x))\,\big] \leq \\ &\leq \beta \sum_{\substack{x=(x_1,x_2)\in\partial\Lambda \\ x_2=-M}} \big[\,\mathbf{E}_{\Lambda_{\text{bot}}}^{\beta,\bar{J}_a,+}(\sigma(x)) - \mathbf{E}_\Lambda^{\beta,\bar{J}_a,+}(\sigma(x))\,\big] \end{aligned} \tag{4.6}$$

Part (ii) of Proposition 2.1 applied to Λ_{bottom} and Λ implies that

$$\text{RHS of (4.6)} \leq \beta(2L+1)2e^{-m(\beta,a,\varepsilon')L}$$

where m is the quantity given in Proposition 2.1 and ε' is such that the height of Λ_{bottom} is given by $\lfloor \varepsilon' L \rfloor$. If we redefine

$$m(\beta,\varepsilon) = \frac{1}{2}\inf_{a\in[\varepsilon',1]} m(\beta,a,\varepsilon')$$

we get (4.5) for L large enough.

If we now combine (4.4), (4.5) and use Proposition 3.1, we find

$$\text{RHS of (4.3)} \geq \exp\big[-9\beta\varepsilon L - \beta e^{-mL}\,\big]\, \frac{Z^{\beta,[0]}(\Lambda)}{Z^{\beta,+}(\Lambda)}\, \mu_\Lambda^{\beta,[0]}(F(L,\varepsilon)) \geq e^{-10\beta\varepsilon L}\, \frac{Z^{\beta,[0]}(\Lambda)}{Z^{\beta,+}(\Lambda)} \tag{4.7}$$

On the other side, we have

$$\frac{Z^{\beta,[0]}(\Lambda)}{Z^{\beta,+}(\Lambda)} \geq e^{-\varepsilon\beta L}\, \frac{Z^{\beta,[k]}(\Lambda)}{Z^{\beta,+}(\Lambda)} \tag{4.8}$$

Finally we use duality together with Proposition 3.7 to get that

$$\limsup_{L\to\infty} -\frac{1}{\beta(2L+1)}\, \log\Big[\frac{Z^{\beta,[k]}(\Lambda)}{Z^{\beta,+}(\Lambda)}\Big] \leq \tau_\beta$$

Thus we can conclude that there exists $L_0(\beta,\varepsilon)$ such that, if $L \geq L_0$, then we have

$$\mu_\Lambda^{\beta,J_\varepsilon,[0]}(F(L,\varepsilon)) \geq e^{-8\beta\varepsilon(2L+1)-11\beta\varepsilon L}\, e^{-\beta\tau_\beta(2L+1)} \geq e^{-\beta(\tau_\beta+14\varepsilon)(2L+1)} \quad \square$$

5. Upper bound on the probability of a large deviation.

5.1. The variational problem.

We start by stating in a precise way the variational problem which is preliminary to the proof of Proposition 2.3

We denote by $\mathcal{D}$ the set of all rectifiable curves $\gamma \in \mathbb{R}^2$ such that γ is either a closed curve inside the unit open square $Q = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1\,,\, 0 < y < 1\}$ or it is an open curve, which, with the exception of its endpoints, is entirely contained in Q. $\mathcal{D}_s$ will denote the set of selfavoiding curves in $\mathcal{D}$. The collection of families of curves in $\mathcal{D}$ ($\mathcal{D}_s$) will be denoted by $\mathcal{D}^*$ ($\mathcal{D}^*_s$).

Clearly any $\gamma \in \mathcal{D}_s$ splits the square Q into two disjoint connected subsets denoted by A_γ, B_γ; we set $V(\gamma) = \min\{|A_\gamma|, |B_\gamma|\}$. We also define, for any curve $\gamma \in \mathcal{D}$, the Wulff functional $W(\gamma)$ as

$$W(\gamma) = \int_\gamma \tau(\vec{n}_s)\, ds$$

where s is the length parameter of the curve γ and $\tau(\vec{n}_s)$ is the surface tension at inverse temperature β in the direction of the normal $\vec{n}_s$ to the curve γ at the point s. We finally define the function $\bar{\varphi}(v)$, $0 < v \le 1$, as

$$\bar{\varphi}(v) \equiv \begin{cases} \inf\limits_{\gamma \in \mathcal{D}_s : V(\gamma) = v} W(\gamma) & \text{if} \quad 0 < v \le \frac{1}{2} \\ \bar{\varphi}(\frac{1}{2}) & \text{if} \quad \frac{1}{2} < v \le 1 \end{cases} \tag{5.1}$$

$\bar{\varphi}(v)$ can be computed exactly (see [Sh] where however there is a mistake in the expression for $\bar{\varphi}(v)$ due to a misprint)

$$\bar{\varphi}(v) = \begin{cases} \frac{1}{2} w \sqrt{v} & \text{if} \quad 0 < v \le v_0 \\ \tau(0) & \text{if} \quad v_0 < v \le 1 \end{cases} \tag{5.2}$$

where the constant w is the value of the Wulff functional $W(\gamma)$ on the Wulff curve $\gamma_F \in \mathbb{R}^2$ enclosing a unit area (see e.g. [Sh] or [DKS]) and the singularity point v_0 satisfies the equation

$$\frac{1}{2} w \sqrt{v_0} \;=\; \tau(0)$$

Notice that, if for any $m \in [0, m^*]$ we set $V(m) = \frac{m^* - m}{2m^*}$, then $\bar{\varphi}(V(m)) = \varphi(m)$, where $\varphi(m)$ has been defined in Proposition 2.3.

For future purposes it is convenient generalize a little bit the definition of $\bar{\varphi}(v)$ by showing that the infimum in (5.1) can be taken not only over a single curve but over a family $\mathcal{G} \in \mathcal{D}^*$. With this goal in mind, given a family $\mathcal{G} \in \mathcal{D}^*$ we fix an arbitrary point $x_0 \in Q$ in such a way that x does not lie on one of the curve of $\mathcal{G}$ and we define the set $A_\mathcal{G}$ as the union of those points $x \in Q$ such that any path connecting x with x_0 and intersecting in a finite number of points the curves in $\mathcal{G}$, intersect them in an odd number of times counting multiplicity. We define *phase volume* of the family $\mathcal{G}$, $V(\mathcal{G})$, as the minimum between $|A_\mathcal{G}|$ and $|Q \setminus A_\mathcal{G}|$. Using the above definitions we have

Lemma 5.1. *Let $W(\mathcal{G}) = \sum_i W(\gamma_i)$. Then, for any $0 < v \le \frac{1}{2}$ the following holds*

$$\bar{\varphi}(v) = \inf_{\substack{\mathcal{G} \in \mathcal{D}^* \\ V(\mathcal{G}) \ge v}} W(\mathcal{G})$$

Proof. It is clearly sufficient to prove that for any $0 < v \le \frac{1}{2}$

$$\bar{\varphi}(v) \le \inf_{\substack{\mathcal{G} \in \mathcal{D}^* \\ V(\mathcal{G}) \ge v}} W(\mathcal{G})$$

Given a family $\mathcal{G}$, we assume, without loss of generality, that its phase volume coincides with $|A_{\mathcal{G}}|$, so that $|A_{\mathcal{G}}| \le \frac{1}{2}$. Let now $\{A^\alpha\}$ be the decomposition of the set $A_{\mathcal{G}}$ into mutually disjoint connected components and let, for any given α, $\gamma_1^\alpha \dots \gamma_{k_\alpha}^\alpha$, $k_\alpha \le +\infty$, be a decomposition of the curve $\partial A^\alpha \cap Q$ into countably many selfavoiding curves in $\mathcal{D}$. Using the fact that $|A^\alpha| \le |A_{\mathcal{G}}| \le \frac{1}{2}$, it is not difficult to check that

$$\sum_{i=1}^{k_\alpha} V(\gamma_i^\alpha) \ge |A^\alpha|$$

so that

$$\sum_\alpha \sum_{i=1}^{k_\alpha} V(\gamma_i^\alpha) \ge |A_{\mathcal{G}}| = V(\mathcal{G}) \ge v$$

Moreover it is obvious that

$$W(\mathcal{G}) = \sum_\alpha \sum_{i=1}^{k_\alpha} W(\gamma_i^\alpha)$$

Thus we immediately conclude that

$$\inf_{\substack{\mathcal{G}\in\mathcal{D}^* \\ V(\mathcal{G})\ge v}} W(\mathcal{G}) \ge \inf_{\substack{\mathcal{G}\in\mathcal{D}_s^* \\ \sum_i V(\gamma_i)\ge v}} W(\mathcal{G}) \tag{5.3}$$

We observe at this point that, because of (5.1), (5.2), one has

$$W(\gamma) \ge \bar{\varphi}(V(\gamma)) \qquad \forall\, \gamma \in \mathcal{D}_s$$

and

$$\varphi(v_1) + \varphi(v_2) \ge \varphi(v_1 + v_2) \qquad \forall\, v_1, v_2$$

Thus the RHS of (5.3) can be bounded from below by

$$\inf_{\substack{\gamma_1\dots\gamma_n;\,\gamma_i\in\mathcal{D}_s \\ \sum_i V(\gamma_i)\ge v}} \sum_i \bar{\varphi}(V(\gamma_i)) \ge \bar{\varphi}(\sum_i V(\gamma_i)) \ge \bar{\varphi}(v)$$

and the lemma follows. $\square$

5.2 Proof of Proposition 2.3.

The proof of this result follows closely the proof given by Ioffe [I2] for the case of + boundary conditions. Before starting we need to introduce some more notation.

In the following Λ will always denote the square Q_L, and, for simplicity we let $N = 2L + 1$ be the length of its sides.

We then choose two real numbers $0 < \nu < b < \frac{1}{4}$ and we say that a contour Γ is $b-$large either if it is closed surrounding an area larger than L^{2b}, or if it is open and it splits the set Q_L into two parts each of which has an area larger than L^{2b}. Here we are adopting the convention that the area of a finite set $A \subset \mathbb{Z}^2$ is simply the number of sites of A. The subscript b in objects like $C_b(\cdot)$, $C_b^*(\cdot)$, $\mathcal{G}_{\Lambda,b}^\emptyset(\sigma)$ means that we consider only $b-$large contours.

Given a sequence $(u_1, \dots, u_n)$ of dual sites in Λ^* we denote by $P = [u_1, \dots, u_n]$ the polygon

$$[u_1, \dots, u_n] = \bigcup_{i=2}^n [u_{i-1}, u_i]$$

If $u_1 = u_n$ P is said to be closed ($\delta P = \emptyset$), otherwise it is open and its boundary is $\delta P = \{u_1, u_n\}$. P is called a $\nu-$*skeleton* if

$$\frac{L^\nu}{12} \le |x_{i+1} - x_i|_2 \le L^\nu \qquad \forall\, i = 1, \ldots, n-1$$

Given now a contour γ we say that γ is *consistent* with a $\nu-$skeleton $P = [u_1, \ldots, u_n]$, and we write $\gamma \sim P$, if

(i) $\delta\gamma = \delta P$

(ii) the vertices of S lie on γ in the order corresponding to the natural order of the vertices of γ

(iii) the Hausdorff distance between any edge $l_i \equiv [x_{i-1}, x_i]$ of P and the part of γ connecting x_{i-1} with x_i is not greater than L^ν.

Using the construction of Lemma 5.11 in [DKS] it is easy to check that, for any L large enough and any contour γ with $\mathrm{diam}(\gamma) \ge L^\nu$, there always exists a $\nu-$skeleton S consistent with γ. In particular, for any L large enough, it will always be possible to associate a particular $\nu-$skeleton S to any $b-$large contour γ. We assume that a definite choice has been made once and forall, and $\hat S(\gamma)$ denotes *the* skeleton of γ. Finally, given a set $\mathcal{S} = \{S_1, \ldots, S_r\}$ of skeletons we denote by $E(\mathcal{S})$ the set of all configurations $\sigma \in \Omega_\Lambda$ such that the family of the $\nu-$ skeletons of their $b-$large contours coincide with $\mathcal{S}$. We also set

$$W(\mathcal{S}) = \sum_{i=1}^{r} W(S_i)\,,$$

where, for any $\nu-$skeleton $S = [x_1, \ldots, x_k]$,

$$W(S) \equiv \sum_{i=2}^{k} \tau(x_i - x_{i-1})$$

Notice that, for a given $\nu-$skeleton S, the number k of vertices of S and the value of the Wulff functional $W(S)$ are related by the following simple inequality which follows from the definition of $W(S)$ and from (1.12)

$$W(S) \ge \frac{\tau(0)L^\nu}{20} k \tag{5.4}$$

Given $0 < d < \varphi(m)$ we define the event

$$K \equiv \{\sigma : W(\hat S(\sigma)) \ge (\varphi(m) - d)L\}\,,$$

where $\hat S(\sigma)$ denotes the collection of the $\nu-$skeletons associated to the $b-$large contours of σ. Clearly $\hat S(\sigma)$ can be also the emptyset. Then we write

$$\mu_\Lambda^{\beta,\emptyset}\{m_\Lambda(\sigma) = m\} \le \mu_\Lambda^{\beta,\emptyset}\{m_\Lambda(\sigma) = m \,|\, K^c\} + \mu_\Lambda^{\beta,\emptyset}(K) \tag{5.5}$$

The second term of (5.5) can bounded from above by adapting to our case the technique of Section 10 of [P]. More precisely we have

Lemma 5.2. *For any $\beta > \beta_c$*

$$\liminf_{L\to\infty} -\frac{1}{\beta N} \log \mu_\Lambda^{\beta,\emptyset}(K) \ge \varphi(m) - d$$

Proof. We write

$$\mu_\Lambda^{\beta,\emptyset}(K) \le \sum_{\substack{\mathcal{S}=\{S_1,\ldots,S_n\}\\ W(\mathcal{S})\ge(\varphi(m)-d)L}} \mu_\Lambda^{\beta,\emptyset}(E(\mathcal{S})) \tag{5.6}$$

In order to estimate the generic term $\mu_\Lambda^{\beta,\emptyset}(E(\mathcal{S}))$ we will use the following lemma (compare with Lemma 10.1 in [P] where + boundary conditions are considered)

Lemma 5.3. *Let $\mathcal{S}=\{S_1,\ldots,S_k\}$ be a collection of $\nu-$skeletons. For any $\beta>\beta_c$*

$$\mu_\Lambda^{\emptyset}(E(\mathcal{S}))\le e^{-\beta W(\mathcal{S})}$$

Before proving Lemma 5.3 we finish the proof of Lemma 5.2. For each collection of skeletons $\mathcal{S}$, we let $N(\mathcal{S})$ be the sum of the number of vertices of all skeletons in $\mathcal{S}$ and A_k is the number of $\mathcal{S}$ with $N(\mathcal{S})=k$. Only terms with $N(\mathcal{S})\le 2|\Lambda^*|$ contribute to the RHS of (5.6), so using (5.4) and Lemma 5.3, we have, for all $\varepsilon\in(0,1)$,

$$\text{RHS of (5.6)}\le e^{-(1-\varepsilon)\beta(\varphi(m)-d)L}\sum_{k=1}^{9L^2}A_k\exp\Big[-\varepsilon\frac{\beta\tau_\beta L^\nu k}{20}\Big]$$

A very rough estimate of A_k if given by

$$A_k\le(2L+2)^{2k}\,k^k\,k!$$

where the first term takes into account the choices of the vertices in Λ^*, the second terms is a bound on the number of possible partitions and the last term accounts for the different arrangements. Since $k\le 9L^2$, we have

$$A_k\le(9L^2)^{3k}\le e^{10k\log L}$$

which implies

$$\text{RHS of (5.6)}\le e^{-(1-\varepsilon)\beta(\varphi(m)-d)L}\sum_{k=1}^{9L^2}\exp\Big[-\Big(\frac{\varepsilon\beta\tau_\beta L^\nu}{20}-10\log L\Big)k\Big]$$

Since ε can be arbitrarily small, we have

$$\liminf_{L\to\infty}-\frac{1}{\beta N}\log\mu_{Q_L}^{\beta,\emptyset}(K)\ge\varphi(m)-d$$

We are then left with the

Proof of Lemma 5.3. To simplify the notations we consider the case of a single $\nu-$skeleton $S=[x_1,\ldots,x_n]$. The proof relies on the following two inequalities

$$\mu_\Lambda^{\beta,\emptyset}(E(S))\le\sum_{\substack{\gamma\in C(\mathcal{E}_\Lambda^*,\delta S)\\ \gamma\sim S}}\frac{\widetilde{Z}(\mathcal{E}_\Lambda^*\setminus\Delta(\gamma))}{\widetilde{Z}(\mathcal{E}_\Lambda^*)}\tag{5.7}$$

and, letting $S'=[x_1,\ldots,x_{n-1}]$,

$$\sum_{\substack{\gamma\in C(\mathcal{E}_\Lambda^*,\delta S)\\ \gamma\sim S}}\widetilde{Z}(\mathcal{E}_\Lambda^*\setminus\Delta(\gamma))\le\mathbf{E}_{\Lambda^*}^{\beta,\emptyset}(\sigma(x_{n-1})\sigma(x_n))\sum_{\substack{\gamma\in C(\mathcal{E}_\Lambda^*,\delta S')\\ \gamma\sim S'}}\widetilde{Z}(\mathcal{E}_\Lambda^*\setminus\Delta(\gamma))\tag{5.8}$$

The proof of (5.8) is almost identical to the proof of Lemma 3.2 so we omit it. Iterating this inequality and using (1.10) one gets Lemma 5.3, if (5.7) holds.

To prove (5.7) we let $\hat{\mathcal{E}}_\Lambda=\mathcal{E}_\Lambda^*\setminus\delta\Lambda$ and we have

$$\mu_\Lambda^{\beta,\emptyset}(E(S))\le\mu_\Lambda^{\beta,\emptyset}\{\sigma\in\Omega_\Lambda:\exists\gamma\in\mathcal{G}_{\Lambda,b}^{\emptyset}(\sigma),\ S(\gamma)=S\}\le\sum_{\substack{\gamma\in C(\hat{\mathcal{E}}_\Lambda,\delta S)\\ \gamma\sim S}}\mu_\Lambda^{\beta,\emptyset}\{\gamma\in\mathcal{G}_\Lambda^{\emptyset}(\sigma)\}\tag{5.9}$$

Then we set $J = J_0^{\square}(\Lambda)$ and, using Proposition 1.3, we write

$$\mu_\Lambda^{\beta,\emptyset}\{\gamma \in \mathcal{G}_\Lambda^{\emptyset}(\sigma)\} \le \sum_{\lambda' \in C(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma), \delta\gamma)} \mu_\Lambda^{\beta,\emptyset}\{\gamma \cup \lambda' \in \mathcal{G}_\Lambda^{+}(\sigma)\} = \sum_{\lambda' \in C(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma), \delta\gamma)} \mu_\Lambda^{\beta,J,+}\{\gamma \cup \lambda' \in \mathcal{G}_\Lambda^{+}(\sigma)\} \tag{5.10}$$

Moreover, if we let $\lambda = \gamma \cup \lambda'$, we have

$$\mu_\Lambda^{\beta,J,+}\{\lambda \in \mathcal{G}_\Lambda^{+}(\sigma)\} = w(\gamma)\, w_J(\lambda')\, \frac{\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus \Delta(\lambda))}{\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma))}\, \frac{\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma))}{\widetilde{Z}^J(\mathcal{E}_\Lambda^*)}$$

(the weight of γ has no J superscript since γ has no edges on $\delta\Lambda$). Even though in general $\Delta(\lambda) \supset \Delta''(\gamma) \cup \Delta(\lambda')$, it is easy see that

$$\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus \Delta(\lambda)) = \widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus (\Delta''(\gamma) \cup \Delta(\lambda')))$$

Using the identity (remember that $\delta\gamma = \{x_1, x_n\}$)

$$\sum_{\lambda' \in C(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma), \delta\gamma)} w_J(\lambda')\, \frac{\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus (\Delta''(\gamma) \cup \Delta(\lambda')))}{\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus \Delta''(\lambda))} = \mathbf{E}^{\beta,J,+}_{\Lambda,\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma)}(\sigma(x_1)\sigma(x_n)) \tag{5.11}$$

and the fact that the above quantity is less than or equal to 1, we arrive at

$$\mu_\Lambda^{\beta,\emptyset}(E(S)) \le \sum_{\substack{\gamma \in C(\hat{\mathcal{E}}_\Lambda, \{x_1, x_n\}) \\ \gamma \sim S}} w(\gamma)\, \frac{\widetilde{Z}^J(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma))}{\widetilde{Z}^J(\mathcal{E}_\Lambda^*)} \tag{5.12}$$

Thanks to Proposition 1.2, we know that we can replace J with $J = 1$ in the RHS of (5.12) obtaining upper bound. In order to complete the proof of (5.7) one has to check that

$$\sum_{\substack{\gamma \in C(\hat{\mathcal{E}}_\Lambda, \{x_1, x_n\}) \\ \gamma \sim S}} w(\gamma)\, \widetilde{Z}(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma)) = \sum_{\substack{\gamma \in C(\mathcal{E}_\Lambda^*, \{x_1, x_n\}) \\ \gamma \sim S}} w(\gamma)\, \widetilde{Z}(\mathcal{E}_\Lambda^* \setminus \Delta(\gamma)) \tag{5.13}$$

This equality follows from the existence of a bijective weight–preserving mapping from the set

$$\{(\gamma, \underline{\lambda}) : \gamma \in C(\hat{\mathcal{E}}_\Lambda, \{x_1, x_n\}),\ \underline{\lambda} \in C^*(\mathcal{E}_\Lambda^* \setminus \Delta''(\gamma))\}$$

onto the set

$$\{(\gamma', \underline{\lambda}') : \gamma' \in C(\mathcal{E}_\Lambda^*, \{x_1, x_n\}),\ \underline{\lambda}' \in C^*(\mathcal{E}_\Lambda^* \setminus \Delta(\gamma'))\}$$

Such a mapping is obtained as follows (see also Fig. 4).

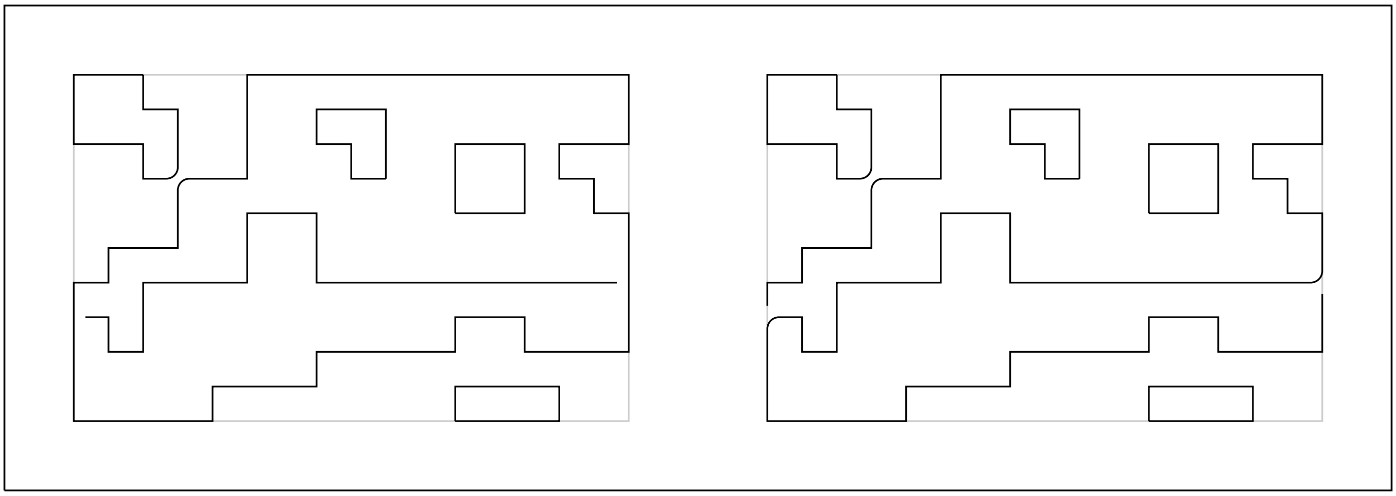

Fig. 4

Let X be the set of all edges contained either in γ or in one of the contours $\lambda \in \underline{\lambda}$. Then X is a set of edges with $\delta X = \{x_1, x_n\}$. Split now X according to the splitting rules. The outcome is a collection of contours one of which is open with boundary $\{x_1, x_n\}$, while the rest are closed. We let γ' to be the unique open contour and $\underline{\lambda}'$ the family of closed contours obtained. One can check that this is actually a bijective mapping between the two sets given above.

This completes the proof of Lemma 5.3. $\square$

In order to finish the proof of Proposition 2.3 it is now sufficient to show that

$$\liminf_{N\to\infty} -\frac{1}{\beta N} \log \mu_\Lambda^{\beta,\emptyset}\{m_\Lambda(\sigma) = m \,|\, K^c\} = +\infty \tag{5.14}$$

We then denote by R the set of all $\underline{\gamma} \in C^*(\hat{\mathcal{E}}_\Lambda, \subset \mathcal{V}(\delta\Lambda))$ such that $W(\hat{S}(\underline{\gamma})) < (\varphi(m) - d)L$ and we write

$$\mu_\Lambda^{\beta,\emptyset}\{m_\Lambda(\sigma) = m \,|\, K^c\} \le \sup_{\underline{\gamma}\in R} \mu_\Lambda^{\beta,\emptyset}\{m_\Lambda(\sigma) = m \,|\, \mathcal{G}^{\emptyset}_{\Lambda,b}(\sigma) = \underline{\gamma}\} \tag{5.15}$$

Bounds on (5.15) when $\underline{\gamma} \neq \emptyset$

We postpone to the end the analysis of the case $\underline{\gamma} = \emptyset$. If $\underline{\gamma} \neq \emptyset$, we can split Λ as a union $\Lambda = \bar{A}(\underline{\gamma}) \cup \bar{B}(\underline{\gamma})$ as follows: let $\underline{\lambda}$ and $\underline{\vartheta}$ be the two elements in $C^*(\mathcal{E}^*_\Lambda)$ which are mapped into $\underline{\gamma}$ by (1.22). Order now the elements of $C^*(\mathcal{E}^*_\Lambda)$ is some arbitrary way and, assuming that $\underline{\lambda}$ precedes $\underline{\vartheta}$ in this order, define (see (1.16))

$$\bar{A}(\underline{\gamma}) = \operatorname{int} \underline{\lambda} \qquad \bar{B}(\underline{\gamma}) = \Lambda \setminus \bar{A}(\underline{\gamma}) = \operatorname{int} \underline{\vartheta}$$

Let also (see (1.15)),

$$\Delta^s_A(\underline{\gamma}) = \Delta^s(\underline{\gamma}) \cap \bar{A}(\underline{\gamma}) \qquad \Delta^s_B(\underline{\gamma}) = \Delta^s(\underline{\gamma}) \cap \bar{B}(\underline{\gamma})$$

and

$$A(\underline{\gamma}) = \bar{A}(\underline{\gamma}) \setminus \Delta^s_A(\underline{\gamma}) \qquad B(\underline{\gamma}) = \bar{B}(\underline{\gamma}) \setminus \Delta^s_B(\underline{\gamma})$$

Then we have

Lemma 5.4. *There exists $d' = d'(\beta, d) > 0$ and $L_0(\beta, d)$ such that for any $L > L_0$ and for any $\underline{\gamma} \in R$ with $\underline{\gamma} \neq \emptyset$, if we let $J = J_0^{\square}(\Lambda)$, we have*

$$\begin{aligned} &\mu_\Lambda^{\beta,\emptyset}\{m_\Lambda(\sigma) = m \,|\, \mathcal{G}_{\Lambda,b}^{\emptyset}(\sigma) = \underline{\gamma}\} \leq \\ &\leq \chi\{|A| \geq d' N^2\}\, \mu_{A,b}^{\beta,J,+}\{|m_A(\sigma) - m^*| \geq d'\} + \chi\{|B| \geq d' N^2\}\, \mu_{B,b}^{\beta,J,+}\{|m_B(\sigma) - m^*| \geq d'\} \end{aligned} \tag{5.16}$$

where the subscript b means that the measure is conditioned not to have any large contour, and we have set $A = A(\underline{\gamma})$, $B = B(\underline{\gamma})$.

Remark. The couplings J have the effect that the + boundary conditions act only on $\delta A \setminus \delta\Lambda$, while we have free boundary conditions on $\delta A \cap \delta\Lambda$ (similarly for B).

Proof. We assume that L and $\underline{\gamma}$ have been chosen satisfying the hypothesis of the Lemma, and we let $\Delta^s = \Delta^s(\underline{\gamma})$ and so on. Let $\mathcal{S} = S(\underline{\gamma})$ be the set of the skeletons of the large contours. Since $W(\mathcal{S}) \leq (\varphi(m) - d)L$ the volume of Δ^s satisfies

$$|\Delta^s| \leq C N^{1+2\nu} \tag{5.17}$$

for a suitable constant C independent of L (see [SS1]). Moreover, using Lemma 5.1, (5.3) and the fact that $\varphi(m) = \bar{\varphi}(\frac{m^* - m}{2m^*})$, we get that there exists $\varepsilon(d) > 0$ such that the phase volume $V(\mathcal{S})$ is bounded by

$$V(\mathcal{S}) \leq (1 - \varepsilon(d))(\frac{m^* - m}{2m^*})N^2 \tag{5.18}$$

for any L large enough. Finally we assume, without loss of generality, that

$$|\bar{A}| \leq \frac{1}{2} N^2$$

Then, using Lemma 5.13 of [DKS] and (5.17), we get that

$$|\,|A| - V(\mathcal{S})\,| \leq C_1 N^{1+2\nu}$$

for a suitable constant C_1, so that, using (5.18),

$$|A| \leq \Big(1 - \frac{\varepsilon(d)}{2}\Big)\Big(\frac{m^* - m}{2m^*}\Big)N^2 \qquad \text{and} \qquad |B| > \frac{1}{2}N^2 \tag{5.19}$$

for any N large enough.

Fix $0 < d' \ll 1$ and let's distinguish between the two cases

(*i*) $\quad |A| \leq d' N^2$

(*ii*) $\quad |A| \geq d' N^2$

In the first case, if we denote by $m_B(\sigma)$ the (normalized) magnetization inside B, we get

$$|m_B(\sigma) - m| \leq 2CN^{2\nu-1} + 2d'$$

and the lemma follows for d' small enough since $2\nu < \frac{1}{2}$.

In the second case, let us suppose that

$$|\,|m_B(\sigma)| - m^*| \leq d' \quad \text{and} \quad |\,|m_A(\sigma)| - m^*| \leq d'$$

Then, using (5.19) and the fact that the total magnetization is equal to $m \geq 0$, it is clear that $m_B \approx m^*$ and $m_A \approx -m^*$ for d' small enough, so that

$$| \, |A| - \frac{(m^* - m)}{2m^*} N^2 | \leq C_2 d' + 2CN^{2\nu - 1}$$

for a suitable constant C_2. Thus, by taking d' small enough independently of L, we get a contradiction with (5.19). The lemma follows from the trivial estimate

$$\mu_A^{\pm, b}\{| \, |m_A| - m^*| \geq d'\} \leq \mu_A^{+, b}\{|m_A - m^*| \geq d'\} \quad \square$$

We are left with the estimate of the two terms in the RHS of (5.16), that is, thanks to (5.17), with the estimate of

$$\sup_{\substack{A \subset Q_L \\ dN^2 \leq |A| \leq N^2 - N^{2b}, \, |\partial^+ A| \leq CN^{1+2\nu}}} \mu_{A,b}^{\beta, J, +}\{|m_A(\sigma) - m^*| \geq d\} \tag{5.20}$$

where $J = J_0^{\square}(\Lambda)$, so that the $+$ boundary conditions act only on $\delta A \setminus \delta \Lambda$. Let, for any $\sigma \in \Omega_A$,

$$\mathcal{C}(\sigma) = \{x \in A : \text{ there exists a path from } x \text{ to } \partial\Lambda \text{ inside } A \text{ such that } \sigma(y) = -1 \text{ along the path } \}$$

Notice that if $u \in A$ is such that $d(u, \Lambda^c) \geq N^{2b}$ then $\mu_{A,b}^{\beta, J, +}$ almost surely, $u \notin \mathcal{C}(\sigma)$, since, otherwise there would be a cluster of minus spins inside A with area larger than N^{2b} and, by consequence there would be a large contour inside A. Hence

$$|\mathcal{C}(\sigma)| \leq 4N^{1+2b} \qquad \mu_{A,b}^{\beta, J, +} - \text{ almost surely} \tag{5.21}$$

We then denote by $\mathcal{C}^*$ the collection of all $\mathcal{C} \subset A$ such that $\mu_{A,b}^{\beta, J, +}\{\mathcal{C}(\sigma) = \mathcal{C}\} > 0$ and we write

$$\mu_{A,b}^{\beta, J, +}\{|m_A(\sigma) - m^*| \geq d\} \leq \sup_{\mathcal{C} \in \mathcal{C}^*} \mu_{A,b}^{\beta, J, +}\{|m_A(\sigma) - m^*| \geq d \, | \, \mathcal{C}(\sigma) = \mathcal{C}\} \tag{5.22}$$

Given now $\mathcal{C} \in \mathcal{C}^*$, let $\bar{\mathcal{C}} = \mathcal{C} \cup \partial^+ \mathcal{C} \cup \partial Q_L$ and let $A' = A \setminus \bar{\mathcal{C}}$. By construction we have that any $\sigma \in \Omega_{A'}$ such that $\mathcal{C}(\sigma) = \mathcal{C}$ satisfies $\sigma(x) = +1 \quad \forall \, x \in \partial^+ A'$. Moreover, thanks to (5.21), we have that

$$|A'| \geq \frac{3d}{4} N^2$$

for any N large enough, so that

$$|m_{A'}(\sigma) - m_A(\sigma)| \geq \frac{d}{2}$$

for any N large enough. Thus we can bound from above a generic term in the RHS of (5.22) by

$$\begin{aligned} \mu_{A,b}^{\beta, J, +}\{|m_A(\sigma) - m^*| \geq d \, | \, \mathcal{C}(\sigma) = \mathcal{C}\} &\leq \mu_{A',b}^{\beta, +}\{|m_{A'}(\sigma) - m^*| \geq \frac{d}{2}\} \leq \\ \leq \mu_{A',b}^{\beta, +}\{m_{A'}(\sigma) \leq m^* - \frac{d}{2}\} &+ \frac{\mu_{A'}^{\beta, +}\{m_{A'}(\sigma) \geq m^* + \frac{d}{2}\}}{\mu_{A'}^{\beta, +}\{ \text{ all contours are small } \}} \end{aligned} \tag{5.23}$$

Using Lemma 3.1 of [I2] (see the appendix of [SS1] for a simplified proof) we have

$$\mu_{A',b}^{\beta, +}\{m_{A'}(\sigma) \leq m^* - \frac{d}{2}\} \leq C \exp\left(-CN^{2-4b}\right) \tag{5.24}$$

for a suitable positive contant C. To bound the other term, since

$$|\partial A'| \le |\partial A| + |\bar{\mathcal{C}}| \le C N^{1+2\nu} + 20 N^{1+4b} + 4N \le N^{\gamma} \qquad \gamma < 2$$

for any N large enough, we can use inequality (4.8) in [I2] and we get

$$\mu_{A'}^{\beta,+}\{m_{A'}(\sigma) \ge m^* + \frac{d}{2}\} \le C_1 \exp\Big(-C_1 N^2\Big) \tag{5.25}$$

for any N large enough a for a suitable constant C_1. We finally estimate the denominator in (5.23). We have

$$\begin{aligned}
&\mu_{A'}^{\beta,+}\{\text{ all contours are small }\} \ge \\
&\ge 1 - \mu_{A'}^{\beta,+}\{\text{ there exists a } *\text{−path of } - \text{ spins with diameter} \ge L^b\ \} \ge && (5.26)\\
&\ge 1 - \mu^{\beta,+}\{\text{ there exists a } *\text{−path of } - \text{ spins with diameter} \ge L^b\ \} && (5.27)
\end{aligned}$$

where we have used the FKG property in the second inequality and $\mu^{\beta,+}$ denotes the infinite volume plus phase. Finally, using the results of [CCS], the RHS of (5.27) tends to zero as $N \to \infty$. If we now combine (5.24), (5.25), (5.26) and (5.27) we get that

$$\liminf_{N\to\infty} -\frac{1}{\beta N} \log \mu_{\Lambda}^{\beta,\emptyset}\{m_{\Lambda}(\sigma) = m \,|\, K^c\} = +\infty$$

and Proposition 2.3 follows. □

Bounds on (5.15) when $\underline{\gamma} = \emptyset$

We finally estimate

$$\mu_{\Lambda,b}^{\beta,\emptyset}\{m_{\Lambda}(\sigma) = m \,|\, \mathcal{G}_b = \emptyset\} \tag{5.28}$$

For this purpose let $\bar{\Lambda} = \{x \in \Lambda : d(x, \partial\Lambda) \ge N^{2b}\}$. Then we have that $\mu_{\Lambda}^{\beta,\emptyset}$ almost surely there exists a closed $*$−path $\mathcal{C} \subset \Lambda \setminus \bar{\Lambda}$ encircling $\bar{\Lambda}$ such that either $\sigma = +1$ on $\mathcal{C}$ or $\sigma = -1$ on $\mathcal{C}$. This is so because on the contrary there would be two paths of spins of opposite sign connecting $\partial\Lambda$ with $\partial\bar{\Lambda}$. Bu this implies the existence of a b−large contour between them. Given now a closed $*$−path $\mathcal{C} \subset \Lambda \setminus \bar{\Lambda}$ encircling $\bar{\Lambda}$ let A be the region enclosed by $\mathcal{C}$. Since by construction $|m_{\Lambda}(\sigma) - m_A(\sigma)| \le 4N^{2b-1}$, we get

$$(5.28) \le \sup_{\tau=\pm} \sup_{\substack{A \subset \Lambda \text{ connected} \\ A \supset \bar{\Lambda}}} \mu_{A,b}^{\beta,\tau}\{|m_A(\sigma) - m| \le 4N^{2b-1}\} \tag{5.29}$$

We can now proceed as in the previous case and obtain

$$\liminf_{N\to\infty} -\frac{1}{\beta N} \log\Big[\sup_{\tau=\pm} \sup_{\substack{A \subset \Lambda \text{ connected} \\ A \supset \bar{\Lambda}}} \mu_{A,b}^{\beta,\tau}\{|m_A(\sigma) - m| \le 4L^{2b-1}\}\Big] = +\infty \quad \square$$

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