On necessary and sufficient conditions for uniform integrability of families of Hamiltonian systems.

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To the memory of Ricardo Mañé

Abstract. In [Po] Ch. V, specially §81, H. Poincaré discussed an obstruction to uniform integrability of families of Hamiltonians. (That is, the existence of changes of variables analytic in the parameter ϵ and in the variables that make the family of Hamiltonians a function of only action variables). We examine his proof and discover that, for non-degenerate systems, this condition is also sufficient for the integrability to first order in the parameter (That is, there exist analytical changes of variables, analytic in ϵ so that the family in these new variables depends only on the action variables up terms which are $o(\epsilon)$.) This leads to the existence of obstructions in higher order. We show that the vanishing of the obstructions to order n is sufficient for the existence of analytic and symplectic changes of variables analytic in the parameter ϵ that reduce the system to integrable up to errors of order ϵ^{n+1}. Moreover, we show that the vanishing of all the obstructions means that the system is uniformly integrable. This answers the question posed by Poincaré at the end of his chapter V. We note that these obstructions have a geometric meaning and they are cohomology obstructions computed on periodic orbits of the system.

1. Introduction

Many problems in Mechanics can be reduced to the study of a family of Hamiltonian systems H_ϵ where H_0 is an integrable system. (By an integrable system we mean one whose phase space admits action-angle variables and such that the Hamiltonian depends only on the actions.)

At the early times in Mechanics — and more incomprehensively, up to recently — there was widespread hope that almost all families H_ϵ could be systematically reduced to integrable by a canonical change of variable. (Found, e.g., solving the Hamiltonian-Jacobi equations.)

That is, there was hope that one could find a family of canonical transformations g_ϵ in such a way that

$$H_\epsilon \circ g_\epsilon = I_\epsilon \tag{1.1}$$

where I_ϵ is only a function of the actions.

The fact that these hopes can not be fulfilled – even in the very weak sense of formal power series – was demonstrated very eloquently in [Po]. The whole of chapter V is devoted to finding obstructions for uniform integrability. In particular, in §81 one can find obstructions for the existence of analytic solutions of (1.1) that depend analytically on ϵ (uniform integrability) and in §84, the conditions are extended slightly and verified for the three-body problem. At the end of [Po] Chapter V (p. 259) the question is raised of whether these obstructions are sufficient. Of course, the more recent K. A. M. theorem shows that provided some non-degeneracy conditions are met, one can find solutions of (1.1) but only on a Cantor set.

The goal of this paper is to reexamine in more modern language the circle of ideas introduced in [Po]. We will show that the criterion of non-integrability introduced in [Po] can be strengthened and that this strengthening is necessary and sufficient for uniform integrability when the unperturbed system is non-degenerate.

We point out that a thorough analysis of the degenerate situation was undertaken in [Ga]. There, the situation is quite different and there are examples with systems integrable to all orders but not uniformly integrable. In [Ga], one can also find sufficient conditions for convergence of perturbation expansions of some degenerate situations.

The obstructions to integrability we will find are integrals over periodic orbits. They are equivalent to those of [Po] even if they are usually stated in terms of Fourier series coefficients. We try to write everything in terms of periodic orbits because the geometric meaning is clearer. In particular, the obstructions can be interpreted as cohomology obstructions for existence of cocycles.

The main technical tool to prove the converse of the criterion will be a result that states that if the obstructions vanish on periodic orbits, a cohomology equation has a solution. This is somewhat reminiscent of the celebrated Livsic theorem [Li] and its use in rigidity results for Anosov diffeomorphisms. Of course, in our case there is no hyperbolic behavior, hence, the proofs are very different. In that respect we call attention to [Ve], where in is shown that cohomology equations for linear maps on the torus can be solved using only obstructions on periodic orbits. The methods, however are very algebraic and do not generalize to our situations.

We will prove two types of results: some that have to do with asymptotic expansions and their feasibility and some more complicated ones that discuss the convergence of these expansions.

Even if the obstructions and asymptotic results need only a finite number of derivations and simple minded calculations, the converse of the full set of obstructions — that is, the theorem that shows that if all the obstructions vanish, the system is uniformly integrable — does require analyticity and, as far as we know, an iterative process similar to those K. A. M. theorem.

The same circle of ideas applies to other related problems that have appeared in the literature. We will also discuss the problem of relative integrability. That is, the existence of a family of canonical transformations g_ϵ and a function of a real variable F_ϵ such that:

$$H_\epsilon \circ g_\epsilon = F_\epsilon(H_0) \tag{1.2}$$

This means that H_ϵ describes the same motions as H_0 up to a change of variables and a reparametrization of the orbits. That is, the orbits of H_ϵ are the same as that of H_0

under a change of variables, even if they may be transversed at different speed. The change of speed depends only on the energy surface.

We emphasize that the results we present here deal exclusively with the problem of uniform integrability. In particular, we do not discuss related problems such as the integrability for open domains of the parameter. We also call attention to the fact that we use integrability in a well defined sense that involves changes of variables that are analytic and with analytic inverses. The literature abounds with weaker notions (multi-valued integrals, integrals with singularities etc.) which are sometimes also referred as integrability. We have little to say in this paper about these notions. It is clear that our methods rely on our precise definitions. When to avoid cluttering the sentences we just refer to "integrability" we mean one of our precise definitions that we hope, will be clear from the context.

2. Definitions, preliminary calculations

Even if several of the calculations we will perform will require only a finite number of derivatives, the main results indeed require analyticity. Hence, we will assume that all the Hamiltonians are analytic in all the parameters.

We will consider a phase space of the form $M = U \times \mathbb{T}^d$ where U is an open set in $\mathbb{R}^d$. (We will also use complexifications of this phase space.) Given a function $\eta : U \times \mathbb{T}^d \to \mathbb{C}$ we will denote the partial Fourier expansion as

$$\eta(A, \varphi) = \sum_{k \in \mathbb{Z}^d} \hat{\eta}_k(A) e^{2\pi i k \varphi}.$$

We denote the symplectic form by γ (and reserve ω for the frequencies). In the action-angle variables we have $\gamma = \sum_{i=1}^d dA_i \wedge d\varphi_i$.

Note that if the phase space admits action-angle variables we have $\gamma = d\theta$ where $\theta = \sum_i A_i d\varphi_i$. We will refer to θ as the symplectic potential. (Not all symplectic manifolds have a symplectic potential, but we are only concerned in this paper with those having action angle variables as above.)

A transformation f is called symplectic when $f_*\gamma = \gamma$. Equivalently, if $d(f_*\theta - \theta) = 0$. If, furthermore, we have $f_*\theta - \theta = dS$, we say that f is exact symplectic. Note that the notion of exact symplectic maps only makes sense for symplectic manifolds in which the symplectic form admits a potential.

We say that a Hamiltonian is integrable if it is a function only of the action variables. Perhaps it would me more appropriate to say that a system is integrable when it can be reduced to one of the above form by a canonical change of variables. However, the above notation is quite extended and we will follow it.

We will also be concerned with integrable systems that are non-degenerate in the following sense:

Definition 2.1. *We say that a Hamiltonian $H_0 : M \to \mathbb{R}$ is non degenerate if the mapping $\omega : U \to \mathbb{R}^d$ defined by $\omega = \frac{\partial H_0}{\partial A}$ is a diffeomorphism in its range and $\omega^{-1} : \omega(U) \to U$ is a uniformly analytic map.*

We will find it convenient to use the deformation calculus for families. If $\mathcal{G}_\epsilon$ is a family of diffeomorphisms we write:

$$\frac{d}{d\epsilon} g_\epsilon = \mathcal{G}_\epsilon \circ g_\epsilon$$

and we will refer to $\mathcal{G}_\epsilon$ as the generator of the family. Note that, for C^1 vector fields, $\mathcal{G}_\epsilon$ and g_0 determine uniquely g_ϵ.

The transformations are symplectic if and only if $g_{\epsilon *}\gamma = \gamma$. This happens if and only if $g_{0*}\gamma = \gamma$ and $\frac{d}{d\epsilon}(g_{\epsilon *}\gamma) = 0$. Using the formulas for the Lie derivative and the fact that $d\gamma = 0$, we see that a family is symplectic if and only if

$$g_{0*}\gamma = \gamma; \quad d(i(\mathcal{G}_\epsilon)\gamma) = 0$$

This motivates the following well known definitions:

Definition 2.2. *We say that a C^1 vector field $\mathcal{F}$ is locally Hamiltonian if and only if*

$$d(i(\mathcal{F})\gamma) = 0$$

We say that a C^1 vector field $\mathcal{F}$ is globally Hamiltonian if there exist a function F such that

$$i(\mathcal{F})\gamma = dF$$

Similarly, we say that g_ϵ is a locally (globally) Hamiltonian isotopy if its generator $\mathcal{G}_\epsilon$ is locally (globally) Hamiltonian. It is a well known result that g_ϵ is a family of symplectic mappings if and only if g_0 is symplectic and is a locally Hamiltonian isotopy. Similarly, it is a family of exact symplectic mappings if g_0 is exact symplectic and g_ϵ is a globally Hamiltonian isotopy.

Note that given a function F, we can compute dF and, because γ is non-degenerate, we can find a unique $\mathcal{F}$ such that $i(\mathcal{F})\gamma = dF$. Conversely, given $\mathcal{F}$, $i(\mathcal{F})\gamma = dF$ determines F up to an additive constant on every connected component.

Definition 2.3. *Let H_0 be an integrable Hamiltonian, H_ϵ be an analytic family agreeing with H_0 when $\epsilon = 0$. We say that H_ϵ is a uniformly integrable family if we can find an analytic family of canonical transformations g_ϵ and a family of integrable Hamiltonians I_ϵ such that*

$$\begin{aligned} &H_0 = I_0 \ , \qquad g_0 = Id \\ &H_\epsilon \circ g_\epsilon = I_\epsilon \end{aligned} \tag{2.1}$$

In the literature, one sometimes finds the notion of relatively integrable family.

Definition 2.4. *We say that H_ϵ is H_0-integrable if we can find a family of canonical transformations g_ϵ and a C^r-family of real valued functions of a real variable F_ϵ in such a way that*

$$\begin{aligned} &g_0 = Id \ , \quad F_0 = Id \\ &H_\epsilon \circ g_\epsilon = F_\epsilon(H_0) \end{aligned} \tag{2.2}$$

We will refer to the g_ϵ as the integrating transformations.

The reason to introduce the concept of relative integrability is that if (2.2) is met, we can understand the dynamics of all the Hamiltonians H_ϵ in terms of the dynamics of H_0. This is advantageous when the dynamics of H_0 is "well understood." Of course, when H_0 is integrable, the dynamics is well understood because of the explicit solution. Nevertheless, a system may have dynamics for which a great deal is known even if there are no explicit formulas.

Note also that the problem of relative integrability is geometrically more natural than the problem of integrability. It does not use the fact that there exist action and angle variables and, hence, it makes sense to pose the problem in any manifold. In particular, there are less stringent notions of action-angle variables (see e.g. [Du]) where much of the ideas here could carry over. In this paper, however, we will only consider the problem of relative integrability with respect to an integrable system. The papers [CEG] and [LMM] consider the problem of relative integrability of Anosov systems, which certainly do not have action angle variables.

In the relatively integrable case, we will use some more geometric structures. Note that for E real – or complex – we can define the energy surface $\Sigma_E = H_0^{-1}(E)$. Recall also that the symplectic form γ defines a volume form γ^d. If the energy surface does not contain critical points of the Hamiltonian, we can introduce a natural $2d-1$ form μ on the energy surface by $\gamma^d = d\mu_E \wedge dH_0$. Since both H_0 and γ are invariant under the flow of H_0, so is μ. We will assume that the Hamiltonians we consider are defined on domains $H_0^{-1}(\Omega)$ where Ω is an interval in $\mathbb{R}$ or a complex domain containing a real interval. We will furthermore assume that there are no critical points in this set and, moreover that the energy surfaces are compact for real values of the energy.

Note that the I_ϵ, g_ϵ that integrate a family of Hamiltonians are not unique. In effect, if $h_\epsilon(A,\varphi) = (A + \Delta_\epsilon, \varphi)$ then,

$$H_\epsilon \circ g_\epsilon \circ h_\epsilon = I_\epsilon \circ h_\epsilon = \widetilde{I}_\epsilon$$

where $\widetilde{I}_\epsilon$ defined by $\widetilde{I}_\epsilon(A) = I_\epsilon(A + \Delta_\epsilon)$ depends only on the actions.

Similarly if $K_\epsilon(A)$ is the generator of k_ϵ, $k_0 = Id$

$$k_\epsilon(A,\varphi) = (A, \varphi + \Gamma_\epsilon(A))$$

(where $\Gamma_\epsilon(A) = \int_0^\epsilon \frac{\partial K_\sigma}{\partial A}(A)\, d\sigma$) and again, $H_\epsilon \circ g_\epsilon \circ k_\epsilon = I_\epsilon \circ k_\epsilon = \widetilde{\widetilde{I}}_\epsilon$ where $\widetilde{\widetilde{I}}_\epsilon$ depends only on the actions.

Similarly, for relatively integrable systems, we observe that if g_ϵ integrates the system and k_ϵ has Hamiltonian $K_\epsilon = K_\epsilon(H_0)$ then $g_\epsilon \circ k_\epsilon$ also integrates the system.

We will use this elementary observations to show that, without loss of generality the integrating transformations may be assumed to satisfy some extra normalizations. This will simplify some of the calculations. Note also that, unless some normalizations are imposed, it will be hard to perform any analysis since these trivial factors may ruin convergence of any limiting process.

Related to the concept of the integrability but better adapted to perturbation expansions is the concept of asymptotic integrability.

Definition 2.5. *We say that an analytic system H_ϵ is "asymptotically integrable to order n", if we can find a g_ϵ and an I_ϵ as before such that:*

$$H_\epsilon \circ g_\epsilon = I_\epsilon(A) + o(\epsilon^n) \tag{2.3}$$

Similarly we say that it is "asymptotically relatively integrable up to order n" if we can find g_ϵ, F_ϵ in such a way that

$$H_\epsilon \circ g_\epsilon = F_\epsilon(H_0) + o(\epsilon^n) \tag{2.4}$$

Note that, if a H_ϵ is asymptotically integrable to order n, and $\tilde{H}_\epsilon$ is another system such that $H_i = \tilde{H}_i$ for $i = 0, \ldots, n$ then $\tilde{H}_\epsilon$ is also integrable and we can use the same g_ϵ. Similarly, if g_ϵ integrates a system asymptotically to order n and $\tilde{g}_\epsilon$ agrees with g_ϵ up to order $n+1$ then $\tilde{g}_\epsilon$ also integrates the system. That is, asymptotic integrability up to order n is a property of the Taylor expansion in ϵ up to order n of H and can be verified using only the Taylor expansion to order n of g.

Since we will have to deal often with truncated Taylor expansions, we introduce the notation for Hamiltonians

$$H_\epsilon^{[\le n]} = \sum_{i=0}^{n} H_i \epsilon^n$$

and, similarly, $g^{[\le n]}$ will be the globally Hamiltonian isotopy starting at g_0 and of Hamiltonian $G_\epsilon^{[\le n]}$. We will also use the notation with other subscripts with inequalities. They are meant to describe the range of the sum in a power series and, for deformations the deformation where the Hamiltonian is a sum of powers of ϵ similarly restricted.

Note that if a system is integrable, then it is asymptotically integrable to order n for all n. The converse is certainly not automatic — it is clearly false for families C^∞ in

ϵ. Take $\widetilde{H}_\epsilon = H_{\exp 1/\epsilon^2}$ where H_ϵ is not integrable. Note also that one system could be integrable to all orders in n but that the transformations that achieve this integrability could be quite different – they are not unique, as we have shown –. Nevertheless one of the main results of this paper is that analytic Hamiltonians satisfying Definition 2.1 that are integrable to all orders are uniformly integrable.

The following proposition will be important to address the non-uniqueness question.

Proposition 2.6. *Assume that H_ϵ is integrable (resp. relatively integrable) to order n. then, it admits an integrating family such that*

$$\int_{\mathbb{T}^d} G^i = 0; \quad 0 \le i \le n$$

$$(\text{resp.} \int_{\Sigma_E} G^i = 0; \quad 0 \le i \le n)$$

Proof. For the uniformly integrable case, note that if $\tilde{g}_\epsilon$ integrates the problem to order n then for any k_ϵ such that $K_\epsilon = K_\epsilon(A)$, $g_\epsilon = \tilde{g}_\epsilon \circ k_\epsilon$ also integrates the problem. We claim that it is possible to choose K_ϵ in such a way that the normalization is satisfied.

Note that the Hamiltonian of g_ϵ is $G_\epsilon = \tilde{G}_\epsilon + K_\epsilon \circ \tilde{g}_\epsilon$. If we expand in powers of *epsilon*, we obtain the the coefficient of ϵ^i has the form $G^i = \tilde{G}^i + K^i + R^i$ where R^i is an expression involving only terms of lower order. Therefore, we can recursively pick $K^i = -\int_{\mathbb{T}^d} \tilde{G}^i + K^i$.

In the relatively integrable case, the same argument tells us that it suffices to pick recursively $K^i = -\int_{\Sigma_E} \tilde{G}^i + K^i$.

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3. Obstructions to uniform integrability and statement of results.

Since the solutions of (2.1), if they exist, are highly non-unique (as illustrated by the remarks after the definition) to derive obstructions, it will be useful to show that if there exist solutions, they can be found in a restricted class.

Proposition 3.1. *Assume that $H_\epsilon \circ g_\epsilon = I_\epsilon$ on $U \times \mathbb{T}^d$, $g_0 = Id$ then, we can find a family t_ϵ of the form*

$$t_\epsilon(A, \varphi) = (A + \Delta_\epsilon, \varphi) \ , \qquad \Delta_0 = 0$$

in such a way that $H_\epsilon \circ g_\epsilon \circ t_\epsilon = I_\epsilon \circ t_\epsilon$ and $g_\epsilon \circ t_\epsilon$ is a globally Hamiltonian isotopy.

In other words, if there is an integrating family defined in a certain domain, we can find a globally symplectic transformation defined in a slightly different domain. Hence, to exclude the existence of integrating transformations, it suffices to exclude the existence of globally Hamiltonian ones.

Proof. We denote by [ ] the cohomology class of a form and by # the operator induced on cohomology by a transformation.

Since

$$\frac{d}{d\epsilon} g_\epsilon \circ t_\epsilon = (\mathcal{G}_\epsilon + g_\epsilon * \mathcal{T}_\epsilon) \circ g_\epsilon \circ t_\epsilon$$

the family $g_\epsilon \circ t_\epsilon$ will be globally Hamiltonian if and only if

$$[i(\mathcal{G}_\epsilon)\gamma] + g_{\epsilon\#}[i(\mathcal{T}_\epsilon)\gamma] = 0 \tag{3.1}$$

Since $g_0 = Id$, g_ϵ is isotopic to the identity and, therefore, $g_{\epsilon\#}$ is the identity.

Moreover, $\mathcal{T}_\epsilon = (\dot{\Delta}_\epsilon, 0)$ and

$$i(\mathcal{T}_\epsilon)\gamma = \sum_i \dot{\Delta}_{\epsilon,i}\, d\varphi_i$$

since $[d\varphi_i]$ are a basis for the first cohomology in our phase space, we can compute $-\dot{\Delta}_{\epsilon,i}$ as the component of $[d\varphi_i]$ of $[i(\mathcal{G}_\epsilon)\gamma]$. Hence, we can compute Δ_ϵ by integration.

Note that, if $\mathcal{G}_\epsilon$ is analytic in ϵ, so is Δ_ϵ.

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To obtain obstructions for asymptotic integrability to order 1 — a fortiori obstructions to uniform integrability —, following [Po] §81, we take derivatives with respect to ϵ on (2.1) and obtain

$$(\dot{H}_\epsilon + \{G_\epsilon, H_\epsilon\}) \circ g_\epsilon = \dot{I}_\epsilon + o(\epsilon) \tag{3.2}$$

where $\{\quad\}$ denotes Poisson brackets and $\dot{H}$ denotes derivatives with respect to ϵ.

Evaluating at $\epsilon = 0$ we obtain $H_1 + \{G_0, H_0\} = I_1$. Furthermore, we note that $\{G_0, H_0\} = L_{\mathcal{G}_0} H_0 = -L_{\mathcal{H}_0} G_0$.

Hence, if β is a periodic orbit of the flow generated by $\mathcal{H}_0$,

$$\int_\beta H_1 + \{G_0, H_0\} = \int_\beta H_1 - L_{\mathcal{H}_0} G_0 = \int_\beta H_1$$

because the integral along a periodic orbit of the derivative along the flow of a function vanishes.

Hence, we have proved, following [Po] §81

Theorem 3.2. *A necessary condition for the family H_ϵ to be asymptotically integrable to order 2 is that*

$$\frac{1}{|\beta|} \int_\beta H_1$$

is the same for all periodic orbits of H_0 with the same action variables.

Moreover, since I_1 is a function of the actions only and the actions are conserved along the flow of H_0, if we denote by $A(\beta)$ the values of the actions and by $|\beta|$ the length of the periodic orbit $\int_\beta I_1 = I_1(A(\beta))|\beta|$. Hence, we should have $I_1 = \frac{1}{|\beta|} \int_\beta H_1$

Note that $\{A\} \times \mathbb{T}^d$ contains a periodic orbit if and only if $\omega(A) = \frac{1}{T} p$ with $p \in \mathbb{Z}^d$. In that case $\{A\} \times \mathbb{T}^d$ is foliated by periodic orbits that can be parameterized by $\mathbb{T}^{d-1}$. If $\frac{1}{|\beta|} \int_\beta H_1$ is independent of all the orbits by averaging over the extra variables

$$I_1(A) = \int_{\mathbb{T}^d} H_1(A, \varphi)\, d\varphi$$

on a torus where there are periodic orbits. We will generalize this observation later.

Similarly, we have

Lemma 3.3. *A necessary condition for relative integrability up to first order is that, for any periodic orbit*

$$\frac{1}{\beta}\int_{\beta} H_1 = F_1(H_0(\beta))$$

Theorem 3.2 is the main result of §81 of [Po]. Even if the ideas presented here are the same, it is quite interesting to compare the notation there. In particular, we call attention to the fact that Poincaré phrases the problem as finding uniform integrals, that is analytic first integrals that depend analytically on the parameter. Of course, due to Liouville-Arnol'd theorem, the existence of integrals in involution is the same as integrability. But the formulation in terms of the number of integrals is more general.

In Chapter V of [Po] we find many variants of this idea and applications to several problems.

At the end of §86, we find

Les conditions énoncées dans ce Chapitre étant neccesaires main non suffisantes, rien ne prouve que cette troisième intégral existe; il convient avant de se prononcer, d'attendre la publication complète des résultats de M$^{\mathrm{me}}$ *de Kowaleski.* The footnote accompanying this paragraph is a one of the few paragraphs in a Mathematics book that are drama. Witness to that the industry – this paper belongs to it– of working out the sketches of these two giants.

Depuis que ces lignes ont été écrites le monde savant a eu à déplorer la morte prématurée de M$^{\mathrm{me}}$ *de Kowaleski. Les notes q'on a retrouvées chez elle sont malhereusement insuffisantes pour permettre de reconstituer ses démonstrations et ses calculs.*

In this paper we will be concerned with stating and proving converses to this obstruction to uniform integrability. The first of the main results of this paper is:

Theorem 3.4. *Assume that H_0, H_1 are defined on $U \times \mathbb{T}^d$ and that, H_0 is non-degenerate, H_1 satisfies the necessary conditions if Theorem 3.2. Then, we can find G_0 defined also in $U \times \mathbb{T}^d$ in such a way that*

$$H_\epsilon^{[\leq 1]} \circ g_\epsilon^{[\leq 0]} = I_\epsilon^{[\leq 1]} + o(\epsilon^2)$$

In other words, when H_0 is non-degenerate, the conditions in Theorem 3.2 are not only necessary but also sufficient for asymptotic integrability to order 2.

The proof of this result will be postponed till section 4, when we have developed some technical results needed in the proof. In that section, we will also show that it is possible to obtain estimates of the analyticity properties of G_0 in terms of the analyticity properties of H_1.

An interesting example that shows that the non-degeneracy conditions in Theorem 3.4 appears in the thesis of E. Meletlidou [Me].

Example 3.5. *Consider $H_\epsilon = H_0 + \epsilon H_1$, with $H_0 = \frac{1}{2}(A_1^2 + A_2^2)$ $H_1 = \cos(2\varphi_1 - \varphi_2)$.*

One can verify that this system has the integrals H_ϵ and $J_1 + 2J_2$, hence, one would be tempted to call it integrable. On the tori invariant by H_0 for which $2A_1 = 2A_2$, $\frac{1}{|\beta|} \int H_1 = \cos(2\varphi_1(0) - \varphi_2(0))$ which is not independent of the starting point.

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The reason why this example is not a contradiction with Theorem 3.4 is that precisely at the tori with $2A_1 = A_2$ the gradient of the second integral fails to be independent of the gradient of the first and, indeed, one can check that the integrating transformation cannot be analytic there.

This example points to the fact that some of the assumptions in the theorem are really necessary and, hence, that talking about "integrability" without attaching it a precise meaning can lead to confusion.

Once that we have necessary and sufficient conditions for asymptotically integrability up to order two, we can investigate integrability up to higher orders.

Clearly, the higher the order, the more necessary conditions will have to be met. It can happen that a system is asymptotically integrable up to a certain order but not to a higher one. (For example, if we take $H_{\epsilon^7} \circ g_\epsilon$ with H_ϵ not integrable to order 2, we obtain a family integrable to order 6 but not to order 7.) As in the criterion to exclude integrability to first order, we can derive necessary conditions for integrability up to high orders which are related to periodic orbits.

Lemma 3.6. *Let n be a natural number bigger than two, H_ϵ a family of Hamiltonians. Assume that there exists a $g_\epsilon^{[\leq n]}$, $I_\epsilon^{[<n]}$ in such a way that:*

$$H_\epsilon \circ g_\epsilon^{[\leq n]} = I_\epsilon^{[\leq n]} + o(\epsilon^{n+1})$$

A necessary condition for the existence of G^{n+1} in such a way that

$$H_\epsilon \circ g_\epsilon^{[\leq n+1]} = I_\epsilon^{[\leq n+2]} + o(\epsilon^{n+2})$$

is that, for any periodic orbit β of H_0 we have:

$$\int_\beta H_{n+1} + \int_\beta R_n = |\beta| I_{n+1}(A(\beta))$$

where R_n is an algebraic expression — described in the proof — involving $H_0, \dots, H_n$, $G_0, \dots, G_n$, Poisson-brackets, sums and products.

Proof. Since for any family of functions A_ϵ,

$$\frac{d}{d\epsilon} A_\epsilon \circ g_\epsilon = (\dot{A}_\epsilon + \{G_\epsilon, A_\epsilon\}) \circ g_\epsilon$$

proceeding by induction, we obtain

$$\left(\frac{d}{d\epsilon}\right)^{n+1} H_\epsilon \circ g_\epsilon = \left[\left(\frac{d}{d\epsilon}\right)^{n+1} H_\epsilon + \left\{ \left(\frac{d}{d\epsilon}\right)^{n} G_\epsilon, H_\epsilon \right\} + R_\epsilon^n \right] \circ g_\epsilon \tag{3.3}$$

where R_ϵ^n is an expression involving derivatives up to order $n-1$ of G and of H up to order n of H_ϵ, Poisson brackets, sums and products.

Evaluating (3.3) at $\epsilon = 0$ we obtain

$$\left(\frac{d}{d\epsilon}\right)^{n+1} H_\epsilon \circ g_\epsilon \bigg|_{\epsilon=0} = (n+1)! H_{n+1} + n! \{G_n, H_0\} + R_0^n$$

If (2.3) is to hold this should equal to $(n+1)!I_{n+1}(A)$.

As in the proof of Theorem 3.2 we observe that if β is a periodic orbit for H_0

$$\int_\beta \{G_n, H_0\} = 0$$

and hence, integrating over a periodic orbit, we obtain the statement of the claim.

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Remark. Note that the proof gives an explicit expression for the R_n. It suffices to evaluate the derivatives in (3.3). For example:

$$\begin{aligned}
R_2 &= \{G_0, H_1\} + \tfrac{1}{2}\{G_0, \{G_0, H_0\}\} \\
R_3 &= \tfrac{1}{6}\{G_0, H_1\} + \tfrac{1}{3}\{G_1, H_1\} + \{G_0, H_2\} + \tfrac{1}{6}\{G_1, \{G_1, H_0\}\} \\
&\quad + \tfrac{1}{6}\{G_1, \{G_0, H_0\}\} + \tfrac{1}{6}\{G_0, \{G_1, H_0\}\} + \tfrac{1}{6}\{G_0, \{G_0, H_1\}\} \\
&\quad + \{G_1, \{G_1, H_0\}\} + \tfrac{1}{3}\{G_0, \{G_0, H_1\}\} + \{G_0, \{G_0, \{G_0, H_0\}\}\}
\end{aligned}$$

Remark. Note that as we have expressed it, the obstruction to integrability to order $n-1$ depends on the integrating transformations up to order n we have selected. Nevertheless, it can be shown by an straightforward – albeit tedious – computation that this is not so. More interesting is to remark that it follows from our results that the vanishing of the obstructions does not depend on the choices. The notion of asymptotic integrability to order n is independent of any choices, but we show in this paper that it is equivalent to the vanishing of the obstructions.

Similarly, we have an statement for relative integrability.

Lemma 3.7. *Let n be a natural number bigger than two, H_ϵ a family of Hamiltonians. Assume that there exists a $g_\epsilon^{[\leq n]}$, $F_\epsilon^{[<n]}$ in such a way that:*

$$H_\epsilon \circ g_\epsilon^{[\leq n]} = F_\epsilon^{[\leq n]}(H_0) + o(\epsilon^{n+1})$$

A necessary condition for the existence of G^{n+1} in such a way that

$$H_\epsilon \circ g_\epsilon^{[\leq n+1]} = F_\epsilon^{[\leq n+2]}(H_0) + o(\epsilon^{n+2})$$

is that, for any periodic orbit β of H_0 we have:

$$\int_\beta H_{n+1} + \int_\beta R_n = |\beta| F_{n+1}(H_0(\beta))$$

where R_n is the same algebraic expression as in Lemma 3.6

The proof of this lemma is identical to that of Lemma 3.6.

■

By an argument very similar to the one that we will use in the proof of Theorem 3.4 we will obtain that, when H_0 is non-degenerate, the necessary conditions in Lemma 3.6 are not only necessary but also sufficient.

Theorem 3.8. *Assume that $H_0, \dots, H_{n+1}$, $G_0, \dots, G_{n-1}$ are defined on $U \times \mathbb{T}^d$ and that then they satisfy the necessary conditions in Lemma 3.6,(resp.Lemma 3.7) then, we can find G_n defined also on $U \times \mathbb{T}^d$ in such a way that*

$$H_\epsilon^{[\leq n+1]} \circ g_\epsilon^{[\leq n]} = I_\epsilon^{[\leq n+1]} + o(\epsilon^{n+2})$$

(resp.

$$H_\epsilon^{[\leq n+1]} \circ g_\epsilon^{[\leq n]} = F_\epsilon^{[\leq n+1]}(H_0) + o(\epsilon^{n+2}) \quad)$$

Moreover, the G_n that makes the above equation true is unique up to addition of a function of A (resp. a function of H_0).

Note that with these results provide us with an algorithm to show that a system is not uniformly (rep. relatively) integrable:

Starting with $n = 1$ compute the obstruction for asymptotic integrability to order $n+1$. If the obstruction does not vanish, we conclude that the system is not integrable up to order $n+1$. If it is, compute the Hamiltonian that integrates it up to order $n+1$ and start checking again.

Our final result shows that this algorithm is optimal for analytic systems. That is, that if it does not fail to exclude a family, then the family is uniformly integrable. We again emphasize that in the definition of family we have included analyticity with respect to the parameters.

Theorem 3.9. *Let H_ϵ be a family defined on $U \times \mathbb{T}^d$. Assume*

i) *H_0 is non-degenerate as in Definition 2.1*

ii) *For every n, we can find a family $g_\epsilon^{(n)}$ and a family of functions of the actions $I_\epsilon^{(n)}(A)$ such that*

$$H_\epsilon \circ g_\epsilon^{(n)} = I_\epsilon^{(n)}(A) + o(\epsilon^{n+1})$$

Then, we can find $\varphi \neq \widetilde{U} \subset U$ and an analytic family g_ϵ defined in $\widetilde{U} \times \mathbb{T}^d$ and such that

$$H_\epsilon \circ g_\epsilon = I_\epsilon(A) \qquad |\epsilon| \leq \epsilon^*$$

with $\epsilon^* > 0$.

We point out that the condition in Theorem 3.9 that H_0 satisfies Definition 2.1 really does belong. In [Ga] p. 369 one can find examples of systems that fail to satisfy Definition 2.1 and are not integrable. A simple example is

$$H_\epsilon(A_1, A_2, \varphi_1, \varphi_2) = \alpha A_1 + A_2 + \epsilon(A_2 + f(\varphi_1) f(varphi_2))$$

where α is the golden mean. The same idea can be used to produce examples where $\text{rank} \frac{\partial^2 H_0}{\partial A \partial A}$ is $1, 2, \cdots, d-1$.

The paper [Ga] studies the problem of finding analogues to Theorem 3.9 when $\text{rank} \frac{\partial^2 H_0}{\partial A \partial A}$ is 1,2. There, one can find conditions – that go beyond those found in [Po] and discussed here – that are also necessary and sufficient.

Two questions that we have not been able to explore are the converses for existence or a reduced number of first integrals. We also do not know whether existence of analytic integrating transformations for all values of ϵ in a complex neighborhood of zero imply that the system is uniformly integrable.

4. Existence and regularity for solutions of cohomology equations

We have shown already that the existence of asymptotic integrating transformations can be reduced to solving a hierarchy of equations of the form

$$\{G_n, H_0\} + I_n = -(n+1)H_{n+1} + \frac{1}{(n+1)!}R_n = \Gamma_n \tag{4.1}$$

where H_0, H_{n+1}, R_n are known and we are supposed to find G_n and I_n. We have already shown that a necessary condition for the existence of (4.1) is that $\int_\beta \Gamma_n$ depends only on the action.

Similarly, the problem of relative integrability reduces to the study of a hierarchy of equations of the form

$$\{G_n, H_0\} + F_n(H_0) = R_n$$

To prove Theorem 3.8, and the similar result for relative integrability, it suffices to show that when the obstructions found in Lemma 3.6, (resp. Lemma 3.7 are met, we can find analytic G_n and I_n. (resp. analytic G_n, F_n.)

We first argue that, we can determine F_n, I_n in such a way that we reduce the problem to considering the equation

$$\{G_n, H_0\} = \Gamma_n$$

with the compatibility conditions $\int_\beta \Gamma_n = 0$ for every periodic orbit of H_0.

We first observe that finding the I_n (for (2.1)) and the F_n's is easy.

In effect, if we consider

$$\Gamma_n^t(A,\varphi) = \frac{1}{t}\int_0^t \Gamma_n(A, \varphi + \omega(A)t)$$

they are equicontinuous functions. Moreover, if $\omega(A)$ is of the form $\frac{2\pi}{T}(p_1, \ldots, p_d)$ with $p_1, \ldots, p_d \in \mathbb{Z}$ they converge to the average over the periodic orbit. If the system is non-degenerate, the set of periodic orbits is dense. By Ascoli-Arzelá theorem, this sequence of functions is converging uniformly to a limit. By our hypothesis on periodic orbits, the limit depends only on the actions.

Hence, if the average on the periodic orbits depends only on the actions, (resp. (on the energy) of a periodic orbit, we obtain that $\Gamma_n^t(A,\varphi)$ converges uniformly as $t \to \infty$ to a function of A (resp. the energy) which, of course, agrees with the averages over periodic orbits. (Hence, we conclude, in particular, that the averages over periodic orbits of Γ_n have to be continuous.)

Now, we proceed to compute a more convenient form of these averages.

Then,

$$\begin{aligned}
\int_{\mathbb{T}^d} d\varphi\ \Gamma_n^t(A,\varphi) &= \frac{1}{t}\int_0^t \int_{\mathbb{T}^d} d\varphi\ \Gamma_n(A,\varphi+\omega(A)t) \\
&= \frac{1}{t}\int_0^t dt \int d\varphi\ \Gamma_n(A,\varphi) \\
&= \int_{\mathbb{T}^d} d\varphi\ \Gamma_n(A,\varphi) \\
\int_{\Sigma_E} d\mu_E\ \Gamma_n^t(A,\varphi) &= \frac{1}{t}\int_0^t \int_{\Sigma_E} d\mu_E\ \Gamma_n(A,\varphi+\omega(A)t) \\
&= \frac{1}{t}\int_0^t \int_{\Sigma_E} d\mu_E\ \Gamma_n \\
&= \int_{\Sigma_E} d\mu_E\ \Gamma_n
\end{aligned}$$

Hence, if Γ_n^t is converging uniformly to a function I_n only of A (resp. F_n only of H_0) we see that this function has to be

$$\begin{aligned}
&I_n(A) = \int_{\mathbb{T}^d} d\varphi\ \Gamma_n(A,\varphi) \\
&(\text{resp.} \qquad F_n(E) = \int_{\Sigma_E} d\mu_E\ \Gamma_n \quad).
\end{aligned} \tag{4.2}$$

These functions have to agree with the average on periodic orbits.

Note that both $I_n(A)$, $F_n(E)$ given by (4.2) are analytic functions of the position if Γ_n is.

Therefore, if we set I_n, (resp. F_n) to be given by (4.2) we reduce our problem to

$$\{G_n, H_0\} = \Gamma_n - I_n$$

(resp.

$$\{G_n, H_0\} = \Gamma_n - F_n \quad)$$

where, in both cases the R.H.S. have average zero over periodic orbits.

Remark. Since the problem of relative integrability is geometrically natural, the above reduction can be worked out in much larger generality. This is done in [LMM], pp. 549–553.

Remark. The I_n's and the F_n's play a role very similar to the averages that need to be taken off in the proofs of K. A. M. theory.

Therefore Theorem 3.4, Theorem 3.8 are reduced to the proof of:

Theorem 4.1. *Let H_0 be an integrable system satisfying Definition 2.1. Let Γ be an analytic function on $U \times \mathbb{T}^d$ satisfying:*

$$\int_\alpha \Gamma = 0$$

for all α periodic orbit of H_0. Then, there exists an analytic function G on $U \times \mathbb{T}^d$ such that

$$\{G, H_0\} = \Gamma$$

. This solution is unique up to addition of a function of the actions.

Proof. Since we will study the cohomology equation

$$\{G, H_0\} \equiv \omega(A) \frac{\partial}{\partial \varphi} G = \Gamma$$

using Fourier analysis, we first work out the implication of the compatibility conditions for the Fourier coefficients.

We observe that if $\omega(A_0) = \frac{1}{T} p$ with $T \in \mathbb{R}$, $p \in \mathbb{Z}^d$, $\{A_0\} \times \mathbb{T}^d$ consists of periodic orbits of period T and, for all $\varphi \in \mathbb{T}^d$,

$$0 = \frac{1}{T} \int_0^T \Gamma(A_0, \omega(A_0)t + \varphi)\, dt \ .$$

Using the partial Fourier expansion, we have

$$0 = \frac{1}{T}\int_0^T \sum_k \hat{\Gamma}_k(A_0) e^{2\pi i \frac{1}{T} p \circ k + k\varphi}\, dt$$
$$= \sum_{k\cdot p = 0} e^{2\pi i k\varphi} \hat{\Gamma}_k(A_0)$$

We denote by $\mathcal{P}$ the set of ω's that can be written as $\frac{1}{T}p$ for some $T \in \mathbb{R}^+$, $p \in \mathbb{Z}^d$. That is, the ω's that give rise to a periodic orbit.

Hence if the compatibility conditions are met,

$$\hat{\Gamma}_k(A_0) = 0 \text{ if } k \cdot \omega(A_0) = 0 \text{ and } \omega(A_0) \in \mathcal{P}$$

Now, we argue that the last condition $\omega(A_0) \in \mathcal{P}$ is superfluous. In effect, note that if $k \in \mathbb{Z}^d$ and $k \neq 0$ — say $k_d \neq 0$ — $\omega \cdot k = 0$ if and only if ω can be written as $(\omega_1, \dots, \omega_{d-1} - \frac{1}{k_d}(k_1\omega_1 + \cdots + k_{d-1}\omega_{d-1}))$. Since given any real numbers $\omega_1, \dots, \omega_{d-1}$ we can approximate then by rationals, we conclude that any $\omega \in \mathbb{R}^d$ with $\omega \cdot k = 0$ can be approximated by one with rational components, which, taking T as the lowest common denominator, is of the form $\frac{1}{T}p$ with $p \in \mathbb{Z}^d$.

We conclude that if $k \in \mathbb{Z}^d$, $k \neq 0$, $\mathcal{P} \cap \{\omega \in \mathbb{R}^d \mid \omega \cdot k = 0\}$ is dense on $\{\omega \in \mathbb{R}^d \mid \omega \cdot k = 0\}$. (Of course the result is also true when $k = 0$.) Since $\hat{\Gamma}_k$ is continuous it vanishes when $\{\omega(A) \cdot k = 0\}$ even if $\omega(A) \notin \mathcal{P}$.

We observe that, in terms of Fourier coefficients, the equation reads

$$\left(\omega(A)\frac{\partial}{\partial\varphi}G\right)^{\wedge}_k = 2\pi i(\omega(A)\cdot k)\hat{G}_k(A) = \hat{\Gamma}_k(A) \tag{4.3}$$

The regularity of the $\hat{G}_k$ and the $\hat{\Gamma}_k$'s is easier to study when we consider them as functions of ω. Since ω^{-1} is uniformly analytic, we can consider $\tilde{\Gamma}_k = \hat{\Gamma}_k \circ \omega^{-1}$, $\tilde{G}_k = \hat{G}_k \circ \omega^{-1}$. Then (4.3) reads

$$\tilde{G}_k(\omega)(\omega\cdot k)2\pi i = \tilde{\Gamma}_k(\omega) \tag{4.4}$$

but we have that $\tilde{\Gamma}_k(\omega) = 0$ when $\omega \cdot k = 0$. Hence, the solution

$$\tilde{G}_k(\omega) = \frac{1}{2\pi i}\begin{cases} (\omega\cdot k)^{-1}\tilde{\Gamma}_k(\omega)\ , & \omega\cdot k \neq 0 \\ \frac{1}{|k|^2}(k\cdot\partial_\omega)\tilde{\Gamma}_k(\omega)\ , & \omega\cdot k = 0 \end{cases} \tag{4.5}$$

is the only continuous solution. Certainly $\tilde{G}_k(\omega)$ is analytic if $\tilde{\Gamma}_k$ is (and hence $\hat{G}_k(A)$ is analytic if $\hat{\Gamma}_k$ is).

Recall that, by Cauchy inequalities we can bound

$$\left|(\partial_\omega)^n \tilde{\Gamma}_k(\omega)\right| \le n! K_\omega a_\omega^n e^{-\delta|k|} \quad \text{where} \quad a < 1, \ \delta > 0$$

It follows from (4.5) that $\tilde{G}_k$ satisfies similar bounds and hence, is analytic.

This finishes the proof of Theorem 4.1 and, hence the proof of Theorem 3.4, Theorem 3.8.

■

The proof of convergence will be obtained, not by estimating the recursion outlined before, but by using a quadratically convergent scheme of the K. A. M. type. This convergent scheme will, of course, require quantitative versions of Theorem 4.1 that not only claim the existence of the solutions.

As it turns out, this convergent scheme will need to be implemented on spaces of families of functions. Hence, our first task is to introduce appropriate spaces of families of functions and formulate quantitative versions of Theorem 4.1 for them.

Definition 4.2. *Given an open set $U \subset \mathbb{R}^d$ we can consider it also as a subset of $\mathbb{C}^d$ and define, for $\delta > 0$*

$$U^\delta = \{ z \in \mathbb{Z}^d \mid d(z, U) \le \delta \}$$

Given a non-degenerate frequency function we will denote $V = \omega(U)$

$$\|\eta\|^{\omega, U, \delta} = \sup_{\substack{z \in V^\delta \\ |\operatorname{Im} \varphi| \le \delta}} |\eta(\omega^{-1} z, \varphi)|$$

The set of functions for which the above norm is finite forms a Banach space which we will denote by $\chi^{\omega, U, \delta}$. Similarly, given an analytic family of non-degenerate frequency functions ω_ϵ we define for

$$\|\Gamma\|^{\omega, U, \delta} = \sup_{\substack{z \in V^\delta \\ |\operatorname{Im} \varphi| \le \delta \\ |\epsilon| \le e^\delta}} |\Gamma_\epsilon(\omega_\epsilon^{-1} x, \varphi)|$$

and denote by $\Phi^{\omega,U,\delta}$ the Banach space of families for which the above norm is finite.

The reason to introduce the norms in Definition 4.2 is that, for the solution of the cohomology equation,(4.5) the variables ω are more natural than the angle variables. This has the inconvenient that the norms depend on the frequency ω, but the dependence is slight as we will prove later.

Note that we denote the radius of the balls of the parameter ϵ by e^{δ}, where δ is the same parameter that measures the complex component in the complex extension of the torus. The exponential notation is justified so that the Cauchy inequalities look the same. The use of the same parameter for the two variables is justified just to keep down the number of parameters even it it leads to wasteful estimates. We also introduce the norms

$$||\eta||_{\infty}^{U,\delta} = \sup_{\substack{z\in U^{\delta} \\ |\operatorname{Im}\varphi|\le\delta}} |\eta(z,\varphi)|$$

$$||\Gamma||_{\infty}^{U,\delta} = \sup_{\substack{z\in U^{\delta} \\ |\operatorname{Im}\varphi|\le\delta}} |\Gamma_{\epsilon})|$$

and denote the corresponding Banach spaces by $\chi_{\infty}^{U,\delta}$, $\Phi_{\infty}^{U,\delta}$.

When the functions take values on matrices or on linear spaces we substitute for the absolute value in the R.H.S. of the above definitions the appropriate norms.

Note that we have that if $V = \omega(U)$ we have

$$||R||^{\omega,U,\delta} = ||R\circ\omega||_{\infty}^{V,\delta} \tag{4.6}$$

We clearly have that

$$||\eta||^{\omega,U,\delta} \le ||\eta||^{\omega,U,\delta+\sigma} \quad \text{for} \quad \sigma > 0$$

since we are taking the supremum over a larger set.

Note that $d(z,U) \le \alpha$ implies

$$d(\omega(z),V) \le \alpha||D\omega||_{\infty}^{U,\alpha}$$

Hence,

$$||\eta||_{\infty}^{U,\delta} \le ||\eta||^{\omega,U,\delta||D\omega||_{\infty}^{U,\alpha}}$$

and since $d(w, V) \le \alpha$ implies $d(\omega^{-1}(w), U) \le \alpha ||D\omega||_\infty^{V,\alpha}$ we have

$$||\eta||^{\omega,U,\delta} \le ||\eta||_\infty^{U,\delta ||D\omega^{-1}||_\infty^{\omega,U,\delta}}$$

The following are well known Cauchy bounds applied to the angular variables and the parameters. (We do not state the much better known inequalities for derivatives with respect to the A.)

Lemma 4.3.

$$\begin{aligned}
||\hat{\eta}||^{\omega,U,\delta} &\le e^{-|k|\delta} ||\eta||^{\omega,U,\delta} \\
||\hat{\eta}_k||_\infty^{U,\delta} &\le e^{-|k|\delta} ||\eta||_\infty^{U,\delta} \\
\left\| \frac{\partial^{|j|}}{\partial\varphi^j} \eta \right\|^{\omega,U,\delta-\sigma} &\le K\sigma^{-|j|} ||\eta||^{\omega,U,\delta} \\
||\eta^n||^{\omega,U,\delta} &\le K e^{-n\delta} ||\eta||^{\omega,U,\delta} \\
||\eta^{[\le N]}||^{\omega,U,\delta-\sigma} &\le K\sigma^{-1} ||\eta||^{\omega,U,\delta} \\
||\eta^{[\ge N]}||^{\omega,U,\delta-\sigma} &\le K\sigma^{-1} e^{-N\delta} ||\eta||^{\omega,U,\delta}
\end{aligned}$$

Before we embark in the proof of Theorem 3.9, let us state another technical lemma that will control the dependence of the norm $|| \, ||^{\omega,U,\delta}$ on ω. The main point of the lemma is that the change of the norms, when we change ω is controlled by the change in ω. We need, however to make sure that the domains match and that the change is not so drastic as to make the new frequency function non-invertible. In view of the observation (4.6) we see that the estimates that we need are just the customary estimates for the composition of functions that appear in almost all the proofs of K. A. M. theorems based in transformations.

Lemma 4.4. *Assume that ω, $\tilde{\omega}$ are non-degenerate frequency functions.*

Assume moreover

i) $\tilde{U}$ is such that $d(\tilde{U}, \mathbb{C}^d - U) > K||\omega - \tilde{\omega}||_\infty^{U,\delta} + K\sigma$

ii $||\omega - \tilde{\omega}||_\infty^{U,\delta} < 1/K$

where K depends only on $||\omega||_\infty^{U,\delta}$, $||\tilde{\omega}||_\infty^{U,\delta}$ $||\omega^{-1}||_\infty^{U,\delta}$, $||\tilde{\omega}^{-1}||_\infty^{U,\delta}$ and can be chosen uniformly when these norms can be bounded uniformly.

Then,

$$\big| ||R||^{\omega,\tilde{U},\delta} - ||R||^{\omega,\tilde{U},\delta} \big| \le C\sigma^{-1} ||R||^{\omega,U,\delta} ||\omega - \tilde{\omega}||_{\infty}^{U,\delta}$$

Analogous results hold for families.

Proof. We have:

$$\big| ||R||^{\omega,\tilde{U},\delta} - ||R||^{\omega,\tilde{U},\delta} \big| \le ||R \circ \omega^{-1} - R \circ \tilde{\omega}^{-1}||_{\infty}^{V,\delta}$$

where V is any domain such that $\omega(U) \subset V$, $\tilde{\omega}(U) \subset V$. If $\tilde{V}$ is such that $\lambda\omega^{-1}(x) + (1-\lambda)\tilde{\omega}^{-1}(x) \in \tilde{V}$ for $x \in V$ and $\lambda \in [0,1]$ we can bound this by

$$||\nabla R||_{\infty}^{\tilde{V},\delta} ||\omega^{-1} - \tilde{\omega}^{-1}||_{\infty}^{V,\delta}$$

Finally, if $\tilde{\tilde{U}}$ is such that $d(\tilde{U},\ \mathbb{C}^d - \tilde{\tilde{U}}) \ge \sigma$ we can use Cauchy bounds to bound the gradient. Moreover, given uniform bounds on the functions we can bound the difference of the inverses by the difference of the functions using the implicit function theorem. We therefore obtain the desired result.

The result for families is obtain by just applying this result for each value of ϵ and then taking suprema.

$\blacksquare$

Since in the proof of the theorem we will have to give up domains in a controlled way repeatedly, we adopt the convention that a domain with a ˜ means a domain related to the original one by a relation such as $i)$ above. The following is the main result of this section. It provides quantitative estimates for the solutions that were shown to exist in Theorem 4.1

Theorem 4.5. *Let I_ϵ be an analytic family of integrable systems with non-degenerate frequency function on U, with U a complex extension of a domain in the reals. Let R_ϵ be an analytic family in $\Phi^{\omega,U,\delta}$ Assume that δ is sufficiently small and that U is sufficiently close to the real and that R satisfies the compatibility conditions $\int_{\beta_\epsilon} R_\epsilon = 0$ for every β_ϵ periodic orbit of I_ϵ.*

Let G_ϵ be the solution of

$$\{I_\epsilon, G_\epsilon\} = R_\epsilon$$

which also satisfies $\int_{\mathbb{T}^d} G_\epsilon(A,\varphi) = 0$. *Then,*

$$\|G\|^{\omega,U,\delta-\sigma} \le \|R\|^{\omega,U,\delta}\sigma^{-d}K$$

where K *depends only on the dimension.*

Remark. Note that a non-degenerate system will not have periodic orbits outside of the real part of U if the extension is small enough. Clearly a frequency with an imaginary frequency will not lead to a periodic orbit. Once we fix a neighborhood, there is an open set of Hamiltonians with the same property. We will assume that all the perturbations are small enough that all our constructions do not leave this neighborhood.

Proof. By Cauchy estimates we have

$$\|\hat{R}_k\|^{\omega,U,\delta} \le \|R\|^{\omega,U,\delta}e^{-|k|\delta}$$

We also recall that the solution G was given by (4.5)

$$\hat{G}_{k\epsilon}\circ\omega_\epsilon^{-1}(\alpha) = \frac{1}{2\pi i}\begin{cases} (\alpha\cdot k)^{-1}\hat{R}_{k,\epsilon}\circ\omega_\epsilon^{-1}(\alpha)\ , & \alpha\cdot k\neq 0\\ \frac{1}{|k|^2}(k\cdot\partial_\alpha)\hat{R}_{k,\epsilon}(\alpha)\ , & \alpha\cdot k = 0\end{cases}$$

Hence, the supremum of $\hat{G}_k\circ\omega^{-1}$ is bounded by a constant times the supremum of $\frac{1}{|k|}\nabla(\hat{R}_k\circ\omega^{-1})$.

This can be bounded as follows. If $\tilde{U}$ is a domain in $\mathbb{C}^d$ such that $\tilde{U}\subset U$ and $d(\tilde{U},\ \mathbb{C}^d - U) > \sigma$ we have:

$$\begin{aligned}\|\hat{G}_k\|^{\omega,\tilde{U},\delta-\sigma} &\le K\frac{1}{|k|}\|\nabla\hat{R}_k\|^{\omega,\tilde{U},\delta-\sigma} \le K\sigma^{-1}\frac{1}{|k|}\|\hat{R}_k\|^{\omega,U,\delta}\\ &\le K\sigma^{-1}\frac{e^{-\sigma|k|}}{|k|}\|R\|^{\omega,U,\delta}\end{aligned}$$

Then

$$\begin{aligned}\|G_k\|^{\omega,U,\delta-\sigma} &\le \sum_{k\in\mathbb{Z}^d-\{0\}} e^{(\delta-\sigma)|k|}\|\hat{G}_k\|^{\omega,U,\delta} \le \sum_{k\in\mathbb{Z}^d-\{0\}} e^{-\sigma|k|}\frac{1}{|k|}\|\hat{R}\|^{\omega,U,\delta}\\ &\le K\sigma^{-1}\sum_{k\in\mathbb{N}-\{0\}} e^{-\sigma k}k^{d-2}\|R\|^{\omega,U,\delta} \le K\sigma^{-d}\|R\|^{\omega,U,\delta}\end{aligned}$$

∎

Remark. It is quite possible that the previous estimates are rather wasteful and that the negative power of σ can be greatly reduced. It seems that since (4.5) only loses one derivative we should only have a factor σ^{-1} in the estimates above. Part of the problem is due to the use of Fourier series and supremum norms in the way that we have done it. Note that if we wanted to estimate the identity operator using the sloppy bounds that we have employed, we would have obtained a $\sigma^{-(d-1)}$ factor, which, obviously does not belong. These issues are important to improve the smoothness conditions in the usual K. A. M. theorem. Since the main results in this paper have no hope of working in any regularity substantially smaller than analytic we will refrain from such improvements.

5. Proof of Theorem 3.9

As we have indicated before the proof will be done using a Newton algorithm to obtain quadratic convergence.

We proceed to describe the inductive steps. At step n we will have a family of Hamiltonians $H_\epsilon^{(n)}$

$$\begin{aligned} H_\epsilon^{(n)} &= I_\epsilon(A)^{(n)} + R_\epsilon^{(n)} \\ H_\epsilon^{(n)} &= F_\epsilon^{(n)}(H_0) + S_\epsilon^{(n)} \end{aligned} \tag{5.1}$$

where $R_\epsilon^{(n)}$ and $S_\epsilon^{(n)}$ are $o(\epsilon^{2^n})$. Since the splitting of the Hamiltonian in these parts would not be unique otherwise, we assume that $I_\epsilon^{(n)}$ contains only terms up to $O(\epsilon^{2^n-1})$ and that $\int_{\mathbb{T}^d} R_\epsilon^{[\leq 2^n-1]} d\varphi = 0$.

The inductive step will consist in determining

$$G_\epsilon^{(n)} = \sum_{i=2^n}^{2^{n+1}-1} G^{(n),i}\epsilon^i$$

(and $I_\epsilon^{(n+1)}$, $F_\epsilon^{(n+1)}$) in such a way that setting $H_\epsilon^{(n+1)} = H_\epsilon^{(n)} \circ g_\epsilon^{(n)}$ we can also find $R_\epsilon^{(n)}$, $I_\epsilon^{(n)}$ (resp. $S_\epsilon^{(n)}$, $F_\epsilon^{(n)}$) in such a way that they satisfy (5.1) with $n+1$ in place of n. We will describe the procedure to determine the $G_\epsilon^{(n)}$ once all the other elements are known and, then, we will develop enough bounds that imply convergence As it is customary in K. A. M. theory, the functions $H_\epsilon^{(n)}$ will be controlled in domains $U^{(n)} \times \mathbb{T}^d$ which will be decreasing as the iteration proceeds, (but which will not become empty). We will denote by $\omega^{(n)}$ the frequency function corresponding to $I^{(n)}$.

It is quite important to remark that because of the way that $H_\epsilon^{(n+1)}$ is produced, it will be integrable to all orders (resp. relatively integrable to all orders) if $H_\epsilon^{(n)}$ is.

The first step of the iteration will be obtained by taking $H_\epsilon^{(0)} = H_{\Lambda\epsilon}$ where Λ is a sufficiently large real number. Note that such $H_\epsilon^{(0)}$ is integrable (resp relatively integrable) to all orders in H_ϵ is. Moreover, by choosing Λ large enough, we can obtain that $||R_\epsilon^{(0)}||^{\omega,U,\delta}$ (resp. $||S_\epsilon^{(0)}||^{\omega,U,\delta}$) is small enough. Of course $H_0^{(0)} = H_0$ so that it is integrable and non-degenerate.

The quadratically convergent procedure will be based in finding a g_ϵ^n in such a way that $H_\epsilon^{(n+1)} \equiv H_\epsilon^{(n)} \circ g_\epsilon^{(n)}$ is much closer to an integrable family.

First, we will derive heuristically what are the equations that the $G_\epsilon^{(n)}$ have to satisfy, show that these equations admit solutions and estimate the new error. Then, it will become quite an standard procedure in K. A. M. theory to show that the $g_\epsilon^{(0)} \circ \cdots \circ g_\epsilon^{(n)}$ converge and the limit integrates the system.

The fact that we can find our $g_\epsilon^{(n)}$ will use the fact that that $H_\epsilon^{(n)}$ is integrable to all orders. Note that, by the definition of $H_\epsilon^{(n+1)}$, it will be integrable to all orders if $H_\epsilon^{(n)}$ is. For the experts in K. A. M. theory we note that, even if it follows from our result that the set of families integrable to all orders is a manifold, we cannot use it in the proof. In particular, we cannot use the formulation of K. A. M. theory as implicit function theorems and just use the fact that all the steps of the iterative procedure respect the set of families that are integrable to all orders. This is very similar to the problem of rotations of the circle as originally solved in in [Ar], where we do not know a priori that the set of maps with the same rotation number is a manifold – it is consequence of the theorem. Similar arguments happen in the papers – much closer to our problem – [CEG] and [Ga].

We will find it convenient to introduce a second auxiliary parameter μ. This may be justified by noting that we are talking about small in two different senses. One is of course, that ϵ is small, but we also have to consider distances between families whose parameter is ϵ.

If we consider the family $H_\epsilon^{(n)} \circ g_{\mu\epsilon}^{(n)}$ for $\mu \in [0,1]$ and apply the mean value theorem $F(1) = F(0) + F'(1) + E$, where $|E| \le \frac{1}{2} \sup |F''|$ we obtain – in the region where the flow does not leave the domain of definition of H_ϵ^n –

(5.2)
$$\begin{aligned} H_\epsilon^{(n)} \circ g_\epsilon^{(n)} &= H_\epsilon^{(n)} + \epsilon \{G^{(n)}, H_\epsilon^{(n)}\} \circ g_\epsilon^{(n)} + E \\ &= H_\epsilon^{(n)} + \epsilon\{G^{(n)}, H_\epsilon^{(n)}\} + [\epsilon\{G^{(n)}, H_\epsilon^{(n)}\} \circ g_\epsilon^{(n)} - \epsilon\{G^{(n)}, H_\epsilon^{(n)}\} \circ g_\epsilon^{(n)}] + E \end{aligned}$$

where E is an error term that, by the mean value theorem can be bounded by the supremum of the second derivative along the flow. We just recall that the second derivative along the flow can be expressed in terms of Poisson brackets

$$\frac{d^2}{d\mu^2} H_\epsilon^{(n)} \circ g_{\mu\epsilon}^{(n)} = \{G_\epsilon^{(n)}, \{G_\epsilon^{(n)}, H_\epsilon^{(n)}\}\} \circ g_{\mu\epsilon}^{(n)} \ . \tag{5.3}$$

Substituting (5.1) in (5.2) we obtain for the problem of uniform integrability

$$\begin{aligned} H_\epsilon^{(n)} \circ g_\epsilon^{(n)} &= I_\epsilon^{(n)} + R_\epsilon^{(n)[\le 2^{n+1}]} + R_\epsilon^{(n)[> 2^{n+1}]} + \epsilon\{G_\epsilon^{(n)}, I_\epsilon^{(n)}\} + \epsilon\{G_\epsilon^{(n)}, R_\epsilon^{(n)}\} + \\ &+ [\epsilon\{G^{(n)}, H_\epsilon^{(n)}\} \circ g_\epsilon^{(n)} - \epsilon\{G^{(n)}, H_\epsilon^{(n)}\}] + E \end{aligned} \tag{5.4}$$

and, for the problem of relative integrability:

(5.5)
$$\begin{aligned} H_\epsilon^{(n)} \circ g_\epsilon^{(n)} &= I_\epsilon^{(n)} + R_\epsilon^{(n)[\le 2^{n+1}]} + S_\epsilon^{(n)[> 2^{n+1}]} + \epsilon\{G_\epsilon^{(n)}, F_\epsilon^{(n)}(H_0)\} + \epsilon\{G_\epsilon^{(n)}, S_\epsilon^{(n)}\} + \\ &+ [\epsilon\{G^{(n)}, H_\epsilon^{(n)}\} \circ g_\epsilon^{(n)} - \epsilon\{G^{(n)}, H_\epsilon^{(n)}\}] + E \end{aligned}$$

Now, we work out the consequences of the system being integrable to all orders.

We note that if there was a g_ϵ that integrated the system, we argued n Proposition 2.6 that we could find one integrating g_ϵ such that $\int_{\mathbb{T}^d} G_\epsilon = 0$. If we apply (5.4) to this g_ϵ, substituting G_ϵ for $G_\epsilon^{(n)}$ in (5.4) we obtain:

$$\{G^0, I_0^{(n)}\} = 0$$

and, that and the normalization condition, allows us to conclude that $G^0 \equiv 0$ by the uniqueness statement in Theorem 4.1 In other words, if $H_\epsilon^{(n)}$ can be integrated – by a system with normalized Hamiltonian, as we showed without loss of generality – then $G^0 = 0$. Similarly, for terms whose order in ϵ is not more that $2^n - 1$ we obtain that

$$\{G^i, I_0^{(n)}\} = \Gamma_i$$

where Γ_i is an expression involving $G^0, \cdots, G^{i-1}$ that vanishes if $G^0, \cdots, G^{i-1}$ vanishes. (This is because the terms involving the second derivatives, looking at (5.3) have order in ϵ at least $2 + 2r$ if the G_ϵ is order r in ϵ.

That is, we have shown that if the system is integrable to all orders then, any integrating normalized transformation has to satisfy: $G^0 = G^1 = \cdots = G^{2^n-1} = 0$.

Once we know that the coefficients in the expansion of order up to $2^n - 1$ have to be zero, equal to zero, we realize that the only terms in (5.4) that contain terms of order less or equal than 2^{n+1} are the first, second and third.

In particular, we obtain that $R_\epsilon^{(n)[\leq 2^{n+1}]} + \epsilon\{G, I_\epsilon^{(n)}\} = I(A) + O(\epsilon^{2^{n+1}+1})$ Proceeding as in Theorem 3.2, integrating along periodic orbits of $I_\epsilon^{(n)}$, we obtain that if the system is integrable to all orders, then, $\frac{1}{|\beta_\epsilon|}\int_{\beta_\epsilon} R_\epsilon^{(n)[\leq 2^{n+1}]}$ is only a function of A.

We note that these are the compatibility conditions for the existence of $\Delta I_\epsilon^{(n)}$ $G_\epsilon^{(n)}$ solving

$$\Delta I_\epsilon^{(n)} + R_\epsilon^{(n)[\leq 2^{n+1}]} + \epsilon\{G_\epsilon^{(n)}, I_\epsilon^{(n)}\} = 0 \tag{5.6}$$

Analogously, we find that if $H_\epsilon^{(n)}$ was relatively integrable to all orders, the compatibility conditions for the equation for the equation

$$\Delta F_\epsilon^{(n)}(H_0) + S_\epsilon^{(n)[\leq 2^{n+1}]} + \epsilon\{G_\epsilon^{(n)}, F_\epsilon^{(n)}(H_0)\} = 0 \tag{5.7}$$

This is the step that makes all the method work because those are the main terms in (5.4) and (5.5). We want to obtain estimates on the resulting $G_\epsilon^{(n)}$'s, then estimate the remainder – in an slightly smaller region – and finally prove that the whole procedure converges.

We will write down with the notation of uniform integrability since the remaining of the procedure is exactly the same for the relative integrability case. As we have shown before, both cases rely on solving the same equation and the two identities (5.4), (5.5) are completely similar.

To obtain estimates for $G_\epsilon^{(n)}$ we observe that by Cauchy estimates Lemma 4.3 we have $||R_\epsilon^{(n)[\leq 2^{n+1}]}||^{\omega,U^{(n)},\delta-\sigma} \leq K\sigma^{-1}||R_\epsilon^{(n)}||^{\omega,U^{(n)},\delta-\sigma}$. Then, applying Theorem 4.5 we obtain:

$$||G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{U^{(n)}},\delta-2\sigma} \leq K\sigma^{-(d+1)}||R_\epsilon^{(n)}||^{\omega,U^{(n)},\delta} \tag{5.8}$$

Also, in an slightly smaller domain we will have:

$$\begin{aligned} ||\nabla G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-3\sigma} &\leq K\sigma^{-(d+2)}||R_\epsilon^{(n)}||^{\omega,U^{(n)},\delta} \\ ||\nabla^2 G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{U^{(n)}}},\delta-3\sigma} &\leq K\sigma^{-(d+3)}||R_\epsilon^{(n)}||^{\omega,U^{(n)},\delta} \end{aligned} \tag{5.9}$$

Since $G_\epsilon^{(n)}$ is chosen in such a way that the main terms in (5.4) cancel, the new remainder will be estimated by estimating systematically all the terms in (5.4) that are not involved in (5.6).

Using (5.9), provided that $\sigma \geq Ke^\delta\sigma^{-(d+2)}||R_\epsilon^{(n)}||^{\omega,U^{(n)},\delta}$ the solutions of the Hamilton equations corresponding to $G_\epsilon^{(n)}$ that start in $\widetilde{\widetilde{\widetilde{U^{(n)}}}} \times |\mathrm{Im}\varphi| \leq \delta - 4\sigma$ will not leave $\widetilde{\widetilde{U^{(n)}}}] \times |\mathrm{Im}\varphi| \leq \delta - 3\sigma$ for a "time" ϵ $|\epsilon| \leq e^\delta$. Then, we can estimate as follows:

$$\begin{aligned} ||\epsilon\{G_\epsilon^{(n)}, R_\epsilon^{(n)}\}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-2\sigma} &\leq K||\nabla G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-2\sigma}||\nabla R_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-2\sigma} \\ &\leq K\sigma^{-(d+3)}\left(||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}\right)^2 \end{aligned} \tag{5.10}$$

Similarly, we have

$$\begin{aligned} ||\nabla\epsilon\{G_\epsilon^{(n)}, H_\epsilon^{(n)}\}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-2\sigma} &\leq K||\nabla^2 G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-2\sigma}||\nabla H_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-2\sigma} \\ &\leq K\sigma^{-(d+5)}||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}||H_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}; \end{aligned}$$

(5.11)

$$||\epsilon\{G^{(n)},H_\epsilon^{(n)}\}\circ g_\epsilon^{(n)} - \epsilon\{G^{(n)},H_\epsilon^{(n)}\}\circ g_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-3\sigma} \le$$
$$\le ||\nabla\epsilon\{G_\epsilon^{(n)},H_\epsilon^{(n)}\}||^{\omega^{(n)},\widetilde{\widetilde{U^{(n)}}},\delta-2\sigma}||g_\epsilon^{(n)}-\mathrm{Id}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-3\sigma}$$
$$\le K\sigma^{-(d+5)}||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}||H_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}||\nabla G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{U^{(n)}}},\delta-2\sigma} \le$$
$$K\sigma^{-(2d+7)}||H_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}\left(||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}\right)^2$$

$$\begin{aligned}
&||\epsilon^2\{G_\epsilon^{(n)},\{G_\epsilon^{(n)},H_\epsilon^{(n)}\}\}\circ g_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-3\sigma}\\
&\qquad\le ||\epsilon^2\{G_\epsilon^{(n)},\{G_\epsilon^{(n)},H_\epsilon^{(n)}\}\}||^{\omega^{(n)},\widetilde{\widetilde{U^{(n)}}},\delta-2\sigma}\\
&\qquad\le K||\nabla^2 G_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{U^{(n)}}},\delta-2\sigma}||\nabla^2 H_\epsilon^{(n)}||^{\omega^{(n)},\widetilde{\widetilde{U^{(n)}}},\delta-2\sigma}\\
&\qquad\le K\sigma^{-(2d+12)}||H_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}\left(||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}\right)^2
\end{aligned} \tag{5.12}$$

$$||R_\epsilon^{(n)[\ge 2^{n+1}]}||^{\omega^{(n)},\widetilde{\widetilde{\widetilde{U^{(n)}}}},\delta-3\sigma} \le Ke^{-\delta 2^{n+1}}||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta} \tag{5.13}$$

Substituting these estimates in (5.4) we obtain

(5.14)
$$||R_\epsilon^{(n+1)}||^{\omega^{(n)},U^{(n+1)},\delta-3\sigma} \le K\sigma^{-a}||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}\left(||R_\epsilon^{(n)}||^{\omega^{(n)},U^{(n)},\delta}+Ke^{-\delta 2^{n+1}}\right)$$

Finally, we note that since

$$\omega^{(n+1)} = \omega^{(n)} + \nabla\int_{\mathbb{T}^d} R^{(n)}$$

we can obtain applying Lemma 4.4 as well as Cauchy bounds – again provided that smallness conditions are satisfied –.

(5.15)
$$||R_\epsilon^{(n+1)}||^{\omega^{(n+1)},U^{(n+1)},\delta-3\sigma} \le K\sigma^{-a}(||R^{(n)}||^{\omega^{(n)},U^{(n)},\delta-\sigma})^2(1+\sigma^{-2}||R^{(n)}||^{\omega^{(n)},U^{(n)},\delta-\sigma})$$

Once we have estimates such as those above, it is quite standard in K. A. M. theory that choosing at each iterative step $\sigma^{(n)} = \sigma_0 2^{-n}$, with σ_0 small enough so that the domain controlled does not reduce to the empty set, if the initial remainder is small enough we obtain a procedure that is converging quadratically. It is also possible to show that the smallness hypothesis that one needs are also satisfied.

Once we have this quadratic convergence, it is also quite straightforward to show that the composition $g_\epsilon^{(0)} \circ g_\epsilon^{(1)} \circ \cdots \circ g_\epsilon^{(n)}$ converges in a non empty domain.

6. Acknowledgements

I thank S. Tompaidis for several comments on a preliminary version of the manuscript and E. Meletlidou for providing Example 3.5. This research has been supported by NSF and TARP grants as well as an AMS Centennial Fellowship and and a URI from U.T. Austin.

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