Stochastic Burgers and KPZ equations from particle systems

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Abstract

We consider the weakly asymmetric exclusion process on the one dimensional lattice. It has been proven that, in the diffusive scaling limit, the density field evolves according to the Burgers equation [8, 19, 14] and the fluctuation field converges to a generalized Ornstein-Uhlenbeck process [8, 10]. We analyze instead the density fluctuations beyond the hydrodynamical scale and prove that their limiting distribution solves the (non linear) Burgers equation with a random noise on the density current. We also study an interface growth model, for which the microscopic dynamics is a Solid-On-Solid type deposition process. We prove that the fluctuation field, if suitably rescaled, converges to the Kardar-Parisi-Zhang equation. This provides a microscopic justification of the so called *kinetical roughening*, i.e. the non Gaussian fluctuations in some nonequilibrium processes. Our main tool is the Cole-Hopf transformation and its microscopic version. We also exploit the (known) connection between the two microscopic models and develop a mathematical theory for the macroscopic equations.

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1 Introduction

The hydrodynamical behaviour of physical systems is usually described by (non linear) PDE's. This description is in most of the cases approximate and, to model various neglected effects, a random forcing term can be added to the macroscopic equation. One is particularly interested in scale invariant forces, of which the space-time white noise is a typical example. This choice however introduces small scale singularities and poses the problem of the existence of the stochastic dynamics even when the deterministic equation is known to have good smoothing properties. Most rigorous results are restricted to one space dimension and several substantial problems appear in higher dimensions, see e.g. [1, 17]. On the other side, the question whether nonlinear stochastic PDE's, at least when they are well defined, are faithful descriptions of the evolution of suitable quantities defined on some particle dynamics is very natural. Moreover, by looking at a particle model, one may hope to understand how to to construct infinite dimensional diffusions associated to some ill–posed stochastic PDE's. The aim of this paper is to derive a stochastic version of the viscous Burgers equation and the Kardar–Parisi–Zhang (KPZ) equation as scaling limits of microscopic particle models.

The Burgers equation with random noise on the density current current is formally written as

$$\partial_t u_t = \frac{1}{2}\Delta u_t - \frac{1}{2}\nabla\left(u_t^2\right) + \nabla \dot{W}_t \tag{1.1}$$

where $u_t = u_t(r)$, $t \in \mathbf{R}^+$, $r \in \mathbf{R}$ is a scalar random field, ∇ is the space gradient, Δ is the Laplacian and $\dot{W}_t = \dot{W}_t(r)$ is space-time white noise, i.e.

$$\mathbf{E}\left(\dot{W}_t(r)\dot{W}_{t'}(r')\right) = \delta(t-t')\delta(r-r') \tag{1.2}$$

The stochastic Burgers equation has been recently analyzed for the case of non-conservative noise, i.e. when the term $\nabla\dot{W}_t$ in (1.1) is replaced by $\dot{W}_t$, and in that case it has been shown that the process has continuous (in time and space) trajectories [3, 7]. Due to the extra spatial gradient in front of the stochastic term of (1.1) the natural state space of the process is a distribution space. This can easily be seen by trying to solve (1.1) as a perturbation of the associated linear equation. The interpretation of the non linearity becomes then a highly non trivial point. In defining an evolution law which corresponds to (1.1) we will profit from specificity of the one–dimensional case, which allows us to avoid the problem of giving a direct meaning to the term u_t^2. We stress that, in spite of the fact that we are in a one dimensional setting, u_t, in the spatial variable, will have the same regularity of the white noise.

We are going to focus on the possibility of deriving the *stochastically perturbed* hydrodynamical equation (1.1), once properly defined, as a scaling limit of a microscopic particles model. The conventional picture in the hydrodynamical limit [29] presents the deterministic equation as a law of large numbers for the density fields and the fluctuations around the hydrodynamic behavior as a central limit theorem. The fluctuations are thereby described by a Gaussian process (generalized Ornstein-Uhlenbeck process). We stress that such a process is the solution of a *linear* stochastic PDE, precisely the linearization of the hydrodynamic PDE, to which a random force term is added (see e.g. [8]). To see the randomness and the nonlinearity in the limiting process one is thus forced to analyze the fluctuations fields beyond the hydrodynamic scale; this possibility has been successfully pursued for one dimensional (nonequilibrium) critical fluctuations in a local mean field theory [4, 13] and for other one dimensional long range models [27]. Here we shall consider a weakly driven lattice gas with hard core interaction.

The particles live in the one dimensional lattice $\mathbf{Z}$ and perform a (continuous time) weakly asymmetric random walk, i.e. they jump to the right with rate $1/2$ and to the left with rate $1/2+\varepsilon$; since at most one particle is allowed for each site the jumps which would violate this condition are suppressed. Let us consider the diffusive scaling $r=\varepsilon\, x$, $t=\varepsilon^2\,\tau$, where $x\in\mathbf{Z}$, $\tau\in\mathbf{R}^+$ are the microscopical coordinates. and assume that the initial distribution of particles approximates a density profile $\rho_0=\rho_0(r)$: in the limit $\varepsilon\to 0$ the empirical density profile evolves then according to the equation

$$\partial_t\rho_t=\frac{1}{2}\Delta\rho_t+\nabla\left(\rho_t(1-\rho_t)\right) \tag{1.3}$$

in which $\Delta\rho$ is due to symmetric part of the evolution and $\nabla(\rho_t(1-\rho_t))$ to the asymmetric drift, which is of order 1 in this time scale since the force is $\sim\varepsilon$. For precise statements and proofs see [8, 14, 19]. It is now natural to look at the particles system to understand the structure of the (random) forcing term which describes the corrections to (1.3) for non zero ε. In [8, 10] the central limit theorem associated with the law of large numbers (1.3) is proven and the behavior of the fluctuations in the limit as $\varepsilon\to 0$ approaches the distribution valued process $\{\xi_t(r)\,,\,t\in\mathbf{R}^+,r\in\mathbf{R}\}$, which solves the linear stochastic PDE

$$\partial_t\xi_t=\frac{1}{2}\,\Delta\xi_t+\nabla\left((1-2\rho_t)\,\xi_t+\sqrt{\rho_t\,(1-\rho_t)}\,\dot W_t\right) \tag{1.4}$$

with suitable initial condition (for example if the initial condition is a product measure, then ξ_0 is equal to W_0 in law). These results, as well as the large deviation results for exclusion dynamics (see [29, 19]), seem to suggest that

$$\partial_t\rho_t^\varepsilon=\frac{1}{2}\,\Delta\rho_t^\varepsilon+\nabla\left(\rho_t^\varepsilon\,(1-\rho_t^\varepsilon)+\sqrt{\varepsilon\,\rho_t^\varepsilon\,(1-\rho_t^\varepsilon)}\,\dot W_t\right) \tag{1.5}$$

could be a reasonable *ansatz* for a first order correction in ε to the hydrodynamic scaling. In fact Equation (1.5) formally reduces to (1.3) when ε vanishes and the evolution law for the fluctuation process $\xi_t^\varepsilon=\varepsilon^{-\frac{1}{2}}(\rho_t^\varepsilon-\rho_t)$, in the same limit, formally reduces to (1.4). Equation (1.5) however cannot be taken too seriously: as it stands it does not make any mathematical sense and this suggests that *for finite* ε the particle process manifests its discreteness and a reduced description for the empirical density, in terms of a diffusion process, is not possible.

In this paper we show that there is a scaling limit in which the particle system exhibits non linear fluctuations, i.e. the time evolution of the fluctuation field is given by a process which formally solves the nonlinear stochastic PDE (1.1). To explain the regime we consider, let $m_t^\varepsilon(r)=\rho^\varepsilon_{\varepsilon^{-2}t}(\varepsilon^{-1}r)$, this corresponds to looking at the empirical density on the longer space-time scale $\tau \sim \varepsilon^{-4}$, $x \sim \varepsilon^{-2}$: recall that the forcing term is $\sim \varepsilon$, so that this new scaling is different from the previous one.The following is a heuristic analysis based on formal manipulations of (1.5). Using the scaling properties of the white noise, we get

$$\partial_t m_t^\varepsilon = \frac{1}{2}\,\Delta m_\varepsilon^t + \nabla\left(\varepsilon^{-1}\, m_t^\varepsilon\,(1-m_t^\varepsilon) + \varepsilon\,\sqrt{m_t^\varepsilon\,(1-m_t^\varepsilon)}\,\dot W_t\right) \tag{1.6}$$

which presents a diverging current. In order to make this term finite we shall consider what has been called, in the contest of the derivation of the Navier-Stokes equation, the *incompressible limit* [11], i.e. we analyze a small perturbation of the global equilibrium of constant density. We assume that m^ε has the form

$$m_t^\varepsilon = \frac{1}{2}\,(1+\varepsilon\, u_t^\varepsilon)\,. \tag{1.7}$$

We are thus considering perturbation of the same order of the noise in (1.6), for this reason we shall see both the nonlinearity and the random force in the scaling limit. With the choice (1.7) the diverging term in (1.6) disappears and, in the limit $\varepsilon \to 0$, the equation which we obtain by substituting (1.7) into (1.6) formally converges to equation (1.1). The stochastic Burgers equation thus should describe the next order correction to a global equilibrium profile.

The real issue behind this heuristic discussion is whether a profile of the type (1.7) is stable under the microscopic evolution. In other words we are asking what happens to a perturbation of order ε (on the space scale ε^{-2}) after a time ε^{-4} and, to derive the non linear equation (1.1), some *propagation of chaos* type result is needed. We note that for the weakly asymmetric exclusion process the *propagation of chaos* has been proven in a very strong form up to the hydrodynamical scale [8, 14], however those results do not hold in our regime; for instance we are exactly at the time scale in which the analysis in [9] breaks down.

Remark. The *asymmetric simple exclusion process*, where the asymmetry is of order one, has an hydrodynamical limit in the hyperbolic scaling $\tau \sim \varepsilon$, $x \sim \varepsilon$ which gives the inviscid Burgers equation, i.e. (1.3) without the Laplacian [24, VIII.5]. This process has been recently analyzed beyond this hydrodynamical scale to obtain the Navier-Stokes correction to the Euler equation (here inviscid Burgers). The *incompressible limit* is studied in [11]: the microscopic dynamics is studied in the diffusive scaling and the mechanism which eliminates the diverging part of the current is analogous to the one above described, formulae (1.6)–(1.7). The resulting macroscopic equation is the deterministic Burgers equation with viscosity given by the Green-Kubo formula. In [23] the microscopic process is still studied in the hyperbolic scaling and a second order differential operator, again given by a Green-Kubo formula, is obtained as the first order correction in ε. Both these results hold only in three or more space dimension when the correction to the hydrodynamic is described by a deterministic PDE. The diffusive scaling limit for the *weakly asymmetric simple exclusion process* is much easier and the viscosity, 1/2 in (1.1) and (1.3), is entirely due to the symmetric part of the microscopic dynamics. On the other hand, by considering a one dimensional model beyond its first non trivial hydrodynamical scale (which is here the diffusive scaling), there is a regime in which the central limit theorem breaks down and the density fluctuations evolve according to the non linear stochastic PDE (1.1). We note that this phenomenon is not expected for the one dimensional *asymmetric simple exclusion process* since the first correction to the hydrodynamics should appear at the time scale $\tau \sim \varepsilon^{-3/2}$ before the random fluctuations become relevant [29, II, 5.1].

Now we turn our attention to a random *growth model*; as an example one may consider a Glauber dynamics [24, IV] for a ferromagnetic model at low enough temperature with an external positive magnetic field: if initially the system shows regions in which there is predominance of plus spins (*favoured phase*) and others in which there is predominance of minus spin (*unfavoured phase*), the favoured phase will expand, invading the regions occupied by the unfavoured phase. The separation layer between the two regions is called interface. Whereas in equilibrium, i.e. with no external field, the interface has Gaussian fluctuations, in some nonequilibrium phenomena non Gaussian fluctuations are expected. We refer to [21] for a more detailed introduction as well as an overview of different models. Moreover we stress that, unless the temperature is zero, the interface cannot be described microscopically as a separation line, because it does have a thickness [21]. We are going to study an effective model in which the interface is sharp and we will not face this problem.

In the spirit of the theory of critical phenomena the long scale behaviour of interface fluctuations in growth models is believed to be *universal*, i.e. independent on the particular model, and given by the so called KPZ equation [18]. The fluctuations of a d dimensional interface are described, in a local coordinate system, by a single valued function $\mathbf{R}^d \ni r \mapsto h(r)$ which evolves according to the formal stochastic PDE

$$dh_t = \frac{1}{2}\left[\Delta h_t - (\nabla h_t)^2\right] dt + dW_t \tag{1.8}$$

Equation (1.8) is heuristically motivated within the *renormalization group* ideas: a general nonlinear local dependence on the gradient ∇h is assumed, but the only *non irrelevant* term is the second order one. We consider here only the one dimensional case $d = 1$ and the link with the stochastic Burgers equation is easily established since (1.1) can be obtained from (1.8) introducing $u = \nabla h$.

In this paper we prove that Equation (1.8) can be derived as the scaling limit of the fluctuation field for a microscopic growth model, the *weakly asymmetric single step solid on solid process*, which can be roughly described as follows. It is an effective model in which the microscopic interface is given as a single valued function $\zeta : \mathbf{Z} \mapsto \mathbf{Z}$. It furthermore obeys the *single step* constraint $|\zeta(x+1) - \zeta(x)| = 1$. The random dynamics is then specified by the following growth rules: local minima become maxima with rate $1/2 + \varepsilon$ whereas local maxima become minima with rate $1/2$. This occurs independently at each site, no other transition is allowed. This process describes a local evaporation/deposition and establishes, for $\varepsilon > 0$ a growth direction.

We note that this model can be obtained from the Metropolis dynamics for a two-dimensional Ising model in the limit in which the external magnetic field and the temperature converge to zero but their ratio is fixed and given by ε, see [21]. From our viewpoint, its main feature is that it can be easily represented in terms of the *weakly asymmetric simple exclusion*, so that we are really proving the same theorem as in the weakly asymmetric exclusion model, with a different physical interpretation. The previous discussion on the microscopic derivation of the stochastic Burgers equation applies then also to the KPZ equation, in particular the interface fluctuations are nonlinear only when considered on a the appropriate space-time scale, in fact they are Gaussian in the hydrodynamical one. We thus establish a scaling limit in which the KPZ equation describes exactly the fluctuation of a microscopic process.

Our results are obtained by using a non linear map, the Cole-Hopf transformation, which reduces (1.1) to a linear equation with multiplicative noise. For the microscopic process a similar transformation has been introduced in [14]; the transformed process solves a semimartingale equation with a linear drift. This technique is peculiar of the model(s) introduced and the type of results we are looking for do not seem to be, at the moment, in the domain of application of more general tools.

Outline of the paper. In the next Section we explain the rigorous meaning of the stochastic Burgers and KPZ equations, recall the definition of the microscopic processes and state our main result on the convergence of the fluctuation fields. In Section 3 we introduce the Cole-Hopf transformation which reduces (1.1) to a linear equation with a multiplicative noise and the corresponding transformation for the microscopic process. The proof of the scaling limit for the *transformed process* is carried out in Section 4, whereas in section 5 it is shown how implies the convergence to the stochastic Burgers equation. Finally in Section 6 we prove some properties of the macroscopic equations.

2 Notation and main results

Throughout all the paper the space of continuous functions on the real line $C(\mathbf{R})$ is equipped with the topology of uniform convergence over compact sets. As test functions we use the space $\mathcal{D}$, i.e. the function in $C_0^\infty(\mathbf{R})$ with the inductive limit topology; its strong topological dual $\mathcal{D}'$ is the space of distributions over $\mathbf{R}$. For $\mathcal{H}$ a topological space and $T > 0$, $D([0,T];\mathcal{H})$ is the the Skorohod space of $\mathcal{H}$-valued functions with cadlag trajectories [25]. Since the Skorohod topology relativized to $C([0,T];\mathcal{H})$ coincides with the uniform topology [5], we may (and sometimes do) consider random functions in $C([0,T];\mathcal{H})$ as elements of $D([0,T];\mathcal{H})$.

The processes we are dealing with are to be considered over an arbitrary but fixed macroscopic time interval. We will be mainly concerned with the weak convergence of sequences of stochastic processes; following the usual notation we denote this convergence by the double arrow $\Longrightarrow$ stressing this notion of convergence depends upon the topology used on the sample space.

Before stating our main result on the convergence of the fluctuation field, we explain the rigorous meaning of the stochastic Burgers equation (1.1).

2.1 Burgers equation with conservative noise

Let W_t, $t \in [0,T]$, be the cylindrical Wiener process on $L^2(\mathbf{R}, dr)$. It is canonically realized as a distribution valued continuous process, i.e. the probability space is given by $(C([0,T];\mathcal{D}'), \mathcal{A}, \mathcal{P})$, here $\mathcal{A}$ is the σ-algebra generated by the cylindrical sets and $\mathcal{P}$ is the Gaussian measure with covariance

$$\int d\mathcal{P}\, W_t(\varphi_1)\, W_s(\varphi_2) = t \wedge s\, (\varphi_1, \varphi_2) \tag{2.1}$$

where $\varphi_1, \varphi_2 \in \mathcal{D}$, are test functions, $a \wedge b := \min\{a, b\}$ and $(\cdot,\cdot)$ is the scalar product in $L^2(\mathbf{R}, dr)$. We denote by $\mathcal{A}_t$ the natural filtration of W_t, i.e. the minimal σ-algebra such that $s \mapsto W_s$ is $\mathcal{A}_t$ measurable for all $s \in [0,t]$.

We next specify the class of initial conditions we are allowing for the stochastic Burgers equation.

Definition 2.1 *The initial datum.*
The initial datum is a random field with trajectories u_0 in $\mathcal{D}'$ and it is assigned by specifying a random field $\mathcal{P}_0$ with trajectories $h_0 \in C(\mathbf{R})$ which is independent on $\mathcal{P}$ and satisfies the following condition: for all $p > 0$ there exists $a = a_p$ such that

$$\sup_{r \in \mathbf{R}} e^{-a|r|} \int d\mathcal{P}_0 \left(e^{-p\, h_0(r)} \right) < \infty \tag{2.2}$$

For $\varphi \in \mathcal{D}$ we then define

$$u_0(\varphi) = -\int dr\, h_0(r)\, \varphi'(r) \tag{2.3}$$

We have thus assumed u_0 be the derivative, in distributional sense, of a continuous trajectory $h_0 = h_0(r)$ which has for every $r \in \mathbf{R}$ exponential moment that grows at most exponentially in r. One can obviously take h_0 to be a deterministic function with at most linear growth.

We shall characterize the solution of the stochastic Burgers equation through a limiting procedure. Accordingly we introduce a regularized version of the cylindrical Wiener process which will define a family of approximating problems.

Let $J \in C_0^\infty(\mathbf{R})$ an even positive function such that $\int dr\, J(r) = 1$. Introduce, for $\kappa > 0$, the mollifier $\delta_r^\kappa(r') := \kappa\, J(\kappa(r - r'))$ and define $W_t^\kappa(r) := W_t(\delta_r^k)$; its covariance is

$$\mathbf{E}\left(W_t^\kappa(r) W_s^\kappa(r')\right) = (t \wedge s)\, C_\kappa(r - r') \qquad C_\kappa(r) := \int dr'\, \delta^\kappa(r - r')\, \delta^\kappa(r') \tag{2.4}$$

We then write the stochastic Burgers equation with this regularized Wiener process against test functions as

$$u_t^\kappa(\varphi) = u_0(\varphi) + \int_0^t ds\, \frac{1}{2} \left\{ u_s^\kappa(\varphi'') + \left[(u_s^\kappa)^2 - C_\kappa(0) \right] (\varphi') \right\} - W_t^\kappa(\varphi') \tag{2.5}$$

which is formally obtained from (1.1) integrating by parts. We have also added the term $C_\kappa(0) \sim \kappa^{-1}$, which corresponds to take the *Wick product* of the nonlinearity $(u^\kappa)^2$, see

Section 2.5 for a physical interpretation in the framework of the interfaces growth. The appearance of the Wick counterterm is somehow natural for distribution valued diffusion [1, 17]; its need can be easily seen trying to solve (1.1) as a perturbation of the associated Ornstein-Uhlenbeck process (defined by the some equation without the nonlinearity).

The limit $\kappa \to \infty$ can be taken according to the following strategy. When the cutoff κ is finite $W_t^\kappa(r)$ is smooth so that (2.5) makes sense in the space of continuous functions; we thus obtain a processes u^κ in $C([0,T];C(\mathbf{R}))$. Since a limiting process will not be continuous in space we have no hope to get a convergent sequence in the topology of $C([0,T];C(\mathbf{R}))$; however $\{u^\kappa\}_{\kappa>0}$ does form a weak convergent family as $\kappa \to \infty$ in the topology of $C([0,T];\mathcal{D}')$.

Theorem 2.2 *Assume u_0 satisfies the hypotheses in Definition 2.1, then*

(i) For all $\kappa > 0$ there exists a unique process $u_t^\kappa = u_t^\kappa(r)$ in $C((0,\infty);C(\mathbf{R}))$, adapted to the filtration $\mathcal{A}_t$, which solves almost surely (2.5) for all $\varphi \in \mathcal{D}$ and every $t > 0$.

(ii) Consider u_t^κ, $t \in [0,T]$ as a random element in $C([0,T];\mathcal{D}')$; then the family $\{u^\kappa\}_{\kappa>0}$ is weakly convergent as $\kappa \to \infty$. The limiting process is denoted by u, i.e. $u^\kappa \Longrightarrow u$ in $C([0,T];\mathcal{D}')$.

Although this result characterizes uniquely the solution of the stochastic Burgers equation through the approximating problems (2.5) it is not completely satisfactory since it avoids the issue of showing that u satisfies a limit equation. In particular we have not defined the Wick product in (2.5) for the limiting process; to our knowledge this problem has been solved only for infinite dimensional diffusions which are either Gaussian or constructed as perturbations of Gaussian measures [1, 17].

2.2 The microscopic process

For notation convenience, i.e. to have centered variables, we describe the particle model in terms of spin variables. The state space of the microscopic process is thus $\Omega := \{-1,1\}^{\mathbf{Z}}$, its elements "spin configurations" are denoted by $\sigma = \{\sigma(x)\ ,\ x \in \mathbf{Z}\}$, where $\sigma(x) = +1$ (-1) is interpreted as the site in x being occupied (resp. empty). Moreover we find more convenient to call $\sqrt{\varepsilon}$ the quantity that in the introduction was denoted by ε (see also Remark 2 at the end of Section 2.3).

The *weakly asymmetric simple exclusion process* (WASEP) is the process generated by

$$L_\varepsilon := \frac{1}{2}L^+ + \left(\frac{1}{2} + \sqrt{\varepsilon}\right)\, L^- \tag{2.6}$$

where $L^\pm$ are the generators of the totally asymmetric exclusion processes, defined by

$$L^\pm f(\sigma) := \sum_x \frac{1+\sigma(x)}{2}\,\frac{1-\sigma(x\pm 1)}{2}\,\left[f(\sigma^{x,x\pm1}) - f(\sigma)\right] \tag{2.7}$$

in which f is a cylindrical function over Ω and given $x, y \in \mathbf{Z}$

$$\sigma^{x,y}(z) \quad := \quad \begin{cases} \sigma(x) & \text{if} \quad z = y \\ \sigma(y) & \text{if} \quad z = x \\ \sigma(z) & \text{otherwise} \end{cases} \tag{2.8}$$

We stress in (2.6) we adopted the (unusual) convention of an asymmetry $\sqrt{\varepsilon}$ toward the left. The details on the construction of the process can be found in Liggett [24]. We consider the WASEP σ_t, $t \geq 0$ canonically realized on the Skorohod space $D(\mathbf{R}^+; \Omega)$ and denote by $\mathbf{P}^\varepsilon_{\mu_\varepsilon}$ its law when the initial distribution is μ_ε, probability on Ω. The expectation with respect to $\mathbf{P}^\varepsilon_{\mu_\varepsilon}$ is denoted by $\mathbf{E}^\varepsilon_{\mu_\varepsilon}$.

Given $\sigma \in \Omega$ set

$$x^0 := \min\{x \in \mathbf{Z} : x \geq 0,\ \sigma(x) = +1\} \tag{2.9}$$

and introduce $\zeta \in \mathbf{Z}^{\mathbf{Z}}$ by

$$\zeta(x) := \begin{cases} \sum_{x^0 < y \leq x} \sigma(y) - x^0 & \text{if} \quad x > x^0 \\ -\sum_{x < y \leq x^0} \sigma(y) - x^0 & \text{if} \quad x < x^0 \\ -x^0 & \text{if} \quad x = x^0 \end{cases} \tag{2.10}$$

We also define $\zeta \in C(\mathbf{R})$ by linear interpolation on the discrete values in (2.10), i.e. for $r \in \mathbf{R}$ we set

$$\zeta(r) := \zeta([r]) + (r - [r])\,(\zeta([r] + 1) - \zeta([r])) \tag{2.11}$$

where $[\cdot]$ is the integer part.

We chose the initial distribution μ_ε to be a small ($\approx \sqrt{\varepsilon}$) perturbation of the global equilibrium profile with density $1/2$.

Definition 2.3 *The initial profile.*

Let ζ be given by (2.10)-(2.11) where σ is distributed according to μ_ε, probability on Ω. It satisfies the following conditions:

(i) *Set $\zeta^\varepsilon(r) := \sqrt{\varepsilon}\zeta(\varepsilon^{-1}r)$. There exists a random function with trajectories $h_0 = h_0(r)$ in $C(\mathbf{R})$ such that*

$$\zeta^\varepsilon \Longrightarrow h_0 \tag{2.12}$$

as $\varepsilon \to 0$ in the topology of $C(\mathbf{R})$.

(ii) *For all $n \in \mathbf{N}$ there are $a = a_n, c = c_n > 0$ such that*

$$\sup_{x \in \mathbf{Z}} e^{-a\,\varepsilon|x|} \int d\mu_\varepsilon \exp\left\{-n\sqrt{\varepsilon}\,\zeta(x)\right\} \leq c \tag{2.13}$$

for all $\varepsilon > 0$;

(iii) *For all $n \in \mathbf{N}$ there are $a' = a'_n, c' = c'_n > 0$ such that*

$$\int d\mu_\varepsilon \left(\sqrt{\varepsilon}\,[\zeta(x) - \zeta(y)]\right)^{2n} \leq c'\, e^{a'\,\varepsilon\,(|x|+|y|)}\, (\varepsilon|x - y|)^n \tag{2.14}$$

for every $x, y \in \mathbf{Z}$ and all $\varepsilon > 0$.

Condition (2.13) entails the random function h_0 defined by (2.12) satisfies the assumption (2.2) of Definition 2.1; the inequality (2.14) implies also h_0 is a.s. Hölder continuous with exponent less than 1/2.

We remark the conditions in Definition 2.3 are satisfied if μ_ε is a product measure on Ω with marginals

$$\mathbf{E}_{\mu_\varepsilon}(\sigma(x)) = \varepsilon^{-\frac{1}{2}}\left[\, m(\varepsilon x) - m(\varepsilon(x-1))\,\right] \tag{2.15}$$

in which $m = m(r)$ is a α–Hölder continuous function ($\alpha \geq 1/2$) on $\mathbf{R}$ which satisfies the following condition: there is $a > 0$ such that for every $r \in \mathbf{R}$, $|m(r)| \leq a(1+|r|)$. The initial datum for the stochastic Burgers equation is then the random element u_0, taking values in $\mathcal{D}'$, such that for all $\varphi \in \mathcal{D}$

$$u_0(\varphi) \stackrel{\text{Law}}{=} \mathcal{N}\left(-m(\varphi')\,,\,(\varphi,\varphi)\right) \tag{2.16}$$

where $m(\varphi') := \int dr\, m(r)\,\varphi'(r)$ and $\mathcal{N}(a, s^2)$ is a Gaussian distribution with mean a and variance s^2. Equation (2.16) means thus $u_0 = \nabla m + w$, where w is the white noise (in the spatial variable).

Admissible initial data include however also deterministic profiles, i.e. the case in which μ_ε is concentrated on a single configuration. For example take μ_ε independent of ε and concentrated on the configuration σ, $\sigma(x) = 1$ if x is even, $\sigma(x) = -1$ otherwise; this gives $u_0 = 0$

2.3 WASEP and stochastic Burgers equation

We introduce the fluctuation field for WASEP with respect to the (global equilibrium) profile with constant density 1/2. We recall that the initial datum μ_ε was chosen to be a small perturbation of it.

Definition 2.4 *Fluctuation field for WASEP.*
For all $\varphi \in \mathcal{D}$, $t \in [0,T]$, let

$$X_t^\varepsilon(\varphi) := \sqrt{\varepsilon}\sum_x \varphi(\varepsilon x)\sigma_{\varepsilon^{-2}t}(x) \tag{2.17}$$

where σ_t is WASEP.
We regard $X^\varepsilon = (X_t^\varepsilon)_{t\in[0,T]}$ as a random element in $D([0,T];\mathcal{D}')$.

In the scaling limit $\varepsilon \to 0$ the distribution of the fluctuation field (2.17) is determined by the Burgers equation with random noise on the current.

Theorem 2.5 *The family $\{X^\varepsilon\}_{\varepsilon>0}$ is weakly convergent in $D([0,T];\mathcal{D}')$ as $\varepsilon \to 0$. Furthermore the weak limit is concentrated on $C([0,T];\mathcal{D}')$ and coincides with the process u constructed in Theorem 2.2, i.e.*

$$X^\varepsilon \Longrightarrow u \tag{2.18}$$

in the topology of $D([0,T];\mathcal{D}')$.

With respect to Theorem 2.2, the convergence in (2.18) can be interpreted as the equivalence of two different regularizations for the Burgers equation with conservative noise. The actual proof will be carried out by showing both the limit of u^κ and X^ε are equal to a third process, see the next Section.

Whereas the approximating problems (2.5) are simply defined trough a space regularization of the white noise, the fluctuation field X^ε has its own microscopic interpretation related to a much studied particle process (WASEP). This might help in proving some properties of the limiting equation, we shall see an example in Section 6.

We note also that the Wick counterterm, which has been introduced *ad hoc* in (2.5) arises naturally in the scaling limit. This possibility was suggested, for the two dimensional Landau-Ginzburg equation with noise, in [16]; see also [6] for a related discussion.

Remark 1. We considered only small perturbations of the profile with constant density 1/2. It is nonetheless possible to study the perturbations of any constant density profile, but one needs to introduce a Galilean transformation [11]. Let the initial profile be a small perturbation of $\bar\rho \in (0,1)$, i.e. for any $\varphi \in \mathcal{D}$

$$\sqrt{\varepsilon}\sum_x \varphi(\varepsilon x)\left(\sigma_0(x) - (2\bar\rho - 1)\right) \Longrightarrow v_0(\varphi)$$

and $v_0 \in \mathcal{D}'$ satisfies the assumptions in Definition 2.1.

The distribution of the fluctuation field

$$Y_t^\varepsilon(\varphi) := \sqrt{\varepsilon}\sum_x \varphi\left(\varepsilon\left[x + \varepsilon^{-3/2}(1-2\bar\rho)\,t\right]\right)\left(\sigma_{\varepsilon^{-2}t}(x) - (2\bar\rho - 1)\right)$$

is then determined, in the limit $\varepsilon \to 0$, by the stochastic Burgers equation. In this case the macroscopic process is obtained as the weak limit of

$$v_t^\kappa(\varphi) = v_0(\varphi) + \int_0^t ds\,\frac{1}{2}\left\{v_s^\kappa(\varphi'') + \left[(v_s^\kappa)^2 - 4\,\bar\rho\,(1-\bar\rho)\,C_\kappa(0)\right](\varphi')\right\} - 2\sqrt{\bar\rho(1-\bar\rho)}\,W_t^\kappa(\varphi') \tag{2.19}$$

instead of (2.5).

Remark 2. If we introduce $\varepsilon_1 := \sqrt{\varepsilon}$, Theorem 2.5 can be stated for the *standard* WASEP generated by $\widetilde{L}_{\varepsilon_1} := (L^+ + L^-)/2 + \varepsilon_1 L^-$, for which the hydrodynamics is studied in [8, 10, 19] and it is given by the Burgers equation with viscosity. With this notation, it is clear we are considering the fluctuation field beyond the hydrodynamical (diffusive) scale, i.e. we are considering the fluctuation field

$$\widetilde{X}_t^{\varepsilon_1}(\varphi) := \varepsilon_1 \sum_x \varphi(\varepsilon_1^2 x)\sigma_{\varepsilon_1^{-4}t}(x) \tag{2.20}$$

2.4 Solid on solid model of growing surfaces

As discussed in the introduction we can reformulate our results in terms of an interface model with a different physical interpretation. The uninterested reader may safely skip this and the following Section.

The solid on solid (SOS) process for growing interfaces (a more appropriate name ought to be *weakly asymmetric single step SOS process*) is defined as follows. On the state space

$$\widehat{\Omega} := \left\{ \zeta \in \mathbf{Z}^{\mathbf{Z}} \;:\; \forall x \in \mathbf{Z} \quad |\zeta(x+1) - \zeta(x)| = 1 \right\}$$

introduce the jump Markov process generated by

$$\widehat{L}_\varepsilon f(\zeta) := \sum_x \left\{ c^+(x,\zeta)\, [f(\zeta + 2\,\delta_x) - f(\zeta)] + c^-(x,\zeta)\, [f(\zeta - 2\,\delta_x) - f(\zeta)] \right\} \tag{2.21}$$

where f is a cylindrical function on $\widehat{\Omega}$, $\delta_x \in \mathbf{Z}^{\mathbf{Z}}$ is defined by $\delta_x(y) := \delta(x-y)$ and

$$\begin{aligned} c^+(x,\zeta) &:= \begin{cases} 1/2 + \varepsilon^{1/2} & \text{if} \quad \zeta(x+1) - 2\,\zeta(x) + \zeta(x-1) = 2 \\ 0 & \text{otherwise} \end{cases} \\ c^-(x,\zeta) &:= \begin{cases} 1/2 & \text{if} \quad \zeta(x+1) - 2\,\zeta(x) + \zeta(x-1) = -\,2 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.22}$$

The structure of the generator (2.21) preserves the single step constraint, $\zeta \in \widehat{\Omega}$ in the dynamical rules. The allowed transition are indeed either when local minima of $\zeta(x)$ are raised to local maxima, $\zeta(x) \mapsto \zeta(x) + 2$, with rate $1/2 + \sqrt{\varepsilon}$ or local maxima are lowered to local minima, $\zeta(x) \mapsto \zeta(x) - 2$, with rate $1/2$.

We shall consider ζ also as a continuous function by linear interpolation on its value on the lattice $\mathbf{Z}$, i.e. for $r \in \mathbf{R}$ $\zeta(r)$ is defined as in (2.11). When viewed in this way the SOS process ζ_t, $t \in \mathbf{R}^+$ is then a random element in $D(\mathbf{R}^+; C(\mathbf{R}))$.

The process above defined is easily obtained from WASEP [21]. Let x_t^0 be the position of the tagged particle for WASEP. We recall that the tagged particle is the particle that at time zero was closest to the origin on the positive half-axis, i.e. given σ_0, x_0^0 is defined by (2.9) and x_t^0 is its position at time t under the WASEP dynamics. It is then a simple check to verify that a version of the SOS process is given by

$$\zeta_t(x) = \begin{cases} \sum_{x_t^0 < y \le x} \sigma_t(y) - x_t^0 & \text{if} \quad x > x_t^0 \\ -\sum_{x < y \le x_t^0} \sigma_t(y) - x_t^0 & \text{if} \quad x < x_t^0 \\ -x_t^0 & \text{if} \quad x = x_t^0 \end{cases} \tag{2.23}$$

where σ_t is WASEP.

We assume the initial distribution of the SOS process satisfies the hypotheses in Definition 2.3, which were actually formulated in terms of the SOS variable ζ.

2.5 Microscopic derivation of Kardar-Parisi-Zhang equation

We first discuss the rigorous meaning of the Kardar-Parisi-Zhang (KPZ) equation (1.8). The situation is essentially the same as for the stochastic Burgers equation. Given the regularized cylindrical Wiener process W_t^κ, $\kappa > 0$ as defined in Section 2.1, we introduce the family of approximating problems

$$h_t^\kappa(\varphi) = h_0(\varphi) + \int_0^t ds\, \frac{1}{2} \left\{ h_s^\kappa(\varphi'') - \left[(\nabla h_s^\kappa)^2 - C_\kappa(0) \right](\varphi) \right\} + W_t^\kappa(\varphi) \tag{2.24}$$

We then have the following result.

Theorem 2.6 *Let h_0 be a random function in $C(\mathbf{R})$ which satisfies the assumption (2.2), then*

(i) For all $\kappa > 0$ there exists a unique process $h_t^\kappa = h_t^\kappa(r)$ in $C((0,\infty); C^1(\mathbf{R}))$, adapted to the filtration $\mathcal{A}_t$, which solves almost surely (2.24) for all $\varphi \in \mathcal{D}$ and every $t > 0$.

(ii) Consider h_t^κ, $t \in [0,T]$ as a random element in $C([0,T]; C(\mathbf{R}))$; then the family $\{h^\kappa\}$ is weakly convergent as $\kappa \to \infty$. The limiting process is denoted by h, i.e. $h^\kappa \Longrightarrow h$ in $C([0,T]; C(\mathbf{R}))$.

Comparing Theorem 2.6 with Theorem 2.2 we note that the solution of KPZ equation h_t is defined as a $C(\mathbf{R})$ valued process; however as the equation involves the square of ∇h, it loses its classical meaning and the renormalazing term $C_\kappa(0)$ has to be added. We note that the need for the Wick product is seen much more clearly in (2.24) than in (2.5), since it formally disappears in the latter.

In the language of growth model the appearance of the Wick product can be physically interpreted as a contribute from the fluctuations to the deterministic growth, so that it has to be eliminated to see finite fluctuations. This can be easily seen from (2.24) by introducing the process $\hat{h}_t^\kappa := h_t^\kappa - C_\kappa(0)t$ which solves the KPZ without renormalization and diverges in the limit $\kappa \to \infty$.

We may now state our result on the convergence of the fluctuation in the SOS process to the KPZ equation.

Theorem 2.7 *For $t \in [0,T]$, $r \in \mathbf{R}$, let*

$$Z_t^\varepsilon(r) := \sqrt{\varepsilon}\, \left(\zeta_{\varepsilon^{-2}t}(\varepsilon^{-1} r) - \upsilon_\varepsilon t \right) \tag{2.25}$$

where $\upsilon_\varepsilon := \frac{1}{2}\varepsilon^{-3/2} + \frac{3}{8}\varepsilon^{-1/2}$ and regard Z^ε as a random element in $D([0,T]; C(\mathbf{R}))$.

The family $\{Z^\varepsilon\}_{\varepsilon>0}$ is weakly convergent as $\varepsilon \to 0$; furthermore the weak limit is concentrated on $C([0,T]; C(\mathbf{R}))$ and coincides with the process h defined in Theorem 2.6, i.e.

$$Z^\varepsilon \Longrightarrow h \tag{2.26}$$

in the topology of $D([0,T]; C(\mathbf{R}))$.

The scaling in equation (2.25) is as follows. Recalling that the driving force is of order $\varepsilon^{1/2}$, the law of large number is obtained for the variable $\varepsilon^{1/2}\zeta_{\varepsilon^{-1}t}(\varepsilon^{-1/2}\cdot)$ since $\zeta_t(x)$ represents the interface height and has to be normalized as the microscopic coordinate

x. On the same space-time scale $t \sim \varepsilon^{-1}$, $x \sim \varepsilon^{-1/2}$, the process ζ_t exhibits Gaussian fluctuations which evolution is given by a linear equation analogous to (1.4). By choosing an initial configuration as specified in Definition 2.3 and looking at longer scales, the fluctuations become non Gaussian and given by the KPZ equation. To obtain this result it is necessary to subtract the (deterministic) interface drift velocity v_ε including the next order correction in ε. This corresponds, for the microscopic process, to the already given interpretation of the Wick counterterm in (2.24): it can be seen as a contribute from the fluctuations to the deterministic growth velocity. We stress for our choice of the initial data the deterministic drift velocity is homogeneous in space. Finally the normalization $\varepsilon^{1/2}$ in (2.25) is the usual (non anomalous) in central limit theorems.

We finally note that Theorem 2.7 gives a rather strong convergence as a process in $D([0,T]; C(\mathbf{R}))$ and not in a distribution space. This is due to the fact that the microscopic process $\zeta_t(x)$ can be written as a function of the empirical average for WASEP, see equation (2.23). In fact a spin flip changes Z^ε by a factor of order $\varepsilon^{1/2}$ so that no space averaging is needed.

3 Non linear transformation method

In this Section we introduce the main tool in proving the stated results. We shall use a non linear map, the so called Cole-Hopf transformation, which reduces the stochastic Burgers equation (and KPZ) to a linear equation with multiplicative noise.

The scaling limit of WASEP (and of the SOS process) is analyzed thought a microscopic analogue of this transformation, introduced by Gärtner in [14]. In this case we do not get a linear equation, but the process obtained is easier to handle and we shall prove it converges to a linear equation in the limit $\varepsilon \to 0$; taking the inverse map this will imply the convergence to the stochastic Burgers equation.

3.1 Cole-Hopf transformation

The heuristic observation is the following. Let u_t be a solution of the stochastic Burgers equation (1.1) and denote by h_t a function such that $u_t = \nabla h_t$; this is always possible in one space dimension (h_t is then a solution of KPZ equation (1.8)). Introduce the process $\theta_t := \exp\{-h_t\}$, it then solves the *linear* equation

$$d\theta_t = \frac{1}{2}\Delta\theta_t dt - \theta_t dW_t \tag{3.1}$$

often called the *stochastic heat equation*.

The rigorous analysis starts by giving a meaning to the above equation when the stochastic differential is interpreted in the Ito sense. We assume the initial datum $\theta_0 = \theta_0(r)$ to be a random function in $C(\mathbf{R})$ with moments of all orders which grows at most exponentially in r; we shall consider $\theta_0(r) := \exp\{-h_0(r)\}$, where h_0 satisfies the hypothesis (2.2).

Introduce the heat kernel

$$G_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} \tag{3.2}$$

and formulate the stochastic heat equation in mild form as

$$\theta_t = G_t * \theta_0 - \int_0^t G_{t-s} * \theta_s dW_s \tag{3.3}$$

where $*$ denotes convolution in space and

$$\int_0^t G_{t-s} * \theta_s dW_s\,(r) := \int_0^t \Big(G_{t-s}(r-\cdot)\theta_s\,,\, dW_s\Big) \tag{3.4}$$

is the Ito integral with respect to the cylindrical Wiener process.

It is convenient to formulate this problem also for the regularized Wiener process, i.e.

$$\theta_t^\kappa = G_t * \theta_0 - \int_0^t G_{t-s} * \theta_s^\kappa dW_s^\kappa \tag{3.5}$$

We finally introduce

$$\mathcal{C}_+(\mathbf{R}) := \{f \in C(\mathbf{R}))\ :\ \forall\, r \in \mathbf{R}\ \ f(r) > 0\} \tag{3.6}$$

Theorem 3.1

(i) For all $\kappa > 0$ there exists a unique process $\theta_t^\kappa = \theta_t^\kappa(r)$ in $C(R^+; C(\mathbf{R}))$, adapted to the filtration $\mathcal{A}_t$, which solves almost surely (3.5) for every $t > 0$.

(ii) For all $p \geq 1$, $\theta_t^\kappa(r) \to \theta_t(r)$ in $L^p(\mathcal{P})$ and $\mathcal{P}$ a.s.; the convergence is uniform for (t,r) in compact subsets of $\mathbf{R}^+ \times \mathbf{R}$. The limiting process is the unique solution of (3.3) in the class of $\mathcal{A}_t$ adapted processes satisfying

$$\sup_{t\in[0,T]} \sup_{r\in\mathbf{R}} e^{-a|r|}\mathbf{E}\left(\theta_t(r)^2\right) < \infty \tag{3.7}$$

for some $a > 0$.

(iii) If $\theta_0(r) > 0$ a.s. then $\theta \in C([0,T]; \mathcal{C}_+(\mathbf{R}))$ a.s.

In the above form this Theorem is proven in [2] where is given also a Feynman-Kac formula for θ_t. The (for us fundamental) positivity property *(iii)* is due to Muller [26]. Since the equation (3.1) is linear this results should not be a big surprise, in [2] they are essentially proven computing explicitly the moments of $\theta_t^\kappa(r)$ which can be expressed thought the local time of independent (one dimensional) Brownian motions.

We are now ready to give an equivalent definition, at this point the third (and last), for the solution of the stochastic Burgers equation.

Theorem 3.2 *Let* $\Psi : C([0,T];\mathcal{C}_+(\mathbf{R})) \mapsto C([0,T];\mathcal{D}')$ *be defined by*

$$f_t \stackrel{\Psi}{\longmapsto} -\nabla \log f_t \qquad \text{where} \quad \nabla \log f_t\,(\varphi) := -\int dr\; \varphi'(r)\, \log f_t(r) \tag{3.8}$$

with $\varphi \in \mathcal{D}$, $t \in [0,T]$.

Let θ_t, $t \in [0,T]$ *be the solution of the stochastic heat equation. Theorem 3.1 entails that* $u := \Psi(\theta)$ *is a.s. defined and continuous; u then coincides (in law) with the process defined in Theorem 2.2.*

Remark. An analogous statement holds for the KPZ equation, in that case we have $h = \hat{\Psi}(\theta)$ where $\hat{\Psi} : C([0,T];\mathcal{C}_+(\mathbf{R})) \mapsto C([0,T];C(\mathbf{R}))$ is defined by $\hat{\Psi}(f_t) = -\log f_t$.

In Section 4.3 we introduce an equivalent formulation of the stochastic heat equation in the form of a martingale problem and we will use this new formulation to identify the limit.

3.2 The Gärtner transformation

We here introduce the microscopic analog of the Cole-Hopf transformation. A similar transformation has been used in [9, 10, 14]. In our case it maps the problem of the convergence of the fluctuation field of WASEP to the stochastic Burgers equation into the problem of the convergence of the transformed process to the stochastic heat equation above discussed.

Let σ_t be WASEP and x_t^0 the tagged particle process. The generator of the joint Markov process (σ_t, x_t^0) is

$$H_\varepsilon := \frac{1}{2}\left(H^+ + (1+2\sqrt{\varepsilon})H^-\right) \tag{3.9}$$

where

$$\begin{aligned} H^\pm f\,(\sigma,x^0) \;\; &:= \;\; \frac{1}{4}\sum_{x\neq x^0}(1+\sigma(x))(1-\sigma(x\pm 1))\left[f(\sigma^{x,x\pm1},x^0)-f(\sigma,x^0)\right] \\ &\quad +\frac{1}{2}(1-\sigma(x^0\pm 1))\left[f(\sigma^{x,x\pm1},x^0\pm 1)-f(\sigma,x^0)\right] \end{aligned} \tag{3.10}$$

and f is a cylindrical function over $\{-1,1\}^{\mathbf{Z}} \times \mathbf{Z}$.

The transformed process $\xi_t = \xi_t(x)$, $x \in \mathbf{Z}$ is defined as

$$\xi_t(x) := \begin{cases} \exp\left\{-\gamma_\varepsilon \sum_{x_t^0<y\le x}\sigma_t(y) + \gamma_\varepsilon\, x_t^0 + \lambda_\varepsilon\, t\right\} & \text{if} \quad x > x_t^0 \\ \exp\left\{\gamma_\varepsilon \sum_{x<y\le x_t^0}\sigma_t(y) + \gamma_\varepsilon\, x_t^0 + \lambda_\varepsilon\, t\right\} & \text{if} \quad x < x_t^0 \\ \exp\left\{\gamma_\varepsilon\, x_t^0 + \lambda\, t\right\} & \text{if} \quad x = x_t^0 \end{cases} \tag{3.11}$$

where

$$\gamma_\varepsilon := \frac{1}{2}\log(1+2\varepsilon^{\frac{1}{2}}) \quad , \quad \lambda_\varepsilon := 1+\varepsilon^{\frac{1}{2}} - \sqrt{1+2\varepsilon^{\frac{1}{2}}} \tag{3.12}$$

We will also consider $\xi_t(r)$ for every $r \in \mathbf{R}$ defined as the piecewise linear continuous extension of (3.11). Equivalently

$$\xi_t(r) = \exp\{-\gamma_\varepsilon \zeta_t(r) + \lambda_\varepsilon t\} \tag{3.13}$$

where $\zeta_t(r)$ is the SOS process.

As $\gamma_\varepsilon = \sqrt{\varepsilon} + O(\varepsilon)$ the first term at the exponent in (3.13) is the fluctuation field so that (3.11) is the microscopic analogue of the Cole-Hopf transformation. The other term $\lambda_\varepsilon t$ has been added in order to obtain the convergence of the process ξ_t; however this term does not affect the inverse map, i.e.

$$\sigma_t(x) \equiv -\frac{1}{\gamma_\varepsilon}\Big[\log \xi_t(x) - \log \xi_t(x-1)\Big] \tag{3.14}$$

Let define the the discrete gradients as

$$\nabla^{\pm} f(x) := \pm\left[f(x \pm 1) - f(x)\right]$$

and the discrete laplacian as

$$\triangle f(x) := \nabla^{+}\nabla^{-} f(x) \equiv f(x+1) + f(x-1) - 2f(x)$$

in which $f : \mathbf{Z} \to \mathbf{R}$.

Let $\mathcal{F}_t$ be the natural filtration of σ_t, i.e. the σ-algebra generated by σ_s, $s \le t$; a direct computation [14] shows that ξ_t satisfies the semimartingale equation

$$d\xi_t(x) = \frac{1}{2}\, e^{\gamma_\varepsilon} \,\triangle\, \xi_t(x)\, dt + dM_t(x) \tag{3.15}$$

where the $\mathcal{F}_t$ martingales $M_t(x)$ have brackets

$$\begin{aligned} &d\langle M(x), M(y)\rangle_t \\ &= \frac{\varepsilon}{2}\,\delta(x-y)\,\xi_t(x)^2 \\ &\quad \cdot \Big[\Big(1+e^{-2\gamma_\varepsilon}\Big)\left(1-\sigma_t(x)\,\sigma_t(x+1)\right) + \Big(1-e^{-2\gamma_\varepsilon}\Big)\left(\sigma_t(x)-\sigma_t(x+1)\right)\Big]\, dt \qquad (3.16) \\ &= \frac{1}{2}\,\delta(x-y)\,\Big[(1-e^{\gamma_\varepsilon})^2\Big(2\,\xi_t(x)^2 + \xi_t(x)\,\triangle\,\xi_t(x)\Big) - \Big(1+e^{2\gamma_\varepsilon}\Big)\,\nabla^{+}\xi_t(x)\,\nabla^{-}\xi_t(x)\Big]\, dt \qquad (3.17) \end{aligned}$$

In the scaling limit the equation (3.15) converges to the stochastic heat equation; the precise statement is the following.

Theorem 3.3 *For $t \in [0,T]$, $r \in \mathbf{R}$, let $\xi^\varepsilon_t(r) := \xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r)$ and regard it as a random element in $D([0,T];C(\mathbf{R}))$. For all $T > 0$, ξ^ε is then weakly convergent as $\varepsilon \to 0$*

$$\xi^\varepsilon \Longrightarrow \theta \tag{3.18}$$

where θ is the solution of the stochastic heat equation with initial datum $\theta_0 = \exp\{-h_0\}$, h_0 as in Definition 2.3.

4 Convergence of the transformed process

In this Section we prove the scaling limit for the transformed process ξ^ε. We first obtain, in Section 4.1, some moments estimates that imply the Höolder continuity of the process. In Section 4.2 we then establish a key estimate on the decay of the correlations. In Section 4.3 we finally complete the proof of Theorem 3.3 by showing compactness of ξ^ε and that any weak limit solves the stochastic heat equation. We use several times some properties of the transition probability for a random walk in $\mathbf{Z}$ that are proven in Appendix A.

We have to introduce some more notations. Let $p_t^\varepsilon(x)$ be the Green function associated with the drift term in equation (3.15), i.e. the function which solves

$$\begin{aligned} \partial_t p_t^\varepsilon(x) &= \frac{1}{2} e^{\gamma_\varepsilon} \triangle p_t^\varepsilon(x) \\ p_0^\varepsilon(x) &= \delta(x) \end{aligned} \tag{4.1}$$

We note that it can be interpreted as the transition probability for a symmetric random walk in continuum time over $\mathbf{Z}$.

The semimartingale equation (3.15) can then be rewritten as

$$\xi_t(x) = p_t^\varepsilon \circ \xi_0\, (x) + N_t^t(x) \tag{4.2}$$

where

$$p_t^\varepsilon \circ \xi_0\, (x) := \sum_y p_t^\varepsilon(x-y)\, \xi_0(y) \tag{4.3}$$

and for $s \le t$

$$N_s^t(x) := \int_0^s p_{t-\tau}^\varepsilon \circ dM_\tau\, (x) \equiv \sum_y \int_0^s p_{t-\tau}^\varepsilon(x-y)\, dM_\tau(y) \tag{4.4}$$

We note that for every fixed $t > 0$, $N_s^t(x)$, $s < t$ is a $\mathcal{F}_s$ martingale with bracket

$$\langle N^t(x)\,,\, N^t(x)\rangle_s = \sum_y \int_0^s p_{t-\tau}^\varepsilon(x-y)^2\, d\langle M(y)\,,\, M(y)\rangle_\tau \tag{4.5}$$

4.1 Moments estimates

In this Section we estimate the moments of $\xi_t(x)$, $\xi_t(x) - \xi_t(y)$ and $\xi_t(x) - \xi_s(x)$. By equation (3.16) we have

$$\langle M(x), M(x)\rangle_t \le c\,\varepsilon\, \xi_t(x)^2 \tag{4.6}$$

for some constant c. This bound allows *to close* the equations for the moments. In fact it is the only property of the martingale term in (3.15) we are going to use here.

Before starting we note that there is the following *a priori* bound

$$\xi_t(x) \le \exp\left\{2\gamma_\varepsilon |x_t^0| + \gamma_\varepsilon |x| + \lambda_\varepsilon t\right\} \tag{4.7}$$

and for all n, ε and t we can find $b = b_\varepsilon$ such that

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\exp\left\{2n\gamma_\varepsilon|x^0_t| + n\gamma_\varepsilon|x| + n\lambda_\varepsilon t\right\}\right) \le b \tag{4.8}$$

This follows directly from the fact that $|x^0_t|$ is stochastically dominated by a Poisson random variable with mean $2t$ (since the jump rates for x^0 are bounded by 2). From the definition (3.11) we obtain that for every m there is c such that

$$|\xi_t(x)^m - \xi_{t^-}(x)^m| \le c\,\varepsilon^{\frac{1}{2}}\,\xi_t(x)^m \tag{4.9}$$

and by using again the fact that the rates are bounded (by 2) we have also that

$$\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\left(|\xi_t(x) - \xi_{t^-}(x)| > 0\right) = 0 \tag{4.10}$$

which, together with (4.7) and (4.8), implies that for all n, m

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(|\xi_t(x)^m - \xi_{t^-}(x)^m|^n\right) = 0 \tag{4.11}$$

We start by proving that the $L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)$ norm of $\xi_t(x)$ is bounded uniformly in ε.

Lemma 4.1 *For any $p > 1$, $T > 0$ there are $a, c > 0$ such that*

$$\sup_{t\in[0,\varepsilon^{-2}T]}\ \sup_{x\in\mathbf{Z}}\ e^{-a\varepsilon|x|}\ \|\xi_t(x)\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \le c \tag{4.12}$$

for all $\varepsilon > 0$.

Proof. For $s < t$ we have

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(p^{\varepsilon}_{t-s}\circ\xi_s(x)\right)^n = \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(p^{\varepsilon}_{t}\circ\xi_0\,(x)\right)^n + \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\int_0^s d\tau\ \left(\partial_\tau + H_\varepsilon\right)\left(p^{\varepsilon}_{t-\tau}\circ\xi_\tau(x)\right)^n \tag{4.13}$$

The first term on the right hand side can be bounded using the hypothesis on the initial condition. Indeed

$$\begin{aligned}\|p^{\varepsilon}_t\circ\xi_0\,(x)\|_{L^n\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} &\le \sum_y p^{\varepsilon}_t(x-y)\ \|\xi_0\,(y)\|_{L^n\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \le c_1\sum_y p^{\varepsilon}_t(x-y)\,e^{a_1\varepsilon|y|}\\ &\le 2\,c_1\,e^{a_1\varepsilon|x|}\sum_y p^{\varepsilon}_t(y)\,e^{a_1\varepsilon y} \le 2\,c_1\,e^{a_1\varepsilon|x|}\ \exp\left\{c_2a_1^2\varepsilon^2t\right\}\end{aligned} \tag{4.14}$$

in which we used (2.13) and the fact that $\exp\{Ax\}$ is an eigenfunction of $\triangle$ with eigenvalue $2(\cosh A - 1)$.

Taking the limit $s\uparrow t$ in (4.13) and expanding the power on the right hand side we get

$$\begin{aligned}\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_t(y)\right)^n &\le c_3e^{na_1\varepsilon|x|}\exp\left\{nc_2a_1^2\varepsilon^2t\right\}\\ &+\ \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\int_0^t d\tau\sum_{y_1,\ldots,y_n}\left(\partial_\tau + H_\varepsilon\right)\prod_{i=1}^n\left(p^{\varepsilon}_{t-\tau}(x-y_i)\,\xi_\tau(y_i)\right)\end{aligned} \tag{4.15}$$

where we used (4.11) to exchange the limit with the expectation and (4.14).

We next observe that

$$\begin{aligned} & H_\varepsilon \xi(x)^n - n\,\xi(x)^{n-1}\,H_\varepsilon \xi(x) \\ = \;& \frac{1}{2}\,\xi(x)^n \left\{ \frac{1+\sigma(x)}{2}\,\left[e^{2n\gamma_\varepsilon} - 1 - n\left(e^{2\gamma_\varepsilon}-1\right)\right] \right. \\ & + \frac{1+\sigma(x+1)}{2}\,e^{2\gamma_\varepsilon}\,\left[e^{-2n\gamma_\varepsilon} - 1 - n\left(e^{-2\gamma_\varepsilon}-1\right)\right] \\ & \left. - \frac{1+\sigma(x)}{2}\,\frac{1+\sigma(x+1)}{2}\,\left[e^{2n\gamma_\varepsilon} - 1 - e^{2\gamma_\varepsilon}\left(e^{-2n\gamma_\varepsilon}-1\right)\right]\right\} \\ \le \;& c_4\,\varepsilon\,\xi(x)^n \end{aligned} \tag{4.16}$$

On the other hand, if we apply the generator to a product of ξ computed at different points, H_ε obeys the Leibniz rule. That is if $x_i \neq x_j$ for $i \neq j$

$$H_\varepsilon\left[\xi(x_1)\cdots\xi(x_n)\right] = \sum_{i=1}^n \xi(x_1)\cdots\left[H_\varepsilon \xi(x_i)\right]\cdots\xi(x_n) \tag{4.17}$$

Recalling that p_t^ε solves (4.1), formulae (4.15), (4.16) and (4.17) imply

$$\begin{aligned} & \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\xi_t(y)\right)^n \le c_3 e^{na_1\varepsilon|x|}\exp\left\{nc_2 a_1^2\varepsilon^2 t\right\} \\ & +\;\; c_4\,\varepsilon\,\mathbf{E}^\varepsilon_{\mu_\varepsilon}\int_0^s d\tau \sum_{y_1,\cdots,y_n\,:\,y_1=y_2}\prod_{i=1}^n p^\varepsilon_{t-\tau}(x-y_i)\,\xi_\tau(y_i) \end{aligned} \tag{4.18}$$

and by using Hölder inequality the second term in the right hand side of (4.18) can be bounded by

$$c_4\,\varepsilon\int_0^s d\tau \sum_{y_1,\cdots,y_n\,:\,y_1=y_2}\prod_{i=1}^n p^\varepsilon_{t-\tau}(x-y_i)\;\left[\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\xi_\tau(y_i)\right)^n\right]^{\frac{1}{n}} \tag{4.19}$$

For $t \le T$, $a = na_1$, define

$$f(t) := \sup_{x\in\mathbf{Z}}\, e^{-a\varepsilon|x|}\,\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\xi_{\varepsilon^{-2}t}(x)^n\right) \tag{4.20}$$

the bounds (4.18) and (4.19) yield

$$f(t) \le c_5 + c_6\,\varepsilon\int_0^{\varepsilon^{-2}t} d\tau \sum_{y_1,\cdots,y_n\,:\,y_1=y_2}\prod_{i=1}^n p^\varepsilon_{\varepsilon^{-2}t-\tau}(x-y_i)\,e^{\frac{a}{n}\varepsilon|x-y_i|}\;\cdot\; f(\varepsilon^2\tau) \tag{4.21}$$

It will be shown in Appendix A that for all $a \in \mathbf{R}^+$ there is c such that for all $\tau \in [0, \varepsilon^{-2}T]$, $\varepsilon > 0$

$$\sup_{y\in\mathbf{Z}} p^\varepsilon_\tau(y) \le c\,\tau^{-1/2} \tag{4.22}$$

By repeating the estimate done in (4.15) we then obtain

$$\sum_{y_1,\cdots,y_n\,:\,y_1=y_2}\prod_{i=1}^n p^\varepsilon_{\varepsilon^{-2}t-\tau}(x-y_i)\,e^{\frac{a}{n}\varepsilon|x-y_i|} \le c_7\left(\varepsilon^{-2}t-\tau\right)^{-\frac{1}{2}} \tag{4.23}$$

Finally, (4.21) and (4.23) imply that for every n there exists $c_8 \in (0,\infty)$ such that

$$f(t) \leq c_8 \left\{ 1 + \int_0^t \frac{f(s)}{\sqrt{t-s}} ds \right\} \tag{4.24}$$

hence the estimate (4.12) follows from the *singular generalized Gronwall's Lemma* (see [15, Lemma 6, pag. 33]) which states that (4.24) implies that for all T, c_8 there is C such that $f(t) \leq C$ for $t \in [0,T]$. $\square$

We now state an Hölder estimate in x for the process $\xi_t(x)$.

Lemma 4.2 *For all $p > 1$, $T > 0$, $\alpha < 1/2$ there are $a, c > 0$ such that*

$$\sup_{t\in[0,\varepsilon^{-2}T]} \|\xi_t(x) - \xi_t(y)\|_{L^p(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon})} \leq c\, e^{a\varepsilon(|x|+|y|)}\, (\varepsilon|x-y|)^\alpha \tag{4.25}$$

for any $x, y \in \mathbf{Z}$ and all $\varepsilon > 0$.

Proof. We shall apply the Burkholder-Davis-Gundy (BDG) inequality to the the martingale N^t_s

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(N^t_s(x)^{2n}\right) \leq c_1\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left([N^t(x), N^t(x)]^n_s\right) \tag{4.26}$$

where $c_1 = c_1(n)$ is a universal constant and

$$\begin{aligned} [N^t(x), N^t(x)]_s &:= \sum_{\tau\leq s}\left(N^t_\tau(x) - N^t_{\tau^-}(x)\right)^2 \\ &= \sum_{\tau\leq s}\sum_y p^{\varepsilon}_{t-\tau}(x-y)^2\,[\xi_\tau(y) - \xi_{\tau^-}(y)]^2 \end{aligned} \tag{4.27}$$

is the quadratic variation of $N^t(x)$, since $N^t(x)$ is a bounded variation martingale. The sum over τ stands for the sum over the jump times τ, i.e. the (a.s. countably many) times for which the term in the sum does not vanish. This form of the BDG inequality can be found in [25, 6 E.3].

We observe that if we could replace the quadratic variation on the right hand side of (4.26) with the bracket process (4.5) then the bound (4.25) would follow from Lemma 4.1 and the Hölder property for $p^{\varepsilon}_t(x)$. We thus first show that the bracket process and the quadratic variation are close. To do this, we will take advantage of the fact that the quadratic variation minus the bracket process is a martingale [25] and we apply the BDG inequality to this martingale. Let us introduce

$$D^t_s(x) := [N^t(x), N^t(x)]_s - \langle N^t(x), N^t(x)\rangle_s \tag{4.28}$$

which quadratic variation is

$$\begin{aligned} [D^t(x), D^t(x)]_s &= \sum_{\tau\leq s}\left(N^t_\tau(x) - N^t_{\tau^-}(x)\right)^4 \\ &= \sum_{\tau\leq s}\sum_y p^{\varepsilon}_{t-\tau}(x-y)^4\,[\xi_\tau(y) - \xi_{\tau^-}(y)]^4 \end{aligned} \tag{4.29}$$

since $\langle N^t(x), N^t(x)\rangle_s$ is continuous and of bounded variation.

Let $m \in \mathbf{N}$ and $k = [t]$, set $t_i = i/m$ and partition $[0,t]$ in the $mk+1$ subintervals $\mathcal{I}_i = [t_i, t_{i+1}]$ for $i = 0, \ldots, mk$ and $\mathcal{I}_{mk} = [k,t]$. Using (4.9) we get

$$\left\| \left[D^t(x), D^t(x)\right]_s \right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c_3\, \varepsilon^2 \sum_y \sum_{i=0}^{mk} \sup_{s\in\mathcal{I}_i} \left[p^\varepsilon_{t-s}(x-y)\right]^4 \cdot \left\| \sum_{\tau\in\mathcal{I}_i} \xi^4_\tau(y) \right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \tag{4.30}$$

Calling Q_i the number of exchanges between the sites y and $y+1$ in the interval $\mathcal{I}_i$, we have

$$\sum_{\tau\in\mathcal{I}_i} \xi_\tau(y)^4 \le \xi_{t_i}(y)^4\, Q_i\, (1 + c\sqrt{\varepsilon})^{4Q_i}\, \exp(4\lambda_\varepsilon/m) \tag{4.31}$$

as it follows from (3.11) and (4.9). But the jump rates are bounded above by 2 and so by the fact that (4.31) is monotonic in Q_i and that the random variable Q_i is stochastically dominated by a Poisson random variable with mean $2/m$ we obtain

$$\begin{aligned} \left\| \sum_{\tau\in\mathcal{I}_i} \xi^4_\tau(y) \right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} &\le \mathbf{E}^\varepsilon_{\mu_\varepsilon} \left\{ \xi^4_{t_i}(y)\, \mathbf{E}^\varepsilon_{\mu_\varepsilon} \left[Q_i^n (1 + c\sqrt{\varepsilon})^{4nQ_i} \exp(4n\lambda_\varepsilon/m) \middle| \mathcal{F}_{t_i} \right] \right\}^{1/n} \\ &\le \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\xi^4_{t_i}\right) \sum_{j=1}^n p_j \left(\frac{2}{m}\right)^j (1 + c\sqrt{\varepsilon})^{4nj} e^{\{(2/m)[1-(1+c\sqrt{\varepsilon})^{4n}] + (4n\lambda_\varepsilon/m)\}} \le \frac{c_4}{m}\, e^{a\varepsilon|y|} \end{aligned} \tag{4.32}$$

where p_j are bounded constants that can be easily computed and we used Lemma 4.1. From (4.30) and (4.32) we get

$$\left\| \left[D^t(x), D^t(x)\right]_s \right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c_5 \varepsilon^2 \sum_y \frac{1}{m} \sum_{i=1}^{mk} \sup_{s\in\mathcal{I}_i} \left(p^\varepsilon_{t-s}(x-y)\right)^4 \exp(a\varepsilon|y|) \tag{4.33}$$

Let now $m \to \infty$ and note that the difference between $\sup_{s\in\mathcal{I}_i} p^\varepsilon_{t-s}(x-y)$ and $p^\varepsilon_{t-t_i}(x-y)$ is $O(1/m)$ uniformly. By using the inequality $\exp(a\varepsilon|y|) \le \exp(a\varepsilon y) + \exp(-a\varepsilon y)$, the fact that $\exp(a\varepsilon y)$ is eigenfunction of $\triangle$ with eigenvalue $2(\cosh(a\varepsilon) - 1)$ and that $p^\varepsilon_t(x) \le 1 \wedge ct^{-1/2}$ (see (4.22)) we finally obtain

$$\begin{aligned} &\left\| \left[D^t(x), D^t(x)\right]_s \right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\ &\le\ c_6 \varepsilon^2 \exp\{c_7 \varepsilon^2 t\} \exp\{a\varepsilon|x|\} \int_0^t \frac{1}{(t-s)^{1-b}} ds \le c_8 \varepsilon^{2-2b} \exp\{a\varepsilon|x|\} \end{aligned} \tag{4.34}$$

where $b > 0$ can be arbitrarily chosen.

By using (3.16), (4.5), (4.26) and (4.34) we have

$$\begin{aligned} &\left\| N^t_s(x) - N^t_s(y) \right\|^2_{L^{2n}\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\ &\le\ c_8\, \varepsilon^{2-2b}\, e^{a\varepsilon(|x|+|y|)} + c_9\, \varepsilon \sum_z \int_0^s d\tau\ \left[p^\varepsilon_{t-\tau}(x-z) - p^\varepsilon_{t-\tau}(y-z)\right]^2 \left\| \xi_\tau(z)^2 \right\|^2_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\ &\le\ c_3\, \varepsilon\, e^{a\varepsilon(|x|+|y|)} + c_{10}\, \varepsilon \sum_z \int_0^s d\tau\ \left[p^\varepsilon_{t-\tau}(x-z) - p^\varepsilon_{t-\tau}(y-z)\right]^2 e^{a\varepsilon|z|} \end{aligned} \tag{4.35}$$

where we used Lemma 4.1.

In Appendix A it is shown that for any $\beta \le 1/2$ there is a constant c such that for all $t \in [0, \varepsilon^{-2}T]$ and any $x, y \in \mathbf{Z}$

$$\sup_{z\in\mathbf{Z}} |p_t^\varepsilon(x-z) - p_t^\varepsilon(y-z)| \le c\,\frac{1}{\sqrt{t}}\,\frac{|x-y|^{2\beta}}{t^\beta} \tag{4.36}$$

Let us now take the limit $s \nearrow t$ in (4.35) keeping in mind that $\lim_{s\uparrow t} |N_s^t(x) - N_t^t(x)| = 0$ $\mathbf{P}^\varepsilon_{\mu_\varepsilon}-$a.s. (and that the same quantity is bounded by $c\sqrt{\varepsilon}\xi_t(x)$, for which all the moments exist). Choose $\beta = \alpha < 1/2$ in (4.36) and $b < 1/2$. We get

$$\begin{aligned} \left\|N_t^t(x) - N_t^t(y)\right\|_{L^{2n}\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} &\le c_{11}\, e^{a\varepsilon(|x|+|y|)} \left(\varepsilon^{1-b} + (\varepsilon|x-y|)^\alpha\right) \\ &\le c_{12}\, e^{a\varepsilon(|x|+|y|)}\ (\varepsilon|x-y|)^\alpha \end{aligned} \tag{4.37}$$

On the other hand, because of our hypothesis (2.14) on the initial measure μ_ε, we have

$$\|p_t^\varepsilon \circ \xi_0(x) - p_t^\varepsilon \circ \xi_0(y)\|_{L^{2n}\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c\, e^{a\varepsilon(|x|+|y|)}\ (\varepsilon|x-y|)^{\frac{1}{2}} \tag{4.38}$$

which used together with (4.37) in (4.2), yields (4.25). $\square$

An analog Hölder estimate holds in the time variable t.

Lemma 4.3 *For all $b > 0$, $p > 1$, $T > 0$, $\alpha < 1/4$ there are $a, c > 0$ such that*

$$\sup_{x\in\mathbf{Z}} e^{-a\varepsilon|x|}\, \|\xi_t(x) - \xi_s(x)\|_{L^p\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c\,\left[\left(\varepsilon^2|t-s|\right)^\alpha + \varepsilon^{1-b}\right] \tag{4.39}$$

for any $t, s \in [0, \varepsilon^{-2}T]$ and all $\varepsilon > 0$.

Proof. Let $h > 0$, in order to analyze the martingale term in (4.2) we write

$$N_{t+h}^{t+h}(x) - N_t^t(x) = \left[N_{t+h}^{t+h}(x) - N_t^{t+h}(x)\right] + \left[N_t^{t+h}(x) - N_t^t(x)\right] \tag{4.40}$$

By repeating the same steps of the previous Lemma we obtain

$$\begin{aligned} &\left\|\lim_{s\uparrow t}\left[N_{s+h}^{t+h}(x) - N_s^{t+h}(x)\right]\right\|^2_{L^{2n}\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\ &\le\ c_1\, e^{a\varepsilon|x|}\,\varepsilon^{2-2b} + c_1\,\varepsilon\sum_y \int_t^{t+h} d\tau\ p^\varepsilon_{t+h-\tau}(x-y)^2\ \left\|\xi_\tau(y)^2\right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\ &\le\ c_1\, e^{a\varepsilon|x|}\,\varepsilon^{2-2b} + c_2\, e^{a\varepsilon|x|}\,\varepsilon \int_t^{t+h} d\tau\ (t+h-\tau)^{-1/2} \le\ c_3\, e^{a\varepsilon|x|}\ \left[(\varepsilon^2 h)^{\frac{1}{2}} + \varepsilon^{2-2b}\right] \end{aligned} \tag{4.41}$$

in which we used Lemma 4.1 and the bound (4.22).

In Appendix A we verify that for all $\beta \le 1$ there is c such that for all $t \in [0, \varepsilon^{-2}T]$, $h > 0$

$$\sup_{x\in\mathbf{Z}} \left|p^\varepsilon_{t+h}(x) - p^\varepsilon_t(x)\right| \le c\frac{1}{\sqrt{t}}\left(\frac{h}{t}\right)^\beta \tag{4.42}$$

In analogy with (4.41), by choosing $\beta = 2\alpha < 1/2$ we thus get

$$\begin{aligned}
&\left\| \lim_{s\uparrow t} \left[N_s^{t+h}(x) - N_s^t(x) \right] \right\|^2_{L^{2n}\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c_4\, e^{a\varepsilon|x|}\, \varepsilon^{2-2b} \\
&\le \; c_4\, e^{a\varepsilon|x|}\, \varepsilon^{2-2b} + \, c_4\, \varepsilon \sum_y \int_0^t d\tau \, \left[p^\varepsilon_{t+h-\tau}(x-y) - p^\varepsilon_{t-\tau}(x-y) \right]^2 \left\| \xi_\tau(y)^2 \right\|_{L^n\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\
&\le \; c_4\, e^{a\varepsilon|x|}\, \varepsilon^{2-2b} + c_5\, e^{a\varepsilon|x|}\, h^{2\alpha}\, \varepsilon \int_0^t d\tau\, (t-\tau)^{-\frac{1}{2}-2\alpha} \le c_6\, e^{a\varepsilon|x|} \left[(\varepsilon^2 h)^{2\alpha} + \varepsilon^{2-2b} \right]
\end{aligned} \tag{4.43}$$

Hence the bound (4.39) is implied by (4.41), (4.43) and the following estimate on the initial datum ξ_0

$$\left\| p^\varepsilon_{t+h} \circ \xi_0(x) - p^\varepsilon_t \circ \xi_0(x) \right\|_{L^{2n}\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c_7\, e^{a\varepsilon|x|} \left(\varepsilon^2 h \right)^\alpha$$

which is a direct consequence of (2.14). $\square$

The next step is to obtain uniform Hölder estimates from the Lemmata we just proved. We will use Kolmogorov Theorem as stated in [28, Th 2.1]. In our notation it becomes

Proposition 4.4 *Let K be a compact subset of $\mathbf{R}^d$ and let $\{X_z\}_{z\in K}$ be a real valued process for which there exist three strictly positive constants α_1, α_2 and α_3 such that*

$$\mathbf{E}\left(|X_z - X_{z'}|^{\alpha_1} \right) \le \alpha_2 |z - z'|^{d+\alpha_3} \tag{4.44}$$

for all $z, z' \in K$. Then for every $\alpha \in [0, \alpha_3/\alpha_1)$, there exists $c \in \mathbf{R}^+$ such that

$$\mathbf{E}\left[\left(\sup_{z\neq z'} \frac{|X_z - X_{z'}|}{|z - z'|^\alpha} \right)^{\alpha_1} \right] \le c \tag{4.45}$$

The additional problem comes from the fact that for $\varepsilon > 0$, $\xi_t(x)$ is not continuous in t, so that (4.44) cannot be satisfied. This problem will be overcame by defining a new process $\overline{\xi}_t(x)$, which is simply the interpolation of the values of the previous process at integer times, and by estimating the distance of ξ and $\overline{\xi}$ uniformly in space and time.

We start by establishing the uniform Hölder estimate in the space variable for fixed time.

Lemma 4.5 *For any $p > 1$, $T > 0$, $b < 1/2$ and any compact set $K \subset \mathbf{R}$ there is $c > 0$ such that*

$$\sup_{t\in[0,T]} \left\| \sup_{r,r'\in K} \frac{|\xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r) - \xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r')|}{|r - r'|^b} \right\|_{L^p\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \le c \tag{4.46}$$

for all $\varepsilon > 0$.

Proof. We recall that $\xi_s(r) = ([r]+1-r)\xi_s([r]) + (r-[r])\xi_s([r]+1)$. It is then straightforward to verify that for all $r_1, r_2 \in \mathbf{R}$ and $b' \in (0,1)$

$$\frac{\xi_s(r_1)-\xi_s(r_2)}{|r_1-r_2|^b} \leq 2^{b'} \frac{\xi_s([r_1])-\xi_s([r_2])}{|[r_1]-[r_2]|^{b'}} + \sum_{i=1,2} |\xi_s([r_i]+1)-\xi_s([r_i])| \tag{4.47}$$

Choose $b' \in (b, 1/2)$. The first term on the right hand side is bounded by using Lemma 4.2 and for the second one we use Lemma 4.1. We then obtain that for all $p>0$ the is a constant c such that for all $r, r' \in K$ we have

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\left|\xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r') - \xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r)\right|^p\right) \leq c|r'-r|^{b'p} \tag{4.48}$$

choose p and such that $b' = b + (1/p) < 1/2$ and apply Proposition 4.4 with $\alpha_1 = p$, $\alpha_2 = c$ and $\alpha_3 = b'p - d = bd$ to get the result. $\square$

Let us consider now the time dependence. To this purpose we introduce a new process $\overline{\xi}_t(r)$, $t \in \mathbf{R}^+$, $r \in \mathbf{R}$, which is defined as

$$\overline{\xi}_t(r) := ([t]+1-t)\,\xi_{[t]}(r) + (t-[t])\,\xi_{[t]+1}(r) \tag{4.49}$$

By repeating the proof of Lemma 4.2 and by using Lemma 4.3 instead of Lemma 4.2 one easily gets that for any $p>1$, $T>0$, $b<1/4$ and any compact set $K \subset \mathbf{R}$ there is $c>0$ such that

$$\sup_{r\in K}\left\| \sup_{t,t'\in[0,T]} \frac{\left|\overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r) - \overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r)\right|}{|t-t'|^b}\right\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \leq c \tag{4.50}$$

for all $\varepsilon > 0$. But we are looking for a stronger result. Precisely

Lemma 4.6 *For any $p>1$, $T>0$, $b<1/4$ and any compact set $K \subset \mathbf{R}$ there is $c>0$ such that*

$$\left\| \sup_{r\in K} \sup_{t,t'\in[0,T]} \frac{\left|\overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r) - \overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r)\right|}{|t-t'|^b}\right\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \leq c \tag{4.51}$$

for all $\varepsilon > 0$.

Remark. We will naturally get the stronger result

$$\left\| \sup_{r,r'\in K} \sup_{t,t'\in[0,T]} \frac{\left|\overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r) - \overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r')\right|}{(|t-t'|+|r-r'|)^b}\right\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \leq c \tag{4.52}$$

Proof. It is very similar to the proof of Lemma 4.5. In fact

$$\frac{|\overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r') - \overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r)|}{(|t'-t|+|r'-r|)^{b'}} \leq \frac{|\overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r') - \overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r)|}{|r'-r|^{b'}} + \frac{|\overline{\xi}_{\varepsilon^{-2}t'}(\varepsilon^{-1}r) - \overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r)|}{|t'-t|^{b'}} \tag{4.53}$$

Choose $b' \in (b, 1/4)$ and use the fact that $\overline{\xi}_t(r) \le \xi_{[t]}(r) + \xi_{[t]+1}$ and $\overline{\xi}_t(r) \le \xi_t([r]) + \xi_t([r]+1)$. Then the result (4.51) follows by applying the decomposition (4.47) to both term in the right hand side of (4.53) and by using Lemmata 4.1, 4.2, 4.3 and Proposition 4.4. □

We are left with proving that ξ and $\overline{\xi}$ are uniformly close, i.e. if K is a compact subset of $\mathbf{R}$, $T, \delta > 0$, we want the measure of the set

$$B(\delta) := \left\{ \sup_{t\in[0,T]} \sup_{r\in K} \left| \overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r) - \xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r) \right| > \delta \right\} \tag{4.54}$$

to be small, i.e. vanishing with ε. We need only a statement in probability, but a stronger statement is available too (see formula (4.58) in the proof below).

Lemma 4.7 *For any $\delta > 0$, $T > 0$ and any compact set $K \subset \mathbf{R}$*

$$\lim_{\varepsilon\to 0} \mathbf{P}^{\varepsilon}_{\mu_\varepsilon}(B(\delta)) = 0 \tag{4.55}$$

Proof. Define $I \subset \mathbf{Z}^+ \times \mathbf{Z}$ as follows

$$I := \{(n,x) \ : \ n = 0, \dots [\varepsilon^{-2}T] \ \ , \ \ \min_{r\in K} |x - \varepsilon^{-1}r| \le 1\} \tag{4.56}$$

and observe that the cardinality of I is bounded by $C_{K,T}\varepsilon^{-3}$, where $C_{K,T} < \infty$ for every $T \in (0,\infty)$ and any compact set K. We claim that for $b \in (0, 1/4)$ there is c such that for all $(n,x) \in I$

$$\left\| \sup_{r\in[x,x+1]} \sup_{t\in[n,n+1]} \left| \overline{\xi}_t(r) - \xi_t(r) \right| \right\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \le c\,\varepsilon^b \tag{4.57}$$

If (4.57) is granted, (4.55) follows because for all $p > 0$

$$\begin{aligned} &\mathbf{E}^{\varepsilon}_{\mu_\varepsilon} \left\{ \sup_{r\in K} \sup_{t\in[0,T]} \left| \overline{\xi}_{\varepsilon^{-2}t}(\varepsilon^{-1}r) - \xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r) \right|^p \right\}_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \\ &\le \ C_{K,T}\,\varepsilon^{-3} \sup_{(n,x)\in I} \mathbf{E}^{\varepsilon}_{\mu_\varepsilon} \left\{ \sup_{r\in[x,x+1]} \sup_{t\in[n,n+1]} \left| \overline{\xi}_t(r) - \xi_t(r) \right|^p \right\} \le C_{K,T}\, c^p\, \varepsilon^{-3+pb} \end{aligned} \tag{4.58}$$

Choose then $p > 3/b$ and apply the Chebychev inequality to obtain (4.55).

We are left with the proof of (4.57). Let us fix (n,x) and start by observing that for $s \in [n, n+1]$ and $r \in [x, x+1]$

$$|\overline{\xi}_s(r) - \xi_s(r)| \le \sum_{i=0,1} \left(|\xi_n(x+i) - \xi_s(x+i)| + |\xi_{n+1}(x+i) - \xi_n(x+i)| \right) \tag{4.59}$$

by using Lemma 4.3 we thus get

$$\left\| \sup_{r\in[x,x+1]} \sup_{t\in[n,n+1]} \left| \overline{\xi}_t(r) - \xi_t(r) \right| \right\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \le c\,\varepsilon^b + \sum_{i=0,1} \left\| \sup_{t\in[n,n+1]} |\xi_n(x+i) - \xi_t(x+i)| \right\|_{L^p\left(\mathbf{P}^{\varepsilon}_{\mu_\varepsilon}\right)} \tag{4.60}$$

In order to bound the second term in the right hand side of formula (4.60) we recall that the jump rates are bounded by 2 so that if $m = [p] + 1$, we obtain

$$\begin{aligned}
&\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\sup_{t\in[n,n+1]} |\xi_n(x+i) - \xi_t(x+i)|^m\right) \le \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_n(x+i)^m \left[e^{\lambda_\varepsilon}(1+c\sqrt{\varepsilon})^Q - 1\right]^m\right) \\
&\le \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left\{\xi_n(x+i)^m \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left\{\left[e^{\lambda_\varepsilon}(1+c\sqrt{\varepsilon})^Q - 1\right]^m \Big| \mathcal{F}_n\right\}\right\} \\
&\le \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left\{\xi_n(x+i)^m\right\} \sum_{q=0}^{\infty} \frac{2^q e^{-2}}{q!}\left[e^{\lambda_\varepsilon}\left(1+c\sqrt{\varepsilon}\right)^q - 1\right]^m \le c_1\, \varepsilon^{m/2} \qquad (4.61)
\end{aligned}$$

in which Q is the number of jumps involving site $x+i$ in the time interval $[n, n+1]$. We have dominated Q (in the stochastic sense) by the number of jumps which occur in a Poisson process with rate 2 in a unit time. The last inequality is finally consequence of Lemma 4.1 and what follows. Given $N \in \mathbf{N}$ we have

$$\sum_{q=0}^{\infty} \frac{2^q e^{-2}}{q!}\left(e^{\lambda}_{\varepsilon}(1+\sqrt{\varepsilon})^q - 1\right)^m \le \sum_{q=0}^{N-1} \frac{2^q e^{-2}}{q!}\left(e^{\lambda}_{\varepsilon}(1+\sqrt{\varepsilon})^q - 1\right)^m + \frac{2^{N(m+1)}}{N!} e^{2^{m+1}-2} \quad (4.62)$$

Choose then $N = 2|\log\varepsilon|$ and $\varepsilon \le \bar{\varepsilon}$, so that $N\sqrt{\varepsilon} \le 1$. Elementary computations and the fact that $e^{\lambda}_{\varepsilon}(1+c\sqrt{\varepsilon})^q \le \exp(\lambda_\varepsilon + q\sqrt{\varepsilon}) - 1 \le e^2 q\sqrt{\varepsilon}$ for all $q < N$ yield that for every m there is c such that

$$\begin{aligned}
\sum_{q=0}^{\infty} \frac{2^q e^{-2}}{q!}\left(e^{\lambda_\varepsilon}(1+\sqrt{\varepsilon})^q - 1\right)^m &\le e^{2m}\varepsilon^{m/2}\left(\sum_{q=0}^{\infty} \frac{2^q e^{-2} q^m}{q!}\right) + \frac{2^{N(m+1)}}{N!} e^{2^{m+1}-2} \\
&\le c\left(\varepsilon^{m/2} + \frac{2^{N(m+1)}}{N!}\right) \qquad (4.63)
\end{aligned}$$

By the choice of N, the proof of (4.61) is completed. The claim (4.57) is then proven by using (4.61) in (4.60). $\square$

4.2 The key estimate

Up to now we did not fully exploit the content of the martingale term in (3.15). In fact we only used the bound (4.6). In order to study the convergence of ξ_t we have to show that, on the space-time scale we are considering, $\langle M(x), M(x)\rangle_t \to \varepsilon\xi_t(x)^2$, and this will require some control over the decay in time of the correlations for ξ_t. This is the key point in the proof of the scaling limit for ξ_t.

Lemma 4.8 *For all $T > 0$, $\delta > 0$, there are $a, c > 0$ such that*

$$\sup_{x\in\mathbf{Z}} e^{-a\varepsilon|x|}\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left|\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\nabla^+\xi_{\varepsilon^{-2}t}(x)\, \nabla^-\, \xi_{\varepsilon^{-2}t}(x)\,\Big|\, \mathcal{F}_{\varepsilon^{-2}s}\right)\right| \le c\,\varepsilon^{\frac{3}{2}-\delta}\,(t-s)^{-\frac{1}{2}} \qquad (4.64)$$

for all $\sqrt{\varepsilon} \le s < t \le T$ and all $\varepsilon > 0$.

The idea of the proof is to express the conditioned expectation in term of the process ξ until one obtains a closed scheme that can be iterated. It is however important to note that the result does not reflect a scaling property: the statement of the lemma would be false if in the left hand side of (4.64) we replace $\nabla^+\xi \ \nabla^- \ \xi$ with $[\nabla^\pm\xi]^2$ and the bound we find ($\varepsilon^{3/2-\delta}$, not optimal) it is the result of a cancellation.

Proof. Let us introduce the notation

$$K_t(x) := \nabla^+ p_t^\varepsilon(x) \ \nabla^- \ p_t^\varepsilon(x) \tag{4.65}$$

For $0 \leq s_1 \leq s_2 < t$, recalling (4.5), we have

$$\begin{aligned} \mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\nabla^+ N_{s_2}^t(x) \ \nabla^- \ N_{s_2}^t(x) \Big| \ \mathcal{F}_{s_1} \right) &= \nabla^+ N_{s_1}^t(x) \ \nabla^- \ N_{s_1}^t(x) \\ + \ \ \mathbf{E}_{\mu_\varepsilon}^\varepsilon & \left(\sum_y \int_{s_1}^{s_2} K_{t-\tau}(x-y) \, d\langle M(y) \, , \, M(y) \rangle_\tau \Bigg| \ \mathcal{F}_{s_1} \right) \end{aligned} \tag{4.66}$$

By using the decomposition

$$\nabla^\pm \xi_t(x) = \nabla^\pm p_t^\varepsilon \circ \xi_0 \, (x) + \nabla^\pm N_t^t(x) \tag{4.67}$$

in (4.66) and noticing that ξ_0 is independent on $\mathcal{F}_{s_1}$, we get

$$\begin{aligned} &\mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\nabla^+ \xi_t(x) \ \nabla^- \ \xi_t(x) \Big| \ \mathcal{F}_s \right) = \nabla^+ p_t^\varepsilon \circ \xi_0 \, (x) \ \nabla^- \ p_t^\varepsilon \circ \xi_0 \, (x) \\ &\quad + \ \ \nabla^+ p_t^\varepsilon \circ \xi_0 \, (x) \ \nabla^- \ N_s^t(x) \ + \ \nabla^+ N_s^t(x) \ \nabla^- \ p_t^\varepsilon \circ \xi_0 \, (x) \\ &\quad + \ \ \nabla^+ N_s^t(x) \ \nabla^- \ N_s^t(x) \ + \ \mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\sum_y \int_s^t K_{t-\tau}(x-y) \, d\langle M(y) \, , \, M(y) \rangle_\tau \Bigg| \ \mathcal{F}_s \right) \end{aligned} \tag{4.68}$$

where we considered the limit $s_2 \uparrow t$ using the fact that, as in (4.11),

$$\mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\nabla^+ \xi_{t^-}(x) \ \nabla^- \ \xi_{t^-}(x) \Big| \ \mathcal{F}_s \right) = \mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\nabla^+ \xi_t(x) \ \nabla^- \ \xi_t(x) \Big| \ \mathcal{F}_s \right) \tag{4.69}$$

The Lemma will be proven by estimating the $L^1\left(\mathbf{P}_{\mu_\varepsilon}^\varepsilon\right)$ norm of the various terms in the left hand side of (4.68).

We start by looking at the terms containing the intial datum ξ_0

$$\begin{aligned} &\mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\left[\nabla^\pm p_t^\varepsilon \circ \xi_0 \, (x) \right]^2 \right) \leq \sum_{y,y'} \nabla^\pm p_t^\varepsilon(x-y) \ \nabla^\pm \ p_t^\varepsilon(x-y') \, \mathbf{E}_{\mu_\varepsilon}^\varepsilon \left(\xi_0(y) \, \xi_0(y') \right) \\ &\leq \ \ c_2 \, e^{a\varepsilon|x|} \left[\sum_y \nabla^\pm p_t^\varepsilon(y) \, e^{a\varepsilon|y|} \right]^2 \leq c_2 \, e^{a\varepsilon|x|} \ \sum_y \left[\nabla^\pm p_t^\varepsilon(y) \right]^2 \ e^{(a+1)\varepsilon|y|} \cdot \sum_y e^{-\varepsilon|y|} \\ &\leq \ \ c_3 \, e^{a\varepsilon|x|} \, \varepsilon^{-1} \, t^{-\frac{3}{2}} \end{aligned} \tag{4.70}$$

where we used the hypothesis on μ_ε, Cauchy-Schwartz inequality and the last inequality follows from

$$\sum_y \left[\nabla^\pm p_t^\varepsilon(y) \right]^2 \ e^{a\varepsilon|y|} \leq c \, t^{-\frac{3}{2}} \tag{4.71}$$

which is verified in Appendix A.

Now we control the martingale terms $\bigtriangledown^{\pm} N^t_s$. From (3.16) and (4.4) we get

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\left[\bigtriangledown^{\pm} N^t_s(x)\right]^2\right) = \mathbf{E}^{\varepsilon}_{\mu_\varepsilon} \sum_y \int_0^s \left[\bigtriangledown^{\pm} p^{\varepsilon}_{t-\tau}(x-y)\right]^2 d\langle M(y)\,,\, M(y)\rangle_\tau$$
$$\leq \; c_4\, \varepsilon \sum_y \int_0^s d\tau\, \left[\bigtriangledown^{\pm} p^{\varepsilon}_{t-\tau}(x-y)\right]^2\, e^{a\varepsilon|y|} \leq c_5\, e^{a\varepsilon|x|}\, \varepsilon\,(t-s)^{-\frac{1}{2}} \tag{4.72}$$

in which we used Lemma 4.1 and (4.71).

In order to estimate the last term in (4.68), we express $\langle M(y), M(y)\rangle_\tau$ as a function of ξ_t as in (3.17); keeping in mind (3.12) we obtain

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\sum_y \int_s^t K_{t-\tau}(x-y)\, d\langle M(y)\,,\, M(y)\rangle_\tau \,\middle|\, \mathcal{F}_s\right)$$
$$= \; \left(1-\sqrt{1+2\varepsilon^{\frac{1}{2}}}\right)^2 \left[I^1_{t,s}(x) + \frac{1}{2}\, I^2_{t,s}(x)\right] + \left(1+\varepsilon^{\frac{1}{2}}\right) I^3_{t,s}(x) \tag{4.73}$$

where

$$\begin{aligned}
I^1_{t,s}(x) \;&:=\; \sum_y \int_s^t d\tau\, K_{t-\tau}(x-y)\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_\tau(y)^2 \,\middle|\, \mathcal{F}_s\right) \\
I^2_{t,s}(x) \;&:=\; \sum_y \int_s^t d\tau\, K_{t-\tau}(x-y)\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_\tau(y)\, \triangle\, \xi_\tau(y) \mid \mathcal{F}_s\right) \\
I^3_{t,s}(x) \;&:=\; \sum_y \int_s^t d\tau\, K_{t-\tau}(x-y)\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\bigtriangledown^{+}\xi_\tau(y)\, \bigtriangledown^{-}\, \xi_\tau(y) \,\middle|\, \mathcal{F}_s\right)
\end{aligned} \tag{4.74}$$

We start by estimating the easiest, which is $I^2_{t,s}(x)$. By the definition (3.11) of $\xi_t(x)$ we have

$$|\triangle \xi_t(x)| \leq c_7\, \varepsilon^{\frac{1}{2}}\, \xi_t(x) \tag{4.75}$$

and, by Lemma 4.1, we then conclude

$$\mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left|I^2_{t,s}(x)\right| \leq c_8\, \varepsilon^{\frac{1}{2}} \sum_y \int_s^t d\tau\, \left|K_{t-\tau}(x-y)\right|\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_\tau(y)^2\right) \leq c_9\, e^{a\varepsilon|x|}\, \varepsilon^{\frac{1}{2}} \tag{4.76}$$

To obtain a useful estimate of $I^1_{t,s}(x)$ is more difficult: we have to exploit a hidden cancellation which is revealed in Lemma A.1. As a consequence of that result $I^1_{t,s}(x)$ can be rewritten as

$$I^1_{t,s}(x) \;=\; \sum_y \int_s^t d\tau\, K_{t-\tau}(x-y)\, \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_\tau(y)^2 - \xi_t(x)^2 \,\middle|\, \mathcal{F}_s\right)$$
$$+\; \mathbf{E}^{\varepsilon}_{\mu_\varepsilon}\left(\xi_t(x)^2 \,\middle|\, \mathcal{F}_s\right) \cdot \sum_y \int_{t-s}^{\infty} d\tau\, K_\tau(x-y) \tag{4.77}$$

By Lemma 4.1 the $L^1\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)$ norm of the second term in the right hand side of (4.77) is bounded by

$$c_{10}\, e^{a\varepsilon|x|} \int_{t-s}^{\infty} d\tau \sum_x \left[\nabla^+ p^\varepsilon_\tau(x)\right]^2 \leq c_{11}\, e^{a\varepsilon|x|} \int_{t-s}^{\infty} d\tau\, \tau^{-\frac{3}{2}} = 2\, c_{11}\, e^{a\varepsilon|x|}\, (t-s)^{-\frac{1}{2}} \tag{4.78}$$

in which we used (4.71). But the $L^1\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)$ norm of the first term on the right hand side of (4.77) is smaller than

$$\begin{aligned} &\sum_y \int_s^t d\tau\ |K_{t-\tau}(x-y)|\ \|\xi_\tau(y)+\xi_t(x)\|_{L^2\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \cdot \|\xi_\tau(y)-\xi_t(x)\|_{L^2\left(\mathbf{P}^\varepsilon_{\mu_\varepsilon}\right)} \\ &\leq\ c_{12}\, e^{a\varepsilon|x|} \sum_y \int_s^t d\tau\ \left|\nabla^+ p^\varepsilon_{t-\tau}(y)\ \nabla^-\ p^\varepsilon_{t-\tau}(y)\right|\ e^{a\varepsilon|y|} \left[(\varepsilon|y|)^{2\alpha} + \left(\varepsilon^2|t-\tau|\right)^\alpha + \varepsilon^{\frac{1}{2}}\right] \end{aligned} \tag{4.79}$$

where $\alpha < 1/4$. The second inequality follows from Lemmata 4.2 and 4.3.

In order to estimate the right hand side of (4.79) we observe that

$$\sum_y \int_0^1 d\tau\ \left|\nabla^+ p^\varepsilon_\tau(y)\ \nabla^-\ p^\varepsilon_\tau(y)\right|\ e^{a\varepsilon|y|}\ \left(|y|^{2\alpha}+\tau^\alpha\right) \leq c_{13}$$

But, by (4.36), we have that

$$\begin{aligned} &\sum_y \int_1^{\varepsilon^{-2}T} d\tau\ \left|\nabla^+ p^\varepsilon_\tau(y)\ \nabla^-\ p^\varepsilon_\tau(y)\right|\ e^{a\varepsilon|y|}\ \left(|y|^{2\alpha}+\tau^\alpha\right) \\ &\leq\ c_{14} \int_1^{\varepsilon^{-2}T} d\tau\, \tau^{-\frac{3}{2}} \sum_{y\in\mathbf{Z}} p^\varepsilon_\tau(y)\, e^{a\varepsilon|y|}\ \left(|y|^{2\alpha}+\tau^\alpha\right) \leq c_{15} \int_1^{\varepsilon^{-2}T} d\tau\, \tau^{-\frac{3}{2}}\, \tau^\alpha \leq c_{16} \end{aligned}$$

Keeping in mind that $\alpha < 1/4$, the right hand side of (4.79) is thus bounded by $c_{17}\, e^{a\varepsilon|x|}\, \varepsilon^{\frac{1}{2}-\delta}$. We have hence proven that, for some $a \in \mathbf{R}^+$ and every $\delta > 0$

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon} \left|I^1_{t,s}(x)\right| \leq c_{18}\, e^{a\varepsilon|x|} \left(\varepsilon^{\frac{1}{2}-\delta} + (t-s)^{-\frac{1}{2}}\right) \tag{4.80}$$

Instead of estimating $I^3_{t,s}$, we note that it is of the same form of the left hand side in (4.68). We have thus obtain a closed inequality. We next show it can be iterated giving the bound (4.64). For $\sqrt{\varepsilon} \leq s < t \leq T$ let us define

$$f_{t,s}(x) := \varepsilon^{-1}\, \mathbf{E}^\varepsilon_{\mu_\varepsilon} \left|\mathbf{E}^\varepsilon_{\mu_\varepsilon} \left(\nabla^+ \xi_{\varepsilon^{-2}t}(x)\, \nabla^-\ \xi_{\varepsilon^{-2}t}(x)\ \middle|\ \mathcal{F}_{\varepsilon^{-2}s}\right)\right|$$

From (4.68), (4.70), (4.72), (4.73), (4.76) and (4.80) we obtain

$$f_{t,s}(x) \leq c_{19}\, e^{a\varepsilon|x|}\, \varepsilon^{\frac{1}{2}-\delta} \left[1+\varepsilon^{\frac{1}{2}}\,(t-s)^{-\frac{1}{2}}\right] + \varepsilon^{-2}\ \left(1+\varepsilon^{\frac{1}{2}}\right) \int_s^t d\tau \sum_{y\in\mathbf{Z}} \left|K_{\varepsilon^{-2}(t-\tau)}(x-y)\right|\, f_{\tau,s}(y) \tag{4.81}$$

and if we define

$$f_a(t,s) := \sup_{x\in\mathbf{Z}}\ e^{-a\varepsilon|x|}\, f_{t,s}(x)$$

the inequality (4.81) implies

$$f_a(t,s) \le c_{20}\,\varepsilon^{\frac{1}{2}-\delta}\left[1+\varepsilon^{\frac{1}{2}}(t-s)^{-\frac{1}{2}}\right]+\varepsilon^{-2}\left(1+\varepsilon^{\frac{1}{2}}\right)\int_s^t d\tau\,\sum_x \left|K_{\varepsilon^{-2}(t-\tau)}(x)\right|\,e^{a\varepsilon|x|}\,f_a(\tau,s) \tag{4.82}$$

Because of the results stated in Lemmata A.3 and A.4, the above inequality can be iterated, for ε sufficiently small, and we obtain a convergent series. The estimate (4.64) is thus proven. □

4.3 Proof of the scaling limit

In this Section we prove the scaling limit for the process ξ as stated in Theorem 3.3. We study the family of random variables $\xi^\varepsilon_t(r)=\xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r)$, $\varepsilon>0$ over $D([0,T];C(\mathbf{R}))$ and we show that such a family is tight. We then prove that the limit is concentrated on $C([0,T];C(\mathbf{R}))$. Taking the limit of ξ^ε along a convergent subsequence we show it satisfies a suitable martingale problem. By the existence of a unique solution to the martingale problem, we finally conclude the proof.

The space $C(\mathbf{R})$ is metrizable and a bounded metric which generates the topology of uniform convergence over compact sets is

$$\varrho(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\left\{1\wedge\max_{x\in[-n,n]}|f(x)-g(x)|\right\} \tag{4.83}$$

We recall that for $a,b\in\mathbf{R}$, $a\wedge b=\min(a,b)$.

Proof of Theorem 3.3.

- *Tightness and support properties*

We use a tightness criterium due to Aldous and Kurtz [22, Th. 2.7], which in our notation becomes

Proposition 4.9 *Let $\xi^\varepsilon\in D([0,T];C(\mathbf{R}))$. Suppose that for each $t\in[0,T]$ the family of random functions $\{\xi^\varepsilon_t\}_{\varepsilon>0}$ is tight in $C(\mathbf{R})$ and that for any $\delta>0$ there exists a process $\{A_\varepsilon(\delta)\}_{\varepsilon>0}$ such that*

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left\{\varrho\left(\xi^\varepsilon_{t+\delta},\xi^\varepsilon_t\right)\Big|\,\mathcal{F}_t\right\}\le\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left\{A_\varepsilon(\delta)\,|\,\mathcal{F}_t\right\} \tag{4.84}$$

and

$$\lim_{\delta\to0}\,\overline{\lim_{\varepsilon\to0}}\,\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left\{A_\varepsilon(\delta)\right\}=0 \tag{4.85}$$

then the family $\{\xi^\varepsilon\}_{\varepsilon>0}$ is tight in $D([0,T];C(\mathbf{R}))$ and hence, by Prohorov Theorem, there is a sequence $\{\varepsilon_n\}_{n=1,2,\ldots}$ such that ξ^{ε_n} converges as $n\to\infty$.

First of all observe that Lemma 4.1 and Lemma 4.5, together with Theorem 8.2 of [5], imply that $\{\xi^\varepsilon_t\}_{\varepsilon>0}$ is tight in $C(\mathbf{R})$.

It is sufficient to choose

$$A_\varepsilon(\delta) = \sup_{t\in[0,T]} \varrho\left(\xi^\varepsilon_{t+\delta}, \xi^\varepsilon_t\right) \tag{4.86}$$

The condition (4.84) is then obvious. We denote by $\mathbf{1}_B$ the characteristic function of the event $B(\delta)$ defined in (4.54) and we choose $K = [-N, N]$ in that definition. For any $\delta, \delta' > 0$ and every $N \in \mathbf{Z}^+$ we have

$$\begin{aligned}
\mathbf{E}^\varepsilon_{\mu_\varepsilon}\{A_\varepsilon(\delta)\} \leq& \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\
&+ \sum_{n=1}^{N} \frac{1}{2^n} \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left\{\sup_{t\in[0,T]}\left[1 \wedge \max_{x\in[-n,n]} |\xi_{\varepsilon^{-2}(t+\delta)}(x) - \xi_{\varepsilon^{-2}t}(x)|\right]\left(\mathbf{1}_{B(\delta')} + \mathbf{1}_{B^c(\delta')}\right)\right\} \\
\leq&\ 2^{-N} + \sum_{n=1}^{N} \frac{1}{2^n}\left\{\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\mathbf{1}_{B(\delta')}\right) + 2\delta' + \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left[\sup_{t\in[0,T]} \max_{x\in[-n,n]} |\overline{\xi}_{\varepsilon^{-2}(t+\delta)}(x) - \overline{\xi}_{\varepsilon^{-2}t}(x)|\right]\right\}
\end{aligned}$$

hence, by Lemmata 4.6 and 4.7

$$\overline{\lim_{\varepsilon\to 0}}\, \mathbf{E}^\varepsilon_{\mu_\varepsilon}\{A_\varepsilon(\delta)\} \leq 2\delta' + c\,\delta^\alpha + 2^{-N}$$

where $\alpha \in (0, 1/4)$. By the arbitrariety of δ' and N we have verified also condition (4.85) and this completes the proof of the tightness.

We observe that by Lemma 4.5 any limit point $\xi_t(r)$ of $\xi^\varepsilon_t(r)$ is a.s. $\alpha-$hölderian, $\alpha \in (0, 1/2)$ in r for every t. By using Lemmata 4.6 and 4.7 it is straightforward to see that $\xi \in C([0,T], C(\mathbf{R}))$.

- *Identification of the limit*

In order to identify the limit of ξ^ε we formulate the stochastic heat equation (3.1) as a martingale problem which will be shown to be equivalent to the mild formulation (3.3). In the following $f_t(r)$ denotes the canonical coordinate in $C(\mathbf{R}^+, C(\mathbf{R}))$.

Definition 4.10 *The martingale problem.*

Let $\mathcal{Q}$ be a probability measure on $C(\mathbf{R}^+, C(\mathbf{R}))$ such that for all $T > 0$

$$\sup_{t\in[0,T]} \sup_{r\in\mathbf{R}} e^{-a|r|} \int f_t(r)^2 d\mathcal{Q} < \infty \tag{4.87}$$

for some $a > 0$ .The measure $\mathcal{Q}$ solves the martingale problem if $f_0 = \theta_0$ $\mathcal{Q}$–a.s. and for all $\varphi \in \mathcal{D}$

$$\begin{aligned}
M_t(\varphi) &:= (f_t, \varphi) - (f_0, \varphi) - \frac{1}{2}\int_0^t ds\,(f_s, \varphi'') &(4.88)\\
\Lambda_t(\varphi) &:= M_t(\varphi)^2 - \int_0^t ds \int dr\, f_s(r)^2 \varphi(r)^2 &(4.89)
\end{aligned}$$

are $\mathcal{Q}$–local martingales.

The connection between the process θ as defined in (3.3) and the martingale problem in Definition 4.10 is made by the following proposition, which is proven in Section 6.

Proposition 4.11 *For every $\theta_0 \in C(\mathbf{R})$, such that $\sup_r \exp(-a|r|)\theta(r)^2 < \infty$ for some $a > 0$, the martingale problem has a unique solution $\mathcal{Q}$. Moreover $\mathcal{Q}$ coincides with the law of the process of the process θ, defined by (3.3).*

By Lemma 4.1 and Fatou's Lemma, we obtain that any limit point ξ of the family $\{\xi^\varepsilon\}_{\varepsilon>0}$ satisfies

$$\sup_{t\in[0,T]} \sup_{r\in\mathbf{R}} e^{-a|r|}\, \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\xi_t(r)^2\right) < \infty \tag{4.90}$$

so that the condition (4.87) is met.

- *Martingale relations.*

In order to analyze the martingale structure of ξ^ε, we introduce the *density field for the transformed process*. For all $t \in [0,T]$, $\varphi \in \mathcal{D}$ let us define

$$Y^\varepsilon_t(\varphi) := \varepsilon \sum_x \varphi(\varepsilon x)\, \xi_{\varepsilon^{-2}t}(x) \tag{4.91}$$

We also introduce

$$\begin{aligned} M^\varepsilon_t(\varphi) &:= Y^\varepsilon_t(\varphi) - Y^\varepsilon_0(\varphi) - \frac{1}{2} e^{\gamma_\varepsilon} \int_0^t ds\, \varepsilon \sum_x \varepsilon^{-2} \,\triangle\, \xi_{\varepsilon^{-2}s}(x)\, \varphi(\varepsilon x) \\ \Lambda^\varepsilon_t(\varphi) &:= M^\varepsilon_t(\varphi)^2 - \int_0^t ds\, \varepsilon \sum_x \xi_{\varepsilon^{-2}s}(x)^2\, \varphi(\varepsilon x)^2 + R^\varepsilon_1(\varphi) + R^\varepsilon_2(\varphi) \end{aligned} \tag{4.92}$$

where

$$\begin{aligned} R^\varepsilon_1(\varphi) &:= \int_0^t ds\, \varepsilon \sum_x \varphi(\varepsilon x)^2 \\ &\qquad \times \left[\left(\varepsilon - (1-e^{\gamma_\varepsilon})^2\right) \xi_{\varepsilon^{-2}s}(x)^2 - \frac{1}{2}\varepsilon^{-1}(1-e^{\gamma_\varepsilon})^2\, \xi_{\varepsilon^{-2}s}(x)\,\triangle\, \xi_{\varepsilon^{-2}s}(x)\right] \\ R^\varepsilon_2(\varphi) &:= \int_0^t ds\, \varepsilon \sum_x \varphi(\varepsilon x)^2\, \frac{1}{2}\varepsilon^{-1}\left(1+e^{2\gamma_\varepsilon}\right) \nabla^-\, \xi_{\varepsilon^{-2}s}(x)\, \nabla^+\, \xi_{\varepsilon^{-2}s}(x) \end{aligned} \tag{4.93}$$

The semimartingale equation (3.15) and (3.17) directly imply that $M^\varepsilon_t(\varphi)$ e $\Lambda^\varepsilon_t(\varphi)$ are $\mathbf{P}^\varepsilon_{\mu_\varepsilon}$–martingales. We are then left with studying the limit (along subsequences) of (4.93).

By Lemma 4.1 follows immediately that for all $\varphi \in \mathcal{D}$ and all $t \in \mathbf{R}^+$ there is c such that

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(R^\varepsilon_1(\varphi)^2\right) \le c\varepsilon \tag{4.94}$$

and so the L^2–norm of $R^\varepsilon_1(\varphi)$ vanishes as ε goes to zero. To prove the same for $R^\varepsilon_2(\varphi)$ we have to use the key Lemma 4.8. We have

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(R^\varepsilon_2(\varphi)^2\right) \;=\; \left(1+e^{2\gamma_\varepsilon}\right)^2 \int_0^t ds \int_0^s ds'\, \varepsilon^2 \sum_{x,y} \varphi(\varepsilon x)^2\, \varphi(\varepsilon y)^2\, \varepsilon^{-2}$$

$$\times \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\nabla^-\xi_{\varepsilon^{-2}s'}(x)\,\nabla^+\xi_{\varepsilon^{-2}s'}(x)\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\nabla^-\xi_{\varepsilon^{-2}s}(y)\,\nabla^+\xi_{\varepsilon^{-2}s}(y)\Big|\,\mathcal{F}_{\varepsilon^{-2}s'}\right)\right)$$

$$\leq \quad c_2\int_0^t ds\int_0^s ds'\,\varepsilon^2\sum_{x,y}\varphi(\varepsilon x)^2\,\varphi(\varepsilon y)^2$$

$$\times \mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\xi_{\varepsilon^{-2}s'}(x)^2\,\varepsilon^{-1}\left|\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\nabla^-\xi_{\varepsilon^{-2}s}(y)\,\nabla^+\xi_{\varepsilon^{-2}s}(y)\Big|\,\mathcal{F}_{\varepsilon^{-2}s'}\right)\right|\right) \tag{4.95}$$

since

$$\left|\nabla^\pm\xi_t(x)\right| \leq c_1\,\varepsilon^{\frac{1}{2}}\,\xi_t(x) \tag{4.96}$$

By Lemma 4.1 and the above inequality, we may assume $s' \geq \sqrt{\varepsilon}$. By Lemma 4.8 for every $\delta > 0$ there are $c_2, a > 0$ such that for any $x, y \in \mathbf{Z}$, $\kappa \in \mathbf{R}^+$

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left\{\mathbf{1}_{\left\{\xi_{\varepsilon^{-2}s'}(x)\leq\kappa\right\}}\,\xi_{\varepsilon^{-2}s'}(x)^2\,\varepsilon^{-1}\,\left|\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\nabla^-\xi_{\varepsilon^{-2}s}(y)\,\nabla^+\xi_{\varepsilon^{-2}s}(y)\Big|\,\mathcal{F}_{\varepsilon^{-2}s'}\right)\right|\right\}$$
$$\leq c_2\,e^{a\varepsilon(|x|+|y|)}\,\kappa^2\,\varepsilon^{-\delta+1/2} \tag{4.97}$$

Moreover by (4.96), Lemma 4.1 and the Cauchy–Schwartz inequality, there exists a constant c_3 such that for all $x, y \in \mathbf{Z}$, $\kappa \in \mathbf{R}^+$

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left\{\mathbf{1}_{\left\{\xi_{\varepsilon^{-2}s'}(x)>\kappa\right\}}\,\xi_{\varepsilon^{-2}s'}(x)^2\,\varepsilon^{-1}\,\left|\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(\nabla^-\xi_{\varepsilon^{-2}s}(y)\,\nabla^+\xi_{\varepsilon^{-2}s}(y)\Big|\,\mathcal{F}_{\varepsilon^{-2}s'}\right)\right|\right\}$$
$$\leq c_3\,e^{a\varepsilon(|x|+|y|)}\,\mathbf{P}^\varepsilon_{\mu_\varepsilon}\left(\xi_{\varepsilon^{-2}s'}(x) > \kappa\right) \tag{4.98}$$

which vanishes as $\kappa \to \infty$ again by Lemma 4.1.

By using the bounds (4.97) and (4.98) in (4.95), letting $\varepsilon \to 0$ and then $\kappa \to \infty$ we have proven that

$$\lim_{\varepsilon\to 0}\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(R^\varepsilon_2(\varphi)^2\right) = 0$$

Fix $\varphi \in \mathcal{D}$ and rewrite $M^\varepsilon_t(\varphi)$ as

$$M^\varepsilon_t(\varphi) = Y^\varepsilon_t(\varphi) - Y^\varepsilon_0(\varphi) - \frac{1}{2}\int_0^t ds\,Y^\varepsilon_s(\varphi'') + R^\varepsilon_3(\varphi) \tag{4.99}$$

where $R^\varepsilon_3(\varphi)$ contains the errors coming from the approximations $\triangle\varphi(\varepsilon x) \sim \varepsilon^2\varphi''(\varepsilon x)$ and $e^{\gamma_\varepsilon} \sim 1$. They are easily controlled.

For all $N \in \mathbf{Z}^+$ let us define the stopping time τ_N on $D([0,\infty); C(\mathbf{R}))$ by

$$\tau_N := \inf\{t\,:\,|M_t(\varphi)| \wedge |\Lambda_t(\varphi)| > N\}$$

with the standard convention that $\tau_N = +\infty$ if $|M_t(\varphi)| \leq N$ and $|\Lambda_t(\varphi)| \leq N$ for all $t \in \mathbf{R}^+$. We denote by $M^{\tau_N}_\cdot(\varphi)$ ($\Lambda^{\tau_N}_\cdot(\varphi)$) the stopped martingale $M_{\cdot\wedge\tau_N}(\varphi)$ (respectively $\Lambda_{\cdot\wedge\tau_N}(\varphi)$).

Let $\mathcal{Q}^\varepsilon$ the law of the canonical coordinate process over $D([0,T], C(\mathbf{R}))$ which has the same law as ξ^ε. We have already proven the tightness of $\{\mathcal{Q}^\varepsilon\}_{\varepsilon>0}$. For all s, t, $0 \leq s < t \leq T$, and every function $F : D([0,T]; C(\mathbf{R})) \to \mathbf{R}$ which is measurable over

$D([0,s];C(\mathbf{R}))$, continuous and bounded if restricted to $C([0,T];C(\mathbf{R}))$, we have that for all converging subsequences $\mathcal{Q}^{\varepsilon_n} \Longrightarrow \mathcal{Q}$

$$\begin{aligned} 0 &= \lim_n \int d\mathcal{Q}^{\varepsilon_n} \left[M_t^{\tau_N}(\varphi) - M_s^{\tau_N}(\varphi)\right] F = \int d\mathcal{Q} \left[M_t^{\tau_N}(\varphi) - M_s^{\tau_N}(\varphi)\right] F \\ 0 &= \lim_n \int d\mathcal{Q}^{\varepsilon_n} \left[\Lambda_t^{\tau_N}(\varphi) - \Lambda_s^{\tau_N}(\varphi)\right] F = \int d\mathcal{Q} \left[\Lambda_t^{\tau_N}(\varphi) - \Lambda_s^{\tau_N}(\varphi)\right] F \end{aligned}$$

which follows from (4.99), (4.93) and [5, 5.2]. In fact $[M_t^{\tau_N}(\varphi) - M_s^{\tau_N}(\varphi)]F$ is continuous and bounded if restricted to $C([0,T];C(\mathbf{R}))$. By the arbitrariety of F, s, t we conclude that $M^{\tau_N}(\varphi)$ and $\Lambda^{\tau_N}(\varphi)$ are $\mathcal{Q}$–martingales. Moreover

$$\lim_{N\to\infty} \tau_N = +\infty \qquad \mathcal{Q} - \text{a.s.}$$

which follows directly from the fact that any limit point $\mathcal{Q}$ is supported by $C([0,T],C(\mathbf{R}))$ for all $T>0$. This implies that $M(\varphi)$ and $\Lambda(\varphi)$ are $\mathcal{Q}$–local martingales. $\square$

5 Scaling limits for WASEP and SOS

In this Section we prove our main results on the convergence of the fluctuation field for WASEP to the solution of the stochastic Burgers equation and the convergence of the (properly renormalized) interface profile of SOS to the solution of the KPZ equation. We shall make use of the convergence of the transformed process.

Proof of Theorem 2.5 and Theorem 2.7.

By the inverse Gärtner transform (3.14) and the Definition 2.4 of the fluctuation field for WASEP, we obtain

$$\begin{aligned} X_t^\varepsilon(\varphi) &= \sqrt{\varepsilon} \sum_x \varphi(\varepsilon x)\, \sigma_{\varepsilon^{-2}t}(x) = -\frac{\sqrt{\varepsilon}}{\gamma_\varepsilon} \sum_x \varphi(\varepsilon x)\ \bigtriangledown^- \log \xi_{\varepsilon^{-2}t}(x) \\ &= \varepsilon \sum_x \varphi'(\varepsilon x)\ \log \xi_{\varepsilon^{-2}t}(x) + R_1^\varepsilon(\varphi) \end{aligned} \tag{5.1}$$

in which the correction $R_1^\varepsilon(\varphi)$ is due to the approximations $\gamma_\varepsilon \sim \sqrt{\varepsilon}$ and $\bigtriangledown^+\varphi(\varepsilon x) \sim \varepsilon\varphi'(\varepsilon x)$, which are very easily controlled.

By Theorem 3.1, $\theta_t(r)$ is almost surely strictly positive and continuous. Hence for every compact set $K \subset \mathbf{R}$ there is a (random) constant $c_1 > 0$ such that

$$\inf_{t\in[0,T]} \inf_{r\in K} \theta_t(r) \ge c_1 \qquad a.s. \tag{5.2}$$

If $\operatorname{supp}\varphi \subset K$, then

$$\begin{aligned} &\mathbf{P}^\varepsilon_{\mu_\varepsilon}\left(\sup_{|t-s|\le h} \left| \varepsilon \sum_x \varphi'(\varepsilon x)\ (\log \xi_{\varepsilon^{-2}t}(x) - \log \xi_{\varepsilon^{-2}s}(x)) \right| \ge \delta_1 \right) \\ &= \mathbf{P}^\varepsilon_{\mu_\varepsilon}\left(\sup_{|t-s|\le h} \left| \varepsilon \sum_x \varphi'(\varepsilon x)\ (\log \xi_{\varepsilon^{-2}t}(x) - \log \xi_{\varepsilon^{-2}s}(x)) \right| \ge \delta_1\,, \inf_{t,r\in[0,T]\times K} \xi_{\varepsilon^{-2}t}(r) \ge \delta_2 \right) \\ &\quad + R_2^\varepsilon(\varphi) \end{aligned} \tag{5.3}$$

where

$$R_2^\varepsilon(\varphi) \;:=\; \mathbf{P}^\varepsilon_{\mu_\varepsilon}\left(\sup_{|t-s|\le h}\left|\varepsilon\sum_x \varphi'(\varepsilon x)\;(\log\xi_{\varepsilon^{-2}t}(x)-\log\xi_{\varepsilon^{-2}s}(x))\right|\ge\delta_1\,,\right.$$
$$\left.\inf_{t,r\in[0,T]\times K}\;\xi_{\varepsilon^{-2}t}(r)<\delta_2\right)$$

We have

$$\begin{aligned}\lim_{\delta_2\to 0}\,\overline{\lim_{\varepsilon\to 0}}\,R_2^\varepsilon(\varphi) &\le \lim_{\delta_2\to 0}\,\overline{\lim_{\varepsilon\to 0}}\,\mathbf{P}^\varepsilon_{\mu_\varepsilon}\left(\inf_{t,r\in[0,T]\times K}\;\xi_{\varepsilon^{-2}t}(r)<\delta_2\right)\\ &= \lim_{\delta_2\to 0}\mathbf{P}\left(\inf_{t,r\in[0,T]\times K}\;\theta_t(r)<\delta_2\right)=0 \end{aligned}\tag{5.4}$$

by the observation made above. Hence by Lemmata 4.6 and 4.7 we obtain

$$\lim_{h\downarrow 0}\,\overline{\lim_{\varepsilon\to 0}}\,\mathbf{P}^\varepsilon_{\mu_\varepsilon}\left(\sup_{|t-s|\le h}\left|\varepsilon\sum_x\varphi'(\varepsilon x)\;(\log\xi_{\varepsilon^{-2}t}(x)-\log\xi_{\varepsilon^{-2}s}(x))\right|\ge\delta_1\right)=0\tag{5.5}$$

Since $\mathcal{D}'$, being the inductive limit of nuclear spaces, is nuclear, see e.g. [31, pp. 293], we can use the general result in [12]: it is enough to verify the tightness of $Y_t^\varepsilon(\varphi)$, as a function on $D([0,T])$, for each fixed $\varphi\in\mathcal{D}$. We can then use the following classical result [5, 15.5], in which $\{\mathcal{P}^\varepsilon\}$ is a family of measures on $D\left([0,T]\right)$ and g_t is the canonical coordinate in that space.

Proposition 5.1 *Let us assume that for all $\delta>0$ there is $K>0$ such that*

$$\mathcal{P}^\varepsilon\left(|g_0|>K\right)\le\delta\tag{5.6}$$

for all $\varepsilon>0$. Furthermore assume that, for all $\delta_1,\delta_2>0$, there are $h>0,\varepsilon_0>0$ such that

$$\mathcal{P}^\varepsilon\left(\sup_{s,t\in[0,T]\,,\,|t-s|\le h}\;|g_t-g_s|>\delta_1\right)\le\delta_2\tag{5.7}$$

for all $\varepsilon<\varepsilon_0$. Then the family $\{\mathcal{P}^\varepsilon\}$ is tight and every limit point is supported in $C([0,T])$.

The tightness of X^ε in $D([0,T];\mathcal{D}')$ follows then from (5.5) and the assumptions in Definition 2.3. We also get that any limit point is concentrated on $C([0,T];\mathcal{D}')$.

To complete the proof of Theorem 2.5 we have to show any limit point of X^ε has the same distribution of the process u defined in Theorem 2.2. Observe that, by Theorem 3.1, for almost every θ_t, given any sequence of continuous functions $\{\theta^n\}_{n\in\mathbf{Z}^+}$, such that $\lim_{n\to\infty}\theta^n=\theta_t$ uniformly over the compact sets, we have

$$\lim_{\varepsilon\to 0}\varepsilon\sum_x\varphi'(\varepsilon x)\log\theta^{[\varepsilon^{-1}]}(\varepsilon x)=\int\varphi'(r)\log\theta_t(r)dr$$

Hence we can apply Theorem 5.5 of [5] and, by recalling formula (5.1) and Theorem 3.3, we conclude that for all $\varphi \in \mathcal{D}$, $X^\varepsilon_t(\varphi)$ converges to $-\nabla \log \theta_t(\varphi)$, defined in (3.8). Apply Theorem 3.2 to conclude the proof of Theorem 2.5.

By equation (3.13) we have

$$Z^\varepsilon_t(\varphi) = \gamma_\varepsilon \sum_x \varphi(\varepsilon x) \, \log \xi_{\varepsilon^{-2}t}(x) \tag{5.8}$$

The proof of Theorem 2.7 is then the same as the one we have just given. We note that the deterministic drift velocity v_ε in (2.25) is given by $\varepsilon^{-2}\lambda_\varepsilon/\gamma_\varepsilon$. $\square$

6 Properties of the macroscopic equations

In this Section we characterize the solution of the stochastic Burgers and KPZ equations as the weak limit of the corresponding approximating problems. This is carried out by using the Cole-Hopf transformation and the existence and uniqueness theorem for the stochastic heat equation. We next prove the martingale formulation of the latter equation which has been used in the identification of the limit for the transformed process. We finally state and prove some consequences of the scaling limit for WASEP: we show the white noise measure is stationary for the stochastic Burgers equation and the existence of the second moment. We stress the latter property is proved only in finite volume.

Proof of Theorems 2.2, 2.6 and 3.2.

Let $\theta^\kappa_t(r)$ be the (unique) solution of (3.5), a straightforward application of Ito calculus, see [3] for a similar computation, shows that $u^\kappa_t(r) := \nabla \log \theta^\kappa_t(r)$ solves

$$u^\kappa_t = u_0 + \frac{1}{2} \int_0^t ds \, \left\{ \Delta u^\kappa_s + \nabla \left[(u^\kappa_s)^2 - C_\kappa(0) \right] \right\} + \nabla W^\kappa_t \tag{6.1}$$

which implies (3.5). We recall C_κ is defined in (2.5). It is in fact easy to verify that $r \mapsto u^\kappa_t(r)$ is a.s. differentiable when the cutoff κ is finite. To complete the proof of the first statement we need only to show (3.5) as a unique solution in the class of continuous (in space and time) and adapted processes. This is the same as proving uniqueness for the deterministic Burgers equation.

Using the already quoted result in [12], the tightness of $\{u^\kappa\}_{\kappa>0}$ in $C([0,T];\mathcal{D}')$ follows from the tightness of $u^\kappa(\varphi) = -\int dr \log \theta^\kappa_t(r) \, \varphi(r)$ in $C([0,T])$ for every $\varphi \in \mathcal{D}$. This in turn follows from Theorem 3.1. The identification of the limit is as (actually easier since $u^\kappa \in C([0,T];\mathcal{D}')$) in the proof of Theorem 2.5.

This argument proves also Theorems 2.6 and 3.2. $\square$

Proof of Proposition 4.11.

The existence result for the martingale problem may be easily deduced from Theorem 3.1). However we do not need it since it follows from the convergence (along subsequences) of ξ^ε.

To prove uniqueness we will show that we can extend the probability space to accommodate a suitable Wiener process W_t together with the martingale solution f_t: on this extended space f_t and W_t solve (3.3) (this is usually referred to as a *Representation Theorem*). The uniqueness of the martingale problem will then follow from the pathwise uniqueness in Theorem 3.1.

To prove the Representation Theorem we follow the line of [20, Lemma 2.4]. By Definition 4.10, there exists a sequence of stopping times $\{\tau_N\}$ such that $\lim_{N\to\infty}\tau_N = +\infty$ $\mathcal{Q}$–a.s., $M^{\tau_N}(\varphi)$ is a square integrable martingale and $\Lambda^{\tau_N}(\varphi)$ is a martingale. Let us then consider the *martingale measure* (see [30, Ch.2]) $M(dsdr)$ associated to $M^{\tau_N}(\varphi)$ and notice that it is an *orthogonal martingale measure*, its quadratic variation measure $\langle M\rangle(dsdx)$ is $(f_s(r))^2 dsdr$.

Possibly by extending the probability space (and consequently the filtration), we introduce a cylindrical Wiener process $\overline{W}_t$, independent of f_t. Let us denote by $\mathcal{Q}'$ the probability measure on the extended space. Set

$$W_t^N(\varphi) := -\int_0^t \int \frac{1}{f_s(r)} \mathbf{1}_{\{f_s(r)\neq 0\}}\varphi(x) M(dsdr) - \int_0^{t\wedge\tau_N} \left(\mathbf{1}_{\{f_s(\cdot)=0\}}\varphi, d\overline{W}_s\right) \tag{6.2}$$

By direct computation one checks that $W_t^N = W_t^{\tau_N}$, where W_t is a cylindrical Wiener process, and that

$$M_t^{\tau_N}(\varphi) = -\int_0^{t\wedge\tau_N} (f_s\varphi, dW_s) \tag{6.3}$$

so that

$$(f_{t\wedge\tau_N}, \varphi) - (f_0, \phi) = \frac{1}{2}\int_0^{t\wedge\tau_N} (f_s, \Delta\varphi) - \int_0^{t\wedge\tau_N} (f_s\varphi, dW_s) \tag{6.4}$$

$\mathcal{Q}'$–a.s.. By letting N go to infinity in (6.3) and using the hypothesis (4.87) one directly shows that

$$\int d\mathcal{Q}' \left[(f_t, \varphi) - (f_0, G_t * f_0) + \int_0^t (f_s(G_{t-s} * \varphi), dW_s)\right]^2 = 0 \tag{6.5}$$

so that the quantity between square brackets is zero $\mathcal{Q}'$–a.s.. By using a countable family $\{\varphi\}$ which separates points in $C^0(\mathbf{R})$ we obtain that f_t and W_t solve the stochastic heat equation (3.3) and by Theorem 3.1, point (ii), we conclude. $\square$

Let ν the white noise measure on $\mathcal{D}'$, i.e. the Gaussian measure with mean zero and covariance

$$\int d\nu\, X(\varphi_1)\, X(\varphi_2) = (\varphi_1\,,\,\varphi_2) \tag{6.6}$$

with $\varphi_i \in \mathcal{D}$.

Proposition 6.1 *The measure ν is stationary for the process u defined in Theorem 2.2.*

Proof. We consider WASEP with initial distribution $\nu_{1/2}$, the Bernoulli measure on Ω with marginals $\nu_{1/2}\,(\sigma(x) = \pm 1) = 1/2$. Let F be a bounded and continuous function on $\mathcal{D}'$, by [24, VIII 2.1], which characterizes the stationary measures for WASEP, we have

$$\mathbf{E}^{\varepsilon}_{\nu_{1/2}}\left(F\left(X^{\varepsilon}_t\right)\right) = \int d\nu_{1/2}\, F\left(X^{\varepsilon}\right) \tag{6.7}$$

Since

$$\lim_{\varepsilon\to 0} \int d\nu_{1/2}\, e^{i\,X^\varepsilon(\varphi)} = \exp\left\{-\frac{1}{2}\,(\varphi\,,\,\varphi)\right\} \tag{6.8}$$

for all $\varphi \in \mathcal{D}$, the right hand side of (6.7) converges to $\int d\nu\, F$ and thus the stationariety of ν follows from Theorem 2.5. $\square$

We next prove that u_t has the second moment. This fact is not apparent from its construction through the Cole-Hopf transformation. In fact, whereas the process θ_t has all moments, this control is lost by taking the logarithm. We prove this result only when the stochastic Burgers equation is considered on the one dimensional torus $\mathbf{T}$ instead on $\mathbf{R}$. In this case Theorem 2.5 holds when WASEP is considered on the periodic lattice $\mathbf{Z}_\varepsilon := \mathbf{Z} \bmod [\varepsilon^{-1}]$.

Proposition 6.2 *Let the initial condition for the stochastic Burgers equation on* $\mathbf{T}$ *be given by (2.16), then*

$$\mathbf{E}\left(u_t(\varphi)^2\right) < \infty \tag{6.9}$$

for all $t \in \mathbf{R}^+$, $\varphi \in C^\infty(\mathbf{T})$.

Proof. Let consider WASEP with initial distribution μ_e as in (2.15). The relative entropy of μ_ε with respect to the Bernoulli measure $\nu_{1/2}$ is

$$S\left(\mu_\varepsilon \,|\, \nu_{1/2}\right) := -\int d\nu_{1/2}\, \frac{d\mu_\varepsilon}{d\nu_{1/2}}\, \log \frac{d\mu_\varepsilon}{d\nu_{1/2}} = -2^{-[\varepsilon^{-1}]} \sum_{\sigma\in\{-1,1\}^{\mathbf{Z}_\varepsilon}} f_\varepsilon(\eta)\, \log f_\varepsilon(\eta) \tag{6.10}$$

where

$$\begin{aligned} f_\varepsilon(\eta) \;\; := \;\; & \prod_{x\in\mathbf{Z}_\varepsilon} \left(\frac{1}{2} - \varepsilon^{-\frac{1}{2}}\,[m(\varepsilon x) - m(\varepsilon(x-1))]\right)^{\frac{1+\sigma(x)}{2}} \\ & \cdot \left(\frac{1}{2} + \varepsilon^{-\frac{1}{2}}\,[m(\varepsilon x) - m(\varepsilon(x-1))]\right)^{\frac{1-\sigma(x)}{2}} \end{aligned}$$

A straightforward computation shows there exists $c > 0$ such that

$$S\left(\mu_\varepsilon \,|\, \nu_{1/2}\right) \le c \tag{6.11}$$

for all $\varepsilon > 0$.

Let $\mu_\varepsilon(t)$ be the distribution of $\sigma_{\varepsilon^{-2}t}$. Since $\nu_{1/2}$ is a stationary measure and the relative entropy is monotone in time, see [24, II 4.2], we have

$$S\left(\mu_\varepsilon(t) \,|\, \nu_{1/2}\right) \le c \tag{6.12}$$

From the basic entropy inequality

$$\mathbf{E}_{\mu_\varepsilon(t)}\,(F) \le \frac{1}{\gamma}\, \log \mathbf{E}_{\nu_{1/2}}\left(e^{\gamma\,F}\right) + \frac{1}{\gamma}\, S\left(\mu_\varepsilon(t) \,|\, \nu_{1/2}\right) \tag{6.13}$$

with $F = X^\varepsilon(\varphi)^2$, we get by using (6.12)

$$\mathbf{E}^\varepsilon_{\mu_\varepsilon}\left(X^\varepsilon_t(\varphi)^2\right) \le \frac{c}{\gamma} + \frac{1}{\gamma}\log \mathbf{E}_{\nu_{1/2}}\left(e^{\gamma X^\varepsilon(\varphi)^2}\right) \tag{6.14}$$

The existence of the second moment thus follows, by choosing γ small enough, from the convergence of the fluctuation field X^ε and the convergence of $\nu_{1/2}$ to the white noise measure ν, see (6.8). $\square$

A Some properties of the transition probability

We here state and prove some elementary properties of $p^\varepsilon_t(x)$, which is the probability kernel for a symmetric random walk on $\mathbf{Z}$ defined in (4.1).

Let $q_t(x)$ be the transition probability kernel defined as

$$\partial_t q_t(x) = \frac{1}{2} \triangle q_t(x) \quad , \quad q_0(x) = \delta(x) \tag{A.1}$$

and let us express it in Fourier series

$$q_t(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, e^{ikx}\, e^{-t(1-\cos k)} \tag{A.2}$$

Hence $p^\varepsilon_t(x) = q_{e^{\gamma_\varepsilon} t}(x)$. By using (A.2) it is straightforward to verify the inequalities (4.22), (4.36) and (4.42). In the same way one easily shows that there is $c > 0$ such that for all $t > 0$

$$\sup_{x\in\mathbf{Z}}\left|\nabla^\pm p^\varepsilon_t(x)\right| \le c\, t^{-\frac{3}{2}} \tag{A.3}$$

and this implies (4.71).

Lemma A.1 *Let $K_t(x)$ be defined as in (4.65), then*

$$\sum_x \int_0^\infty dt\, K_t(x) = 0 \tag{A.4}$$

Proof. It follows from a straightforward computation by using the representation of $p^\varepsilon_t(x)$ in Fourier series (A.2). $\square$

Lemma A.2 *Let $q_t(x)$ be defined as in (A.1), then*

$$\sum_x \int_0^\infty dt\, \left|\nabla^+ q_t(x)\, \nabla^- q_t(x)\right| < 1 \tag{A.5}$$

Proof. By the Cauchy-Schwartz inequality

$$\sum_x \left|\bigtriangledown^- q_t(x)\, \bigtriangledown^+ q_t(x)\right| < \left\{\sum_x \left[\bigtriangledown^- q_t(x)\right]^2\right\}^{\frac{1}{2}} \left\{\sum_x \left[\bigtriangledown^+ q_t(x)\right]^2\right\}^{\frac{1}{2}}$$

and we stress that the inequality is strict because $\bigtriangledown^+ q_t(x) \neq \bigtriangledown^- q_t(x)$.

By summing by parts and using (A.1), we obtain

$$\sum_x \int_0^\infty dt\, \left[\bigtriangledown^\pm q_t(x)\right]^2 = -\int_0^\infty dt\, \sum_x \partial_t \left(q_t(x)^2\right) = 1$$

and (A.5) is proven. □

Lemma A.3 *For every $T > 0$, $a \geq 0$, there exist $\varepsilon_0 > 0$ and $\beta < 1$ such that*

$$\sum_x \int_0^{\varepsilon^{-2}T} dt\, |K_t(x)|\, e^{a\varepsilon|x|} \leq \beta \tag{A.6}$$

for every $\varepsilon < \varepsilon_0$.

Proof. Let $\kappa > 0$ and start by considering

$$\int_\kappa^{\varepsilon^{-2}T} dt\, \sum_x |K_t(x)|\, e^{a\varepsilon|x|} \tag{A.7}$$

By using (A.3) it can be bounded by

$$c_1 \int_\kappa^{\varepsilon^{-2}T} dt\, t^{-\frac{3}{2}} \sum_x p_t^\varepsilon(x)\, e^{a\varepsilon|x|} \leq 2\, c_1 \int_\kappa^{\varepsilon^{-2}T} dt\, t^{-\frac{3}{2}}\, e^{a^2\varepsilon^2 t} \leq 4\, c_1\, e^{a^2 T}\, \kappa^{-\frac{1}{2}} = c_2\, \kappa^{-\frac{1}{2}} \tag{A.8}$$

since

$$\sum_x p_t^\varepsilon(x) e^{a\varepsilon x} \leq c_3\, \exp\left\{c_4\, a^2\, \varepsilon^2\, t\right\}$$

On the other side

$$\lim_{\varepsilon\to 0} \int_0^\kappa dt\, \sum_x |K_t(x)|\, e^{a\varepsilon|x|} = \int_0^\kappa dt\, \sum_x \left|\bigtriangledown^+ q_t(x)\, \bigtriangledown^- q_t(x)\right| \tag{A.9}$$

by the dominated convergence Theorem.

Choose κ sufficiently large and then ε_0 small. Inequality (A.6) is then a consequence of (A.8) and Lemma A.2. □.

Lemma A.4 *For every $T > 0$, $a \geq 0$, there is $c > 0$ such that*

$$\sum_x \int_0^{\varepsilon^{-2}T} dt\, |K_t(x)|\, e^{a\varepsilon|x|}\, \left(T - \varepsilon^2 t\right)^{-\frac{1}{2}} \leq c \tag{A.10}$$

for every $\varepsilon > 0$.

Proof. Let us start by considering

$$\sum_x \int_{\varepsilon^{-2}T/2}^{\varepsilon^{-2}T} dt\ |K_t(x)|\ e^{a\varepsilon|x|}\ \Big(T-\varepsilon^2 t\Big)^{-\frac{1}{2}}$$

which can be bounded by proceeding as in (A.8). We obtain

$$\int_{\varepsilon^{-2}T/2}^{\varepsilon^{-2}T} dt\, t^{-\frac{3}{2}}\, e^{a^2\,T}\ \Big(T-\varepsilon^2 t\Big)^{-\frac{1}{2}} \le c_1\,\varepsilon$$

On the other side

$$\sum_x \int_0^{\varepsilon^{-2}T/2} dt\ |K_t(x)|\ e^{a\varepsilon|x|}\ \Big(T-\varepsilon^2 t\Big)^{-\frac{1}{2}} \le (T/2)^{-\frac{1}{2}} \sum_x \int_0^{\varepsilon^{-2}T/2} dt\ |K_t(x)|\ e^{a\varepsilon|x|} \le c_2 \tag{A.11}$$

By Lemma A.3. □

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