SECOND $N = 1$ SUPERANALOG OF COMPLEX STRUCTURE

Steven Duplij[*†‡]
Department of Physics, University of Kaiserslautern,
Postfach 3049, D-67653 Kaiserslautern, Germany

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Abstract

We found another $N = 1$ odd superanalog of complex structure (the even one is widely used in the theory of super Riemann surfaces). New $N = 1$ superconformal-like transformations are similar to anti-holomorphic ones of nonsupersymmetric complex function theory. They are dual to the ordinary superconformal transformations subject to the Berezinian addition formula presented, noninvertible, highly degenerated and twist parity of the tangent space in the standard basis. They also lead to the "mixed cocycle condition" which can be used in building noninvertible objects analogous to super Riemann surfaces. A new parametrization for the superconformal group is presented which allows us to extend it to a semigroup and to unify the description of old and new transformations.

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[*] Alexander von Humboldt Fellow

[†] On leave of absence from: Theory Division, Nuclear Physics Laboratory, Kharkov State University, KHARKOV 310077, Ukraine

[‡] E-mail: duplij@physik.uni-kl.de

The idea of superconformal symmetry is exceptionally important in the theory of super Riemann surfaces [1] and in two-dimensional superconformal field theories [2]. The main and fundamental ingredient of the idea is a special class of reduced mappings of two-dimensional $(1|1)$ complex superspace, namely superconformal transformations [3]. In the local approach to super Riemann surfaces represented as collections of open superdomains the superconformal transformations are used as gluing transition functions [3, 4]. ¿From another side they appear as a result of the special reduction of the structure supergroup [5]. Here we consider an alternative tangent space reduction, which leads to new transformations (see also [6] and [7]).

We use the functional approach to superspace [8] which admits existence of nontrivial topology in odd directions [9] and can be suitable for physical applications [10]. Also we exploit the coordinate language which is more physically transparent and adequate in constructing objects having new features.

Locally $(1|1)$-dimensional superspace $\mathrm{C}^{1|1}$ is described by $Z=(z,\theta)$, where z is an even coordinate and θ is an odd one. The most intriguing peculiarity of the functional definition of superspace [8] is the existence of soul parts in the even coordinate $z=z_{body}+z_{soul},\ z_{body}=\epsilon\left(z\right),\ z_{soul}\overset{def}{=}z-z_{body}$, where ϵ is a body map [8] vanishing all nilpotent generators. The body map acts on the coordinates as follows $\epsilon\left(z\right)=z_{body},\ \epsilon\left(\theta\right)=0$. This allows one to consider non-trivial soul topology in even directions on a par with odd ones [9].

A superanalytic (SA) transformation $\mathrm{T}_{SA}:\mathrm{C}^{1|1}\rightarrow\mathrm{C}^{1|1}$ is

$$\begin{aligned} \tilde{z} &= f\left(z\right)+\theta\cdot\chi\left(z\right), \\ \tilde{\theta} &= \psi\left(z\right)+\theta\cdot g\left(z\right), \end{aligned} \tag{1}$$

where four component functions $f\left(z\right),g\left(z\right):\mathrm{C}^{1|0}\rightarrow\mathrm{C}^{1|0}$ and $\psi\left(z\right),\chi\left(z\right):\mathrm{C}^{1|0}\rightarrow\mathrm{C}^{0|1}$ satisfy some supersmooth conditions generalizing C^{∞} [8], and simultaneously they can be noninvertible [6] (here and in the following we denote even functions and variables by Latin letters and odd ones by Greek letters, point is a product in Grassmann algebra). The set of invertible and noninvertible SA transformations (1) form a semigroup of superanalytic transformations $\mathcal{T}_{SA}$ [6]. The invertible transformations are in its subgroup, while the noninvertible ones are in an ideal (see [6] and [11] for details).

The invertibility of the superanalytic transformation (1) is determined first of all by invertibility of the even functions $f(z)$ and $g(z)$, because odd functions are noninvertible by definition. In case $\epsilon(g(z)) \neq 0$ for SA transformations (1) the superanalog of a Jacobian, the Berezinian [12], can be determined $\mathrm{Ber}\left(\tilde{Z}/Z\right) = \frac{f'(z)}{g(z)} + \frac{\chi(z)\psi'(z)}{g^2(z)} + \theta\left(\frac{\chi(z)}{g(z)}\right)'$, where prime is a differentiation by argument (or by z). Therefore, we can classify the transformations (1) in the following way: 1) the Berezinian exists and invertible ($\epsilon(g(z)) \neq 0,\ \epsilon(f(z)) \neq 0$); 2) the Berezinian exists and noninvertible ($\epsilon(g(z)) \neq 0,\ \epsilon(f(z)) = 0$); 3) the Berezinian does not exist ($\epsilon(g(z)) = 0,\ \epsilon(f(z)) = 0$). First type of SA transformations form a subgroup of the superanalytic semigroup, while second two types are in an ideal of the semigroup [6].

The tangent superspace in $\mathrm{C}^{1|1}$ is defined by the standard basis $\{\partial,\ D\}$, where $D = \partial_\theta + \theta\partial$, $\partial_\theta = \partial/\partial\theta$, $\partial = \partial/\partial z$. The dual cotangent space is spanned by 1-forms $\{dZ,\ d\theta\}$, where $dZ = dz + \theta d\theta$ (the signs as in [3]). In these notations the supersymmetry relations are $D^2 = \partial$, $dZ^2 = dz$. The semigroup of SA transformations acts in the tangent and cotangent superspaces by means of the tangent space matrix P_A as $\begin{pmatrix} \partial \\ D \end{pmatrix} = P_A \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix}$ and $\begin{pmatrix} d\tilde{Z}, & d\tilde{\theta} \end{pmatrix} = \begin{pmatrix} dZ, & d\theta \end{pmatrix} P_A$, where

$$P_A = \begin{pmatrix} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta} \\ D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix}. \tag{2}$$

In case of invertible SA transformations the matrix P_A defines structure of a supermanifold for which these transformations play the part of transition functions, and $\mathrm{Ber}\left(\tilde{Z}/Z\right) = \mathrm{Ber}\, P_A$. Therefore different reductions of the matrix P_A give us various additional supermanifold structures [5]. It was shown in [7] that there exist two nontrivial reductions of any supermatrix P_A. Indeed, if $\epsilon\left(D\tilde{\theta}\right) \neq 0$ we observe that $\mathrm{Ber}\, P_A = \frac{\partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta}}{D\tilde{\theta}} + \frac{\left(D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta}\right)\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}$. Then using the Berezinian addition theorem [7] we obtain the formula

$$\mathrm{Ber}\, P_A = \mathrm{Ber}\, P_S + \mathrm{Ber}\, P_T, \tag{3}$$

where

$$P_S \stackrel{def}{=} \begin{pmatrix} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}, \tag{4}$$

$$P_T \stackrel{def}{=} \begin{pmatrix} 0 & \partial\tilde{\theta} \\ D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix}. \tag{5}$$

Denote sets of the matrices (4) and (5) by $\mathbf{P}_S$ and $\mathbf{P}_T$ respectively. Then their intersection $\mathbf{P}_D = \mathbf{P}_S \cap \mathbf{P}_T$ is a set of the degenerated matrices P_D of the form

$$P_D \stackrel{def}{=} \begin{pmatrix} 0 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}, \tag{6}$$

which depend on the odd coordinate θ transformation only. The degenerated matrix of the shape (6) can be obtained by projection from P_S and P_T matrices using the following equations

$$Q \stackrel{def}{=} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} = 0, \tag{7}$$

$$\Delta \stackrel{def}{=} D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} = 0 \tag{8}$$

correspondingly. It means that, if the transformation of the odd sector (second line in (1)) is given, i.e. the functions $\psi(z)$ and $g(z)$ are fixed, the conditions (7) and (8) determine behavior of the even sector (functions $f(z)$ and $\chi(z)$). In this case, since the degenerated matrix P_D depends on the odd sector transformation only, we obtain

$$P_D = P_S|_{Q=0} = P_T|_{\Delta=0}. \tag{9}$$

An opposite situation occurs if we apply the conditions (7) and (8) to the matrices P_S and P_T in a reverse order. Then we derive

$$P_{SCf} \stackrel{def}{=} P_S|_{\Delta=0} \tag{10}$$

$$P_{TPt} \stackrel{def}{=} P_T|_{Q=0}. \tag{11}$$

The condition $\Delta = 0$ (8) gives us superconformal (SCf) transformations T_{SCf} [3] and the reduced matrix P_{SCf} (10) is a result of the standard reduction of structure supergroup (in the invertible case [5]). Another condition $\Delta = 0$ (7) leads to the degenerated transformations T_{TPt} twisting parity of the standard tangent space (TPt) [6]. The alternative reduction [7] of the tangent space supermatrix P_A gives us the supermatrix P_{TPt} (11). The dual role of SCf and TPt transformations is clearly seen from the Berezinian addition theorem (3) (see [7]) and the projections (10) and (11). Since SCf transformations give us a superanalog of complex structure [13, 14], we can treat TPt transformations as another odd $N = 1$ superanalog of complex structure in a certain extent[1].

Using (10) and (11) with the obvious relation $\operatorname{Ber} P_D = 0$ we can project the Berezinian addition equality (3) to T_{SCf} and T_{TPt} as follows

$$\operatorname{Ber} P_A = \begin{cases} \operatorname{Ber} P_{SCf}, & \Delta = 0, \\ \operatorname{Ber} P_{TPt}, & Q = 0. \end{cases} \tag{12}$$

A general relation between Q and Δ is $Q - D\Delta = \left(D\tilde{\theta}\right)^2$. After corresponding projections we have

$$Q|_{\Delta=0} = \left(D\tilde{\theta}\right)^2, (SCf), \tag{13}$$

$$\Delta|_{Q=0} \equiv \Delta_0 = \partial_\theta \tilde{z} - \partial_\theta \tilde{\theta} \cdot \tilde{\theta}, (TPt). \tag{14}$$

It is remarkable to notice the similarity of (7) and (14). Using (13) one obtains [5]

$$P_{SCf} = \begin{pmatrix} \left(D\tilde{\theta}\right)^2 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}. \tag{15}$$

If $\varepsilon\left(D\tilde{\theta}\right) \neq 0$ then $\operatorname{Ber} P_{SCf}$ can be determined and it is

$$\operatorname{Ber} P_{SCf} = D\tilde{\theta}. \tag{16}$$

[1]It is more natural to call TPt transformations anti-SCf transformations due to the following analogy with the nonsupersymmetric case. For an ordinary 2×2 matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we obviously have the following identity $\det P = \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \det P_{Diag} + \det P_{Antidiag}$, which can be called a "determinant addition formula". In the complex function theory the first matrix describes the tangent space matrix of holomorphic mappings and the second one—of antiholomorphic mappings. In supersymmetric case the supermatrices P_S and P_T play the role similar to one of the nonsupersymmetric diagonal and antidiagonal matrices in ordinary theory as it is seen from (3). Therefore, if P_{SCf} generalizes the tangent space matrix of holomorphic mappings, supermatrices P_{TPt} could be considered as respective generalization for antiholomorphic mappings.

In case $\varepsilon\left(D\tilde{\theta}\right)=0$ the Berezinian cannot be defined, but we can accept (16) as a definition of the Jacobian of noninvertible SCf transformations (see [6] and [15]).

¿From (14) we derive

$$P_{TPt}=\begin{pmatrix} 0 & \partial\tilde{\theta} \\ \partial_\theta\tilde{z}-\partial_\theta\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix} \tag{17}$$

(cf. (4)). If $\varepsilon\left(D\tilde{\theta}\right)\neq 0$ the Berezinian of P_{TPt} can be determined as

$$\operatorname{Ber} P_{TPt}=\frac{\Delta_0\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}. \tag{18}$$

¿From (14) it follows that $D\Delta_0=-\left(D\tilde{\theta}\right)^2$ and, therefore, $\partial\Delta_0=-2\cdot D\tilde{\theta}\cdot\partial\tilde{\theta}$, which gives $\operatorname{Ber} P_{TPt}=\frac{\partial\Delta_0\cdot\Delta_0}{2\left(D\tilde{\theta}\right)^3}$. Since Δ_0 is odd and so nilpotent, $\operatorname{Ber} P_{TPt}$ is also nilpotent and pure soul. The Berezinian can be also presented as

$$\operatorname{Ber} P_{TPt}=D\left(\frac{D\tilde{z}}{D\tilde{\theta}}\right) \tag{19}$$

which should be remarkably compared with (16).

The most intriguing peculiarity of TPt transformations is twisting the parity of tangent and cotangent spaces in the standard basis, viz.

$$SCf:\begin{cases} D &= \left(D\tilde{\theta}\right)\cdot\tilde{D}, \\ d\tilde{Z} &= \left(D\tilde{\theta}\right)^2\cdot dZ, \end{cases} \qquad TPt:\begin{cases} \partial &= \partial\tilde{\theta}\cdot\tilde{D}, \\ d\tilde{Z} &= \Delta_0\cdot d\theta. \end{cases} \tag{20}$$

The reduction conditions (7) and (8) fix 2 of 4 component functions form (1) in each case. Usually [3] SCf transformations T_{SCf} are parametrized by $\begin{pmatrix} f \\ \psi \end{pmatrix}$, while other functions are found from (7) and (8). However, the latter can be done for invertible transformations only. To avoid this difficulty we introduce an alternative parametrization by the pair $\begin{pmatrix} g \\ \psi \end{pmatrix}$, which allows us to consider SCf and TPt transformations in a unified way and include noninvertibility. Indeed, fixing $g\left(z\right)$ and $\psi\left(z\right)$ we find for other component functions of (1) the equations

$$\begin{cases} f_n'(z) & = & \psi'(z)\,\psi(z) + \frac{1+n}{2} g^2(z)\,, \\ \chi_n'(z) & = & g'(z)\,\psi(z) + n g(z)\,\psi'(z)\,, \end{cases} \tag{21}$$

where $n = \begin{cases} +1, & SCf, \\ -1, & TPt, \end{cases}$ can be treated as a projection of some "reduction spin" switching the type of transformation. So the reduced transformation of the even coordinate (see (1)) should contain this additional index, i.e. $z \to \tilde{z}_n$ (at this point the analogy with complex structure is most transparent). Since $f_{-1}'(z) = \psi'(z)\,\psi(z)$ is nilpotent, TPt transformations are always noninvertible and high degenerated after the body mapping. The unified multiplication law is

$$\begin{pmatrix} h \\ \varphi \end{pmatrix}_n * \begin{pmatrix} g \\ \psi \end{pmatrix}_m = \begin{pmatrix} g \cdot h \circ f_m + \chi_m \cdot \psi \cdot h' \circ f_m + \chi_m \cdot \varphi' \circ f_m \\ \varphi \circ f_m + \psi \cdot h \circ f_m \end{pmatrix}, \tag{22}$$

where $(*)$ is transformation composition and $(\circ)$ is function composition. For "reduction spin" projections we have only two definite products $(+1) * (+1) = (+1)$ and $(+1) * (-1) = (-1)$. The first formula is a consequence of $\mathbf{P}_S \cdot \mathbf{P}_S \subseteq \mathbf{P}_S$ (see (4)), which is simple manifestation of the fact that SCf transformations T_{SCf} form a substructure [5], i.e. a subsemigroup $\mathcal{T}_{SCf}$ of SA semigroup $\mathcal{T}_{SA}$ (in the invertible case—a subgroup [3]). From $\mathbf{P}_S \cdot \mathbf{P}_S \subseteq \mathbf{P}_S$ it also follows the standard (for component functions too) cocycle condition [3] $\tilde{\mathrm{T}}_{SCf} * \mathrm{T}_{SCf} = \tilde{\tilde{\mathrm{T}}}_{SCf}$ (having identical arrows, i.e. (SCf) actions) on triple overlaps $U \cap \tilde{U} \cap \tilde{\tilde{U}}$, where U, $\tilde{U}$, $\tilde{\tilde{U}}$ are open superdomains and $\mathrm{T} : U \to \tilde{U}$, $\tilde{\mathrm{T}} : \tilde{U} \to \tilde{\tilde{U}}$, $\tilde{\tilde{\mathrm{T}}} : U \to \tilde{\tilde{U}}$. In the invertible SCf case the cocycle condition leads to the definition of a super Riemann surface as a holomorphic $(1|1)$-dimensional supermanifold equipped with an additional one-dimensional subbundle [3, 5, 13], which grounds on the cocycle relation $D\tilde{\tilde{\theta}} = D\tilde{\theta} \cdot \tilde{D}\tilde{\tilde{\theta}}$ and the formula (16). Unfortunately, TPt transformations T_{TPt} form a subsemigroup only providing additional conditions on component functions [6]. However, they have also another important abstract meaning: using the unrestricted relation $\mathbf{P}_T \cdot \mathbf{P}_S \subseteq \mathbf{P}_T$ we obtain a "mixed cocycle condition" $\tilde{\mathrm{T}}_{SCf} * \mathrm{T}_{TPt} = \tilde{\tilde{\mathrm{T}}}_{TPt}$ (having different arrows). Then we derive the "mixed cocycle relation"

$$\partial\tilde{\tilde{\theta}} = \partial\tilde{\theta} \cdot \tilde{D}\tilde{\tilde{\theta}} \tag{23}$$

which can be exploited in constructing new objects analogous to super Riemann surfaces. It is remarkable that under the degenerated (Deg) transformations defined by (9) the both cocycle relations hold valid simultaneously. Also, Deg transformations form a subsemigroup $\mathcal{T}_{Deg}$ in $\mathcal{T}_{SA}$, because of $\mathbf{P}_D \cdot \mathbf{P}_D \subseteq \mathbf{P}_D$. Moreover, $\mathcal{T}_{Deg}$ is an ideal in $\mathcal{T}_{SA}$, $\mathcal{T}_{SCf}$ and $\mathcal{T}_{TPt}$ since $\mathbf{P}_D \cdot \mathbf{P}_A \subseteq \mathbf{P}_D$, $\mathbf{P}_D \cdot \mathbf{P}_S \subseteq \mathbf{P}_D$ and $\mathbf{P}_D \cdot \mathbf{P}_T \subseteq \mathbf{P}_D$. The degenerated transformations are characterized by one odd function $\psi(z)$ only and by the absence of the θ-dependence of the transformation $Z \to \tilde{Z}$ (see (14)), so that $\tilde{z}_{Deg} = f(z)$, $\tilde{\theta}_{Deg} = \psi(z)$ and $f'(z) = \psi'(z)\,\psi(z)$. The multiplication in $\mathcal{T}_{Deg}$ coincides with the second row of (22).

We conclude that thorough consideration of invertibility, while supergeneralizing standard constructions of string theory, leads to nontrivial consequences and possibility of building new objects analogous, for instance, to super Riemann surfaces, which could give additional contributions to fermionic string amplitude. It would be also interesting to work out sequences of noninvertible functions and corresponding bundles or their generalizations.

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