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How Hot Is the de Sitter Space?

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Abstract

We show that the unique invariant locally Minkowskian state of quantum fields in de Sitter space M has for an observer moving along with a Killing vector field a temperature $\frac{1}{2\pi}\sqrt{\frac{1}{R^2}+|a|^2}$ where R is the radius of M and a his acceleration. States with another temperature cannot be locally Minkowskian all over M.

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1 Introduction

The Unruh effect [1], namely that the vacuum looks to an accelerated observer like a thermal state is astonishing since it combines the completely different area of geometry with that of statistical mechanics where the thermal states are consequences of molecular chaos. It was pointed out by Bekenstein [2], Hawking and Gibbons [3] that in general relativity some geometrical configurations produce states of a certain temperature. Sewell [4] observed the strong relation of the Hawking radiation to the Unruh effect and its mathematically precise formulation as a Bisognano–Wichmann theorem [5,6].

It has been emphasized by Figari, Höegh-Krohn and Nappi [17] and recently by Bros and Moschella [7] that also in de Sitter space with radius R the most natural state — the one which is invariant under the full de Sitter group and that has the best analyticity properties and thus corresponds to the vacuum state over Minkowski space — locally satisfies a KMS condition with the natural temperature $1/2\pi R$ (if $\hbar = c = k = 1$). In this note we want to refine this result by discussing what an accelerated and a geodesic observer experiences in the course of time in a de Sitter space and compare it with a accelerated and non–accelerated observer in Minkowski space.

There is no doubt that in non–relativistic quantum mechanics the precise definition of the temperature $T = \beta^{-1}$ is given by the KMS condition. There it is linked to the time evolution τ_t of the algebra $\mathcal{A}$ of observables by requiring for the temperature state ω

$$\omega(a\tau_t b) = \omega(b\tau_{-t+i\beta}a) \qquad \forall\, a, b \in \mathcal{A}. \tag{1}$$

In relativity there is no preferred time and in a curved space M time may not even be globally definable. In a pseudo–Riemannian space any timelike vector field X can be used to define a time locally and if we normalize it to $\langle X|X\rangle = 1$ the parameter t of its flow measures the physical proper time of the lines of flow $\tau_t^* : M \ni z \to z(t)$. However the corresponding transformation of a quantum field Φ,

$$\tau_t \Phi(z) = \Phi(\tau^*_{-t}(z)) \tag{2}$$

will in general not be an automorphism of $\mathcal{A}$ even when restricted to a neighbourhood Λ of z. The best one can hope for in the general case is that there exists a shrinking sequence $\Lambda^{(n)}$, $\bigcap_n \Lambda^{(n)} = z$ and an automorphism τ_t of $\mathcal{A}_{\Lambda^{(1)}}$ such that $\tau_t \mathcal{A}_{\Lambda^{(n)}} \subset \mathcal{A}_{\bar{\Lambda}_t^{(n)}}$ for some $\bar{\Lambda}_t^{(n)}$ such that $\bigcap_n \bar{\Lambda}_t^{(n)} = z(t)$. This minimal requirement can be met in relativistic quantum field theory by a geometrical transformation (2). However for those theories the algebraic structure of $\mathcal{A}$ and the metric structure of M are so tightly bound together that (2) will give an automorphism of $\mathcal{A}$ only if τ^* is an isometry and thus if X is a Killing vector field. Fortunately in the case we are interested in, namely maximally symmetric spaces, there are plenty of Killing vector fields and even for each geodesic $z(t)$ there is a Killing vector field X which has $z(t)$ as a flow line. Thus the time defined by X includes the proper time experienced by one freely falling observer. However our Killing vector fields will not be geodesic so that its flow lines except this special one describe accelerated motion. Futhermore $\langle X|X\rangle \neq$ constant so that the flow parameter t does not describe the proper time of the accelerated observer and has to be renormalized because the physical temperature is related to the physical time. With these precautions we find the following.

For each geodesic trajectory in de Sitter space there is a unique Killing vector field X such that this trajectory is a flow line of X. In Minkowski space this Killing vector field is a translation and is geodesic everywhere. In de Sitter space it is not, in fact it is timelike only in a certain region (a "wedge"). Thus the other flow lines have a certain acceleration a and an observer on them experiences a temperature $\frac{1}{2\pi}\sqrt{\frac{1}{R^2}+|a^2|}$. Only the geodesic observer has $a=0$ and feels the Figari, Höegh-Krohn, Nappi, Gibbons–Hawking–Bros–Moschella $1/2\pi R$. The result of Minkowski space where a geodesic observer feels temperature 0 is reached in the limit $R\to\infty$. Our result can be interpreted as an Unruh effect in the ambient 5–dimensional Minkowski space. There even the geodesics of de Sitter space have an acceleration a_5 and generally $|a_5^2| = \sqrt{\frac{1}{R^2}+|a|^2}$. Thus an observer feels like moving in a 5–dimensional space though the fifth dimension has otherwise no physical reality. Since the de Sitter universe is homogeneous and isotropic the temperature of this background depends only on the acceleration of the observer and not on his (or her) position or velocity. In particular there is neither red shift nor Doppler shift in contradistinction to our background radiation.

Another important difference to Minkowski space is that whereas the latter supports KMS states which are globally regular for any temperature in de Sitter space states with another temperature than the natural one become irregular on the horizon. Since the horizon is observer dependent we have the remarkable situation that if the globally regular structure of the geometry is to be respected by the state it dictates what the temperature has to be.

Since the verification of these claims draws on results from various branches of mathematics at the risk of boring some experts we first collect these facts for the convenience of the reader. Our results are more or less direct consequences of known facts, see f.i. [6], [19], [20].

2 Vector fields on a (pseudo–) Riemannian manifold M

We shall denote vector fields by G, K, X, Y, Z and their scalar products by $\langle X|Y\rangle$, etc. The Lie (resp. the covariant) derivative in direction of X is denoted by L_X (resp. D_X) and they are related by $D_XY = D_YX + L_XY$. A Killing vector field K is characterized by $L_K\langle Y|Z\rangle = \langle L_KY|Z\rangle + \langle Y|L_KZ\rangle \;\forall\; Y, Z$ and a geodesic vector field G by $D_GG = 0$. On the contrary, $D_X\langle Y|Z\rangle = \langle D_XY|Z\rangle + \langle Y|D_XZ\rangle$, $L_XY = -L_YX$ and therefore $L_XX = 0$ holds $\forall\; X, Y, Z$. For $K =$ Killing denote by H_c, $c \in \mathbf{R}$ the submanifold $\{x \in M : \langle K|K\rangle(x) = c\}$ so that its tangent space $T(H_c)$ is spanned by the $\{X : L_X\langle K|K\rangle = 0\}$.

Proposition I

(i) $X \in T(H_c) \Longleftrightarrow \langle X|D_KK\rangle = 0$

(ii) $K_1, K_2 = \text{Killing} \Rightarrow \langle K_1|D_{K_1}K_2\rangle = 0$ (in particular $K \in T(H_c)$)

(iii) If $K_1, K_2 = \text{Killing}$ and $L_{K_1}K_2 = \lambda K_2$, $\lambda \in \mathbf{R}$ then $\langle D_{K_1}K_1|K_2\rangle = \lambda\langle K_1|K_2\rangle$

(iv) $G = \text{geodesic}$, $\langle G|G\rangle \neq 0 \Longleftrightarrow \dfrac{G}{|\langle G|G\rangle|^{1/2}} = \text{geodesic}$

(v) If $K = \text{Killing}$ then $K = \text{geodesic} \Longleftrightarrow \langle K|K\rangle = \text{const.}$

(vi) If $K = fG$ on a K–invariant submanifold h for some function f then $D_K K|_h = \kappa K|_h$, $\kappa = \text{const.}$ on flowlines of K.

Remarks

1. $K \in T(H_c)$ means that $\langle K|K\rangle$ remains constant along the flow lines of K. H_c may have the dimension of M, then K is geodesic on H_c or its dimension may be less by 1 and K is not geodesic.

2. The situation (vi) arises in a space with Minkowski signature $(1, -1, \ldots, -1)$ of the metric if $N = \{x : K(x) = 0\} \subset H_0$ is spacelike and $\dim M = \dim H_0 + 1 = \dim N + 2$. Then $\forall\ x \in N\ \exists$ exactly 2 vectors $T_x(H_0) \ni G_{1,2}(x) \in T_x(N)^\perp$ which are lightlike and future–directed. Denote by $G_{1,2}$ the geodesic vector fields whose flowlines through x go in direction $G_{1,2}(x)$. Since the Killing vector field K preserves the properties which characterize these geodesics it shifts them into themselves. This means $K = f_1G_1$ or f_2G_2 on H_0.

3. (iii) means that $K_{1,2}$ give a realization of the Anosov group [8] and λ is the Lyapunov exponent. If $D_{K_1}K_1 = \kappa K_1$ (which according to (vi) happens only on H_0) and $\langle K_1|K_2\rangle \neq 0$ then λ equals the "surface gravity κ of the Killing horizon H_0".

Proof

(i)

$$\begin{aligned} L_X\langle K|K\rangle &= 0 = D_X\langle K|K\rangle = 2\langle D_XK|K\rangle = 2\langle D_KX|K\rangle + 2\langle L_XK|K\rangle \\ &= 2(D_K - L_K)\langle X|K\rangle - 2\langle X|D_KK\rangle = -2\langle X|D_KK\rangle. \end{aligned}$$

(ii)

$$\langle K_1|D_{K_1}K_2\rangle = \langle K_1|D_{K_2}K_1\rangle - \langle K_1|L_{K_2}K_1\rangle = \frac{1}{2}(D_{K_2} - L_{K_2})\langle K_1|K_1\rangle = 0$$

and

$$0 = \langle K|D_KK\rangle = \frac{1}{2}D_K\langle K|K\rangle = \frac{1}{2}L_K\langle K|K\rangle.$$

(iii) According to (ii)

$$\langle D_{K_1}K_1|K_2\rangle = D_{K_1}\langle K_1|K_2\rangle = L_{K_1}\langle K_1|K_2\rangle = \langle K_1|L_{K_1}K_2\rangle = \lambda\langle K_1|K_2\rangle.$$

(iv)

$$D_G \frac{G}{|\langle G|G\rangle|^{1/2}} = \frac{D_G G}{|\langle G|G\rangle|^{1/2}} - G\frac{\langle D_G G|G\rangle}{|\langle G|G\rangle|^{3/2}} = 0.$$

(v) $\forall\, Y$ we have

$$D_K\langle K|Y\rangle = \langle D_K K|Y\rangle + \langle K|D_K Y\rangle = \langle D_K K|Y\rangle + \langle K|D_Y K\rangle + \langle K|L_K Y\rangle$$

but also $= L_K\langle K|Y\rangle = \langle K|L_K Y\rangle$. Thus

$$\langle D_K K|Y\rangle = -\frac{1}{2}D_Y\langle K|K\rangle = -\frac{1}{2}L_Y\langle K|K\rangle$$

and therefore $D_K K = 0 \Leftrightarrow \langle K|K\rangle = \text{const.}$

(vi) Since h is K–invariant we get with $fD_G = D_{fG}$ after restriction to h

$$\begin{aligned} 0 &= L_K(K - fG) = -L_K(f)G - fL_K G = -L_K(f)G - fD_K G + fD_G K \\ &= -\frac{1}{f}L_K(f)K - f^2 D_G G + D_K K = -L_K(\ln f)K + D_K K \Rightarrow D_K K = \kappa K \end{aligned}$$

with

$$\kappa = L_K(\ln f).$$

Taking again L_K of $D_K K = \kappa K$ and observe $L_K D_K = D_{L_K K} + D_K L_K$, we see $L_K \kappa = 0$.

To a timelike Killing vector field X one can intrinsically associate an acceleration vector field

$$a = \langle X|X\rangle^{-1} D_X X. \tag{3}$$

The motivation is the following. To any vector field corresponds a flow

$$\frac{dz^i(t)}{dt} = X^i(z(t)) \quad \text{and thus} \quad \frac{d^2 z^i}{dt^2} = X^i{}_{,k}\frac{dz^k}{dt}.$$

Now in Riemann normal coordinates where the connection vanishes at $z(t)$ we have

$$(D_X X)^i(z(t)) = X^i{}_{,k}X^k(z(t)), \quad \text{thus} \quad \frac{d^2 z^i(t)}{dt^2} = (D_X X)^i(z(t)).$$

Now mind that the flow parameter is not the proper time s of an observer on this trajectory. s is normalized by $\langle dz/ds|dz/ds\rangle = 1$, that is to say $(ds/dt)^2 = \langle X|X\rangle$. If X is Killing then according to (ii) this is constant along the flow lines and thus

$$\frac{D_X X}{\langle X|X\rangle} = \left(\frac{dt}{ds}\right)^2 \frac{d^2 z}{dt^2}$$

has the significance of the acceleration felt physically. Furthermore

$$0 = L_X\langle X|X\rangle = D_X\langle X|X\rangle = 2\langle D_X X|X\rangle,$$

thus $D_X X$ is spacelike if X is timelike if the metric is Minkowskian.

3 Maximally symmetric spaces

A maximally symmetric space M of dimension m is locally isomorphic to the submanifold $\{x_i \in \mathbf{R}^{m+1}, x_i x_k \eta^{ik} = \pm R^2, \eta^{ik} = 1$ for $i = k = 0, 1, \ldots, n-1, = -1$ for $i = k = n, n+1, \ldots, m$ and zero otherwise$\} \subset \mathbf{R}^{m+1}$. The metric of M is η^{ik} restricted to M. The $SO(n, m+1-n)$ group generated in $\mathbf{R}^{m+1}$ by the vector fields $L_{ik} = x_i \partial_k - x_k \partial_i$, $k > i = 0 \ldots m$ leaves M and η invariant so that one can speak of the restriction of L_{ik} to M and there they define $m(m+1)/2$ Killing vector fields.

Proposition II In the de Sitter space ($n = 1$, $m = 4$) there are no timelike Killing vector fields which are also geodesic.

Proof: Since the Killing vector fields are a linear space any element can be written $K = v_{ik} x_i \partial_k$, $v_{ik} = -v_{ki} \in \mathbf{R}$. According to Prop. I, (v) we only have to see whether $\langle K|K \rangle = -x_i v_{ik} \eta_{km} v_{mn} x_n$ can be a constant > 0 on M. This happens iff $v \eta v = c \eta$, $c \in \mathbf{R}^+$, which implies $(\det v)^2 = c^5 > 0$. But since v is a real antisymmetric 5×5 matrix it has one eigenvalue 0 and hence $\det v = 0$.

Remark This is not a general feature of curved maximally symmetric spaces since on S^3 ($n = 0$, $m = 3$) the matrix

$$v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$v^2 = -1$ generates a K which is geodesic.

In the de Sitter case the K which are in some region timelike are of the form L_{0i}, $i = 1, \ldots, 4$ which is timelike only in the two wedges $|x_i| > |x_0|$. In one, say $H_0^+ \equiv \{x_i > |x_0|\}$ time flows upwards, in $H_0^- \equiv \{x_i < -|x_0|\}$ it flows downwards. This makes one–particle states unstable in theories with interaction [16]. A particle may tunnel through to the other side of M where it appears with the opposite energy which causes some instability. The intersections of M with planes going through the origin of M are the geodesics. The Killing vector field which leaves this plane invariant has this geodesic as flow line. In contradistinction to the Killing vector fields there are globally defined timelike geodesic vector fields. They are not Killing and to these observers the universe appears to be first contracting and then expanding. (The corresponding Hamiltonian is time–dependent [9].)

The wedge is the union of the future light cones emerging from the points of the geodesic intersected with the union of its past light cones. This set is the same for all timelike curves in the wedge. Thus one can say that the wedge is the part of the de Sitter space with which an observer can communicate. This means it consists of the points where one can receive a message from the observer and an immediate reply can still reach him.

The covariant derivative D_X in M and the one $\bar{D}_X$ in the ambient $\mathbf{R}^{m+1}$ are related via the second fundamental form S,

$$\bar{D}_X Y = D_X Y + S(X, Y). \tag{4}$$

Since M has constant curvature and all sectional curvatures are the same Gauss' famous "Theorema egregium" tells us that for M we simply have

$$S(X, X) = \frac{\langle X|X\rangle}{R}\nu$$

where ν is the unit vector field $\perp T(M)$. Thus the acceleration a_4 of X in M and a_5 in $\mathbf{R}^5$ are related by

$$a_5 = a_4 + \frac{\nu}{R}$$

or since a_4 and ν are spacelike and $a_4 \perp \nu$

$$|\langle a_5|a_5\rangle|^{1/2} = \sqrt{|\langle a_4|a_4\rangle| + \frac{1}{R^2}}.$$

Since the ambient space is flat we have $D_X X = X^k{}_{,\ell} X^\ell \partial_k$. For the Killing vector field $L_{01} = x_0\partial_1 - x_1\partial_0$ we find $\bar{D}_X X = x_0\partial_0 - x_1\partial_1$ and $\langle \bar{D}_X X|\bar{D}_X X\rangle = -\langle X|X\rangle$ on M. Thus

$$|a_5^2| = \frac{1}{\langle X|X\rangle}. \tag{5}$$

Furthermore the Laplace–Beltrami operator $\Box$ on M is the angular part of the one on the flat ambient $\mathbf{R}^5$ [10]. Finally a time flow given by a L_{0i}, $i = 1, 2, 3, 4$ gives a realization of the Anosov group (Prop. I, (iii)) if the transversal shift is identified with $L_{0i} - L_{ij}$, $j = 1, \ldots, 4$. Since on the Killing horizon for $K = L_{01}$ we have $D_K K = \kappa K$ we have here "surface gravity" $\kappa =$ Lyapunov exponent.

4 KMS states

A faithful state ω over an algebra $\mathcal{A}$ with a continuous automorphism group τ_t is said to be τ–KMS if $\omega(b\tau_t a)$ is analytic for $0 < \operatorname{Im} t < \beta$, continuous on the boundary and $\omega(ab) = \omega(b\tau_{i\beta}(a))\ \forall\ a, b \in \mathcal{A}$.

Proposition III

(i) If ω is τ and σ–KMS then $\tau_t = \sigma_t\ \forall\ t$.

(ii) If ω is τ–KMS and invariant under some automorphism σ, $\omega \circ \sigma = \omega$, then $[\tau, \sigma] = 0$.

Proof

i)
$$F(t) := \omega(\tau_t(a)\sigma_t(b)) = \omega(\sigma_t(b)\tau_{t+i\beta}(a)) = \omega(\tau_{t+i\beta}(a)\sigma_{t+i\beta}(b) = F(t+i\beta)$$

for all a, b in $\mathcal{A}$. For a dense set $\widetilde{\mathcal{A}}$ $F(t)$ is entire and can be continued periodically in Im (t) and since it is bounded in the strip it is bounded in all of $\mathbf{C}$. But for an analytic function $|F(t)| < M \ \forall\ t \in \mathbf{C}$ implies $F(t) = \text{const}$. By the same argument ω is invariant under σ such that $\omega(\sigma_t^{-1}\tau_t(a)b) = \omega(ab)\ \forall\ t \in \mathbf{R}$, $a, b \in \widetilde{\mathcal{A}}$. Since ω is faithful this implies $\sigma_t^{-1}\tau_t = id$.

(ii) Consider $\bar{\tau}_t = \sigma^{-1}\tau_t\sigma$.

$$\omega(b\bar{\tau}_{i\beta}a) = \omega(b\sigma^{-1}\tau_{i\beta}\sigma a) = \omega(\sigma(b)\tau_{i\beta}\sigma a) = \omega(\sigma a \cdot \sigma b) = \omega(ab).$$

Thus ω is also $\bar{\tau}$–KMS and because of (i) this implies $\bar{\tau} = \tau$ or $\sigma\tau = \tau\sigma$.

Conclusion: Because of (ii) a state invariant under the full de Sitter group $SO(1,4)$ cannot be KMS for any 1–parameter subgroup τ_t since $SO(1,4)$ has trivial center and τ_t would have to commute with all elements of the invariance group.

This conclusion is not so sad since we have no global time automorphism either. The only candidates are given by Killing vector fields L_{0i} which are timelike only in the wedges $H_0^\pm$. The L_{0i} gives an automorphism τ_t of the subalgebra $\mathcal{A}_{H_0^\pm}$ but the other L_{kj} do not leave $H_0^\pm$ invariant and thus do not create automorphisms of $\mathcal{A}_{H_0^\pm}$. Hence there is no contradiction in the restriction $\omega|_{\mathcal{A}_{H_0^\pm}}$ being τ_t–KMS. Because of (i) it cannot be KMS for another automorphism and here it is not the state but the subalgebra $\mathcal{A}_{H_0^\pm}$ which singles out a distinguished time.

If $\mathcal{A}$ contains operators such that their commutator (or anticommutator) is a multiple of the identity (as for free bosons or fermions) then this fixes their correlation function in a KMS state ω_β with an arbitrary temperature β^{-1}. By taking the Fourier transform of (1) with respect to t we deduce [11]

$$\omega_\beta(ab) = \int_{-\infty}^{\infty} \frac{d\nu dt}{2\pi} e^{i\nu t} \frac{e^{\beta\nu}}{e^{\beta\nu}-1} \omega_\beta([\tau_t(a), b]) = \int_{-\infty}^{\infty} \frac{d\nu dt}{2\pi} e^{i\nu t} \frac{1}{e^{-\beta\nu}+1} \omega_\beta([\tau_t(a), b]_+) \quad (6)$$

and verify

$$\begin{aligned} \omega_\beta([a,b]) &= \int_{-\infty}^{\infty} \frac{d\nu dt}{2\pi} \cos \nu t \left(\frac{e^{\beta\nu}}{e^{\beta\nu}-1} + \frac{e^{-\beta\nu}}{e^{-\beta\nu}-1} \right) \omega_\beta([\tau_t(a), b]) \\ &= \int_{-\infty}^{\infty} dt \delta(t) \omega_\beta([\tau_t(a), b]). \end{aligned}$$

Relation (7) holds under the assumption that such a state ω_β exists and will in fact be used to show that the de Sitter space supports no other than the natural temperature if the state is to be locally regular on all of M.

5 Quantum fields in de Sitter space

We shall concentrate on real free scalar fields Φ since they show already the relevant features. (For spinors things work the same way, see [12]. Since our main argument is based on the properties of the de Sitter group it should carry over to the interacting case but these theories have not yet been constructed.) If we use as coordinates the Euclidean coordinates of $\mathbf{R}^5$ and dx denotes the invariant measure

$$\int d^5x\delta(x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 + R^2)$$

supported on M, then $\mathcal{A}$ is generated linearly by the Weyl operators

$$W(f) = \exp[i\int dx f(x)\Phi(x)], \qquad f \in \mathbf{C}_0^\infty(\mathbf{R}^5) \text{ real.} \tag{7}$$

The algebraic properties are characterized by a symplectic form σ,

$$W(f)W(g) = e^{-i\sigma(f,g)/2}W(f+g) \tag{8}$$

where $\sigma(f,g) = \int dxdx' f(x)\Delta(x-x')g(x')$ and Δ is the real odd function

$$[\Phi(x), \Phi(x')] = i\Delta(x-x'). \tag{9}$$

If the de Sitter group is to be realized by geometrical automorphisms Δ can depend only on the invariant distance $(x-x')^2 = -2R^2 - 2xx'$. Thus Δ admits a type of Lehmann–Källen representation

$$\Delta(x) = i\int_0^\infty da\rho(a)\int d^5k e^{ikx}\delta(k^2-a)\varepsilon(k_0). \tag{10}$$

If Φ were to obey a Klein–Gordon equation $(\Box - m^2)\Phi(x) = 0$ on M, m would be related to the degree of homogenity of ρ as the radial part of $\Box^5$ gives on a function of radial degree of homogeneity ν an additional $\nu(\nu+3)$. Since our argument is independent of the mass we shall not pursue this further and work with a generalized free field.

A quasifree state ω over $\mathcal{A}$ is characterized by [13]

$$\omega(W(f)) = e^{-\sigma(f,Jf)/4}. \tag{11}$$

for some operator J such that

$$\sigma(f, Jg) = \int dxdx' f(x)\Delta^{(1)}(x,x')g(x')$$

where the real symmetric function $\Delta^{(1)}(x,x')$ is

$$\omega([\Phi(x), \Phi(x')]_+) = \Delta^{(1)}(x,x').$$

If $J^2 = -1$ one can in the usual way introduce creation and annihilation operators and represent $\mathcal{A}$ irreducibly in a Fock space so that ω is pure. If ω is to be invariant then

$\Delta^{(1)}(x,x')$ should depend only on $(x-x')^2$ and to meet reality and evenness $\Delta^{(1)}$ and positivity of ω, J must be in k–space multiplication by $ci\varepsilon(k_0)$, $c \in \mathbf{R}^+$, $\varepsilon(x) = x/|x|$. If $c = 1$ we have $J^2 = -1$ and ω is pure. Furthermore in this case

$$\Delta^{(1)}(x) = \int da\rho(a) \int d^5k e^{ikx} \delta(k^2 - a) \tag{12}$$

such that

$$\omega(\Phi(x)\Phi(x')) = \int da\rho(a) \int d^5k e^{ik(x-x')} \delta(k^2 - a)\Theta(k_0), \qquad \Theta = \frac{1+\varepsilon}{2} \tag{13}$$

satisfies for small $x - x'$, $(x, x'$ near $(0,0,0,0,R))$ by suitable normalization of $\int da\rho(a)$ the principal of local definiteness, i.e. it tends to $\lim_{\varepsilon\downarrow 0}(x - x' - i\varepsilon)^{-2}(2\pi)^{-2}$ where ε is in the upper lightcone. There are other invariant states [14] but these requirements single out the "Euclidean vacuum".

For this state the work of Bisognano and Wichmann has been carried over to M by Bros and Moschella and they showed that $\omega|_W$ is τ–KMS if τ_t is generated by L_{01} and the wedge $H_0^+ = \{x \in M : x_1 > |x_0|\}$ and $\beta = 2\pi$.

The argument boils down to the following. L_{01} generates $\tau^*_{i\alpha}(x) = (x_0c + ix_1s, x_1c + ix_0s, x_\perp)$ with $\begin{matrix} c \\ s \end{matrix} = \begin{matrix} \cos \\ \sin \end{matrix}\alpha$ and $x_\perp = x_{2,3,4}$. Since $\omega(\Phi(x')\tau_{i\alpha}(\Phi(x))) = \Delta^+((x' - \tau^*_{i\alpha}(x))^2)$ and the singularity of Δ^+ sits at the origin we have to see whether we can continue α from 0 to 2π without that $(x' - \tau^*_{i\alpha}(x))^2$ touches 0. By invariance we may assume in the wedge $x_0 = 0$ then

$$(x' - \tau^*_{i\alpha}(x))^2 = (x'_0 - ix_1s)^2 - (x'_1 - x_1c)^2 - (x'_\perp - x_\perp)^2$$

and

$$\begin{aligned} \mathrm{Re}\ (x' - \tau^*_{i\alpha}(x))^2 &= {x'_0}^2 - {x'_1}^2 - x_1^2 + 2x'_1x_1c - (x'_\perp - x_\perp)^2 \\ \mathrm{Im}\ (x' - \tau^*_{i\alpha}(x))^2 &= -2x'_0x_1s. \end{aligned}$$

Since $x_1 > 0$ in H_0^+ Im $= 0$ only if $s = 0 \Leftrightarrow \alpha = \pi$ or $x'_0 = 0$. In both cases Re < 0 and thus in H_0^+ we can move α from 0 to 2π without encountering the singularity. When we approach 2π the only difference is that now the real axis approached from the other side. This corresponds exactly to the difference between $\Delta^+(x - x')$ and $\Delta^+(x' - x)$ and hence

$$\omega(\Phi(x'), \tau_{2\pi i}\Phi(x)) = \omega(\Phi(x)\cdot\Phi(x'))$$

which is just the KMS condition we have been looking for. It holds $\forall\ x, x' \in H_0^+$ and the only difference to Bisognano–Wichmann is that in M τ^*_t also generates one geodesic line and thus also a freely falling observer sees a temperature.

Remarks

1. The wedge H_0^+ cannot be invariant under another automorphism of the de Sitter group not commuting with L_{01}. This would contradict Prop. II, (ii) since ω is invariant under all of them. Thus the apparent contradiction can be avoided only if these transformations do not create automorphisms of $\mathcal{A}_{H_0^+}$.

2. One might ask whether there are states with a different temperature for $\mathcal{A}_{H_0^+}$. In fact (7) gives an explicit representation for states for arbitrary β. However the analysis of [11] carries directly over and shows that at the edge of the horizon where $x_0, x_1 \to 0$ these states do not satisfy the principle of local definiteness. This requires that in the small distance limit the propagator has the same singularity as in Minkowski space. As we have seen in Sect. 5 the commutator is by construction the same as in Minkowski space and what happens is that the expectation value of the anticommutator has in front a factor $1/2\pi\beta$ instead of $1/(2\pi)^2$. The same effect with $\beta < 2\pi$ occurs if one goes in Minkowski space by a Bogoliubov transformation into a wrong vacuum. For $\beta > 2\pi$ positivity of the state gets lost since $\Delta^{(1)}$ becomes too small compared with Δ. This means that for T larger than the "natural" temperature there are infinitely many negative energy quanta in the tangent space of the edge of the wedge and smaller temperatures do not exist at all. To see how this happens here one can directly take over the analysis of [11] where with the notation of their Sect. 3 we have when projected from $\mathbf{R}^5$ to M

$$\begin{aligned}\Delta(\tau_1+\tau, x_1|\tau_2, x_2) &= \int d^5k\delta(k^2-m^2)\varepsilon(k_0)e^{ik_u(u_1e^{\tau}-u_2)+ik_v(v_1e^{-\tau}-v_2)+ik^{\perp}(x_1^{\perp}-x_2^{\perp})}\cdot \\ &\cdot\delta(u_1v_1+x_1^{\perp 2}-R^2)\delta(u_2v_2+x_2^{\perp 2}-R^2).\end{aligned}$$

   The only change is that $x^{\perp}$ has an extra dimension. This is however fixed by the δ–functions and the main point is not affected. Thus only for $\beta = 2\pi$ the state can be extended beyond the horizon without violating local definiteness.

3. The KMS property is not the only equilibrium feature of ω. Because of the Anosov property mentioned in Sect. 3 and the invariance of ω we know from general arguments that we get sensitive dependence on initial conditions and exponential decay in t of the correlation functions [8].

4. A light source which keeps shining and never exhausts itself seems strange and one might wonder whether it could not be used to construct a perpetuum mobile. That this is not so shows another characterization of KMS states, the passivity [18]. It states that by an external perturbation depending periodically on time one can only invest energy into the system but never extract any from it.

Everything which has been said about the wedge H_0^+ can also be said about subalgebras supported in $H_c^+ = \{x \in H_c, x_1 > 0\}$. H_c^+ is invariant under the flow of L_{01} and therefore the associated automorphism leaves these subalgebras invariant. Thus the wedge algebra has plenty of invariant subalgebras, its Anosov property excludes only finite-dimensional invariant subalgebras. We shall now show that because of Haag duality they all have the same strong closure and their consideration does not give an essentially new information.

Definition (5.1) Let $\mathcal{A}$ be an algebra localized in a space-time region Δ and denote by $\mathcal{A}$; its commutant and by $\mathcal{A}^\perp$ the strong closure of the algebra localized in the causal complement of Δ. We have the inclusions

(i) $\mathcal{A}'' \supset \mathcal{A}$, equality holds if $\mathcal{A}$ is strongly closed (von Neumann's theorem).

(ii) $\mathcal{A}^{\perp\perp} \supset \mathcal{A}$, equality holds if Δ is causally closed.

(iii) $\mathcal{A}' \supset \mathcal{A}^\perp$ by causality, equality is called Haag duality.

(iv) $\mathcal{B}' \supset \mathcal{A}'$, $\mathcal{B}^\perp \supset \mathcal{A}^\perp$ for any subalgebra $\mathcal{B}$ of $\mathcal{A}$.

Proposition (5.2)

(i) Let $\mathcal{A}$ be causally closed. Then for Haag duality $\mathcal{A}'' = \mathcal{A}^{\perp\perp} = \mathcal{A}^{\perp\prime} = \mathcal{A}^{\prime\perp} = \mathcal{A}$ is necessary and $\mathcal{A}'' = \mathcal{A}^{\perp\prime}$ is sufficient.

(ii) If $\mathcal{B}$ is Haag dual and $\mathcal{A}^\perp = \mathcal{B}^\perp$, then $\mathcal{B}$ is strongly dense in $\mathcal{A}$.

(iii) If in addition $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}$ is also Haag dual.

Proof:

(i) Necessity: From (5.1) follow the inclusions $\mathcal{A}^{\prime\perp} \subset \mathcal{A}^{\perp\perp} \supset \mathcal{A} \subset \mathcal{A}'' \subset \mathcal{A}^{\perp\prime}$. If $\mathcal{A}$ is causally (and therefore strongly) closed the inclusions in the middle are equalities and so are the ones at the ends if Haag dualoty holds.

Sufficiency: $\mathcal{A}' = \mathcal{A}''' = \mathcal{A}^{\perp\prime\prime} = \mathcal{A}^\perp$ since $\mathcal{A}'$ and $\mathcal{A}^\perp$ are strongly closed.

(ii) $\mathcal{B}'' = \mathcal{B}^{\perp\prime} = \mathcal{A}^{\perp\prime} \supset \mathcal{A}''$.

(iii) $\mathcal{A}' \supset \mathcal{A}^\perp = \mathcal{B}^\perp = \mathcal{B}' \supset \mathcal{A}'$.

Corollary. For the quasifree bosons we are considering Haag duality holds and the causal complements of all H_c^+ is H_0^-. Thus all $\mathcal{A}_{H_c^+}$, $c > 0$, are strongly dense in $\mathcal{A}_{H_0^+}$.

6 Conclusions

Our first postulate is that the physical temperature is linked to the physical time. After all one measures the temperature of astrophysical objects by measuring their frequency spectrum. Furthermore we work with a given metric which is supposed to measure the physical time. This postulate has been extended by Connes and Rovelli [15] to the generally covariant situation of quantum gravity and our more modest application illustrates some of the points made there. Secondly we argued that the physical timeflow has to be represented by an automorphism group of the observables. To call the parameter of transformations which do not preserve the structure of the laws of nature the time seems illusory since this will not be what is measured by a clock. If these transformations are to be geometrical they have to be flows of Killing vector fields. In the case of de Sitter space there are as many Killing vector fields as in Minkowski space and there a time automorphism exists at least locally though not globally. If there were no Killing vector fields one would have to withdraw to the infinitesimal level by some scaling limit. This is presumably the way how the background radiation in our universe has to be interpreted. Also in de Sitter space M one could project a KMS state with respect to the x_0–shift of the 5–dimensional Minkowski field theory onto the field theory on M. This state will be faithful and therefore defines a global modular automorphism which will not be geometric. The corresponding geometrical generator of the x_0–shift is x_0–dependent and thus does not generate a group of unitaries. However to the extent that the universe in some coordinate system is only adiabatically expanding one may then define by some scaling limit a local temperature in the tangent space of each point. This temperature will then be different at different points and may show red shift and Doppler shift. The algebraic structure of free quantum fields is fixed by the geometrical setting but the multitude of states requires a selection principle. We adopt the principle of local definiteness which requires that at the infinitesimal level the state looks like a Minkowski vacuum. This singles out one among the invariant states and then our claims follow immediately. For each Killing vector field $K = L_{0i}$ the state restricted to the algebra in the wedge where K is timelike is KMS for $\beta = 2\pi$. Since this is measured in the parameter t of the flow and not the proper time s we have to use from Sect. 2 that along a trajectory $(s/t)^2 = \langle K|K\rangle$ and thus $\beta_{\text{phys}} = 1/T = 2\pi\langle K|K\rangle^{1/2}$. Now we recognize with Sect. 3 $\langle K|K\rangle^{-1/2}$ as the 5–dimensional acceleration $|a_5|$ and when expressed in the intrinsic acceleration a_4 we come to our conclusion

$$T = \frac{|a_5|}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{R^2} + |a_4^2|}.$$

Though Minkowski and de Sitter space belong both to the class of maximally symmetric spaces the curvature changes physics quite a bit. Therefore we shall conclude by confronting the various properties of the two spaces how they are reflected by free quantum fields.

Minkowski	de Sitter
Vector fields	
There are globally timelike geodesic vector fields which are also Killing.	There are globally timelike geodesic vector fields but they are not Killing. There are global Killing vector fields but they are only in some regions timelike. Each has only a single flow line which is geodesic.
States	
There is one invariant locally definite state. For each timelike global Killing vector field there are for all temperatures KMS states which are everywhere locally definite. For all geodesic obserserves the state of zero temperature is the distinguished invariant state.	There is one invariant locally definite state. Each Killing vector field which is locally timelike has one geodesic observer. For him the distinguished invariant state has a temperature $1/2\pi R$. There is no state which is everywhere locally definite which would have another temperature for him.

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