NEW CLASSES OF TODA SOLITON SOLUTIONS

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Abstract. We provide a detailed investigation of limits of N–soliton solutions of the Toda lattice as N tends to infinity. Our principal results yield new classes of Toda solutions including, in particular, new kinds of soliton–like (i.e., reflectionless) solutions. As a byproduct we solve an inverse spectral problem for one–dimensional Jacobi operators and explicitly construct tri–diagonal matrices that yield a purely absolutely continuous spectrum in $(-1,1)$ and give rise to an eigenvalue spectrum that includes any prescribed countable and bounded subset of $\mathbb{R}\backslash[-1,1]$.

1. Introduction

Our principal aim in this paper is a detailed study of N–soliton solutions $(a_N(t,n), b_N(t,n))$ of the Toda lattice equations

$$\begin{aligned} &\tfrac{d}{dt}a(t,n) = a(t,n)[b(t,n)-b(t,n+1)],\\ &\tfrac{d}{dt}b(t,n) = 2[a(t,n-1)^2 - a(t,n)^2], \qquad (t,n)\in\mathbb{R}\times\mathbb{Z} \end{aligned} \tag{1.1}$$

in the limit $N\to\infty$. In particular, if $(a_\infty(t,n), b_\infty(t,n))$ denotes the limit of $(a_N(t,n), b_N(t,n))$ as $N\to\infty$, we shall undertake a careful investigation of the spectral (and scattering) properties of the associated Jacobi operator $H_\infty(t) = a_\infty(t)S^+ + S^- a_\infty(t) + b_\infty(t)$ in $\ell^2(\mathbb{Z})$ (where $(S^\pm f)(n) = f(n\pm1),\ f\in\ell^\infty(\mathbb{Z})$ are the usual shift operators).

Our principal techniques are based on a Hilbert space approach to Toda systems (as studied in detail in [5]), the use of Fredholm determinants in the context of discrete evolution equations (see also [1], [25]), and Weyl-Titchmarsh spectral theory (see, e.g., [11]).

In order to describe our approach in some detail we briefly recall some of the basic facts of N–soliton solutions $(a_N(t,n), b_N(t,n))$ of the Toda system (1.1). They can be represented as (cf. Section 2 for more details)

$$a_N(t,n) = \frac{\{\det[1_N + C_N(t,n+1)]\det[1_N + C_N(t,n-1)]\}^{\frac12}}{2\det[1_N + C_N(t,n)]}, \qquad n\in\mathbb{Z}, \tag{1.2}$$

$$\begin{aligned} b_N(t,n) = {} & \tfrac12(k+k^{-1})\\ & - \frac{k}{2}\frac{\det[1_N + D_N(k,t,n+1)]\det[1_N + C_N(t,n-1)]}{\det[1_N + D_N(k,t,n)]\det[1_N + C_N(t,n)]}\\ & - \frac{1}{2k}\frac{\det[1_N + D_N(k,t,n-1)]\det[1_N + C_N(t,n)]}{\det[1_N + D_N(k,t,n)]\det[1_N + C_N(t,n-1)]}, \qquad n\in\mathbb{Z},\\ \lambda = {} & \tfrac12(k+k^{-1}) \le \inf\{\lambda_j = \tfrac12(\kappa_j+\kappa_j^{-1})\}_{j=1}^N, \end{aligned} \tag{1.3}$$

where

$$C_N(t,n) = \left[c_j(t)c_l(t)\frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l} \right]_{j,l=1}^N, \tag{1.4}$$

$$A_N(k) = \left[\frac{\kappa_j - k}{\kappa_j^{-1} - k}\delta_{j,l} \right]_{j,l=1}^N, \tag{1.5}$$

$$D_N(t,n) = A_N(k)C_N(t,n-1). \tag{1.6}$$

The corresponding self–adjoint Jacobi operator in $\ell^2(\mathbb{Z})$ defined by

$$H_N(t) = a_N(t)S^+ + S^- a_N(t) + b_N(t), \qquad \mathfrak{D}(H_N(t)) = \ell^2(\mathbb{Z}), \qquad t \in \mathbb{R} \tag{1.7}$$

then has the t–independent spectrum

$$\sigma(H_N(t)) = \{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j=1}^N \cup [-1,1], \tag{1.8}$$

the essential spectrum $[-1,1]$ of $H_N(t)$ being purely absolutely continuous. The scattering matrix $S_N(\lambda)$ associated with the pair $(H_N(t), H_0)$, $H_0 = \frac{1}{2}(S^+ + S^-)$ is t–independent and reflectionless, i.e.,

$$S_N(\lambda) = \begin{pmatrix} T_N(k) & 0 \\ 0 & T_N(k) \end{pmatrix},$$

$$T_N(k) = \prod_{j=1}^N \operatorname{sgn}(\kappa_j)\frac{1-k\kappa_j}{\kappa_j - k}, \qquad \lambda = \tfrac{1}{2}(k+k^{-1}) \in [-1,1]. \tag{1.9}$$

In order to illustrate the limit $N \to \infty$ one first observes that $C_N(t,n) > 0$ and hence $\mathrm{Tr}[C_N(t,n)] = \|C_N(t,n)\|_1$ ($\|\cdot\|_1$ denotes the trace norm). Upon embedding $\mathbb{C}^N$ into $\ell^2(\mathbb{N})$ (viewing $C_N(t,n)$ as an operator in $\ell^2(\mathbb{N})$) one then shows that

$$\{\kappa_j\}_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N}), \qquad 0 < \kappa_0 \le |\kappa_j| < 1,$$
$$\{c_j^2(1-\kappa_j^2)^{-1}\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \tag{1.10}$$

is a natural hypothesis such that $C_N(t,n)$ converges to $C_\infty(t,n)$ in $\mathfrak{B}_1(\ell^2(\mathbb{N}))$–norm ($\mathfrak{B}_1(\cdot)$ the set of trace class operators) and hence the determinants $\det[1_N + C_N(t,n)]$ converge to Fredholm determinants $\det_1[1_\infty + C_\infty(t,n)]$ as $N \to \infty$. Given Hypotheses (1.10) we shall prove in our principal Theorem 4.1 that the corresponding Jacobi limit operator $H_\infty(t)$ in $\ell^2(\mathbb{Z})$ is bounded and self–adjoint with t–independent spectrum

$$\sigma_{ess}(H_\infty(t)) = \{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}'_{j\in\mathbb{N}} \cup [-1,1], \tag{1.11}$$

$$\sigma_{ac}(H_\infty(t)) = [-1,1], \tag{1.12}$$

$$\{\sigma_p(H_\infty(t)) \cup \sigma_{sc}(H_\infty(t))\} \cap (-1,1) = \emptyset, \tag{1.13}$$

$$\{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}} \subseteq \sigma_p(H_\infty(t)) \subseteq \overline{\{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}}}. \tag{1.14}$$

That is, the spectrum of $H_\infty(t)$ in $(-1,1)$ is purely absolutely continuous and its point spectrum contains the bounded set $\{\frac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}}$. This set, however, besides being bounded (and of course countable), needs to satisfy no further restrictions if the constants $c_j > 0$ satisfy the conditions in (1.10). In particular, these eigenvalues $\{\lambda_j = \frac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}}$ of $H_\infty(t)$ can be dense in any subinterval of $(-\frac{1}{2}(\kappa_0 + \kappa_0^{-1}), -1) \cup (1, -\frac{1}{2}(\kappa_0 + \kappa_0^{-1}))$. Hence Theorem 4.1 can be viewed as a solution of the following inverse spectral problem: *Given any bounded and countable subset* $\{\lambda_j\}_{j\in\mathbb{N}}$ *of* $(-\infty,-1) \cup (1,\infty)$, *construct a tri–diagonal matrix whose* *absolutely continuous spectrum equals* $[-1,1]$ *and whose set of eigenvalues includes the prescribed set* $\{\lambda_j\}_{j\in\mathbb{N}}$. (In (1.11)–(1.14) $\sigma_{ess}(\cdot)$, $\sigma_{ac}(\cdot)$, $\sigma_{sc}(\cdot)$ and $\sigma_p(\cdot)$ denotes the essential, absolutely continuous, singularly continuous and point spectrum (i.e., the set of eigenvalues), respectively and Σ' denotes the derived set of $\Sigma \in \mathbb{R}$, i.e., the set of accumulation points of Σ.)

By inspection, the product in the transmission coefficient $T_N(k)$ in (1.9) converges absolutely as $N\to\infty$ iff

$$\{1-|\kappa_j|\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}). \tag{1.15}$$

Therefore, assuming (1.15) in addition to (1.10) allows one to study scattering theory for the pair $(H_\infty(t), H_0(t))$. Theorem 4.2 is devoted to the detailed treatment of this case.

In Section 2 we briefly review the necessary prerequisites on N–soliton solutions of the Toda lattice. Except, perhaps, for our representation of $b_N(t,n)$ in (2.7), this material is standard. Since the spectrum of $H_N(t)$ is t–independent we restrict ourselves to the stationary (i.e., t–independent) case in the following Sections 3 and 4. Section 3 contains the main technical results on convergence properties of various quantities as $N\to\infty$. Section 4 contains our principal results on spectral properties of H_∞ outlined above. Section 5 finally returns to the t–dependent case and yields the construction of new classes of Toda soliton solutions associated with $H_\infty(t)$.

We emphasize that our techniques are by no means restricted to the Toda system but apply equally well to integrable systems of the AKNS class. The particular case of the Korteweg de Vries (KdV) equation has been worked out in detail in [11] (see also [12]). Further results on one–dimensional, generalized reflectionless potentials can be found in [4],[16]–[19], and [22].

2. Reflectionless Jacobi operators and Toda N–soliton solutions

This section briefly summarizes reflectionless short–range Jacobi operators and Toda N–soliton solutions. Since a complete bibliography on this subject is nearly impossible due to the extensive literature available, we restrict ourselves to some of the key references from which the following material has been taken [3],[6]–[10],[13],[24].

We start with the stationary (i.e., t–independent) case and consider Toda solutions at the end of this section. Define the matrices

$$C_N(n) = \Big[c_j c_l \frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l}\Big]_{j,l=1}^N, \tag{2.1}$$

$$A_N(k) = \Big[\frac{\kappa_j - k}{\kappa_j^{-1}-k}\delta_{j,l}\Big]_{j,l=1}^N, \tag{2.2}$$

$$D_N(k,n) = A_N(k)C_N(n-1), \tag{2.3}$$

where

$$0<|\kappa_j|<1,\ \kappa_j\in\mathbb{R},\ c_j>0,\ 1\le j\le N,\ N\in\mathbb{N},\ k\in\mathbb{C}\backslash\{\kappa_j^{-1}\}_{j=1}^N. \tag{2.4}$$

Reflectionless Jacobi operators H_N in $\ell^2(\mathbb{Z})$ are then defined by

$$H_N = a_N S^+ + S^- a_N + b_N, \qquad \mathfrak{D}(H_N)=\ell^2(\mathbb{Z}), \tag{2.5}$$

where

$$a_N(t,n) = \frac{\{\det[1_N + C_N(n+1)]\det[1_N + C_N(n-1)]\}^{\frac{1}{2}}}{2\det[1_N + C_N(n)]}, \qquad n \in \mathbb{Z}, \tag{2.6}$$

$$\begin{aligned} b_N(t,n) &= \tfrac{1}{2}(k+k^{-1}) \\ &\quad - \frac{k}{2}\frac{\det[1_N + D_N(k,n+1)]\det[1_N + C_N(n-1)]}{\det[1_N + D_N(k,n)]\det[1_N + C_N(n)]} \\ &\quad - \frac{1}{2k}\frac{\det[1_N + D_N(k,n-1)]\det[1_N + C_N(n)]}{\det[1_N + D_N(k,n)]\det[1_N + C_N(n-1)]}, \qquad n \in \mathbb{Z}, \end{aligned} \tag{2.7}$$

$$\lambda = \tfrac{1}{2}(k+k^{-1}) \leq \inf\{\lambda_j = \tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j=1}^N,$$

$S^{\pm}$ denote the shift operators in $\ell^\infty(\mathbb{Z})$,

$$(S^{\pm}f)(n) = f(n \pm 1), \qquad n \in \mathbb{Z},\ f = \{f(n)\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \tag{2.8}$$

and

$$1_N = \{\delta_{j,l}\}_{j,l=1}^N\,. \tag{2.9}$$

Since

$$(f, C_N(n)f)_{\mathbb{C}^N} = \sum_{j,l=1}^N \overline{f(j)} c_j c_l \frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l} f(l) = \sum_{m=n+1}^\infty \Big| \sum_{j=1}^N f_j c_j \kappa_j^m \Big|^2 > 0, \quad f = \{f(j)\}_{j=1}^N \in \mathbb{C}^N, \tag{2.10}$$

one infers

$$C_N(n) > 0 \tag{2.11}$$

and (2.6), (2.7) are well–defined. One verifies the spectral properties of the self–adjoint operator H_N,

$$\begin{aligned} &\sigma_{ess}(H_N) = \sigma_{ac}(H_N) = [-1,1], \qquad \sigma_{sc}(H_N) = \emptyset, \\ &\sigma_d(H_N) = \sigma_p(H_N) = \{\lambda_j = \tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j=1}^N, \end{aligned} \tag{2.12}$$

where $\sigma_{ess}(\cdot)$, $\sigma_{ac}(\cdot)$, $\sigma_{sc}(\cdot)$, $\sigma_d(\cdot)$, and $\sigma_p(\cdot)$ denotes the essential, absolutely continuous, singularly continuous, discrete, and point spectrum (i.e., the set of eigenvalues), respectively.

While the representation (2.6) for $a_N(n)$ is quite standard, the one in (2.7) follows, for instance, by inserting the solution $f_{N,+}(k,n)$ from (2.22) into $b_N(n) = z-(a_N(n)f_{N,+}(k,n+1)+a_N(n-1)f_{N,+}(k,n-1))f_{N,+}(k,n)^{-1}$ (using the difference equation (2.21)). One can show that

$$|a_N(n)| \leq 2^{-\frac{1}{2}}\|H_N\| = 2^{-\frac{3}{2}}(\kappa_0 + \kappa_0)^{-1}, \tag{2.13}$$

$$|b_N(n)| \leq \|H_N\| = \tfrac{1}{2}(\kappa_0 + \kappa_0^{-1}), \tag{2.14}$$

where

$$\kappa_0 = \sup|\kappa_j|_{j=1}^N. \tag{2.15}$$

The normalized eigenfunctions $\psi_{N,j}$ are explicitly given by

$$H_N\psi_{N,j} = \lambda_j\psi_{N,j}, \qquad \|\psi_{N,j}\|_2 = 1, \qquad 1 \leq j \leq N, \tag{2.16}$$

where

$$\Psi_N(n) = (\psi_{N,1}(n), \dots, \psi_{N,N}(n))^T, \qquad \Psi_N^0(n) = (c_1\kappa_1^n, \dots, c_N\kappa_1^N)^T,$$

$$\Psi_N(n) = \Big\{\frac{\det[1_N + C_N(n)]}{\det[1_N + C_N(n-1)]}\Big\}^{\frac{1}{2}} \Big[1_N + C_N(n)\Big]^{-1} \Psi_N^0(n), \qquad n \in \mathbb{Z}. \tag{2.17}$$

In addition to (2.7), b_N also admits the trace formula representation [9],

$$b_N(n) = -\tfrac{1}{2}\sum_{j=1}^{N}(\kappa_j - \kappa_j^{-1})\psi_{N,j}(n)^2, \qquad n \in \mathbb{Z}. \tag{2.18}$$

Next, we consider Jost and scattering wavefunctions of H_N defined by

$$f_{N,+}(k,n) = \bigg\{\Big\{\frac{\det[1_N + C_N(n)]}{\det[1_N + C_N(n-1)]}\Big\}^{\frac{1}{2}} - \sum_{j=1}^{N} c_j\kappa_j^n \frac{k\kappa_j}{1-k\kappa_j}\psi_{N,j}(n)\bigg\}k^n, \qquad |k|<1, \tag{2.19}$$

$$\psi_{N,-}(k,n) = f_{N,+}(k^{-1},n), \qquad |k|<1, \tag{2.20}$$

satisfying

$$H_N f_{N,+}(k) = z f_{N,+}(k), \qquad H_N\psi_{N,-}(k) = z\psi_{N,-}(k), \qquad z = \tfrac{1}{2}(k+k^{-1}) \tag{2.21}$$

in the weak sense. One verifies

$$\begin{aligned} f_{N,+}(k,n) &= k^n \frac{\det[1_N + D_N(k,n)]}{\{\det[1_N + C_N(n)]\det[1_N + C_N(n-1)]\}^{\frac{1}{2}}}, \\ \psi_{N,-}(k,n) &= k^{-n} \frac{\det[1_N + D_N(k^{-1},n)]}{\{\det[1_N + C_N(n)]\det[1_N + C_N(n-1)]\}^{\frac{1}{2}}}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} f_{N,+}(k,n) &= \begin{cases} k^n, & n\to+\infty \\ T_N(k)^{-1}k^n, & n\to-\infty \end{cases}, \\ \psi_{N,-}(k,n) &= \begin{cases} k^{-n}, & n\to+\infty \\ T_N(k)k^{-n}, & n\to-\infty \end{cases}, \end{aligned} \tag{2.23}$$

where $T_N(k)$ denotes the corresponding transmission coefficient

$$T_N(k) = \prod_{j=1}^{N} \operatorname{sgn}(\kappa_j)\frac{1-k\kappa_j}{\kappa_j - k} \tag{2.24}$$

(note that $T_N(k)^{-1} = T_N(k^{-1})$). The asymptotic relations (2.23) prove the reflectionless property of H_N,

$$R_N^{r,l}(k) = 0, \tag{2.25}$$

(with $R^r(\cdot)$, $R^l(\cdot)$ denoting the reflection coefficient from right and left incidence, respectively) and hence yield the following unitary scattering matrix $S_N(\lambda)$ in $\mathbb{C}^2$ associated with the pair (H_N, H_0), $H_0 = \frac{1}{2}(S^+ + S^-)$,

$$S_N(\lambda) = \begin{pmatrix} T_N(k) & 0 \\ 0 & T_N(k)\end{pmatrix}, \qquad \lambda = \tfrac{1}{2}(k+k^{-1}) \in [-1,1]. \tag{2.26}$$

We also note that

$$f_{N,+}(\kappa_j,n)=c_j^{-1}\psi_{N,j}(n), \tag{2.27}$$

$$\operatorname*{Res}_{k=\kappa_j}\psi_{N,-}(k,n)=-c_j\kappa_j\psi_{N,j}(n),\quad 1\le j\le N. \tag{2.28}$$

Finally, we briefly consider N–soliton solutions of the Toda lattice (in Flaschka's variables) given by

$$\begin{aligned}&\tfrac{d}{dt}a(t,n)=a(t,n)[b(t,n)-b(t,n+1)],\\ &\tfrac{d}{dt}b(t,n)=2[a(t,n-1)^2-a(t,n)^2)],\qquad (t,n)\in\mathbb{R}\times\mathbb{Z}.\end{aligned} \tag{2.29}$$

Replacing

$$c_j\to c_j(t)=c_je^{\beta_jt},\qquad \beta_j=\tfrac{1}{2}(\kappa_j-\kappa_j^{-1}),\ 1\le j\le N,\ t\in\mathbb{R} \tag{2.30}$$

in (2.6) and (2.7), denoting the result by $a_N(t,n)$, $b_N(t,n)$, $H_N(t)$, etc., then yields the N–soliton solutions of (2.29). Using the standard Lax pair for the Toda lattice equations (2.29) then proves that $H_N(t)$ is unitarily equivalent to $H_N(0)$ for all $t\in\mathbb{R}$.

3. Convergence results as $N\to\infty$

This is the main technical section in which we investigate various limits of $a_N(n)$, $b_N(n)$, H_N, $C_N(n)$, $f_{N,+}(k,n)$, $\psi_{N,-}(k,n)$, and $T_N(k)$ as $N\to\infty$.

Throughout the remainder of this paper we shall consider sequences $\{\kappa_j\}_{j\in\mathbb{N}}$, $\{c_j\}_{j\in\mathbb{N}}$ subject to the following hypothesis:

(H.3.1) Assume that $\{\kappa_j\}_{j\in\mathbb{N}}$ satisfies $\kappa_0\le|\kappa_j|<1$, $\kappa_j\in\mathbb{R}$, $j\in\mathbb{N}$ for some $\kappa_0>0$, and that $\{c_j\}_{j\in\mathbb{N}}$ satisfies $c_j>0$, $j\in\mathbb{N}$, $\{c_j^2(1-\kappa_j^2)^{-1}\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})$.

Whenever we are interested in scattering theory we shall assume the stronger hypothesis

(H.3.2) In addition to (H.3.1) suppose that $\{1-|\kappa_j|\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})$

motivated by the absolute convergence of the product for $T_N(k)$ as $N\to\infty$ in (2.24).

We first derive general convergence results assuming (H.3.1) only. The scattering case in connection with (H.3.2) will be treated at the end of this section.

Define

$$C_\infty(n)=\Big[c_jc_l\frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l}\Big]_{j,l\in\mathbb{N}},\qquad n\in\mathbb{Z}, \tag{3.1}$$

$$A_\infty(k)=\Big[\frac{\kappa_j-k}{\kappa_j^{-1}-k}\delta_{j,l}\Big]_{j,l\in\mathbb{N}},\quad k\in\mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}}, \tag{3.2}$$

$$D_\infty(k,n)=A_\infty(k)C_\infty(n-1),\qquad k\in\mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}},\qquad n\in\mathbb{Z}. \tag{3.3}$$

Basic properties of $C_\infty(n)$ and $D_\infty(k,n)$ are listed in the following

Lemma 3.3. *Assume (H.3.1) and $n \in \mathbb{Z}$, $N \in \mathbb{N}$, $k \in \mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}}$. Then for all $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$*

$$0 \leq C_{\mathcal{M}}(n) \in \mathcal{B}_1(\ell^2(\mathbb{N})), \quad \|C_{\mathcal{M}}(n)\|_1 \leq \|C_\infty(n)\|_1 = \sum_{j\in\mathbb{N}} c_j^2 \frac{\kappa_j^{2n+2}}{1-\kappa_j^2}, \tag{3.4}$$

$$\det[1_N + C_N(n)] \xrightarrow[N\to\infty]{} \det{}_1[1_\infty + C_\infty(n)], \tag{3.5}$$

$$\det{}_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)] \xrightarrow[n\to+\infty]{} 1 \text{ uniformly with respect to } \mathcal{M}, \tag{3.6}$$

$$D_{\mathcal{M}}(k,n) \in \mathcal{B}_1(\ell^2(\mathbb{N})),$$

$$\|D_{\mathcal{M}}(k,n)\|_1 \leq \|D_\infty(k,n)\|_1 \leq const.(k)\|C_\infty(n)\|_1, \tag{3.7}$$

$$\det[1_N + D_N(k,n)] \xrightarrow[N\to\infty]{} \det{}_1[1_\infty + D_\infty(k,n)], \tag{3.8}$$

$$\det{}_1[1_{\mathcal{M}} + D_{\mathcal{M}}(k,n)] \xrightarrow[n\to+\infty]{} 1 \text{ uniformly with respect to } \mathcal{M}. \tag{3.9}$$

(Here $\mathcal{B}_1(\cdot)$ denotes the set of trace class operators, $\|\cdot\|_1$ the corresponding trace norm, $\det_1[1_\infty + \ \cdot\ ]$ the associated Fredholm determinant, and 1_∞ the identity in $\ell^2(\mathbb{N})$.)

Proof. Let $f = \{f(p)\}_{p\in\mathbb{N}} \in \ell^2(\mathbb{N})$ then

$$(f, C_{\mathcal{M}}(n)f)_{\ell^2(\mathbb{N})} = \sum_{m=n+1}^{\infty} \Big|\sum_{j=1}^{\mathcal{M}} f(j)c_j\kappa_j^m\Big|^2 \geq 0 \tag{3.10}$$

and thus $\|C_{\mathcal{M}}\|_1 = \mathrm{Tr}[C_{\mathcal{M}}(n)] \leq \mathrm{Tr}[C_\infty(n)]$ yield (3.4) given (H.3.1). Since also $C_\infty(n) - P_N C_\infty(n) P_N \geq 0$, we infer

$$\|C_\infty(n) - P_N C_\infty(n) P_N\|_1 = \sum_{j=N+1}^{\infty} c_j^2 \frac{\kappa_j^{2n+2}}{1-\kappa_j^2} \xrightarrow[N\to\infty]{} 0 \tag{3.11}$$

and hence (3.5) since

$$\begin{aligned}\big|\det{}_1[1_\infty + C_\infty(n)] - \det[1_N + C_N(n)]\big| \leq &\|C_\infty(n) - P_N C_\infty(n) P_N\|_1 \\ &\times \exp\big\{\|C_\infty(n)\|_1 + 1\big\}\end{aligned} \tag{3.12}$$

by [23], p.66. Here P_N, $N \in \mathbb{N}$ denotes the projection

$$P_N : \begin{cases} \ell^2(\mathbb{N}) & \to \quad \ell^2(\mathbb{N}) \\ f = (f_1, \dots, f_N, f_N + 1, \dots) & \mapsto \quad (f_1, \dots, f_N, 0, 0, \dots) \end{cases} \tag{3.13}$$

which embeds $\mathbb{C}^N$ into $\ell^2(\mathbb{N})$ and $P_\infty = 1_\infty$ denotes the identity operator in $\ell^2(\mathbb{N})$. The fact that $\|P_{\mathcal{M}} C_\infty(n) P_{\mathcal{M}}\|_1 \leq \|C_\infty(n)\|_1 \xrightarrow[n\to+\infty]{} 0$ by the Weierstrass test for all $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$ together with

$$\det{}_1[1_\infty + P_{\mathcal{M}} C_\infty(n) P_{\mathcal{M}}] \leq \exp\{\|C_\infty(n)\|_1\} \tag{3.14}$$

(cf. [23], p.47) then proves (3.6). The corresponding results for $D_{\mathcal{M}}(k,n)$, $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$ then follow from (3.3) since $A_\infty(k)$ is a bounded operator in $\ell^2(\mathbb{N})$ with

$$\|A_\infty(k)\| = \sup_{j\in\mathbb{N}} \Big|\frac{\kappa_j - k}{\kappa_j^{-1} - k}\Big| < \infty, \qquad k \in \mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}} \tag{3.15}$$

and

$$\begin{aligned}
\|A_{\mathcal{M}}(k)C_{\mathcal{M}}(n-1)\|_1 &\leq \|A_{\mathcal{M}}(k)\|\|C_{\mathcal{M}}(n-1)\|_1,\\
P_N A_\infty(k)P_N C_\infty(n-1)P_N &= A_\infty(k)P_N C_\infty(n-1)P_N,\\
\|D_\infty(k,n)-P_N D_\infty(k,n)P_N\|_1 &\leq \|A_\infty(k)\|\|C_\infty(n-1)-P_N C_\infty(n)P_N\|_1.
\end{aligned}$$

$\square$

Lemma 3.4. *Assume (H.3.1) and $n\in\mathbb{Z}$, $N\in\mathbb{N}$, $k\in\mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}}$. Then*

$$\det[C_N(n)] = \sum_{j=1}^N c_j^2\kappa_j^{2n+2}\prod_{\substack{l,m=1\\ m>l}}^N(\kappa_l-\kappa_m)^2\prod_{r,s=1}^N(1-\kappa_r\kappa_s)^{-1}, \tag{3.16}$$

$$\det[D_N(n)] = \prod_{j=1}^N\frac{\kappa_j-k}{\kappa_j^{-1}-k}\ \det[C_N(n-1)]. \tag{3.17}$$

For $\mathcal{M}\in\mathbb{N}\cup\{\infty\}$,

$$\det{}_1[1_{\mathcal{M}}+C_{\mathcal{M}}(n)] = \sum_{I\in\mathcal{P}_{\mathcal{M}}} a_I x_I^n, \tag{3.18}$$

$$\det{}_1[1_{\mathcal{M}}+D_{\mathcal{M}}(k,n)] = \sum_{I\in\mathcal{P}_{\mathcal{M}}} a_I p_I(k) x_I^{n-1}, \tag{3.19}$$

where $\mathcal{P}_N$ is the power set of $\{1,\dots,N\}$, $\mathcal{P}_\infty$ is the set of all finite subsets of $\mathbb{N}$,

$$\begin{aligned}
&x_I=\prod_{j\in I}\kappa_j^2>0, \qquad p_I(k)=\prod_{j\in I}\frac{\kappa_j-k}{\kappa_j^{-1}-k},\\
&a_I=\prod_{j\in I}c_j^2\kappa_j^2\prod_{\substack{l,m\in I\\ m>l}}(\kappa_l-\kappa_m)^2\prod_{r,s\in I}(1-\kappa_r\kappa_s)^{-1}>0, \qquad I\subset\mathbb{N}.
\end{aligned} \tag{3.20}$$

In particular, this yields the monotonicity property

$$1<\det{}_1[1_{\mathcal{M}}+C_{\mathcal{M}}(n+1)]<\det{}_1[1_{\mathcal{M}}+C_{\mathcal{M}}(n)], \qquad \mathcal{M}\in\mathbb{N}\cup\{\infty\}. \tag{3.21}$$

Proof. (3.16) follows from [20], p.92 and (3.17) is then clear from (3.2) and (3.3). For $N\in\mathbb{N}$ we expand

$$\begin{aligned}
\det[1_N+C_N(n)] =&1+\det[C_N(n)]+\sum_{j_1=1}^N\det[C_N^{j_1}(n)]+\sum_{\substack{j_1,j_1=1\\ j_1<j_2}}\det[C_N^{j_1,j_2}(n)]\\
&+\ \cdots\ +\sum_{\substack{j_1,j_2,\dots,j_{N-1}=1\\ j_1<j_2<\dots<j_{N-1}}}\det[C_N^{j_1,j_2,\dots,j_{N-1}}(n)],
\end{aligned} \tag{3.22}$$

where $C_N^{j_1,j_2,\dots,j_k}(n)$ denotes the $(N-k)\times(N-k)$ matrix obtained by deleting the $j_1,\dots,j_k$ rows and columns of $C_N(n)$. Each of these matrices is of the same form as the original matrix $C_N(n)$, in particular, the analog of (3.16) applies to all $C_N^{j_1,\dots,j_k}(n)$. Since each summand in (3.22) corresponds to one element $I\in\mathcal{P}_N$, we proved (3.18) for $\mathcal{M} \in \mathbb{N}$. Since all terms in (3.22) are positive,

$$\det[1_N + C_N(n)] = \sum_{I\in\mathcal{P}_N} a_I x_I^n \leq \sum_{I\in\mathcal{P}_{N+1}} a_I x_I^n = \det[1_{N+1} + C_{N+1}(n)]$$
$$\leq \exp[\|C_\infty(n)\|_1], \tag{3.23}$$

and hence

$$\det{}_1[1_\infty + C_\infty(n)] = \lim_{N\to\infty} \det[1_N + C_N(n)] = \lim_{N\to\infty} \sum_{I\in\mathcal{P}_N} a_I x_I^n \tag{3.24}$$

proves (3.18). (3.21) is an immediate consequence of (3.16), $a_I > 0$ and $0 < x_I < 1$. (3.19) is proved similarly. (Even though the terms in (3.19) are not necessarily positive, the $p_I(k)$ are uniformly bounded with respect to I by a constant depending on k.) □

Next we define

(3.25)
$$a_\infty(n) = \frac{\{\det_1[1_\infty + C_\infty(n+1)]\det_1[1_\infty + C_\infty(n-1)]\}^{\frac{1}{2}}}{2\det_1[1_\infty + C_\infty(n)]}, \qquad n\in\mathbb{Z},$$

$$b_\infty(n) = \tfrac{1}{2}(k+k^{-1}) - \frac{k}{2}\frac{\det_1[1_\infty + D_\infty(k,n+1)]\det_1[1_\infty + C_\infty(n-1)]}{\det_1[1_\infty + D_\infty(k,n)]\det_1[1_\infty + C_\infty(n)]}$$
$$- \frac{1}{2k}\frac{\det_1[1_\infty + D_\infty(k,n-1)]\det_1[1_\infty + C_\infty(n)]}{\det_1[1_\infty + D_\infty(k,n)]\det_1[1_\infty + C_\infty(n-1)]}, \tag{3.26}$$
$$n\in\mathbb{Z},\ 0<|k|\leq\kappa_0,\ k\in\mathbb{R}.$$

We note that the conditions on k in (3.26) imply $\frac{\kappa_j - k}{\kappa_j^{-1}-k} \geq 0$ and hence $p_I \geq 0$ in (3.20) implying

$$\det{}_1[1_\mathcal{M} + D_\mathcal{M}(k,n)] \geq 1 \tag{3.27}$$

(cf. (3.19)).

Theorem 3.5. *Assume (H.3.1). Then for $n\in\mathbb{Z}$,*

$$a_N(n) \xrightarrow[N\to\infty]{} a_\infty(n), \qquad b_N(n) \xrightarrow[N\to\infty]{} b_\infty(n), \tag{3.28}$$

$$\tfrac{1}{2} \leq a_\infty(n) \leq 2^{-\frac{3}{2}}(\kappa_0 + \kappa_0^{-1}), \qquad |b_\infty(n)| \leq \tfrac{1}{2}(\kappa_0+\kappa_0^{-1}). \tag{3.29}$$

In addition, the operator H_∞ on $\ell^2(\mathbb{N})$ defined by

$$H_\infty = a_\infty S^+ + S^- a_\infty + b_\infty, \quad \mathfrak{D}(H_\infty) = \ell^2(\mathbb{Z}) \tag{3.30}$$

is self–adjoint and

$$\operatorname*{s-lim}_{N\to\infty} H_N = H_\infty, \tag{3.31}$$

or equivalently, H_N converges to H_∞ in the strong resolvent sense.

Proof. (3.28) follows from (3.5) and (3.8) and the fact that $\det_1[1_\mathcal{M}+D_\mathcal{M}(k,n)] \geq 1$ for all $n\in\mathbb{Z}$, $0<|k|<\kappa_0$, $k\in\mathbb{R}$, $\mathcal{M}\in\mathbb{N}\cup\{\infty\}$. Together with (2.13) and (2.14) this yields the upper bounds in (3.29). In order to prove $a_\infty(n) \geq \frac{1}{2}$ one considers

(3.32)
$$\begin{aligned} 4a_\infty(n)^2 &= \sum_{I\in\mathcal{P}_\infty} a_I x_I^{n+1} \sum_{K\in\mathcal{P}_\infty} a_K x_K^{n-1} \Big(\sum_{L\in\mathcal{P}_\infty} a_L x_L^n \Big)^{-2} \\ &= \sum_{I,K\in\mathcal{P}_\infty} a_I a_K (x_I x_K)^{n-1} x_I^2 \Big(\sum_{R,S\in\mathcal{P}_\infty} a_R a_S x_R^n x_S^n \Big)^{-1} \\ &= \tfrac{1}{2} \sum_{I,K\in\mathcal{P}_\infty} a_I a_K (x_I x_K)^{n-1} (x_I^2 + x_K^2) \Big(\sum_{R,S\in\mathcal{P}_\infty} a_R a_S x_R^n x_S^n \Big)^{-1} \\ &= 1 + \tfrac{1}{2} \sum_{I,K\in\mathcal{P}_\infty} a_I a_K (x_I x_K)^{n-1} (x_I - x_K)^2 \Big(\sum_{R,S\in\mathcal{P}_\infty} a_R a_S x_R^n x_S^n \Big)^{-1} \geq 1. \end{aligned}$$

(The reordering of the terms in (3.32) being justified due to absolute convergence of the sums involved.) (3.28) and (3.29) then prove (3.31). ☐

The convergence properties in Lemmas 3.3, 3.4 and Theorem 3.5 immediately imply the following result for the wave functions of H_∞.

Lemma 3.6. *Assume (H.3.1) and $k \in \mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}}$. Then*

(3.33)
$$f_{N,+}(k,n) \underset{N\to\infty}{\longrightarrow} f_{\infty,+}(k,n) = k^n \frac{\det_1[1_\infty + D_\infty(k,n)]}{\{\det_1[1_\infty + C_\infty(n)] \det_1[1_\infty + C_\infty(n-1)]\}^{\frac{1}{2}}},$$

$$\psi_{N,-}(k,n) \underset{N\to\infty}{\longrightarrow} \psi_{\infty,n}(k,n) = f_{\infty,+}(k^{-1},n) \tag{3.34}$$

pointwise with respect to n, in particular,

$$\psi_{N,j}(n) = c_j f_{N,+}(\kappa_j,n) \underset{N\to\infty}{\longrightarrow} \psi_{\infty,j}(n) = c_j f_{\infty,+}(\kappa_j,n) \tag{3.35}$$

pointwise and

$$\begin{aligned} H_\infty f_{\infty,+}(k) &= z f_{\infty,+}(k), \\ H_\infty \psi_{\infty,-}(k) &= z\psi_{\infty,-}(k), \end{aligned} \qquad z = \tfrac{1}{2}(k+k^{-1}), \tag{3.36}$$

$$H_\infty \psi_{\infty,j} = \lambda_j \psi_{\infty,j}, \qquad \lambda_j = \tfrac{1}{2}(\kappa_j + \kappa_j^{-1}) \tag{3.37}$$

in the weak sense.

Unfortunately, Lemma 3.6 sheds no light on whether or not $\psi_{\infty,j} \in \ell^2(\mathbb{Z})$ and $\psi_{\infty,j} \not\equiv 0$, i.e., whether or not $\lambda_j \in \sigma_p(H_\infty)$. In order to clarify this point we need to pursue a more sophisticated strategy. We introduce some notation first.

Let

$$\hat{\mathfrak{H}}_a = \ell^2(\mathbb{N}) \otimes \ell^2([a,\infty)), \qquad a \in \mathbb{Z} \tag{3.38}$$

and introduce the following identity operators

$$\begin{aligned} &1_N \text{ in } \mathbb{C}^N, \\ &1_\infty \text{ in } \ell^2(\mathbb{N}), \\ &\hat{1} \text{ in } \ell^2([a,\infty)), \\ &\hat{1}_N = 1_N \otimes \hat{1} \text{ in } \mathbb{C}^N \otimes \ell^2([a,\infty)), \\ &\hat{1}_\infty = 1_\infty \otimes \hat{1} \text{ in } \ell^2(\mathbb{N}) \otimes \ell^2([a,\infty)) = \hat{\mathfrak{H}}_a. \end{aligned} \tag{3.39}$$

Moreover, embed $\mathbb{C}^N$ and $\mathbb{C}^N \otimes \ell^2([a,\infty))$ into $\ell^2(\mathbb{N})$ and $\hat{\mathfrak{H}}_a$ in the obvious way and let $\hat{P}_N$ be the projection from $\hat{\mathfrak{H}}_a$ onto $\mathbb{C}^N \otimes \ell^2([a,\infty))$, i.e.,

$$\hat{\mathfrak{H}}_a = [\mathbb{C}^N \otimes \ell^2([a,\infty))] \oplus (\hat{1}_\infty - \hat{P}_N)\hat{\mathfrak{H}}_a, \qquad \hat{P}_N = \begin{pmatrix} \hat{1}_N & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.40}$$

and denote by $\hat{C}_N$ and $\hat{C}_\infty$ the operators in $\mathbb{C}^N \otimes \ell^2([a,\infty))$ and $\hat{\mathfrak{H}}_a$ induced by $C_N(\cdot)$ and $C_\infty(\cdot)$, i.e.,

$$\Big(\hat{C}_{\mathcal{M}} f\Big)_j(n) = \sum_{l=1}^{\mathcal{M}} c_j c_l \frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l} f_l(n), \qquad n \in \mathbb{Z},\ 1 \le j \le \mathcal{M},\ \mathcal{M} \in \mathbb{N} \cup \{\infty\}. \tag{3.41}$$

Lemma 3.7. *Assume (H.3.1). Then*

$$\hat{C}_\infty \ge 0, \tag{3.42}$$

$$\|\hat{C}_\infty\| < \infty, \tag{3.43}$$

$$\operatorname*{s-lim}_{N\to\infty} \begin{pmatrix} (\hat{1}_N + \hat{C}_N)^{-1} & 0 \\ 0 & 0 \end{pmatrix} = (\hat{1}_\infty + \hat{C}_\infty)^{-1}. \tag{3.44}$$

Proof. (3.42) is shown as in the proof of of Lemma 3.3. (3.43) follows from

$$\begin{aligned} \|\hat{C}_\infty f\|^2_{\hat{\mathfrak{H}}_a} &= \sum_{n=a}^\infty \sum_{j=1}^\infty \Big| \sum_{l=1}^\infty c_j c_l \frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l} f_l(n)\Big|^2 \\ &\le \sum_{n=a}^\infty \sum_{j=1}^\infty \sum_{l=1}^\infty c_j^2 c_l^2 \frac{(\kappa_j\kappa_l)^{2n+2}}{(1-\kappa_j\kappa_l)^2} \sum_{k=1}^\infty |f_k(n)|^2 \\ &\le \sum_{n=a}^\infty \Big(\sum_{j=1}^\infty c_j^2 \frac{\kappa_j^{2n+2}}{1-\kappa_j^2}\Big)^2 \sum_{k=1}^\infty |f_k(n)|^2 \le \Big(\sum_{j=1}^\infty c_j^2 \frac{\kappa_j^{2a+2}}{1-\kappa_j^2}\Big)^2 \|f\|^2_{\hat{\mathfrak{H}}_a}. \end{aligned} \tag{3.45}$$

Finally, to prove (3.44), we first note that $\|(\hat{1}_\infty + \hat{C}_\infty) f\|_{\hat{\mathfrak{H}}_a} \ge \|f\|_{\hat{\mathfrak{H}}_a}$, and hence it suffices to estimate

$$\begin{aligned} &\Big\|(\hat{1}_\infty + \hat{C}_\infty)\Big[(\hat{1}_\infty + \hat{C}_\infty)^{-1} g - \begin{pmatrix} (\hat{1}_N + \hat{C}_N)^{-1} & 0 \\ 0 & 0 \end{pmatrix} g\Big]\Big\|_{\hat{\mathfrak{H}}_a} \\ &\quad = \Big\| g - \begin{pmatrix} \hat{1}_N & 0 \\ 0 & 0 \end{pmatrix} g - \begin{pmatrix} 0 & 0 \\ (\hat{1}_\infty - \hat{P}_N)\hat{C}_\infty \hat{P}_N & 0 \end{pmatrix} g \Big\|_{\hat{\mathfrak{H}}_a} \\ &\quad \le \|g - \hat{P}_N g\|_{\hat{\mathfrak{H}}_a} + \|(\hat{1}_\infty - \hat{P}_N)\hat{C}_\infty \hat{P}_N f_N\|_{\hat{\mathfrak{H}}_a}, \end{aligned} \tag{3.46}$$

where we abbreviated $f_N = \left(\begin{smallmatrix} (\hat{1}_N + \hat{C}_N)^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right) g$ (note that $\|f_N\|_{\hat{\mathfrak{H}}_a} \le \|g\|_{\hat{\mathfrak{H}}_a}$). Since $\operatorname{s-lim}_{N\to\infty} \hat{P}_N = \hat{1}_\infty$, the first term on the right–hand–side of (3.46) tends to zero. As for the second term one estimates

$$\begin{aligned} \|(\hat{1}_\infty - \hat{P}_N)\hat{C}_\infty \hat{P}_N f_N\|^2_{\hat{\mathfrak{H}}_a} &= \sum_{j=N+1}^\infty \sum_{n=a}^\infty \Big| \sum_{l=1}^N c_j c_l \frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l} f_{N,l}(n)\Big|^2 \\ &\le \Big(\sum_{j=N+1}^\infty c_j^2 \frac{\kappa_j^{2n+2}}{1-\kappa_j^2}\Big)\Big(\sum_{l=1}^\infty c_l^2 \frac{\kappa_l^{2n+2}}{1-\kappa_l^2}\Big) \|f_N\|^2_{\hat{\mathfrak{H}}_a} \xrightarrow[N\to\infty]{} 0. \end{aligned} \tag{3.47}$$

□

Hence defining

$$\Psi_\infty(n) = (\{\psi_{\infty,j}(n)\}_{j\in\mathbb{N}})^T, \qquad \Psi^0_\infty(n) = (\{c_j\kappa_j^n\}_{j\in\mathbb{N}})^T,$$
$$\Psi_\infty(n) = \Big\{\frac{\det_1[1_\infty + C_\infty(n)]}{\det_1[1_\infty + C_\infty(n-1)]}\Big\}^{\frac{1}{2}}\Big[1_\infty + C_\infty(n)\Big]^{-1}\Psi^0_\infty(n), \tag{3.48}$$

one obtains the following result.

Theorem 3.8. *Assume (H.3.1) and let $a\in\mathbb{Z}$. Then*

$$\|\Psi_\infty - \Psi_N\|_{\hat{\mathfrak{H}}_a} \xrightarrow[N\to\infty]{} 0, \tag{3.49}$$

in particular,

$$\|\psi_{\infty,j} - \psi_{N,j}\|_{\ell^2([a,\infty))} \xrightarrow[N\to\infty]{} 0 \tag{3.50}$$

implying

$$\|\psi_{\infty,j}\|_2 \le 1, \qquad j\in\mathbb{N}. \tag{3.51}$$

Proof. By Lemma 3.3, the ratio of determinants in (2.17) converges to that in (3.48). Moreover, (3.11) shows that the convergence is uniform for $n\in[a,\infty)$. By the previous Lemma 3.7 one then concludes (3.49) and (3.50). (3.51) then follows from $\|\psi_{\infty,j}\|_{\ell^2([a,\infty))} = \lim_{N\to\infty}\|\psi_{N,j}\|_{\ell^2([a,\infty))} \le 1$, for all $a\in\mathbb{Z}$ by taking $a\to-\infty$. ☐

The fact that $\psi_{\infty,j}$ defined in (3.48) coincide with the ones in (3.35) is clear from the pointwise convergence in (3.35). It remains to prove that $\psi_{\infty,j}\not\equiv 0$. In order to achieve this it suffices to study the asymptotics of $\psi_{\infty,j}(n)$ as $n\to+\infty$.

Lemma 3.9. *Assume (H.3.1) and $k\in\mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}}$, $j\in\mathbb{N}$. Then*

$$\lim_{n\to+\infty} k^{-n} f_{\infty,+}(k,n) = 1, \tag{3.52}$$

$$\lim_{n\to+\infty} c_j^{-1}\kappa_j^{-n}\psi_{\infty,j}(n) = 1. \tag{3.53}$$

Proof. This is clear from (3.6), (3.9), (3.33) and (3.35). ☐

Finally we study the trace formula (2.18) in the limit $N\to\infty$.

Lemma 3.10. *Assume (H.3.1) and $n\in\mathbb{Z}$. Then*

$$b_\infty(n) = -\tfrac{1}{2}\sum_{j\in\mathbb{N}}(\kappa_j - \kappa_j^{-1})\psi_{\infty,j}(n)^2. \tag{3.54}$$

Proof. This follows from (2.18), $b_N(n) \xrightarrow[N\to\infty]{} b_\infty(n)$ in (3.28) and

$$\begin{aligned}
&\Big|\sum_{j=1}^N(\kappa_j - \kappa_j^{-1})\psi_{N,j}(n)^2 - \sum_{j=1}^\infty(\kappa_j - \kappa_j^{-1})\psi_{\infty,j}(n)^2\Big| \\
&\quad\le \tfrac{1}{2}(\kappa_0^{-1} - \kappa_0)\sum_{j=1}^\infty|\psi_{N,j}(n)^2 - \psi_{\infty,j}(n)^2| \\
&\quad\le \tfrac{1}{2}(\kappa_0^{-1} - \kappa_0)\|\Psi_N(n) + \Psi_\infty(n)\|_{\ell^2(\mathbb{N})}\|\Psi_N(n) - \Psi_\infty(n)\|_{\ell^2(\mathbb{N})}.
\end{aligned} \tag{3.55}$$

The first factor on the right–hand–side of (3.55) is bounded by (3.49) and the second tends to zero again by (3.49). ☐

This completes the technical convergence part under the general hypothesis (H.3.1). In the remainder of this section we assume the stronger hypothesis (H.3.2) motivated by scattering theory. In this case we will be able to prove convergence of the a_N and b_N in $\ell^1(\mathbb{Z})$–norm. To do this, we first have to get a handle on the behavior of various ratios of determinants as $n \to -\infty$.

Lemma 3.11. *Assume (H.3.2) and $k \in \mathbb{C}\backslash\overline{\{\kappa_j^{-1}\}_{j\in\mathbb{N}}}$. For all $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$, $l \leq \mathcal{M}$, $l \in \mathbb{N}$ and for all $n \in \mathbb{Z}$,*

$$\frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]} > x_{\mathbb{N}}, \tag{3.56}$$

$$\Big|\frac{\det_1[1_{\mathcal{M}} + D_{\mathcal{M}}(\kappa_l, n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]}\Big| \leq const.(l)\kappa_l^{-2n+2} \tag{3.57}$$

with the constant being independent of $\mathcal{M}$ and n. Furthermore, for all $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$, $n_\varepsilon \in \mathbb{Z}$ such that for all $\mathcal{M} > N_\varepsilon$, $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$ and all $n < n_\varepsilon$, $n \in \mathbb{Z}$,

$$\frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]} - x_{\mathbb{N}} < \varepsilon, \tag{3.58}$$

$$\Big|\frac{\det_1[1_{\mathcal{M}} + D_{\mathcal{M}}(k, n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]} - p_{\mathbb{N}}\Big| < \varepsilon, \tag{3.59}$$

where

$$x_{\mathbb{N}} = \prod_{j\in\mathbb{N}} \kappa_j^2 > 0, \qquad p_{\mathbb{N}}(k) = \prod_{j\in\mathbb{N}} \frac{\kappa_j - k}{\kappa_j^{-1} - k} \neq 0. \tag{3.60}$$

Proof. By (H.3.2), $x_{\mathbb{N}}$ and $p_{\mathbb{N}}(k)$ exist since $x_{\mathbb{N}} = \prod_{j\in\mathbb{N}}[1-(1-\kappa_j^2)]$, $p_{\mathbb{N}}(k) = \prod_{j\in\mathbb{N}}[1 - \frac{1-\kappa_j^2}{1+\kappa_j k}]$. (3.56) is obvious from (3.18) and $x_I > x_{\mathbb{N}}$ for $I \subsetneq \mathbb{N}$. In order to prove (3.57) we first note that $p_I(k)$ as given by (3.20) is equal to zero if $k = \kappa_l$ and $l \in I$. (3.20) also implies that $a_{I\cup\{l\}} \geq \tilde{c}_1(l)a_I$ with some constant $\tilde{c}_1(l) > 0$ independent of I. Similarly, $|p_I(\kappa_l)| \leq \tilde{c}_2(l)$ with another constant $\tilde{c}_2(l)$. Finally, observing $x_{I\cup\{l\}} = x_I\kappa_l^2$, we get

$$\begin{aligned}\Big|\frac{\det_1[1_{\mathcal{M}} + D_{\mathcal{M}}(\kappa_l, n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]}\Big| &= \Big|\Big(\sum_{K\in\mathcal{P}_{\mathcal{M}}} a_K x_K^{n-1}\Big)^{-1} \sum_{I\in\mathcal{P}_{\mathcal{M}}} a_I p_I(\kappa_l) x_I^{n-1}\Big| \\ &\leq \Big|\Big(\sum_{\substack{K\in\mathcal{P}_{\mathcal{M}}\\ l\notin K}} a_{K\cup\{l\}} x_{K\cup\{l\}}^{n-1}\Big)^{-1} \sum_{\substack{I\in\mathcal{P}_{\mathcal{M}}\\ l\notin I}} a_I p_I(\kappa_l) x_I^{n-1}\Big| \leq const.(l)\kappa_l^{-2n+2}.\end{aligned} \tag{3.61}$$

In order to prove (3.58), choose $N_\varepsilon \in \mathbb{N}$ such that

$$\Big|\Big(\prod_{j=1}^{\mathcal{M}} \kappa_j^2\Big) - x_{\mathbb{N}}\Big| < \frac{\varepsilon}{4} \qquad \text{for all } \mathcal{M} > N_\varepsilon. \tag{3.62}$$

By (3.18),

$$\Big|\frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]} - x_{\mathbb{N}}\Big| = \Big(\sum_{K\in\mathcal{P}_{\mathcal{M}}} a_K x_K^{n-1}\Big)^{-1} \sum_{I\in\mathcal{P}_{\mathcal{M}}} a_I x_I^{n-1}(x_I - x_{\mathbb{N}}).$$

Next we split $\mathcal{P}_{\mathcal{M}}$ into two disjoint parts,

$$\mathcal{P}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M},1,\varepsilon} \cup \mathcal{P}_{\mathcal{M},2,\varepsilon}, \qquad \begin{aligned} \mathcal{P}_{\mathcal{M},1,\varepsilon} &= \{I \in \mathcal{P}_{\mathcal{M}} \mid (x_I - x_{\mathbb{N}}) > \tfrac{\varepsilon}{2}\}, \\ \mathcal{P}_{\mathcal{M},2,\varepsilon} &= \{I \in \mathcal{P}_{\mathcal{M}} \mid (x_I - x_{\mathbb{N}}) \leq \tfrac{\varepsilon}{2}\} \end{aligned}$$

and estimate

$$\begin{aligned} &\Big(\sum_{K\in\mathcal{P}_{\mathcal{M}}} a_K x_K^{n-1}\Big)^{-1} \sum_{I\in\mathcal{P}_{\mathcal{M}}} a_I x_I^{n-1}(x_I - x_{\mathbb{N}}) \\ &\leq \frac{\varepsilon}{2} + \Big(\sum_{K\in\mathcal{P}_{\mathcal{M},2,\varepsilon}} a_K x_K^{n-1}\Big)^{-1} \sum_{I\in\mathcal{P}_{\mathcal{M},1,\varepsilon}} a_I x_I^{n-1} \\ &\leq \frac{\varepsilon}{2} + \Big(\sum_{K\in\mathcal{P}_{\mathcal{M},2,\varepsilon}} a_K \big[x_K(x_{\mathbb{N}} + \tfrac{\varepsilon}{2})^{-1}\big]^{n-1}\Big)^{-1} \sum_{I\in\mathcal{P}_{\mathcal{M},1,\varepsilon}} a_I \big[x_I(x_{\mathbb{N}} + \tfrac{\varepsilon}{2})^{-1}\big]^{n-1}. \end{aligned} \tag{3.63}$$

For $I \in \mathcal{P}_{\mathcal{M},1,\varepsilon}$, $[x_I(x_{\mathbb{N}} + \frac{\varepsilon}{2})^{-1}] > 1$ and hence $[x_I(x_{\mathbb{N}} + \frac{\varepsilon}{2})^{-1}]^{n-1} \to 0$ as $n \to -\infty$. Similarly, for $K \in \mathcal{P}_{\mathcal{M},2,\varepsilon}$, $[x_K(x_{\mathbb{N}} + \frac{\varepsilon}{2})^{-1}] \leq 1$ implies $[x_K(x_{\mathbb{N}} + \frac{\varepsilon}{2})^{-1}]^{n-1} \geq 1$ as $n \to -\infty$. Thus the denominator on the right–hand–side of (3.63) is bounded from below by a positive constant independently of $\mathcal{M}$ since $\mathcal{P}_{\mathcal{M}+1,2,\varepsilon} \supseteq \mathcal{P}_{\mathcal{M},2,\varepsilon}$. For the numerator in (3.63) one infers

$$\sum_{I\in\mathcal{P}_{\mathcal{M},1,\varepsilon}} a_I \big[x_I(x_{\mathbb{N}} + \tfrac{\varepsilon}{2})^{-1}\big]^{n-1} \leq \sum_{I\in\mathcal{P}_{\infty,1,\varepsilon}} a_I \big[x_I(x_{\mathbb{N}} + \tfrac{\varepsilon}{2})^{-1}\big]^{n-1} \xrightarrow[n\to-\infty]{} 0 \tag{3.64}$$

by the Weierstrass test since $\sum_{I\in\mathcal{P}_\infty} a_I = \det_1[1_\infty + C_\infty(0)] < \infty$ by Lemma 3.4. This proves (3.58). Since $x_I - x_{\mathbb{N}} \leq \delta$ implies $|p_I(k) - p_{\mathbb{N}}(k)| \leq const.(k)\delta$, (3.59) is proven along the same lines. □

Lemma 3.12. *Assume (H.3.2), $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$, $j \in \mathbb{N}$, $j \leq \mathcal{M}$. Then for all $n \in \mathbb{Z}$,*

$$\psi_{\mathcal{M},j}(n) \leq const.(j)\kappa_j^{|n|}, \tag{3.65}$$

with the constant being independent of $\mathcal{M}$ and n.

Proof. This is obvious from (3.33) and (3.35) (resp. (2.22) and (2.27) for finite $\mathcal{M}$) and Lemmas 3.3 (respectively 3.11) for $n \to +\infty$ (respectively $n \to -\infty$). □

This then allows us to prove

Lemma 3.13. *Assume (H.3.2). Then for all $\mathcal{M} \in \mathbb{N} \cup \{\infty\}$ and for all $n \in \mathbb{Z}$*

$$\|a_{\mathcal{M}} - \tfrac{1}{2}\|_1 \leq const., \qquad \|b_{\mathcal{M}}\|_1 \leq const., \tag{3.66}$$

with constants being independent of $\mathcal{M}$. Moreover, for all $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $\mathcal{M} \in \mathbb{N} \cup \{\infty\}$,

$$\|a_{\mathcal{M}} - \tfrac{1}{2}\|_{\ell^1((-\infty,-n_\varepsilon])} < \varepsilon, \qquad \|a_{\mathcal{M}} - \tfrac{1}{2}\|_{\ell^1([n_\varepsilon,\infty))} < \varepsilon, \tag{3.67}$$

$$\|b_{\mathcal{M}}\|_{\ell^1((-\infty,-n_\varepsilon])} < \varepsilon, \qquad \|b_{\mathcal{M}}\|_{\ell^1([n_\varepsilon,\infty))} < \varepsilon. \tag{3.68}$$

Proof. In order to estimate $a_{\mathcal{M}}$ we note that $a_{\mathcal{M}}(n) > \frac{1}{2}$ by (3.29) and $\frac{\det_1[1_{\mathcal{M}}+C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}}+C_{\mathcal{M}}(n+1)]} > 1$. Thus

$$\begin{aligned}(3.69)\quad \sum_{n=n_1}^{n_2} |a_{\mathcal{M}}(n) - \tfrac{1}{2}| &\le \sum_{n=n_1}^{n_2} \Big\{ \frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n+1)]} \Big\}^{\frac{1}{2}} (a_{\mathcal{M}}(n) - \tfrac{1}{2}) \\ &= \frac{1}{2} \sum_{n=n_1}^{n_2} \Big\{ \Big\{ \frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n-1)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]} \Big\}^{\frac{1}{2}} - \Big\{ \frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n+1)]} \Big\}^{\frac{1}{2}} \Big\} \\ &= \frac{1}{2} \Big\{ \Big\{ \frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n_1-1)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n_1)]} \Big\}^{\frac{1}{2}} - \Big\{ \frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n_2)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n_2+1)]} \Big\}^{\frac{1}{2}} \Big\}, \qquad n_1, n_2 \in \mathbb{Z}.\end{aligned}$$

By Lemma 3.3,

$$\frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n+1)]} \xrightarrow[n\to+\infty]{} 1$$

uniformly in $\mathcal{M}$ and by Lemma 3.11

$$x_{\mathbb{N}}^{-1} = \prod_{j\in\mathbb{N}} \kappa_j^{-2} > \frac{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n)]}{\det_1[1_{\mathcal{M}} + C_{\mathcal{M}}(n+1)]} > (x_{\mathbb{N}} + \varepsilon)^{-1}$$

uniformly in $\mathcal{M} > N_\varepsilon$ for some N_ε, which proves the first part of (3.66) and both statements in (3.67) for all $\mathcal{M}$ sufficiently large. But for any fixed finite $\mathcal{M}$ (3.67) clearly holds as well.

In order to estimate the norm of $b_{\mathcal{M}}$ we use the trace formula (2.18), respectively (3.54),

$$\begin{aligned}(3.70)\qquad \sum_{n=n_1}^{n_2} |b_{\mathcal{M}}(n)| &\le \tfrac{1}{2} \sum_{n=n_1}^{n_2} \sum_{j=1}^{\infty} (|\kappa_j^{-1}| - |\kappa_j|) \psi_{\mathcal{M},j}(n)^2 \\ &\le \tfrac{1}{2}(\kappa_0^{-1} - \kappa_0) \sum_{j=1}^{\tilde{N}} \sum_{n=n_1}^{n_2} \psi_{\mathcal{M},j}(n)^2 + \tfrac{1}{2} \sum_{j=\tilde{N}+1}^{\mathcal{M}} (|\kappa_j^{-1}| - |\kappa_j|), \\ &\qquad\qquad n_1, n_2 \in \mathbb{Z},\ \tilde{N} \in \mathbb{N}.\end{aligned}$$

The first equality in (3.70) together with $\|\psi_{M,j}\|_2 \le 1$ (cf. (3.51)) proves the second part of (3.66). The last sum in (3.70) can be made arbitrarily small by choosing $\tilde{N}$ large enough, thus leaving just a finite sum over the $\ell^2([n_1, n_2])$–norm of $\psi_{\mathcal{M},j}(n)$. Together with Lemma 3.12 this completes the proof. □

Theorem 3.14. *Assume (H.3.2). Then for all* $j \in \mathbb{N}$

$$(3.71)\qquad \|\psi_{N,j} - \psi_{\infty,j}\|_2 \xrightarrow[N\to\infty]{} 0, \qquad \|\psi_{\infty,j}\|_2 = 1,$$

$$(3.72)\qquad \|a_N - a_\infty\|_1 \xrightarrow[N\to\infty]{} 0, \qquad \|b_N - b_\infty\|_1 \xrightarrow[N\to\infty]{} 0.$$

$$(3.73)\qquad \sum_{n\in\mathbb{Z}} b_\infty(n) = -\tfrac{1}{2} \sum_{j\in\mathbb{N}} (\kappa_j - \kappa_j^{-1}).$$

Proof. (3.71) follows from

$$(3.74)\quad \|\psi_{N,j} - \psi_{\infty,j}\|_2^2 \le \sum_{n=-\infty}^{a-1} (\psi_{N,j}(n)^2 + \psi_{\infty,j}(n)^2) + \|\psi_{N,j} - \psi_{\infty,j}\|^2_{\ell^2([a,\infty))},$$

Theorem 3.8, and Lemma 3.12. In order to prove (3.72) we use

(3.75)
$$\|a_N - a_\infty\|_1 \le \sum_{n=-\infty}^{n_1} \Big(\big|a_N(n) - \tfrac{1}{2}\big| + \big|a_\infty(n) - \tfrac{1}{2}\big| \Big) + \sum_{n=n_1+1}^{n_2} \big|a_N(n) - a_\infty(n)\big|$$
$$+ \sum_{n=n_2+1}^{\infty} \Big(\big|a_N(n) - \tfrac{1}{2}\big| + \big|a_\infty(n) - \frac{1}{2}\big| \Big), \qquad n_1, n_2 \in \mathbb{Z},$$

the pointwise convergence (Theorem 3.5), and Lemma 3.13. $\|b_N - b_\infty\|_1$ can be estimated analogously. (3.73) follows from (3.54) and (3.71). □

This yields

Theorem 3.15. *Assume (H.3.2). Then H_N converges to H_∞ in trace norm resolvent sense,*

$$\|(H_N - z)^{-1} - (H_\infty - z)^{-1}\|_1 \xrightarrow[N\to\infty]{} 0, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.76}$$

Proof. This follows immediately from

(3.77)
$$\|(H_N - z)^{-1} - (H_\infty - z)^{-1}\|_1 \le \|(H_N - z)^{-1}\| \|H_N - H_\infty\|_1 \|(H_\infty - z)^{-1}\|$$
(3.78)
$$\le \|(H_N - z)^{-1}\| \|(H_\infty - z)^{-1}\| \{2\|a_N - a_\infty\|_1 + \|b_N - b_\infty\|_1\} \xrightarrow[N\to\infty]{} 0.$$

(We emphasize that H_N and H_∞ are not trace class separately, but their difference $H_N - H_\infty$ is.) □

Lemma 3.11 also implies

Lemma 3.16. *Assume (H.3.2) and $k \in \mathbb{C}\backslash\{\pm 1\}$, $|k| = 1$, $n \in \mathbb{Z}$. Then*

$$f_{\infty,+}(k,n) = \begin{cases} k^n, & n \to +\infty \\ T_\infty(k)^{-1} k^n, & n \to -\infty, \end{cases}$$
$$\psi_{\infty,-}(k,n) = \begin{cases} k^{-n}, & n \to +\infty \\ T_\infty(k) k^{-n}, & n \to -\infty \end{cases} \tag{3.79}$$

with transmission coefficient

$$T_\infty(k) = \prod_{j=1}^{\infty} \operatorname{sgn}(\kappa_j) \frac{1 - k\kappa_j}{\kappa_j - k} = \prod_{j=1}^{\infty} \frac{\kappa_j^{-1} - k}{\kappa_j - k}. \tag{3.80}$$

Moreover, H_∞ is reflectionless, i.e.,

$$R_\infty^{r,l}(k) = 0. \tag{3.81}$$

Proof. It suffices to combine Lemma 3.3 (for $n \to +\infty$), Lemma 3.11 (for $n \to -\infty$), and Lemma 3.6. □

4. Spectral properties of H_∞

This section describes our principal results concerning spectral properties of the limit operator H_∞. As in Section 3 we distinguish two cases governed by hypotheses (H.3.1) and (H.3.2), respectively. We start by assuming (H.3.1).

Theorem 4.1. *Assume (H.3.1). Then H_∞ defined in $\ell^2(\mathbb{Z})$ by*

$$H_\infty = a_\infty S^+ + S^- a_\infty + b_\infty, \qquad \mathfrak{D}(H_\infty) = \ell^2(\mathbb{Z}) \tag{4.1}$$

is self–adjoint, and

$$\sigma_{ess}(H_\infty) = [-1,1] \cup \{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}'_{j\in\mathbb{N}} \tag{4.2}$$

(here Σ' denotes the derived set of $\Sigma \in \mathbb{R}$, i.e., the set of accumulation points of Σ),

$$\sigma_{ac}(H_\infty) = [-1,1], \tag{4.3}$$

$$\{\sigma_p(H_\infty) \cup \sigma_{sc}(H_\infty)\} \cap (-1,1) = \emptyset, \tag{4.4}$$

$$\{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}} \subseteq \sigma_p(H_\infty) \subseteq \overline{\{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}}}. \tag{4.5}$$

The spectral multiplicity of H_∞ on $(-1,1)$ is two while $\sigma(H_\infty)\backslash[-1,1]$ has multiplicity one. In addition,

$$0 \not\equiv \psi_{\infty,j} \in \ell^2(\mathbb{Z}), \qquad H_\infty \psi_{\infty,j} = \tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\psi_{\infty,j}, \qquad j \in \mathbb{N} \tag{4.6}$$

and

$$H_\infty f_{\infty,+}(k^{\pm 1}) = \tfrac{1}{2}(k + k^{-1}) f_{\infty,+}(k^{\pm 1}), \qquad k \in \mathbb{C}\backslash\overline{\{\kappa_j^{\pm 1}\}_{j\in\mathbb{N}}} \tag{4.7}$$

in the weak sense. If $\{\frac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}})$ is a discrete subset of $(\infty,-1) \cup (1,\infty)$ (i.e., if ± 1 are its only limit points), then

$$\sigma_{sc}(H_\infty) = \emptyset, \tag{4.8}$$

$$\sigma(H_\infty) \cap [(-\infty,-1) \cup (1,\infty)] = \sigma_d(H_\infty) = \{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}}. \tag{4.9}$$

More generally, if $\{\kappa_j\}'_{j\in\mathbb{N}}$ is countable, then (4.8) holds.

Proof. We shall use Weyl m–function techniques to prove Theorem 4.1. In the notation of [13], the Weyl m–functions associated with H_∞ are given by

$$m_{\infty,\pm}(z) = -f_{\infty,+}(k^{\pm 1}, 1)[a_\infty(0) f_{\infty,+}(k^{\pm 1}, 0)]^{-1}, \qquad z = \tfrac{1}{2}(k + k^{-1}), \quad |k| < 1. \tag{4.10}$$

($m_{\infty,\pm}(z)$ are unique since H_∞ is in the limit point case at $\pm\infty$ (i.e., self–adjoint) by Theorem 3.5.) (4.10) is obtained from the corresponding Weyl m–functions $m_{N,\pm}(z)$ of H_N by

$$m_{\infty,\pm}(z) = \lim_{N\to\infty} m_{N,\pm}(z), \qquad z \in \mathbb{C}\backslash\mathbb{R} \tag{4.11}$$

with uniform convergence for z in compact subsets of $\mathbb{C}\backslash\mathbb{R}$. (4.11) is clear for $m_{\infty,+}(z)$ since $f_{\infty,+}(k,.)$, $f_{N,+}(k,.) \in \ell^2((a,\infty))$, $a \in \mathbb{Z}$. In order to obtain the corresponding result (4.11) for $m_{\infty,-}(z)$ one utilizes $\psi_{N,-}(k,.) = f_{N,+}(k^{-1},.) \in \ell^2((-\infty,a))$, $N \in \mathbb{N}$, $a \in \mathbb{Z}$ and the fact that

$$\psi_{N,-} = const.(z)(H^D_{N,-} - z)^{-1}\delta_{n,1} \in \ell^2((-\infty,a)), \qquad a \in \mathbb{Z} \tag{4.12}$$

($H^D_{N,-}$ the operator H_N in $\ell^2((-\infty,1))$ with a Dirichlet boundary condition at $n=1$) together with strong resolvent convergence of H_N to H_∞ (cf. Theorem 3.5). Combining (2.19), (3.33) and (4.10) yields the fundamental property

$$\overline{m_{\infty,+}(\lambda+i0)} = m_{\infty,-}(\lambda+i0), \qquad \lambda\in(-1,1), \tag{4.13}$$

where we used the obvious notation, $m_\pm(\lambda+i0)=\lim_{\varepsilon\downarrow 0} m_\pm(\lambda+i\varepsilon)$ in connection with the branch cut $k=z+(z^2-1)^{\frac{1}{2}}$, $|k|\le 1$ approaching $|k|=1$ nontangentially from inside the unit k–disk. The corresponding Weyl $M-$ matrix $M_\infty(z)$ and (self–adjoint and right–continuous) spectral matrix $\rho_\infty(\lambda)$ of H_∞ then read

$$\begin{aligned} M_\infty(z) = {} & \frac{1}{a^2_\infty(0)[m_{\infty,+}(z)-m_{\infty,-}]} \\ & \times \begin{pmatrix} 1 & -a_\infty(0)m_{\infty,+}(z) \\ -a_{\infty,+}(0)m_{\infty,+}(z) & a^2_\infty(0)m_{\infty,+}(z)m_{\infty,-}(z)\end{pmatrix}, \\ & z\in\mathbb{C}\backslash\sigma(H_\infty), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \rho_{\infty,p,q}(\lambda)-\rho_{\infty,p,q}(\mu) = -\frac{1}{\pi}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\mu+\delta}^{\lambda+\delta} d\nu\, Im[M_{\infty,p,q}(\nu+i\varepsilon)], \\ \lambda,\mu\in\mathbb{R},\ 1\le p,q\le 2. \end{aligned} \tag{4.15}$$

One computes

$$\begin{aligned} m_{\infty,+}(\lambda+i0)-m_{\infty,-}(\lambda+i0) &= \frac{W(f_{\infty,+}(k),f_{\infty,+}(k^{-1}))(0)}{a_\infty(0)^2 f_{\infty,+}(k,0)f_{\infty,+}(k^{-1},0)} \\ = \lim_{n\to+\infty}\frac{W(f_{\infty,+}(k),f_{\infty,+}(k^{-1}))(n)}{a_\infty(0)^2 f_{\infty,+}(k,0)f_{\infty,+}(k^{-1},0)} &= \frac{1}{2}\frac{k-k^{-1}}{a_\infty(0)^2 f_{\infty,+}(k,0)f_{\infty,+}(k^{-1},0)} \\ &\ne 0, \qquad \lambda=\tfrac{1}{2}(k+k^{-1})\in(-1,1), \end{aligned} \tag{4.16}$$

where $W(f,g)(n)=a_\infty(n)[f(n)g(n+1)-f(n+1)g(n)]$ denotes the (modified) Wronskian of f and g. Given these preliminaries we can now follow the strategy of proof in Theorem 5.9 in [11]. (4.6) and (4.7) follow from (3.51), (3.53) and the strong resolvent convergence of H_N to H_∞. Applying Lemma 5.8 of [11] yields that

$$\lim_{\varepsilon\downarrow 0} m_{\infty,-}(\lambda+i\varepsilon) \text{ exists and is real–valued for a.e. } \lambda\in\mathbb{R}\backslash[-1,1]. \tag{4.17}$$

Since $m_{\infty,+}(z)$ also shares property (4.17), one concludes that

$$\sigma_{ac}(H_\infty)\cap[(-\infty,-1)\cup(1,\infty)]=\emptyset. \tag{4.18}$$

Moreover, (4.13), (4.15), and (4.16) show that $\rho_\infty(\lambda)$ has rank two on $(-1,1)$ implying

$$(-1,1)\in\sigma_{ac}(H_\infty) \tag{4.19}$$

and spectral multiplicity two of H_∞ in $(-1,1)$. Together with (4.6) and the strong resolvent convergence of H_N to H_∞ this proves (4.2)—(4.5). (4.8) and (4.9) are just special cases and multiplicity one of $\sigma(H_\infty)\backslash[-1,1]$ follows from simplicity of $\sigma_p(H_\infty)$ (since H_∞ is in the limit point case at $\pm\infty$) and of $\sigma_{sc}(H_\infty)$, if any (cf. [14], [15]). ☐

The analogous result in the scattering case governed by hypothesis (H.3.2) then reads as follows.

Theorem 4.2. *Assume (H.3.2). Then*

$$\sigma_{ess}(H_\infty) = \sigma_{ac}(H_\infty) = [-1,1], \qquad \sigma_p(H_\infty) \cap (-1,1) = \sigma_{sc}(H_\infty) = \emptyset, \tag{4.20}$$

$$\sigma_d(H_\infty) = \{\tfrac{1}{2}(\kappa_j + \kappa_j^{-1})\}_{j\in\mathbb{N}}. \tag{4.21}$$

H_∞ has spectral multiplicity two on $(-1,1)$, $\lambda_j = \frac{1}{2}(\kappa_j + \kappa_j^{-1})$, $j \in \mathbb{N}$ are simple eigenvalues of H_∞ with (normalized) eigenfunctions $\psi_{\infty,j}$

$$\|\psi_{\infty,j}\|_2 = 1, \qquad H_\infty \psi_{\infty,j} = \lambda_j \psi_{\infty,j}, \qquad j \in \mathbb{N}. \tag{4.22}$$

The weak solutions $f_{\infty,+}(k,n)$, $\psi_{\infty,-}(k,n) = f_{\infty,+}(k^{-1},n)$ of H_∞ satisfy

$$H_\infty f_{\infty,+}(k^{\pm 1}) = \tfrac{1}{2}(k + k^{-1}) f_{\infty,+}(k^{\pm 1}), \qquad k \in \mathbb{C}\backslash\{+1,-1\},\ |k| \le 1 \tag{4.23}$$

and

$$f_{\infty,+}(k,n) = \begin{cases} k^n, & n \to +\infty, \\ T_\infty(k)^{-1} k^n, & n \to -\infty, \end{cases}$$

$$\psi_{\infty,-}(k,n) = \begin{cases} k^{-n}, & n \to +\infty, \\ T_\infty(k) k^{-n}, & n \to -\infty, \end{cases} \qquad k \in \mathbb{C}\backslash\{+1,-1\},\ |k| \le 1, \tag{4.24}$$

where the transmission coefficient $T_\infty(k)$ of H_∞ holomorphically extends to

$$T_\infty(k) = \prod_{j=1}^{\infty} |\kappa_j| \frac{\kappa_j^{-1} - k}{\kappa_j - k}, \qquad k \in \mathbb{C}\backslash\overline{\{\kappa_j\}_{j\in\mathbb{N}}}. \tag{4.25}$$

The wave operators

$$\Omega_\pm(H_{\mathcal{M}}, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_{\mathcal{M}}} e^{-itH_0}, \qquad \mathcal{M} \in \mathbb{N} \cup \{\infty\} \tag{4.26}$$

exist in $\ell^2(\mathbb{Z})$ and are strongly asymptotically complete, i.e.,

$$\operatorname{Ran}\big(\Omega_\pm(H_{\mathcal{M}}, H_0)\big) = \mathfrak{H}_{ac}(H_{\mathcal{M}}) = \operatorname{Ran}\big(E_{H_{\mathcal{M}}}((0,\infty))\big), \qquad \mathcal{M} \in \mathbb{N} \cup \{\infty\}. \tag{4.27}$$

(Here $\mathfrak{H}_{ac}(\cdot)$, $E_H(\cdot)$ denote the absolutely continuous spectral subspace and the family of spectral projections of a self–adjoint operator H.) The scattering operators in $\ell^2(\mathbb{Z})$,

$$S(H_{\mathcal{M}}, H_0) = \Omega_+(H_{\mathcal{M}}, H_0)^* \Omega_-(H_{\mathcal{M}}, H_0), \qquad \mathcal{M} \in \mathbb{N} \cup \{\infty\} \tag{4.28}$$

are unitary, and $\Omega_\pm(H_N, H_0)$ and $S(H_N, H_0)$ are strongly continuous as $N \to \infty$,

$$\operatorname*{s-lim}_{N\to\infty} \Omega_\pm(H_N, H_0) = \Omega_\pm(H_\infty, H_0), \qquad \operatorname*{s-lim}_{N\to\infty} S(H_N, H_0) = S(H_\infty, H_0). \tag{4.29}$$

In addition, the fibers $S_N(\lambda)$, $\lambda \in (-1,1)$ in $\mathbb{C}^2$ of $S(H_N, H_0)$ converge pointwise to the fibers $S_\infty(\lambda)$ of $S(H_\infty, H_0)$,

$$\lim_{N\to\infty} \|S_N(\lambda) - S_\infty(\lambda)\| = 0, \qquad \lambda \in (-1,1). \tag{4.30}$$

In particular, H_∞ is reflectionless, i.e.,

$$S_\infty(\lambda) = \begin{pmatrix} T_\infty(\lambda) & 0 \\ 0 & T_\infty(\lambda) \end{pmatrix}, \qquad R^{r,l}(\lambda) = 0, \qquad \lambda \in (-1,1). \tag{4.31}$$

Proof. The spectral properties (4.20)—(4.23) are a special case of Theorem 4.1; (4.24) and (4.25) have been discussed in Lemma 3.16. Trace norm resolvent convergence of H_N to H_∞ (cf. Theorem 3.15) then yields continuity of the wave and scattering operators along the lines of [2], [21], p.27, 387. ∎

Remark 4.3. Under the additional condition $\{c_j^2(1 \mp \kappa_j)^{-2}\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$ or $\{c_j(1 \mp \kappa_j)^{-1}(1-|\kappa_j|)^{-\frac{1}{2}}\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$ one can prove that there is no square summable solution ψ_∞ of $H_\infty\psi_\infty = \pm\psi_\infty$, i.e., either one of these hypotheses guarantees that $\pm 1 \notin \sigma_p(H_\infty)$.

5. A new class of Toda soliton solutions

In our final section we return to the Toda lattice equations (2.29) and construct a new class of Toda soliton solutions obtained from N–soliton solutions as $N \to \infty$.

Assuming (H.3.1) in this section, we introduce the substitution (cf. (2.30))

$$c_j \to c_j(t) = c_j e^{\beta_j t}, \qquad \beta_j = \tfrac{1}{2}(\kappa_j - \kappa_j^{-1}),\ (j,t) \in \mathbb{N}\times\mathbb{R}, \tag{5.1}$$

which renders all quantities in Sections 3 and 4 t–dependent. In obvious notation we denote the resulting objects by $a_\infty(t,n)$, $b_\infty(t,n)$, $H_\infty(t)$, $\psi_{\infty,j}(t,n)$, $C_\infty(t,n)$, $C_N(t,n)$, etc.. Our main result then reads as follows.

Theorem 5.1. *Assume (H.3.1). Then $H_\infty(t)$ is unitary equivalent to $H_\infty(0)$ for all $t\in\mathbb{R}$ and $(a_\infty(t,n), b_\infty(t,n))$ satisfies the Toda lattice equations (2.29), i.e.,*

$$\begin{aligned} &\tfrac{d}{dt}a_\infty(t,n) = a_\infty(t,n)[b_\infty(t,n) - b_\infty(t,n+1)],\\ &\tfrac{d}{dt}b_\infty(t,n) = 2[a_\infty(t,n-1)^2 - a_\infty(t,n)^2], \qquad (t,n)\in\mathbb{R}\times\mathbb{Z}. \end{aligned} \tag{5.2}$$

Proof. The standard Lax representation of (2.29) proves unitary equivalence of $H_\infty(t)$ and $H_\infty(0)$, $t\in\mathbb{R}$. (5.2) will follow from the results of Sections 2 and 3 if we can prove that

$$\lim_{N\to\infty} \tfrac{d}{dt}\det[1_N + C_N(t,n)] = \tfrac{d}{dt}\det{}_1[1_\infty + C_\infty(t,n)], \qquad (t,n)\in\mathbb{R}\times\mathbb{Z}. \tag{5.3}$$

Indeed, (5.3) together with the obvious convergence of $\det[1_N + C_N(t,n)] \xrightarrow[N\to\infty]{} \det_1[1_\infty + C_\infty(t,n)]$, $(t,n)\in\mathbb{R}\times\mathbb{Z}$ then yields

$$\tfrac{d^m}{dt^m}a_N(t,n) \xrightarrow[N\to\infty]{} \tfrac{d^m}{dt^m}a_\infty(t,n), \tag{5.4}$$

$$\tfrac{d^m}{dt^m}[1_N + C_N(t,n)]^{-1} \xrightarrow[N\to\infty]{s} \tfrac{d^m}{dt^m}[1_\infty + C_\infty(t,n)]^{-1}, \tag{5.5}$$

$$\|\tfrac{d^m}{dt^m}\Psi_N(t,n) - \tfrac{d^m}{dt^m}\Psi_\infty(t,n)\|_{\ell^2(\mathbb{N})} \xrightarrow[N\to\infty]{} 0, \tag{5.6}$$

$$\tfrac{d^m}{dt^m}b_N(t,n) \xrightarrow[N\to\infty]{} \tfrac{d^m}{dt^m}b_\infty(t,n) \tag{5.7}$$

for all $(t,n)\in\mathbb{R}\times\mathbb{Z}$, $m=0,1$ and hence (5.2) using the fact that $(a_N(t,n), b_N(t,n))$ satisfy (2.29). In order to prove (5.3) one expands

$$\begin{aligned} \tfrac{d}{dt}\det[1_N + C_N(t,n)] &= \sum_{j=1}^{N} \det[(1_N + \widetilde{C_N(t,n)})^{(j)}]\\ &= \det[1_N + C_N(t,n)]\mathrm{Tr}\big\{[1_N + C_N(t,n)]^{-1}[B_N C_N(t,n) + C_N(t,n)B_N]\big\}, \end{aligned} \tag{5.8}$$

where $(1_N + \widetilde{C_N(t,n)})^{(j)}$ results by replacing the j–th column by its derivative and

$$B_N = \{\beta_j\delta_{j,l}\}_{j,l=1}^N, \tag{5.9}$$

$$\|B_N\| \le \tfrac{1}{2}(\kappa_0^{-1} - \kappa_0). \tag{5.10}$$

Using $\|[1_N + C_N(t,n)]^{-1}\| \le 1$ one infers from (5.8),

$$|\tfrac{d}{dt}\det[1_N + C_N(t,n)]| \le (\kappa_0^{-1} - \kappa_0)\det[1_N + C_N(t,n)]\|C_N(t,n)\|_1. \tag{5.11}$$

In particular, (5.11) extends to $N = \infty$ and

(5.12)

$$\begin{aligned}
&\big|\tfrac{d}{dt}\det[1_N + C_N(t,n)] - \tfrac{d}{dt}\det_1[1_\infty + C_\infty(t,n)]\big| \\
&\le \det[1_N + C_N(t,n)]\Big|\sum_{j=1}^{\infty}\Big\{\Big(\begin{pmatrix}[1_N + C_N(t,n)]^{-1} & 0\\ 0 & 0\end{pmatrix} - [1_\infty + C_\infty(t,n)]^{-1}\Big)\phi^{(j)}(n)\Big\}_j\Big| \\
&\quad + \big|\det[1_N + C_N(t,n)] - \det_1[1_\infty + C_\infty(t,n)]\big| \\
&\quad \times \operatorname{Tr}\big\{[1_\infty + C_\infty(t,n)]^{-1}[B_\infty C_\infty(t,n) + C_\infty(t,n)B_\infty]\big\},
\end{aligned}$$

where

$$\phi^{(j)}(n) = \Big(\Big((\beta_j + \beta_l)c_j(t)c_l(t)\frac{(\kappa_j\kappa_l)^{n+1}}{1-\kappa_j\kappa_l}\Big)_{l=1}^{\infty}\Big)^T \in \ell^2(\mathbb{N}), \qquad j \in \mathbb{N}.$$

While the second term in (5.12) clearly tends to zero as $N \to \infty$ the first can be estimated by

$$\begin{aligned}
(5.13)\quad &\tilde c(t,n)\Big\{\Big|\sum_{j=1}^{N_1}\Big\{\Big(\begin{pmatrix}[1_N + C_N(t,n)]^{-1} & 0\\ 0 & 0\end{pmatrix} - [1_\infty + C_\infty(t,n)]^{-1}\Big)\phi^{(j)}(n)\Big\}_j\Big| \\
&\quad + \Big|\sum_{j=N_1+1}^{\infty}\Big\{\Big(\begin{pmatrix}[1_N + C_N(t,n)]^{-1} & 0\\ 0 & 0\end{pmatrix} - [1_\infty + C_\infty(t,n)]^{-1}\Big)\phi^{(j)}(n)\Big\}_j\Big|
\end{aligned}$$

for some constant $\tilde c(t,n) > 0$ since $\det[1_N + C_N(t,n)]$ is bounded with respect to N. Next, choosing $N_1 \in \mathbb{N}$ sufficiently large, the second term in (5.13) can be made arbitrarily small. Since for a given N_1 the first term in (5.13) converges to zero as $N \to \infty$ by the strong convergence of

$$\begin{pmatrix}[1_N + C_N(t,n)]^{-1} & 0\\ 0 & 0\end{pmatrix} \xrightarrow[N\to\infty]{s} [1_\infty + C_\infty(t,n)]^{-1},$$

the expression in (5.13) and hence in (5.12) tends to zero as $N \to \infty$. Since the whole t–dependence comes from the constants $c_j(t) = c_j e^{\beta_j t}$, and $e^{-\frac{1}{2}(\kappa_0^{-1}-\kappa_0)s}c_j(t) \le c_j(t+s) = c_j(t)e^{\beta_j s} \le e^{\frac{1}{2}(\kappa_0^{-1}-\kappa_0)s}c_j(t)$, $\sum_{j\in\mathbb{N}}\frac{c_j(t)^2}{1-\kappa_j^2}$ converges uniformly with respect to t for t in compact intervals. Thus $\frac{d}{dt}\det[1_N + C_N(t,n)]$ converges locally uniformly in t and hence (5.3) follows. ☐

Since Theorem 5.1 trivially extends to all equations of the Toda hierarchy we omit further details at this point. Moreover, utilizing the commutation formalism developed in [10] (see also [3]), all results of this paper can be transferred to the modified Toda lattice or Kac–van Moerbeke lattice in a straightforward manner.

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