Infinite dimensional SRB measures

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Abstract

We review the basic steps leading to the construction of a Sinai-Ruelle-Bowen (SRB) measure for an infinite lattice of weakly coupled expanding circle maps, and we show that this measure has exponential decay of space-time correlations. First, using the Perron-Frobenius operator, one connects the dynamical system of coupled maps on a d-dimensional lattice to an equilibrium statistical mechanical model on a lattice of dimension $d+1$. This lattice model is, for weakly coupled maps, in a high-temperature phase, and we use a general, but very elementary, method to prove exponential decay of correlations at high temperatures.

1 Introduction.

One of the landmarks in the modern theory of dynamical systems was the introduction of the concept of Sinai-Ruelle-Bowen (SRB) measure [27, 28, 25, 2, 11]. This is, roughly speaking, a measure that is supported on an attractor and that describes the statistics of the long-time behaviour of the orbits, for almost every initial condition (with respect to Lebesgue measure) in the corresponding basin of attraction. On the other hand, one expects the asymptotic behaviour of solutions of a class of partial differential equations, defined on some spatial domain, to be described by an attractor whose dimension increases to infinity with the size of the domain [29]. It seems therefore natural to try to extend the notion of SRB measure to infinite dimensional dynamical systems.

One often replaces differential equations by iterated maps, i.e. recursions of the form:

$$x(t+1) = F(x(t)).$$

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Likewise, in order to simplify the analysis of partial differential equations, one may discretize space and time, and consider instead coupled map lattices. One replaces the field described by the partial differential equation by a set $\mathbf{x} = (x_i)_{i \in \mathbf{Z}^d}$. The dynamics is defined by a recursion of the form:

$$x_i(t+1) = F_i(\mathbf{x}(t))$$

i.e. $x_i(t+1)$ is determined by the values taken by $\mathbf{x}$ at time t (usually on the sites in a neighborhood of i) [18, 19].

In the study of SRB measures, a major role is played by the thermodynamic formalism: these measures are realized as equilibrium states for a "spin system", where the spin configurations correspond to a coding (via a Markov partition) of the phase space of the dynamical system. But for smooth, finite dimensional, dynamical systems, the corresponding spin system is always one-dimensional, with exponentially decaying interactions. And these systems do not undergo phase transitions, which then implies that the SRB measure is unique.

When one considers coupled maps on a d-dimensional lattice, the thermodynamic formalism relates them to a spin system on a $d+1$-dimensional lattice, for which a phase transition is at least possible. Some concrete models of such transitions have been suggested [22, 24]. Unfortunately, all rigorous studies of coupled maps have dealt, so far, with the case of weakly coupled maps, where the corresponding spin system is its high-temperature phase [6, 23, 30, 31, 32, 16, 21, 3, 4]. However, it turns out that the "interactions" between the spins are somewhat unusual, and even the high-temperature phase requires a careful analysis. The investigation of a possible "low temperature" phase is an open problem.

The goal of this paper is to provide a pedagogical introduction to the methods used in [4] to construct the SRB measure for coupled maps, and to analyze its space-time decay properties. We believe that the construction presented here is quite simple.

First of all, we give a unified presentation of high-temperature expansions for spin systems. At high temperatures, one generally expects a statistical mechanical system to have three properties: a unique phase, exponentially small correlations between distant variables and analyticity of the thermodynamic functions with respect to parameters such as temperature, external field etc... We start by showing how to prove the first two of those properties, starting with the simple example of the Ising model (using an argument that goes back to Fisher [13]), and then proceeding by successive generalizations. Our method is closely related to the high temperature expansions for disordered systems developed in [10]. Eventually, we show how to analyze a small, but rather general, perturbation of a lattice of one-dimensional systems defined by subshifts of finite type. This result is slightly different from those of [4].

Then we show in detail how to use the Perron-Frobenius operator to construct a "spin" representation of a dynamical system, for a *single* map. We also explain how the time correlations of the dynamical system are translated into spatial correlations for the corresponding spin system. The construction of an SRB measure for the coupled maps follows then, in a rather straightforward way, from the combination of those two ingredients.

2 High-temperature expansions.

Consider an Ising model on a d-dimensional lattice $\mathbf{Z}^d$: at each site $i \in \mathbf{Z}^d$, we have a variable $s_i = \pm 1$. Let $s_\Lambda = (s_i)_{i\in\Lambda}$, for $\Lambda \subset \mathbf{Z}^d$. In the simplest case, we have the Hamiltonian

$$\mathcal{H}(s_\Lambda) = -J \sum_{\langle ij \rangle} s_i s_j \tag{1}$$

with $J > 0$, and the sum runs over nearest-neighbor ($n.n.$) pairs. Consider the two point correlation function $\langle s_0 s_k \rangle_\Lambda$ where $0, k \in \Lambda \subset \mathbf{Z}^d$ and the expectation $\langle \cdot \rangle_\Lambda$ is taken with respect to the distribution

$$Z(\Lambda)^{-1} \exp(-\beta \mathcal{H}(s_\Lambda)) \tag{2}$$

where $\mathcal{H}(s_\Lambda)$ is defined as in (1), but with the sum restricted to $\langle ij \rangle \subset \Lambda$ (open boundary conditions), and $Z(\Lambda)$ normalizes (2) into a probability distribution. So,

$$\langle s_0 s_k \rangle_\Lambda = \frac{\sum_{s_\Lambda} s_0 s_k \exp(-\beta \mathcal{H}(s_\Lambda))}{\sum_{s_\Lambda} \exp(-\beta \mathcal{H}(s_\Lambda))}. \tag{3}$$

We write

$$\exp(-\beta \mathcal{H}(s_\Lambda)) = (\cosh \beta J)^{|B(\Lambda)|} \prod_{\langle ij \rangle \subset \Lambda} (1 + s_i s_j \tanh \beta J) \tag{4}$$

where $|B(\Lambda)|$ is the number of nearest-neighbor pairs in Λ. Next, we insert (4) in the numerator and the denominator of (3), cancel the common factor $(\cosh \beta J)^{|B(\Lambda)|}$ and expand the product over $\langle ij \rangle \subset \Lambda$. Using $\sum_{s_i = \pm 1} s_i = 0$, $\sum_{s_i = \pm 1} 1 = 2$, we get

$$\langle s_0 s_k \rangle_\Lambda = \frac{\sum_{\mathcal{X}: \partial \mathcal{X} = \{0,k\}} (\tanh \beta J)^{|\mathcal{X}|}}{\sum_{\mathcal{X}: \partial \mathcal{X} = \emptyset} (\tanh \beta J)^{|\mathcal{X}|}} \tag{5}$$

where the sums run over sets of $n.n.$ pairs $\mathcal{X}$ and $\partial \mathcal{X}$ is the set of sites which appear an odd number of times in the pairs of $\mathcal{X}$. Now, each $\mathcal{X}$ with $\partial \mathcal{X} = \{0, k\}$ contains a subset $P = \{\langle i_t, i_{t+1} \rangle\}_{t=0}^{n-1}$ where $i_0 = 0$, $i_n = k$ (otherwise, $\partial \mathcal{X}$ would contain points other than 0, k). By erasing loops in P, one can even choose P to be a self-avoiding path, i.e. such that all the i_t's are different (but we shall not use this last fact in any essential way below). So, each $\mathcal{X}$ in the numerator of (5) can be written as $\mathcal{X} = P \cup \mathcal{X}'$ where P is as above, and $\mathcal{X}'$ satisfies $\partial \mathcal{X}' = \emptyset$. This decomposition of $\mathcal{X}$ is not necessarily unique, but, given an arbitrary choice of a path P for each $\mathcal{X}$ in the numerator of (5), we may write:

$$\sum_{\mathcal{X}: \partial \mathcal{X} = \{0,k\}} (\tanh \beta J)^{|\mathcal{X}|} = \sum_{P: 0 \to k} (\tanh \beta J)^{|P|} \sum_{\mathcal{X}', \partial \mathcal{X}' = \emptyset}^{P} (\tanh \beta J)^{|\mathcal{X}'|} \tag{6}$$

where $\sum^P$ runs over the sets of pairs such that P is the chosen path for $\mathcal{X} = P \cup \mathcal{X}'$. Clearly, since $\tanh \beta J \geq 0$, the sum over $\mathcal{X}'$ in (6) is less than the unconstrained sum over $\mathcal{X}$ in the denominator of (5), and we get

$$\langle s_0 s_k \rangle_\Lambda \leq \sum_{P: 0 \to k} (\tanh \beta J)^{|P|} \tag{7}$$

Recall that $P = \{\langle i_t, i_{t+1}\rangle\}_{t=0}^{n-1}$ where $i_0 = 0$, $i_n = k$ and all the i_t's are different. Given i_t, we can choose at most $2d-1$ nearest-neighbor sites $i_{t+1} \neq i_{t-1}$ and, if

$$(2d-1)\tanh \beta J < 1 \tag{8}$$

we have, since all P in (7) satisfy $|P| \geq |k|$ (with $|P|$ being the number of pairs in P),

$$\langle s_0 s_k \rangle \leq \frac{2d}{2d-1} \sum_{n \geq |k|} ((2d-1)\tanh \beta J)^n \leq C \exp(-m|k|) \tag{9}$$

for some $m > 0$ and $C < \infty$ (the factor $2d$ comes from the first step in P). This proves the exponential decay of the two-point correlation function $\langle s_0 s_k \rangle$, for β small.

Remark. Here and below, we shall denote by C or c a constant that may vary from place to place.

Let us now extend this argument in several directions. First, it is instructive to abstract the argument and apply it to more general correlation functions. Let F depend on the variables $s_A = (s_i)_{i\in A}$ and G depend on $s_B = (s_i)_{i\in B}$, where $A, B \subset \Lambda$. Then, for β small,

$$|\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda \langle G\rangle_\Lambda| \leq C||F||\,||G|| \exp(-md(A,B)) \tag{10}$$

for some $C < \infty$ (depending on $|A|$ and $|B|$) and $m > 0$, where $\|F\|$ is the sup norm and $d(A,B)$ is the distance between A and B. The constants C and m are independent of Λ.

To prove (10), one introduces duplicate variables, i.e. one associates to each lattice site a pair of identically distributed random variables (s_i^1, s_i^2) whose common distribution is given by (2). Using this trick, one may write

$$\begin{aligned} & 2(\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda \langle G\rangle_\Lambda) \\ = \; & Z(\Lambda)^{-2} \sum_{(s_i^1, s_i^2)_{i\in\Lambda}} (F(s_A^1) - F(s_A^2))(G(s_B^1) - G(s_B^2)) \exp(-\mathcal{H}(s_\Lambda^1) - \mathcal{H}(s_\Lambda^2)). \end{aligned} \tag{11}$$

Then, we use the following identity, which is analogous to (4):

$$\exp(\beta J(s_i^1 s_j^1 + s_i^2 s_j^2)) = e^{-2\beta J}(1 + \exp(\beta J(s_i^1 s_j^1 + s_i^2 s_j^2 + 2)) - 1) \equiv e^{-2\beta J}(1 + f_{ij}) \tag{12}$$

where, using $e^x - 1 \leq xe^x$, for $x \geq 0$, we have

$$0 \leq f_{ij} \leq 4\beta J(1 + f_{ij}). \tag{13}$$

Now we insert (12) in the numerator and denominator of (11), cancel the common factor $e^{-4\beta J|B(\Lambda)|}$, and expand $\prod_{\langle ij\rangle}(1 + f_{ij})$ in the numerator; we obtain:

$$\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda \langle G\rangle_\Lambda = \frac{\sum_{\mathcal{X}} \langle \tilde{F}\tilde{G} f_{\mathcal{X}}\rangle^0}{\langle \prod_{\langle ij\rangle \subset \Lambda}(1 + f_{ij})\rangle^0} \tag{14}$$

where $\tilde{F} = F(s_A^1) - F(s_A^2)$, $\tilde{G} = G(s_B^1) - G(s_B^2)$, $f_{\mathcal{X}} = \prod_{\langle ij\rangle \in \mathcal{X}} f_{ij}$, the sum runs over sets $\mathcal{X}$ of $n.n.$ pairs and $\langle f\rangle^0$ means that we sum f over $s_i^1, s_i^2 = \pm 1, i \in \Lambda$.

It is easy to see that the terms in the numerator of (14) for which $\mathcal{X}$ does not contain a path $P = \{\langle i_t, i_{t+1}\rangle\}_{t=0}^{n-1}$ with $i_0 \in A, i_n \in B$ vanish. Indeed, if there is no such path, consider the set of sites $\mathcal{X}(A)$ connected to A by a path in $\mathcal{X}$. By assumption, $B \cap \mathcal{X}(A) = \emptyset$. Now exchange s_i^1 and s_i^2 for all $i \in \mathcal{X}(A)$. The expectation $\langle\cdot\rangle^0$ and each f_{ij}, for $\langle ij\rangle \in \mathcal{X}$, are even under such an exchange, while $\tilde{F}$ is odd and $\tilde{G}$ is even since $B \cap \mathcal{X}(A) = \emptyset$. Hence, the corresponding term vanishes.

So we can choose a path P connecting A and B for each non-zero term in the numerator of (14). We bound this numerator as follows:

$$\sum_P \sum_{\mathcal{X}}{}^P \langle f_P \tilde{F}\tilde{G} f_{\mathcal{X}\setminus P}\rangle^0 \leq \|\tilde{F}\|\,\|\tilde{G}\| \sum_P (4\beta J)^{|P|} \sum_{\mathcal{X}}{}^P \langle f_{\mathcal{X}\setminus P} \prod_{\langle ij\rangle \in P} (1+f_{ij})\rangle^0 \tag{15}$$

where we use (13). Observe that, by the positivity of f_{ij}, the sum $\sum_{\mathcal{X}}^P$ is bounded by the sum over sets of $n.n.$ pairs in Λ not belonging to P and so,

$$\sum_{\mathcal{X}}{}^P f_{\mathcal{X}\setminus P} \leq \prod_{\langle ij\rangle \subset \Lambda, \langle ij\rangle \notin P} (1+f_{ij}).$$

Therefore,

$$\sum_{\mathcal{X}}{}^P \langle f_{\mathcal{X}\setminus P} \prod_{\langle ij\rangle \in P} (1+f_{ij})\rangle^0 \leq \langle \prod_{\langle ij\rangle \subset \Lambda} (1+f_{ij})\rangle^0. \tag{16}$$

Since the RHS of (16) is the denominator of (14) and since all P's in (15) satisfy $|P| \geq d(A,B)$, we can use an estimate like (9) to prove (10) for β small.

A similar argument also shows that there is a unique Gibbs state (i.e. a unique phase). To prove that, one introduces boundary conditions. Define

$$\mathcal{H}(s_\Lambda|\hat{s}_{\Lambda^c}) = -J \sum_{\langle ij\rangle \subset \Lambda} s_i s_j - J \sum_{\langle ij\rangle, i\in\Lambda, j\notin\Lambda} s_i \hat{s}_j \tag{17}$$

for any fixed configuration $\hat{s}_{\Lambda^c}$ outside Λ, and introduce the corresponding expectation $\langle\cdot\rangle_{\Lambda,\hat{s}}$. Uniqueness of the Gibbs state means that for any two boundary conditions s, s' and any function F as in (10), $\langle F\rangle_{\Lambda,s} - \langle F\rangle_{\Lambda,s'} \to 0$ as $\Lambda \uparrow \mathbf{Z}^d$.

Introducing duplicate variables, we write

$$\begin{aligned}\langle F\rangle_{\Lambda,s} - \langle F\rangle_{\Lambda,s'} &= Z(\Lambda|s_{\Lambda^c})^{-1} Z(\Lambda|s'_{\Lambda^c})^{-1} \times \\ &\sum_{(s_i^1, s_i^2) i\in\Lambda} (F(s_A^1) - F(s_A^2)) \exp(-\beta(\mathcal{H}(s_\Lambda^1|s_{\Lambda^c}) + \mathcal{H}(s_\Lambda^2|s'_{\Lambda^c}))).\end{aligned} \tag{18}$$

Now, if we perform the same expansion as in (14), we see that any non-zero term must contain a path connecting A and the boundary of Λ. Hence, repeating the arguments that led to (15, 16), we get that (18) is bounded in absolute value by $C\exp(-md(A,\Lambda^c))$ for some $m > 0$.

It is easy to extend this high-temperature expansion to two-body long range interactions, with Hamiltonian $H = -\sum_{i,j} J_{ij} s_i s_j$ and

$$\sum_j |J_{ij}| e^{\gamma|i-j|} < \infty, \tag{19}$$

for some $\gamma > 0$. Now, the analogue of the connected paths entering (15) are sets of pairs $P = \{(i_t, j_t)\}_{t=0}^n$ with $i_0 \in A$, $j_n \in B$ and $j_t = i_{t+1}$. More generally, for interactions that have only power-law decay, i.e. such that $\sum_y |J_{ij}||i-j|^\alpha < \infty$ some $\alpha > 0$, one gets $\sum_k |\langle s_0 s_k \rangle||k|^\alpha < \infty$ for β small. The signs of J_{ij} do not matter. We can always write

$$\begin{aligned}&\exp(\beta J_{ij}(s_i^1 s_j^1 + s_i^2 s_j^2)) = \exp(-2\beta|J_{ij}|) \exp(\beta(J_{ij}(s_i^1 s_j^1 + s_i^2 s_j^2) + 2|J_{ij}|)) \\ &= \exp(-2\beta|J_{ij}|)(1 + f_{ij})\end{aligned} \tag{20}$$

instead of (12), so that the factors f_{ij} are all positive, and, using (19), are bounded by $C\beta e^{-\gamma|i-j|}$.

However, this does not work for *complex* interactions. Actually, if we want to prove analyticity (e.g. in β) of the thermodynamic functions, we have to use polymer or cluster expansions [5, 26]. While standard, these methods are more involved combinatorically than the arguments given above.

This dichotomy between expansions that prove exponential clustering for real interactions and methods that imply analyticity becomes more relevant when we turn to many-body interactions (involving arbitrarily large number of spins). These interactions appear quite naturally, for example as effective interactions after performing a renormalization group transformation, or in dynamical systems, as we shall see in the next sections. To discuss these interactions, we need some definitions. At each site of the lattice, we have a spin $s_i \in \{0, \cdots, k-1\}$ (or, more generally, s_i belongs to a compact metric space). An interaction is defined by a family $\Phi = (\Phi_X)$ of (continuous) functions of the spin variables in X indexed by finite subsets X of $\mathbf{Z}^d$. We let $\|\Phi_X\|$ denote the sup norm of Φ_X. Given $\Lambda \subset \mathbf{Z}^d$, $|\Lambda| < \infty$, and a configuration $s'_{\Lambda^c} = (s'_i)_{i\in\Lambda^c}$ in Ω_{Λ^c}, the Hamiltonian in Λ (with boundary conditions s'_{Λ^c}) is defined as

$$\mathcal{H}(s_\Lambda | s'_{\Lambda^c}) = -\sum_{X\cap\Lambda\neq\emptyset} \Phi_X(s_{X\cap\Lambda} \vee s'_{X\cap\Lambda^c}) \tag{21}$$

where, for $X \cap Y = \emptyset$, $s_X \vee s'_Y$ is the obvious configuration in $X \cup Y$. The associated (finite volume) Gibbs measure is the probability distribution :

$$\nu(s_\Lambda | s'_{\Lambda^c}) = Z^{-1}(\Lambda | s'_{\Lambda^c}) \exp(-\mathcal{H}(s_\Lambda | s'_{\Lambda^c})) \tag{22}$$

(we put a minus sign in (21) for convenience). See e.g. [25, 26, 12] for more details on the theory of Gibbs states.

Once one introduces many-body interactions, one has to distinguish carefully between different convergence conditions (norms) replacing (19) for two-body interactions. Since we have not put β explicitely in (22), high temperatures (= small β) will mean small norm. Here are some of the most common norms and the results that can be proven about the corresponding interactions.

1) If
$$\|\Phi\|_1 = \sum_{0\in X} |X| \|\Phi_X\| \tag{23}$$
is small enough, then the Gibbs state is unique [8, 9, 26]

2) If, for some $\gamma > 0$,
$$\|\Phi\|_2 = \sum_{0 \in X} e^{\gamma d(X)} \|\Phi_X\|, \tag{24}$$
where $d(X)$ is the diameter of X, is small enough, then the Gibbs state is unique and its correlation functions decay exponentially [14].

3) If, for some $\gamma > 0$,
$$\|\Phi\|_3 = \sum_{0 \in X} e^{\gamma |X|} \|\Phi_X\| \tag{25}$$
is small enough, then the Gibbs state is unique and its correlation functions are analytic [15].

Note that for two-body interactions, (24) reduces to (19), and requiring the other norms to be small is a much weaker condition in that case. But, in general, as wediscussed in [4], $\|\Phi\|_2$ small is not sufficient to prove analyticity of the thermodynamic functions. For that, one basically needs $\|\Phi\|_3$ small. To see that $\|\Phi\|_3$ small does not necessarily imply $\|\Phi\|_2$ small (or even finite), at least on lattices of more than one dimension, observe that for X being a large square or a large cube, $|X|$ is much larger than $d(X)$. On the other hand, it is rather easy to extend the arguments given above to prove exponential decay of the correlation functions for a system with $\|\Phi\|_2$ small enough (see Theorem 1 below).

For the applications to dynamical systems, one needs more than results for interactions with a small norm. High temperature expansion means that one perturbs around a completely decoupled system (i.e. the one defined by the expectation $\langle\cdot\rangle^0$ in (14)). However, it is well-known that one-dimensional systems (say, with finite range interactions, i.e. with $\Phi_X = 0$ if $d(X) > R$ for some finite R) do not undergo phase transitions, i.e. they remain in the high-temperature regime for all temperatures. This results follows from the Perron-Frobenius theorem applied to the transfer matrix of the system (see Proposition 1 below). One might expect that this property is stable under small perturbations. For reasons which will be explained in the next section, we shall need to consider interactions which are the sum of two terms: a one dimensional finite range interaction, whose norm is not necessarily small, and a more general interaction, which may be of infinite range, but whose $\|\Phi\|_2$ norm is small.

Concretely, we will be interested in the following setup. Consider a lattice $\mathbf{Z}^{d+1}$ where we want to single out a particular ("time") direction. Let us write $\alpha = (t, i)$ for $\alpha \in \mathbf{Z}^{d+1}$, $t \in \mathbf{Z}$ and $i \in \mathbf{Z}^d$. At each site α, we have a spin $s_\alpha \in \{0, \cdots, k-1\}$. The phase space is a product of topological Markov chains (or subshifts of finite type):
$$\Omega_{\mathcal{A}} = \{\mathbf{s} = (s_\alpha)_{\alpha \in \mathbf{Z}^{d+1}} | \mathcal{A}(s_{(t,i)}, s_{(t+1,i)}) = 1 \ \forall t \in \mathbf{Z}, \ \forall i \in \mathbf{Z}^d\}$$
and $\mathcal{A}(s, s')$ is a k by k matrix whose entries take value 0 or 1, and which is transitive i.e., $\exists n < \infty$ such that $\mathcal{A}^n(s, s') > 0 \ \forall s, s' \in \{0, \cdots, k-1\}$. With this phase space, the expectation values are defined as before, except that in (22), we restrict ν to s_Λ's such that $s_\Lambda \vee s'_{\Lambda^c}$ belongs to $\Omega_{\mathcal{A}}$, and, in $Z^{-1}(\Lambda|s'_{\Lambda^c})$, the sum runs over those s_Λ's. From the point of view of statistical mechanics, $\mathcal{A}$ defines a nearest-neighbor hard-core along

the time direction, where the forbidden configurations are those where a pair satisfies $\mathcal{A}(s_{(t,i)}, s_{(t+1,i)}) = 0$.

Let Φ^0 be a finite range one-dimensional interaction, i.e. $\exists R < \infty$ such that $\Phi^0_X = 0$ if diam $(X) > R$ and $\Phi^0_X = 0$ unless $X \subset \mathbf{Z} \times \{i\}$ for some $i \in \mathbf{Z}^d$, i.e. unless X is a subset of a one-dimensional sublattice of $\mathbf{Z}^{d+1}$ along the "time" direction. Without loss of generality, we may assume that $R > n$.

Theorem 1 *Let $\Phi = \Phi^0 + \Phi^1$, where Φ^0 is as above. Then, there exist $\epsilon > 0$, $m > 0$, $C < \infty$ such that, if $\|\Phi^1\|_2 \leq \epsilon$, there is a unique Gibbs state μ for Φ and the correlation functions satisfy, for all F, G, with $F : \Omega_A \to \mathbf{R}$, $G : \Omega_B \to \mathbf{R}$, where $A, B \subset \mathbf{Z}^{d+1}$ are finite:*

$$|\langle FG \rangle - \langle F \rangle \langle G \rangle| \leq C \min(|A|, |B|) \|F\| \|G\| e^{-md(A,B)}, \tag{26}$$

where $d(A, B)$ is the distance between the sets A and B and $\langle F \rangle = \int F d\mu$.

Remark 1. We refer to [4] for an alternative version of this Theorem, and for various extensions. In [4], we took Φ^0 to be any interaction which is in a suitably defined high-temperature regime. Note, however, that in [4], we had $\mathcal{A}(s, s') = 1$ for all s, s'.

Remark 2. Since Φ^0 has no coupling in spatial directions and has range R in the time direction we get the following decomposition of the corresponding partition function Z^0. Given $V \subset \mathbf{Z}^{d+1}$, we let, for $i \in \mathbf{Z}^d$, $J_i = (\mathbf{Z} \times i) \cap V$ so that $V = \cup_i J_i$. We shall call the J_i's the time intervals of V. Then,

$$Z^0(V|s) = \prod_i Z^0(J_i|s). \tag{27}$$

The transfer matrix formalism gives the following representation for $Z^0(J|s)$:

Proposition 1 *There exist $\gamma > 0$, $C < \infty$ independent of R such that, if J is an interval of the form $[(i, t), (i, t + \ell R)]$, $\ell \in \mathbf{N}$, then*

$$Z^0(J|s) = \lambda^\ell W(s_-) W(s_+)(1 + g_J(s_-, s_+)) \tag{28}$$

where $s_+ = s_{((i,t+\ell R),(i,t+(\ell+1)R)]}$ and $s_- = s_{[(i,t-R),(i,t))}$ with

$$|g_J(s_-, s_+)| \leq C e^{-\frac{\gamma}{R}|J|}. \tag{29}$$

This last Proposition is rather standard in statistical mechanics; for a proof, see for example the Appendix of [4], where one has to replace $\mathcal{T}$, P by $\mathcal{T}^n$, P^n, to take into account the presence of the matrix $\mathcal{A}$.

Remark 3. We could write $s_\pm(J)$ to indicate that the configurations $s_\pm$ are indexed by sites situated on both sides of the interval J. Keeping that in mind, we write $s_\pm$, for simplicity.

Proof of Theorem 1. We shall sketch the proof of (26), using the ideas explained in this section, and refer the reader to [4] for more details. First, we write the LHS (the expectation are taken with respect to a finite volume Gibbs state with open boundary conditions) of (26) using duplicate variables:

$$\begin{aligned} & 2(\langle FG\rangle_\Lambda - \langle F\rangle_\Lambda \langle G\rangle_\Lambda) \\ = \; & Z(\Lambda)^{-2} \sum_{s^i_\Lambda, i=1,2} \tilde{F}\tilde{G} \exp(-\mathcal{H}(s^1_\Lambda) - \mathcal{H}(s^2_\Lambda)) \end{aligned} \tag{30}$$

where $\tilde{F} = F(s^1_\Lambda) - F(s^2_\Lambda), \tilde{G} = G(s^1_\Lambda) - G(s^2_\Lambda)$.

We may replace Φ^1_X by $\Phi^1_X - \inf_{s_X} \Phi^1_X$, by adding a constant to the Hamiltonian. Thus, we may, without loss of generality, assume that $\Phi^1_X \geq 0$ for all X, and that Φ^1 still satisfies $\|\Phi^1\|_2 \leq 2\epsilon$. Then, we perform a high-temperature expansion on the Φ^1 part of $\mathcal{H}$, as in (14):

$$\exp(\sum_{X, i=1,2} \Phi^1_X(s^i_X)) = \sum_{\mathcal{X}} \prod_{X\in\mathcal{X}} f_X \tag{31}$$

where the sum runs over sets $\mathcal{X}$ of subsets of Λ, and

$$f_X = \exp(\Phi^1_X(s^1_X) + \Phi^1_X(s^2_X)) - 1 \tag{32}$$

satisfies:

$$0 \leq f_X, \tag{33}$$

$$\sum_{0\in X} e^{\gamma d(X)} \|f_X\| \leq C\epsilon \tag{34}$$

(using the positivity of Φ^1_X, (24) and $\|\Phi^1\|_2 \leq \epsilon$).

It is convenient to cover $\mathbf{Z}^{d+1}$ by disjoint R-intervals, i.e. intervals of length R parallel to the time axis. Now, insert (31) in (30) and, for each term in (31), define $V = V(\mathcal{X}) = \underline{A} \cup \underline{B} \cup \underline{\mathcal{X}}$, where $\underline{\mathcal{X}} = \cup_{X\in\mathcal{X}} \underline{X}$ and for any $X \subset \mathbf{Z}^{d+1}$, $\underline{X}$ is the set of R-intervals intersected by X. We have, by summing, in (30), first over s^i_α, for $\alpha \in \Lambda \backslash V$, and then over s^i_α, $\alpha \in V$,

$$(30) = Z(\Lambda)^{-2} \sum_{\mathcal{X}} \sum_{s^i_V, i=1,2} \tilde{F}\tilde{G} \prod_{X\in\mathcal{X}} f_X \exp(-\mathcal{H}^0_V) \prod_{i=1,2} Z^0(\Lambda\backslash V | s^i_V) \tag{35}$$

where $\mathcal{H}^0_V = -\sum_{X\subset V} (\Phi^0_X(s^1_X) + \Phi^0_X(s^2_X))$, and $Z^0(\Lambda\backslash V|s^i_V)$ is the partition function with interaction Φ^0, s^i_V boundary condition in V, and open boundary conditions in Λ^c. Assuming Λ to be a union of R-intervals (it is easy to extend the proof to the general case), $\Lambda\backslash V$ is also such a union, and we may use (27,28) for $Z^0(\Lambda\backslash V|s^i_V)$. Since $Z^0(J|s)$ does not vanish for J being a union of R-intervals (we chose $R > n$), we may define $1 + \tilde{g}_J(s^1_-, s^1_+, s^2_-, s^2_+) = \prod_{i=1,2} \frac{1+g_J(s^i_-, s^i_+)}{\min_{s^i_-, s^i_+}(1+g_J(s^i_-, s^i_+))}$ so that $\tilde{g}_J(s^i_-, s^i_+)$ is positive and still satisfies (29) (with a different C):

$$0 \leq \tilde{g}_J(s^i_-, s^i_+) \leq C e^{-\frac{\gamma}{R}|J|}. \tag{36}$$

We get:

$$Z^0(\Lambda\backslash V|s^i_V) = \mathcal{W}(s^i_V) \prod_J (1 + \tilde{g}_J(s^i_-, s^i_+))$$

where

$$\mathcal{W}(s_V^i) = \prod_J \lambda^{2|J|} \prod_{i=1,2} W(s_-^i)W(s_+^i) \min_{s_-^i, s_+^i}(1 + g_J(s_-^i, s_+^i)). \tag{37}$$

The product over J runs over all the time intervals of $\Lambda \backslash V$ (so that the variables $s_\pm^1$, $s_\pm^2$ are indexed by sites in V).

Let us insert this representation of $Z^0(\Lambda\backslash V|s_V^i)$ in (35) and then expand the product over J of $(1 + \tilde{g}_J(s_-^i, s_+^i))$. The result is:

$$(35) = Z(\Lambda)^{-2} \sum_{\mathcal{X},\mathcal{J}} \sum_{s_V^i, i=1,2} \tilde{F}\tilde{G} \prod_{X\in\mathcal{X}} f_X \prod_{J\in\mathcal{J}} \tilde{g}_J \exp(-\mathcal{H}_V^0)\mathcal{W}(s_V^i) \tag{38}$$

where the sum over $\mathcal{J}$ runs over families of intervals $J \subset \Lambda\backslash V$.

¿From now on, we can proceed as we did previously: the main observation is again that if, for each term in (38), we decompose $V \cup (\cup_{\mathcal{J}} J)$ into connected components (where connected is defined in an obvious way: any two sets can be joined by a "connected path" $P = (Z_i)_{i=1}^n$ where each Z_i is either an $\underline{X}$ or a J and the distance between Z_{i+1} and Z_i is less than 1, $\forall i = 1, \cdots, n-1$), and if A and B belong to different components, then that term vanishes. Let us check this: as before, we interchange s_α^1 and s_α^2 for each α in the connected component containing A. $\tilde{F}$ is odd under such an interchange, while $\tilde{G}$ is even (if A and B belong to different components), and f_X, $\tilde{g}_J$ are obviously even. Next, observe that $\mathcal{W}(s_V^i)$ can be factorized into a product of functions, each of which depends only of s_α^i for α belonging to a connected component of V. Observe also that $\mathcal{H}_V^0$ does not contain terms where X intersects different connected components of V (since Φ_0 has range R and V is defined as a union of R-intervals). So, the two last factors in (38) factorize over connected components of V, and are therefore also even under our interchange of s_α^1 and s_α^2.

Hence, for each non-zero term in (38), we can choose a connected path, as defined above, $P = (Z_i)_{i=1}^n$ where $Z_1 = \underline{A}, Z_n = \underline{B}$. Then, using the positivity of f_X, $\tilde{g}_J$, we bound the sum in (38) by a sum over such paths, and control that sum essentially as in (9). The exponential decay comes from combining (34), when Z_i is an $\underline{X}$ and (36) when Z_i is a J. The uniqueness of the Gibbs state is proven as in (18); for details, see [4]. □

3 SRB measures for expanding circle maps.

We start by recalling the standard theory of invariant measures for smooth expanding circle maps, in a formulation that will be used later. To describe the dynamics, we first fix a map $F : S^1 \to S^1$. We take F to be an expanding, orientation preserving $C^{1+\delta}$ map with $\delta > 0$ (i.e. F is differentiable and its derivative is Hölder continuous of exponent δ). We describe F in terms of its lift to $\mathbf{R}$, denoted by f and chosen, say, with $f(0) \in [0, 1[$. We assume that

$$f'(x) > \lambda^{-1} \tag{1}$$

where $\lambda < 1$. Note that there exists an integer $k > 1$ such that

$$f(x+1) = f(x) + k \quad \forall x \in \mathbf{R}. \tag{2}$$

A probability measure μ on S^1 is called an SRB measure if it is F-invariant and absolutely continuous with respect to the Lebesgue measure. The following results are well-known for maps F as above (see [28, 7, 20]):

(a) There is a unique SRB measure μ.

(b) For any absolutely continuous probability measure ν, and any continuous function G,

$$\int G \circ F^N d\nu \to \int G d\mu \tag{3}$$

as $N \to \infty$.

(c) There exists $C < \infty$, $m > 0$, such that $\forall G \in L^\infty(S^1), \forall H \in \mathcal{C}^\delta(S^1)$,

$$|\int G \circ F^n H d\mu - \int G d\mu \int H d\mu| \leq C \|G\|_\infty \|H\|_\delta e^{-mn}, \tag{4}$$

where $\mathcal{C}^\delta(S^1)$ denotes the space of Hölder continuous functions, with the norm

$$\|H\|_\delta = \|H\|_\infty + \sup_{x,y} \frac{|H(x) - H(y)|}{|x-y|^\delta}.$$

Remark. There are different ways to define an SRB measure. In [11], they are introduced as measures whose restriction on the expanding directions is absolutely continuous with respect to the Lebesgue measure. Since here the whole phase space S^1 is expanding, our definition is natural (besides, with this definition, the SRB measure is unique). But, as we mentioned in the introduction, one of the most interesting properties of the SRB measure is that it describes the statistics of the orbits of almost every point, which means that

$$\frac{1}{N} \sum_{i=0}^{N-1} G \circ F^i(x) \to \int G d\mu, \tag{5}$$

for almost every x, and every continuous function G. Note that, if we integrate (5) with an absolutely continuous probability measure ν, we obtain the Cesaro average of (3). So, both properties are related to each other.

Let us sketch now the construction of μ. In doing so, we shall establish the connection with the statistical mechanics of one-dimensional spin systems. This way of constructing μ may not be the simplest one in the present context, but the connection to statistical mechanics will be essential in the analysis of coupled maps (see [28, 7, 20] for different approaches, although the one below is close to [28]).

The Perron-Frobenius operator P for F is defined by

$$\int G \circ F H dm = \int G P H dm \tag{6}$$

for $G \in L^\infty(S^1)$, $H \in L^1(S^1)$, and dm being the Lebesgue measure. Let us work in the covering space $\mathbf{R}$ and replace G, H by periodic functions denoted $g, h : g(x+n) = g(x)$, $\forall n \in \mathbf{Z}$. We get

$$\int_{[0,1]} g \circ f h dx = \int_{[0,1]} g(x) P h(x) dx. \tag{7}$$

More explicitely,

$$Ph(x) = \sum_s \frac{h(f^{-1}(x+s))}{f'(f^{-1}(x+s))} \tag{8}$$

where $s \in \{0, \cdots, k-1\}$ (and k was introduced in (2)). Note that P maps periodic functions into periodic functions because the sum is periodic even if the summands are not: indeed, (2) implies that f' is periodic and that $f^{-1}(x+1+k-1) = f^{-1}(x+k) = f^{-1}(x)+1$ (so that, if we add 1 to x, it amounts to a cyclic permutation of s).

By (7), the density $h_\mu(x)$ of the absolutely continuous invariant measure $d\mu = h_\mu(x)dx$ satisfies $Ph_\mu = h_\mu$. We shall construct h_μ as the limit, as $N \to \infty$, of $P^N 1$. $P^N 1$ has a direct statistical mechanical interpretation which we now derive.

First, iterating (8), we get

$$(P^N 1)(x) = \sum_{s_1, \cdots, s_N} \prod_{t=1}^N [f'(f_{s_t}^{-1} \circ \cdots \circ f_{s_1}^{-1}(x))]^{-1} \tag{9}$$

where $f_s^{-1}(x) \equiv f^{-1}(x+s)$.

¿From now on, we shall consider $x \in [0,1]$. We introduce a convenient notation: $x \in [0,1]$ and $s_1, \cdots, s_N$ in (9) collectively define a configuration on a lattice $\{0, \cdots, N\}$. To any subset $X \subset \mathbf{Z}_+$ associate the configuration space $\Omega_X = \times_{t \in X} \Omega_t$ where Ω_t equals [0,1] if $t = 0$, and equals $\{0, \cdots, k-1\}$ if $t > 0$. We could use the existence of a Markov partition for F to write x as a symbol sequence, as is usually done, e.g. in [6], but we shall not use explicitely this representation.

Let $\mathbf{s} = (x, s_1, \cdots, s_N) \in \Omega_N$. Then (9) reads

$$(P^N 1)(x) = \sum_{s_1 \cdots s_N} e^{-\mathcal{H}_N(\mathbf{s})} \tag{10}$$

with $e^{-\mathcal{H}_N(\mathbf{s})}$ being the summand in (9). And we want to construct the limit:

$$\int_{[0,1]} g(x) d\mu = \lim_{N \to \infty} \int_{[0,1]} g(x)(P^N 1)(x) dx = \lim_{N \to \infty} \sum_{s_1 \cdots s_N} \int_{[0,1]} g(x) e^{-\mathcal{H}_N(\mathbf{s})} dx \tag{11}$$

for any continuous function g.

This is the statistical mechanical representation we want to use. In that language, $d\mu$ is the restriction to the "time zero" phase space of the Gibbs state determined by $\mathcal{H}$. One can also rewrite the time correlation functions (3) as follows: let $d\nu = h_\nu(x)dx$; then, replacing again G by a periodic function g and F by its lift,

$$\int_{[0,1]} g \circ f^N h_\nu dx = \int_{[0,1]} g P^N h_\nu dx = \sum_{(s_i)_{i=1}^N} \int_{[0,1]} g(x) \exp(-\mathcal{H}_N(\mathbf{s})) h_\nu(\mathbf{s}) dx \tag{12}$$

where

$$h_\nu(\mathbf{s}) = h_\nu(f_{s_N}^{-1} \circ \cdots \circ f_{s_1}^{-1}(x)),$$

and the last equality in (12) follows by iterating (8).

On the other hand, since $\int Phdx = \int hdx$, by definition of P (use (7) with $g = 1$), one has

$$\sum_{(s_i)_{i=1}^N} \int_{[0,1]} \exp(-\mathcal{H}_N(\mathbf{s}))h_\nu(\mathbf{s})dx = \int_{[0,1]} P^N h_\nu dx = \int_{[0,1]} h_\nu dx = 1, \tag{13}$$

since $d\nu$ is a probability measure. So, using (11, 12), one sees that (3) is translated, in the statistical mechanics language, into

$$\begin{aligned}
& \lim_{N\to\infty} (\int G \circ F^N d\nu - \int G d\mu) \\
= & \lim_{N\to\infty} (\int_{[0,1]} g P^N h_\nu dx - \int_{[0,1]} g P^N 1 dx) \\
= & \lim_{N\to\infty} (\sum_{(s_i)_{i=1}^N} \int_{[0,1]} g(x) e^{-\mathcal{H}_N(\mathbf{s})} h_\nu(\mathbf{s}) dx \\
- & (\sum_{(s_i)_{i=1}^N} \int_{[0,1]} g(x) e^{-\mathcal{H}_N(\mathbf{s})} dx)(\sum_{(s_i)_{i=1}^N} \int_{[0,1]} e^{-\mathcal{H}_N(\mathbf{s})} h_\nu(\mathbf{s}) dx)) \\
= & \ 0
\end{aligned} \tag{14}$$

where we used (13) to insert the last factor (which equals one). We shall see below that the last equality expresses the decay of correlation functions for the Gibbs state determined by $\mathcal{H}$. A similar observation holds for (4).

One advantage of these representations is that one may use the statistical mechanics formalism to control the limit. To make the connection with statistical mechanics more explicit, it is convenient to write $\mathcal{H}_N$ in terms of many-body interactions. First of all, from (9, 10), we get

$$\mathcal{H}_N = \sum_{t=1}^N V_t \tag{15}$$

with

$$V_t(\mathbf{s}) = \log(f'(f_{s_t}^{-1} \circ \cdots \circ f_{s_1}^{-1}(x))). \tag{16}$$

We may localize V_t by writing it as a telescopic sum:

$$V_t(\mathbf{s}) = \sum_{l=0}^t \Phi_{[l,t]}(\mathbf{s}) + V_t(\mathbf{0}_{[0,t]}) \tag{17}$$

where, for $l \neq 0, t$,

$$\Phi_{[l,t]}(\mathbf{s}) = V_t(\mathbf{s}_{[l,t]} \vee \mathbf{0}_{[0,l-1]}) - V_t(\mathbf{s}_{[l+1,t]} \vee \mathbf{0}_{[0,l]}) \tag{18}$$

and $\mathbf{0}_{[0,l]}$ denotes the configuration equal to 0 for all $i \in [0, l]$ (note that 0 belongs to the phase space for all i's). For $l = 0, t$, $\Phi_{[l,t]}(\mathbf{s})$ is given by a similar formula, where the intervals that would appear in (18) as $[0, -1]$ and $[t+1, t]$ are replaced by the empty set.

Combining (15, 17), we may write the Hamiltonian as a sum of many-body interactions:

$$\mathcal{H}_N(\mathbf{s}) = \sum_{t=1}^{N} \sum_{l=0}^{t} \Phi_{[l,t]}(\mathbf{s}) + C \tag{19}$$

where the constant $C = \sum_{t=1}^{N} V_t(\mathbf{0}_{[0,t]})$.

The main point of (19) is that $\Phi_{[l,t]}(\mathbf{s})$ depends on $\mathbf{s}$ only through $\mathbf{s}_{[l,t]}$. In the statistical mechanics language, these are many-body interactions coupling all the variables in the interval $[l,t]$. The next Proposition shows that these interactions decay exponentially with the size of the interval $[l,t]$.

Proposition 2 *There exists $C < \infty$, such that*

$$|\Phi_{[l,t]}(\mathbf{s})| \le C\lambda^{\delta(t-l)}. \tag{20}$$

Proof. This combines two bounds: First, since F is $C^{1+\delta}$, one has

$$|\log f'(x) - \log f'(y)| \le C|x-y|^{\delta} \tag{21}$$

and, by (1),

$$|f_s^{-1}(x) - f_s^{-1}(y)| \le \lambda|x-y|. \tag{22}$$

Then, iterating (22), one gets:

$$|f_{s_t}^{-1} \circ \cdots \circ f_{s_{l+1}}^{-1}(x) - f_{s_t}^{-1} \circ \cdots \circ f_{s_{l+1}}^{-1}(y)| \le \lambda^{t-l}$$

since $|x-y| \le 1$. Then (20) follows from this and (21), since the s variables in both terms of (18) (see (16)) coincide in the first $t-l$ places. ∎

We can formulate the system here in the language of Section 2 as follows: we write the interaction as the sum of an interaction Φ^0 of finite range R, which does not necessarily have a small norm plus a long range "tail" Φ^1 whose norm can be made as small as we wish by choosing R large enough. Concretely, choose now R to be the smallest integer such that

$$\lambda^{\frac{\delta R}{2}} < \epsilon. \tag{23}$$

Then, we define Φ^0 as grouping all the $\Phi_{[l,t]}$'s with $t-l \le R$ and Φ^1 to collect all the longer range $\Phi_{[l,t]}$'s. Since, for an interval $X=[l,t]$, $d(X) = t-l$, we easily have (for all $s \in \mathbf{Z}_+$) the bound:

$$\sum_{X \ni s} e^{\gamma d(X)} \|\Phi_X^1\| \le C\epsilon \tag{24}$$

for γ small enough (e.g. so that $e^{\gamma} \le \lambda^{\delta/2}$) , and where C depends on λ^{δ}. Note also that, here, $d(X) = |X|-1$, so that, in this one-dimensional situation, the norms (2.24) and (2.25) are equivalent. However, we cannot use Theorem 1 directly, because the way this Theorem is stated, ϵ depends on Φ^0, i.e. on R, and, here, we choose R in (23) in an ϵ-dependent way. It turns out that all we would need in the proof of Theorem 1 is that $R^{d+1}\epsilon$ is small enough (with $d=0$ here), and that is compatible with (23), for R large (see [4] for details). Of course, in this example, one can also apply directly the transfer matrix formalism to infinite-range interactions decaying as in (20), see [25].

In order to prove the decay of the correlation functions (3,4) one proceeds as follows. First, note that we can approximate the L^1 function h_ν, in the L^1 norm, by a smoother function, $\tilde{h}_\nu$, e.g. by a Hölder continuous function of exponent δ. Since G in (3,14) is bounded, this means that we have the following approximation, uniformly in N:

$$|\int G \circ F^N h_\nu dx - \int G \circ F^N \tilde{h}_\nu dx| \leq \|G\|_\infty \|h_\nu - \tilde{h}_\nu\|_1.$$

So, it is enough to prove (14) when h_ν is Hölder continuous.

Thus, we may write a telescopic sum, as in (17):

$$h_\nu(\mathbf{s}) = \sum_{l=0}^{N} h_{\nu,[l,N]}(\mathbf{s}) + h_\nu(\mathbf{0}_{[0,N]}) \tag{25}$$

and one has an exponential decay of the form:

$$|h_{\nu,[l,N]}(\mathbf{s})| \leq C\lambda^{\delta(N-l)} \tag{26}$$

as in Proposition 2, since h_ν is Hölder continuous. Now insert (25) in (14), observe that $g(x)$ depends only on the "time zero" variable x, while $h_{\nu,[l,N]}(\mathbf{s})$ depends only on $\mathbf{s}_{[l,N]}$. Since (14) has the form of correlation function, we can use the exponential decay of the Gibbs state determined by $\mathcal{H}$, i.e. (2.26) with $A = 0$ and $B = [l, N]$, hence $d(A, B) = l$. Combining this exponential decay with the exponential decay of $h_{\nu,[l,N]}(\mathbf{s})$ and with

$$\sum_{l=0}^{N} e^{-ml} \lambda^{\delta(N-l)} \leq Ce^{-m'N} \tag{27}$$

for $m' < \min(m, \delta|\log \lambda|)$, one proves (3).

If h_ν is not Hölder continuous, the limit in (3) is still reached (via our approximation argument), but not necessarily exponentially. The proof of (4) is similar. One sees also why, in (4), one requires H to be Hölder continuous, while G is only bounded: in order to prove exponential decay, we had to use (25, 26).

4 Coupled map lattices.

We consider now a lattice of coupled expanding circle maps. The phase space $\mathcal{M} = (S^1)^{\mathbf{Z}^d}$ i.e. $\mathcal{M}$ is the set of maps $\mathbf{z} = (z_j)_{j \in \mathbf{Z}^d}$ from $\mathbf{Z}^d$ to the circle.

To describe the dynamics, we first consider a map $F : S^1 \to S^1$ as in Section 3. We let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ denote the Cartesian product $\mathcal{F} = \mathrm{X}_{i \in \mathbf{Z}^d} F_i$ where F_i is a copy of F. $\mathcal{F}$ is called the uncoupled map.

The second ingredient in the dynamics is given by the coupling map $A : \mathcal{M} \to \mathcal{M}$. This is taken to be a small perturbation of the identity in the following sense. Let A_j

be the projection of A on the j^{th} factor and let a_j denote the lift of A_j : $A_j = e^{2\pi i a_j}$. We take, for example,

$$a_j(\mathbf{x}) = x_j + \epsilon \sum_k g_{|j-k|}(x_j, x_k)$$

where g is a periodic $C^{1+\delta}$ function in both variables, with exponential falloff in $|j-k|$. We shall come back later on the reasons for considering this somewhat unusual model (see Remark 3 below). More general examples of such $A's$ can be found in [4, 3] (note, however, that in [3], we restricted ourselves to analytic maps).

The coupled map $T : \mathcal{M} \to \mathcal{M}$ is now defined by

$$T = A \circ \mathcal{F}.$$

We are looking for "natural" T-invariant measures on $\mathcal{M}$. For this, write, for $\Lambda \subset \mathbf{Z}^d$, $\mathcal{M}_\Lambda = (S^1)^\Lambda$, and let m_Λ be the product of Lebesgue measures.

Definition 1 *A Borel probability measure μ on $\mathcal{B}$ is a SRB measure if*

(a) μ is T-invariant

(b) The restriction μ_Λ of μ to $\mathcal{B}_\Lambda$ is absolutely continuous with respect to m_Λ for all $\Lambda \subset \mathbf{Z}^d$ finite.

Remark 1. This is a natural extension to infinite dimensions of the notion of SRB measure, given in section 3, since each S^1 factor can be regarded as an expanding direction. However, unlike the situation for single maps, we do not show that the SRB measure is unique (although we expect it to be so). In [4], we prove a weaker result, namely that there is a unique "regular" SRB measure. We also show that (3.3) holds for ν being a "regular" measure, but we have not extended (3.5). The extension of (3.4) is given below.

Our main result is:

Theorem 2 *Let F and A satisfy the assumptions given above. Then there exists $\epsilon_0 > 0$ such that, for $\epsilon < \epsilon_0$, T has an SRB measure μ. Furthermore, μ is invariant and exponentially mixing under the space-time translations: there exists $m > 0$, $C < \infty$, such that, $\forall B, D \subset \mathbf{Z}^d, |B|, |D| < \infty$ and $\forall G \in L^\infty(\mathcal{M}_B), \forall H \in \mathcal{C}^\delta(\mathcal{M}_D)$,*

$$|\int G \circ T^n H d\mu - \int G d\mu \int H d\mu| \leq C \|G\|_\infty \|H\|_\delta e^{-m(n+d(B,D))}, \tag{1}$$

where $d(B, D)$ is the distance between B and D and C depends on $d(B), d(D)$.

Remark 2. The proof combines the ingredients from the previous two sections. We first derive a formula for the Perron-Frobenius operator of T which is similar to (3.9, 3.10). And we express the Hamiltonian in terms of potentials as in (3.19), using a telescopic sum. The decay of the potentials is proven again using the Hölder continuity of the differential of T and the expansivity of F. We may write the potential Φ as a sum of two terms, as in Theorem 1, $\Phi^0 + \Phi^1$, with Φ^0 one-dimensional and of finite range and $\|\Phi^1\|_2$ small, but on a "space-time" $\mathbf{Z}_+^{d+1}$ lattice. And, using Theorem 1 (which can trivially be extended to this lattice), we construct the SRB measure and prove the exponential decay of correlations. For a discussion of previous work on this problem, see [4].

Remark 3. One would like to extend this Theorem to coupled maps of the interval $[0, 1]$ into itself, where the uncoupled map is not smooth, but, say, of bounded variation. Indeed, all examples were phase transitions are expected to occur are of this form (see e.g. [22, 24]). Moreover, the theory for a single map can easily be extended to maps of bounded variation [7]. Also, one would like to consider more general couplings A, like the standard diffusive coupling.

However, such extensions seem rather difficult, because even if the uncoupled map happens to have a Markov partition, the couplings tend to destroy these partitions. This is basically the reason for considering circle maps instead of expanding maps of the interval. We did not use explicitely the existence of a Markov partition, but we used it implicitely because no characteristic functions appeared in the formula (3.9) for the Perron-Frobenius operator (compare with the formula for P in [7]). The reader should not be misled by the fact that, in the statistical mechanics part of the argument (Section 2), we could handle a general transitive matrix $\mathcal{A}$, defining a subshift. Indeed this is a short-range hard-core interaction, in the statistical mechanics language, while the appearance of characteristic functions in the Perron-Frobenius operator may give rise to an infinite range hard core, and this is much more difficult to control.

Note, however, that existence results on SRB measures in this more general context were obtained in [21]. But there are no results on the exponential decay of correlation functions. Also, Blank has constructed examples of "pathological" behaviour for coupled non-smooth maps with arbitrarily weak coupling [1].

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