LOCAL ANALYTICITY AND BOUNDS ON THE TRUNCATED CORRELATION FUNCTIONS IN DISORDERED SYSTEMS

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Abstract

We consider Gibbs fields with local and unbounded random interactions in the regime of Griffiths' singularities. We prove that the singularities are not worse than that of the randomly diluted ferromagnet and obtain upperbounds ($\sim C^k(k!)^2$) for the semi–invariants of order k, for high enough temperature. To obtain this result, we extend the notion of complete analyticity to that of local analyticity for non–translation–invariant systems.

Keywords : Griffiths' singularities, spin glasses, semi–invariants, local analyticity

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1 Introduction

The concept of complete analyticity is playing a central role in the analysis of the uniqueness regime (large temperature, large external field, ...) of statistical mechanical systems with uniformly bounded interactions. Not only does it unify various equivalent conditions on the interaction (such as demonstrated in Dobrushin–Shlosman (1985) and (1987)), but it has turned out to be of great value in the analysis of relaxation to equilibrium (e.g. in Martinelli–Olivieri (1994) and Stroock–Zegarlinski (1992)). For the moment no such a theory, comparable in power and general validity, exists for non–translation–invariant systems with unbounded interactions. This however is what one typically has to deal with in the context of disordered systems. A prime example are spin glasses with unbounded nearest–neighbor coupling coefficients such as in the Edwards–Anderson model (See Edwards–Anderson (1975)).

Although several papers have been devoted to parts of this problem over the last decade, it was only recently that relativily simple arguments have been found to prove such elementary facts as the uniqueness of a Gibbs measure, exponential decay of the truncated correlation functions and infinite differentiability of the free energy with respect to a real external field. The two most recent and most powerful approaches (von Dreifus, Klein and Perez (1994) and Gielis–Maes (1995)) have in common that they use a percolation picture to control the unboundedness of the non–translation–invariant interaction.

The goal of the present paper is to continue this analysis in the direction of a unifying scheme similar to the one for completely analytic interactions. Our main results concern the location of zero's of the partition function and upper bounds for the semi–invariants. The picture that emerges is that the disordered system enjoys the same kind of analyticity as the translation invariant

interactions as long as the complex perturbations are smaller than a stochastic bound which takes into account the "badness" of the region in which the perturbation is employed. In other words, the region of analyticity has become random. Estimates on the size of these regions then gives information about the growth of the semi–invariants. The conclusion is that disordered systems with unbounded interactions at high temperature are perhaps not a completely "happy family" (see the opening lines of Dobrushin–Shlosman (1987)), but still, the "happiness" of the individual components is as large as the local environment permits it to be. Therefore we think it is justified to introduce the term "local analyticity".
Note that this does not contradict the result of Griffiths that predicts that the free energy is not an analytic function of a *uniform* complex small magnetic field. (See Griffiths (1969).)

The next Section contains definitions and the main results. Section 3 reviews some aspects of diluted ferromagnets. We show in Section 4 that the stochastic–geometric aspects of these random ferromagnets remain essentially unchanged in the case of the more general spin systems, e.g. the Griffiths' singularities for the Edwards–Anderson spin glass are not stronger than for Griffiths' diluted model.

2 Definition and Main Results

The results and the proofs will be presented for a specific model of a disordered system. This is mostly to avoid a too complicated notation and to be able to present the underlying ideas in the most simple case of what should be referred to as specification valued random fields.
We consider spins $\sigma(x) = \pm 1, x \in \mathbb{Z}^d$, in a d–dimensional lattice interacting according to the Edwards–Anderson Hamiltonian (see Edwards–Anderson

(1975)):

$$H(\sigma) = -\sum_{\langle xy\rangle} J_{xy}\sigma(x)\sigma(y). \tag{1}$$

The couplings $\{J_{xy}\}$ are a family of independent and identically distributed random variables associated to the bonds $\langle xy\rangle$ (nearest neighbor edges) of the lattice. The distibution of the $\{J_{xy}\}$ will be denoted by $\mathbf{Q}$, and the expectation by $\mathbf{E}[\cdot]$. The J_{xy} 's are real valued, but $\mathbf{Q}$ can also put a non–zero weight to the values $J_{xy} = \pm\infty$.

When we have a finite region $V \subset \mathbb{Z}^d$ and boundary condition $\eta \in \Omega = \{-1,-1\}^{\mathbb{Z}^d}$, we define the finite volume Hamiltonian as

$$H_V(\sigma|\eta) = -\sum_{\langle xy\rangle\cap V\neq\emptyset} J_{xy}\xi(x)\xi(y),$$
$$\text{where } \begin{cases} \xi(x) = \sigma(x) \text{ if } x \in V \\ \xi(x) = \eta(x) \text{ if } x \notin V \end{cases} \tag{2}$$

for any configuration $\sigma \in \Omega$.

The perturbed system at an inverse temperature β is described by

$$\beta\tilde{H}_V(\sigma|\eta) = \beta H_V(\sigma|\eta) - \sum_{x\in V} h_x\sigma(x), \tag{3}$$

where the $\{h_x\}_{x\in V}$ is a complex valued magnetic field. The corresponding partition function is denoted by

$$Z_V(h|\eta) \equiv \sum_{\substack{\sigma(x)=\pm1\\ x\in V}} \exp[-\beta\tilde{H}_V(\sigma|\eta)] \tag{4}$$

and has to be considered as a function of the complex variables $\{h_x\}_{x\in V}$ for each fixed $V \subset \mathbb{Z}^d$ and $\eta \in \Omega$. Clearly this function is analytic.

¿From (2) we can calculate the single site conditional distribution for $x \in \mathbb{Z}^d$,

$$\begin{aligned} \mathrm{p}_x(\eta) \equiv \mathrm{Prob}_x[\sigma(x) = +1 \quad & |\sigma(y) = \eta(y), y \sim x] \\ & = (1 + \exp[-2\beta \sum_{y\sim x} J_{xy}\eta(y)])^{-1}, \end{aligned} \tag{5}$$

when we specify the configuration η on the nearest neighbors $y \sim x$ of x. The random variable

$$q_x = \max_{\eta,\overline{\eta}} |\mathrm{p}_x(\eta) - \mathrm{p}_x(\overline{\eta})| \tag{6}$$

then measures the maximal variation in the distribution of $\sigma(x)$ when we change the configuration on the nearest neighbors of x. From (2) we can easily see that

$$q_x = \tanh\beta \sum_{y\sim x} |J_{xy}| \tag{7}$$

The $q_x \in [0,1], x \in \mathbb{Z}^d$ can now be considered as the densities of an independent site percolation process. That is, we have independent 0–1 valued random variables $s = \{s(x), x \in \mathbb{Z}^d\}$ distributed according to the product measure $\nu_{\{q_x\}}[\cdot]$ on $\{0,1\}^{\mathbb{Z}^d}$:

$$\nu_{\{q_x\}}[s(y) = 1, \forall y \in A] = \prod_{y\in A} q_y \tag{8}$$

for any finite $A \subset \mathbb{Z}^d$. A site $x \in \mathbb{Z}^d$ for which $s(x) = 1(0)$ is called open (closed). An open path is a sequence of nearest neighbor open sites. $G(x,y)$ is the probability to have such a path from x to y. The cluster $C(x)$ of a site x is the set of all sites that are connected to x via an open path. $|C(x)|$ is the size of the cluster with expectation

$$S(x) = \nu_{\{q_x\}}[|C(x)|]. \tag{9}$$

Note that $S(x) = \sum_{y\in \mathbb{Z}^d} G(x,y)$ depends on the realization $\{J_{xy}\}$ of the disorder.

Finally we define the random set

$$\Lambda_x = \{y \in \mathbb{Z}^d : G(x,y) > \mathrm{e}^{-\lambda|x-y|}\} \tag{10}$$

for some $\lambda > 0$ small enough, with $|x-y| = \max_{i=1\ldots d} |x^i - y^i|$ the lattice distance between the sites $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$.

Define for $x \in \mathbb{Z}^d$

$$\tilde{S}(x) = \max_{y \in \Lambda_x} S(y) \tag{11}$$

and

$$\tilde{S}(x) = 1 \text{ if } \Lambda_x = \emptyset. \tag{12}$$

Proposition 1 *If* $\mathbf{E}[q_x]$ *is small enough, then*

$$\mathbf{E}[e^{\delta \tilde{S}(x)}] < \infty \tag{13}$$

for all $\delta > 0$*, small enough.*

This tells us that when the temperature is high enough, the size of the region $\tilde{S}(x)$ of strongly interacting sites $\tilde{S}(x)$ is at most exponentially distributed. Note however that $\sup_x \tilde{S}(x) = \infty$ because the $\{J_{xy}\}$ are unbounded.

We now state our result about the zero's of the partition function. Let $V_0 = \{x \in V, h_x \neq 0\}$ be the region in V to which we add via (3) a non–zero magnetic field.

Theorem 1 *When*

$$\mathbf{E}[q_x] < \frac{1}{(2d-1)^2} \tag{14}$$

there is a constant $\epsilon > 0$ *so that whenever*

$$0 < |h_x|^\alpha < \epsilon/(\tilde{S}(x))^{-1}, x \in V_0 \tag{15}$$

for some $0 < \alpha < 1$*, then the functions* $Z_V(h|\eta)$ *are non–vanishing uniformly in* V *and* η*. Moreover, there is a constant* $C = C(\epsilon) < \infty$ *so that*

$$|\ln[Z_V(h|\eta)/Z_V(h=0|\eta)]| \leq C|V_0|. \tag{16}$$

This Theorem says that if we locally adjust the strength of the perturbing magnetic field (taking into account the random environment as in (15)), then the perturbed system is a small perturbation of the original one in the sense that an analytic continuation of $\ln Z_V(h|\eta)$ is possible in the random region described by (15). This coincides with our intuition. Indeed, where $\tilde{S}(x)$ is large (what corresponds to a region with strong interactions), the local behavior of the system tends to the low temperature behavior, what is expressed by a smaller region of analyticity.

Let us now consider the semi–invariants or truncated correlation functions:

$$\langle \sigma_{x_1}^{;k_1}, \sigma_{x_2}^{;k_2}, \ldots, \sigma_{x_m}^{;k_m} \rangle = \frac{\partial^{k_1+k_2+\ldots+k_m}}{\partial h_{x_1}^{k_1} \partial h_{x_2}^{k_2} \ldots \partial h_{x_m}^{k_m}} \ln Z_V(h|\eta) \Big|_{h=0} \tag{17}$$

for $k_i > 0, i = 1, \ldots, m$ and sites $\{x_1, \ldots, x_m\} = V_0$.

Denote by $\lceil r \rceil, r \in \mathbb{R}$ the smallest integer larger than r.

Theorem 2 *Under the assumption (14) of Theorem 1 the semi–invariants (17) are bounded from above by*

$$|(17)| \leq m {C_1}^{k_1+\ldots+k_m} k_1! k_2! \ldots k_m! \tilde{S}(x_1)^{\frac{k_1}{\alpha}} \ldots \tilde{S}(x_m)^{\frac{k_m}{\alpha}}, \tag{18}$$

for some constant $C_1 = C_1(\alpha) < \infty$ for all $0 < \alpha < 1$.

Moreover, for every m there exists a constant $p_0 = p_0(m) > 0$ so that if $\mathbf{E}(q_x) < p_0$, averaging over the disorder gives,

$$\mathbf{E} |\langle \sigma_{x_1}^{;k_1}, \sigma_{x_2}^{;k_2}, \ldots, \sigma_{x_m}^{;k_m} \rangle| \leq {C_2}^{k_1+\ldots+k_m} \left(\left\lceil \frac{k_1}{\alpha} \right\rceil ! \ldots \left\lceil \frac{k_m}{\alpha} \right\rceil ! \right)^2, \tag{19}$$

for some $C_2 = C_2(\alpha) < \infty$ for all $0 < \alpha < 1$.

(18) gives a bound which is the usual one for analytic functions, except that the domain of analiticity (see 15) is random. The bound (18) is optimal in the sense that there is a model (see next Section) for which this bound is actually also obtained in the sense of a lower bound.

Remark 1 Given (18), we do not expect (19) to hold for small enough β, uniformly in m.

Remark 2 It is also true that all truncated correlation functions (17) decay spatially exponentially fast. For example,

$$|\mathbf{E}\langle\sigma_{x_1}^{;k_1}, \sigma_{x_2}^{;k_2}, \ldots, \sigma_{x_m}^{;k_m}\rangle| \leq C\exp(-\lambda \mathrm{dist}\{x_1, x_2, \ldots x_m\}) \qquad (20)$$

with $\mathrm{dist}\{x_1, x_2, \ldots x_m\}$ the length of the shortest tree that connects all this sites and $C < \infty$. This follows from an easy argument (see e.g. in von Dreifus, Klein and Perez (1994)) once it is established that all correlation functions of the form $\langle f, g\rangle - \langle f\rangle\langle g\rangle$ decay exponentially fast as the distance between the supports of the local functions f and g increases. (See Gielis–Maes (1995)). This argument however gives very bad upper bounds on the prefactor C in (20) which depends on the order $(k_1 + \ldots + k_m)$ of the semi–invariant.

3 Griffiths' cluster model

We give here a version of Griffiths' model (see Griffiths (1969)) in which Griffiths' singularities are explicitly visible. It is a diluted ferromagnet which has a stochastic–geometric representation which is very close to the one we will obtain in the next Section for the Edwards–Anderson model.

Let ν_p be the Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ with density $p < p_c(\mathbb{Z}^d)$ the threshold for independent site percolation on $\mathbb{Z}^d$. The random set $\{x \in \mathbb{Z}^d, s(x) = 1\}$ of open sites is the disjoint union of finite clusters. Given this realization of open and closed sites we define a finite volume ($V \subset \mathbb{Z}^d$) spin model, $\{\sigma(x), x \in V\}, \sigma(x) = \pm 1$ in the following way. Put all the spins $\sigma(x), x \in C$ belonging to the same open cluster C equal, fluctuating together and taking the values ± 1 with equal probability. If $C \cap V^c \neq \emptyset$ we put all spins in C equal to either all plus or all minus depending on the boundary conditions. If $s(x) = 0$ (a closed site), we put $\sigma(x) = \pm 1$ with probability $1/2$. Since $p < p_c$ all clusters are finite, and hence with ν_p–probability one these finite volume spin measure converges exponentially fast to a unique infinite volume measure as $V \uparrow \mathbb{Z}^d$. Let us now add a uniform complex magnetic field. Taking free boundary conditions, the finite volume partition function equals

$$Z_V(h) = 2^n \prod_{i=1}^{n} \cosh h|C_i| \tag{21}$$

where n is the number of open clusters in V when we put all sites closed outside V, plus the number of closed sites in V. $|C_i|$ is the size of the i–th open cluster. When $s(x) = 0$, we put the corresponding $|C_i| = 1$. Clearly (21) is non–vanishing if $|h| < \pi/2 \max_i |C_i|$, but when V grows this upperbound will go to zero with ν_q–probability one. Some further analysis (see Griffiths (1969)) then shows that the free energy

$$f(h) = \lim_{V \uparrow \mathbb{Z}^d} \frac{\log Z_V(h)}{|V|} \tag{22}$$

is *not* analytic around $h = 0$. Still, if we look at local observables (such as correlation functions) for a fixed disorder (realizations of open and closed sites) we do have analyticity. For example, the magnetization at the origin is

$$\langle \sigma(0) \rangle (h) = \tanh h|C(0)|. \tag{23}$$

This implies that the magnetization is analytic in the circle $|h| < \pi/2|C(0)|$. Moreover, it can be explicitely shown that the truncated correlation function

$$|\langle \sigma_{x_1}^{;k_1}, \sigma_{x_2}^{;k_2}, \ldots, \sigma_{x_m}^{;k_m} \rangle| \simeq (k_1!k_2! \ldots k_m!)^2 |C|^{k_1+k_2+\ldots+k_m} I\{x_i \in C, i : 1 \ldots m\} \tag{24}$$

when $k_1 + k_2 + \ldots + k_m$ is even. $I\{x_i \in C, i : 1 \ldots m\} = 1$ if $\{x_i \in C, i : 1 \ldots m\}$, with $C = C(x_1)$, and zero otherwise.

This implies that the bound in (18) is optimal. In fact, the essence of the proofs of Theorem 1 and 2 is to dominate the disordered spin model by a stochastic–geometric model which is very close to the randomly diluted ferromagnet discussed here. It basically amounts to replacing the density p (in ν_p) by $\mathbf{E}[q_x]$ (see (7)–(9)). The difference of the geometric model obtained for the Edwards–Anderson model with the one discussed in this Section for Griffiths' model, is that in the Edwards–Anderson model spins on distinct open clusters are *not* independent. This is also the reason why in (18) there is $\tilde{S}(x)$ and not simply $S(x)$. Moreover, we do not have the analyticty of $\langle\sigma(0)\rangle(h)$ in a *uniform* magnetic field, but we must work with a spatially dependent perturbation in order to obtain local analyticity for a fixed disorder.

Remark: From the above arguments and the behavior (24) for Griffiths' diluted model, it is seen that the free energy $f(h)$ is also not quasi–analytic (i.e. having a unique extension), thereby answering a question raised by J. Fröhlich (1984).

4 Proofs

In what follows we consider finite volumes $V \subset \mathbb{Z}^d$ (which however can be arbitrarily large) with boundary

$$\partial V = \{x \in V^c = \mathbb{Z}^d \setminus V : x \sim y \text{ for some } \ y \in V\}. \tag{25}$$

We consider the finite volume Gibbs–state μ_V^η obtained as usual from (2) with expectations

$$\langle f(\sigma)\rangle_V^\eta = \frac{1}{Z_V(\eta)} \sum_{\substack{\sigma(x)=\pm 1\\ x\in V}} f(\sigma) \mathrm{e}^{-\beta H_V(\sigma|\eta)}. \tag{26}$$

The first Lemma is about the ratio's of partition functions of the form (4).

Lemma 1 *Let $y \in \partial V$ and $\eta, \overline{\eta}$ be two boundary configurations with $\eta(z) = \overline{\eta}(z)$ for $z \neq y$. For any $0 < \alpha < 1$, there are (small) constants $\epsilon_1, \epsilon_2 > 0$ such that if*

$$\begin{aligned} &|h_x| < \epsilon_1, &&\forall x \in \mathbb{Z}^d \\ &\sum_{x \in \mathbb{Z}^d} |h_x|^\alpha G(x,y) < 2\epsilon_2, &&\forall y \in \mathbb{Z}^d \end{aligned} \tag{27}$$

and $h_x \neq 0, x \in V$, then

$$\frac{\left\langle e^{\sum_{x\in V} h_x \sigma(x)} \right\rangle_V^\eta}{\left\langle e^{\sum_{x\in V} h_x \sigma(x)} \right\rangle_V^{\overline{\eta}}} = 1 + \theta \sum_{x \in V} |h_x|^\alpha G_V(x,y) \tag{28}$$

with $\theta = \theta_V(\eta, \overline{\eta})$ a complex number with modulus $|\theta| < 1$.

$G_V(x,y)$ is as defined in Chapter 2, except that we consider only paths inside V.

Proof The proof is a generalization of Lemma 3.1 in Dobrushin–Shlosman (1985) extended to the non–translation invariant case. It uses an induction argument.

Let $B(y,r)$ be the ball or radius $r > 1$ around y:

$$B(y,r) = \{x \in \mathbb{Z}^d : |x - y| \leq r\} \tag{29}$$

and define the subvolume $\tilde{V} = V \setminus B(y,r)$.

We assume that (28) holds for $\tilde{V}$.

We define $\tilde{\eta}$ a (boundary) configuration associated to $\tilde{V}$ by putting

$$\tilde{\eta}(z) = \eta(z) \text{ if } z \in \partial V \cap \partial \tilde{V} \tag{30}$$

and $\tilde{\eta}(z), z \in \partial\tilde{V} \setminus \partial V$ chosen so that

$$\left| \left\langle e^{\sum_{x\in \tilde{V}} h_x \sigma(x)} \right\rangle_{\tilde{V}}^{\tilde{\eta}} \right| = \max_{\xi : \xi = \eta \text{on } \partial V \cap \partial \tilde{V}} \left| \left\langle e^{\sum_{x\in \tilde{V}} h_x \sigma(x)} \right\rangle_{\tilde{V}}^{\xi} \right|. \tag{31}$$

With this choice we rewrite the left hand side of (28) as

$$\mathrm{LHS}(28) = 1 + \frac{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\tilde\eta}}{\left\langle \mathrm{e}^{h_V\cdot\sigma}\right\rangle_V^{\overline{\eta}}}\left[\langle g(\xi)\rangle_V^{\eta} - \langle g(\xi)\rangle_V^{\overline{\eta}}\right], \tag{32}$$

where we use the notation $h_V\cdot\sigma = \sum_{x\in V} h_x\sigma(x)$ and

$$g(\xi) = \mathrm{e}^{h_{V\setminus\tilde V}\cdot\xi}\frac{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\xi}}{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\tilde\eta}}. \tag{33}$$

To estimate the right hand site of (32) we start by noting that

$$\frac{\left\langle \mathrm{e}^{h_V\cdot\sigma}\right\rangle_V^{\overline{\eta}}}{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\tilde\eta}} = \left\langle \mathrm{e}^{h_{V\setminus\tilde V}\cdot\xi}\frac{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\xi}}{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\tilde\eta}}\right\rangle_V^{\overline{\eta}}, \tag{34}$$

where we average over the ξ–spins. Clearly (in (34)) $\xi = \tilde\eta$ on $\partial\tilde V\cap\partial V$ because $\xi = \overline{\eta} = \eta$ on that set.

For such a ξ, using the induction hypothesis

$$\left|\frac{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\xi}}{\left\langle \mathrm{e}^{h_{\tilde V}\cdot\sigma}\right\rangle_{\tilde V}^{\tilde\eta}}\right| \geq \prod_{z\in\partial\tilde V\setminus\partial V}\left[1-\sum_{x\in\tilde V}|h_x|^\alpha G_{\tilde V}(x,z)\right]. \tag{35}$$

Since $|V\setminus\tilde V|$ and $|\partial\tilde V\setminus\partial V|$ are bounded by some constant (only depending on the radius r and the dimension d) it follows from (34) and (35) that we can choose ϵ_1 and ϵ_2 in Lemma (1) small enough so that

$$|34|^{-1} \leq (1+K_1\sum_{x\in\tilde V}|h_x|^\alpha G_{\tilde V}(x,z^*))(1+K_2\sum_{x\in V\setminus\tilde V}|h_x|), \tag{36}$$

for some $z^*\in\partial\tilde V\setminus\partial V$ and $K_i = K_i(\epsilon_1,\epsilon_2), i=1,2$ constants larger than one.

To estimate the factor with the difference in (32) we use the coupling between μ_V^{η} and $\mu_V^{\overline{\eta}}$ as introduced in van den Berg–Maes (1994) (see also Gielis–Maes

(1995)). We then get

$$\left|\langle g(\xi)\rangle_V^{\eta} - \langle g(\xi)\rangle_V^{\overline{\eta}}\right| \leq \sum_{z\in V\setminus\tilde{V}} \delta_z g G_V(z,y), \tag{37}$$

Now,

$$\delta_z g = \frac{1}{\left|\left\langle \mathrm{e}^{h_{\tilde{V}}\cdot\sigma}\right\rangle_{\tilde{V}}^{\tilde{\eta}}\right|} \max_{\xi:\xi=\eta\text{on } \partial V\cap\partial\tilde{V}} \left|\mathrm{e}^{h_{V\setminus\tilde{V}}\cdot\xi^z}\left\langle \mathrm{e}^{h_{\tilde{V}}\cdot\sigma}\right\rangle_{\tilde{V}}^{\xi^z} - \mathrm{e}^{h_{V\setminus\tilde{V}}\cdot\xi}\left\langle \mathrm{e}^{h_{\tilde{V}}\cdot\sigma}\right\rangle_{\tilde{V}}^{\xi}\right| \tag{38}$$

where we have used that $\xi^z(x) = \xi(x)$ if $z \neq x$, $\xi^z(z) = -\xi(z)$. Because we choose $\tilde{\eta}$ according to (31) we get for every $z \in V \setminus \tilde{V}$

$$\delta_z g \leq \left| \frac{\left\langle \mathrm{e}^{h_{\tilde{V}}\cdot\sigma}\right\rangle_{\tilde{V}}^{\xi_*^z}}{\left\langle \mathrm{e}^{h_{\tilde{V}}\cdot\sigma}\right\rangle_{\tilde{V}}^{\xi_*}} \mathrm{e}^{h_{V\setminus\tilde{V}}\cdot\xi_*^z} - \mathrm{e}^{h_{V\setminus\tilde{V}}\cdot\xi_*}\right| \tag{39}$$

with ξ_* the maximizer in (38). If $z \in V \setminus (\partial\tilde{V} \cup \tilde{V})$ then $\xi_*^z = \xi_*$ on $\partial\tilde{V}$ and hence (39) becomes

$$\delta_z g \leq \left|\mathrm{e}^{h_{V\setminus\tilde{V}}\cdot\xi_*^z} - \mathrm{e}^{h_{V\setminus\tilde{V}}\cdot\xi_*}\right|. \tag{40}$$

If this is not the case we can again use the induction hypothesis (on $\tilde{V}$) and combined with (27) we get

$$\begin{aligned}(37) \leq \left[1 + K_3 \sum_{V\setminus\tilde{V}} |h_x|\right] \; & \left[\sum_{z\in\partial\tilde{V}\cap\partial V} \sum_{x\in\tilde{V}} |h_x|^{\alpha} G_{\tilde{V}}(x,z) G_V(z,y)\right. \\ & \left. +2 \sum_{z\in V\setminus(\partial\tilde{V}\cup\tilde{V})} \sinh|h_z| G_V(z,y)\right],\end{aligned} \tag{41}$$

for some finite $K_3 = K_3(\epsilon_1)$.
Now we note that in (41), for $x \in \tilde{V}$

$$\sum_{z\in\partial\tilde{V}\cap\partial V} G_{\tilde{V}}(x,z) G_V(z,y) \leq G_V(x,y) \tag{42}$$

as follows from the FKG–inequality (after we split up $G(x,y)$ in a sum of probabilities of disjoint events corresponding to the first site $z \in B(y,r)$ on the path from x to y).

Now we combine (34)–(36) with (41)–(42) to finally estimate (32). Depending on $\alpha < 1$ and the radius r we can choose ϵ_1 and ϵ_2 in Lemma 1 so small that

$$|LHS(32) - 1| \leq \sum_{x \in V} |h_x|^\alpha G_V(x, y) \tag{43}$$

which is sufficient to conclude the proof of (28). ■

Lemma 2 *Suppose that $V_0 \subset V$ is the subvolume of V in which $h_x \neq 0, x \in V_0$. Then under the assumptions of Lemma 1, for every boundary condition η,*

$$|\ln Z_V(h_{V_0}|\eta) - \ln Z_V(h = 0|\eta)| \leq C|V_0| \tag{44}$$

with $C = C(\epsilon_1, \epsilon_2) < \infty$ some constant.

Proof Clearly

$$LHS(44) = \left| \ln \left\langle \left\langle \mathrm{e}^{h_{V_0} \cdot \sigma} \right\rangle_{V_0}^{\eta^0} \right\rangle_V^\eta \right| \tag{45}$$

where we average with μ_V^η over the η_0 on ∂V_0. Define a sequence of volumes $V \supset V_0 \supset \ldots \supset V_k$ such that $V_k = \emptyset, V_{i+1} = V_i \setminus B(y_i, r)$ with $y_i \in \partial V_i$. Correspondingly, we define boundary conditions η_i on ∂V_i by putting

$$\begin{aligned} \eta_0(x) &= \eta(x) && \text{for } x \in \partial V \cap \partial V_0 \\ \eta_0(x) &= +1 && \text{for } x \notin \partial V \cap \partial V_0 \\ \eta_i(x) &= \eta_{i-1}(x) && \text{for } x \in \partial V_i \cap \partial V_{i-1} \end{aligned} \tag{46}$$

and such that

$$\left| \left\langle \mathrm{e}^{h_{V_i} \cdot \sigma} \right\rangle_{V_i}^{\eta_i} \right| = \max_{\xi : \xi = \eta_{i-1} \text{on } \partial V_{i-1} \cap \partial V_i} \left| \left\langle \mathrm{e}^{h_{V_i} \cdot \sigma} \right\rangle_{V_i}^{\xi} \right|. \tag{47}$$

Summarizing, (V_i, η_i) is to $(V_{i-1}, \eta_{i-1}, y_{i-1})$ what $(\tilde{V}, \tilde{\eta})$ is to (V, η, y) in the construction of Lemma 1. Since Lemma 1 holds for all V, we can apply the same ideas to obtain

$$\left|\frac{\left\langle \mathrm{e}^{h_{V_i}\cdot\sigma}\right\rangle_{V_i}^{\eta_i}}{\left\langle \mathrm{e}^{h_{V_{i+1}}\cdot\sigma}\right\rangle_{V_{i+1}}^{\eta_{i+1}}}\right| \le \prod_{z\in\partial V_i\cap V_{i+1}} \left[1+\sum_{x\in V_{i+1}} |h_x|^\alpha G(x,z)\right]. \tag{48}$$

Therefore,

$$\begin{aligned}(45) = &\left|\ln\left\langle\left\langle\prod_{i=0}^{k-1}\frac{\left\langle \mathrm{e}^{h_{V_i}\cdot\sigma}\right\rangle_{V_i}^{\eta_i}}{\left\langle \mathrm{e}^{h_{V_{i+1}}\cdot\sigma}\right\rangle_{V_{i+1}}^{\eta_{i+1}}}\right\rangle\right\rangle_V^{\eta}\right| \\ &\le C\sum_{i=0}^{k-1}\sum_{z\in\partial V_i\cap V_{i+1}}\sum_{x\in V_{i+1}}|h_x|^\alpha G(x,z)\end{aligned} \tag{49}$$

for some finite constant C. Rearranging the sums in (49) and using (27) for $x\in V_0$ and the bound (15) for h_x we have

$$\begin{aligned}(49) \quad &\le C\sum_{x\in V_0}\sum_{z\in\mathbb{Z}^d}|h_x|^\alpha G(x,z) \\ &\le C\sum_{x\in V_0}|h_x|^\alpha|C(x)| \\ &\le C|V_0|\end{aligned} \tag{50}$$

which we wanted to prove. ■

Proof of Theorem 1

The assumption (27) is satisfied for $|h_x|^\alpha < 1/\tilde{S}(x)$. Indeed, because $z\in\Lambda_x$ when $x\in\Lambda_z$, and so $S(z)\le\tilde{S}(x)$, we have that

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}|h_x|^\alpha G(x,z) \quad &\le \epsilon_2\sum_{x\in\Lambda_z}\frac{G(x,z)}{\tilde{S}(x)}+\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\lambda|x-z|} \\ &\le \epsilon_2 + C(\lambda) \\ &\le 2\epsilon_2,\end{aligned} \tag{51}$$

when λ is small enough. Hence, Theorem 1 is a consequence of Lemma's 1 and 2. ■

Proof of Theorem 2 and Proposition 1 (18) is a consequence of the Cauchy inequality applied to (17) using Theorem 1 (in particular that(15) implies (16)). For (19) we must take an additional average over the disorder. For this we need information about the joint distibution of the $\tilde{S}(x), x \in V_0$ for some finite set $V_0 = \{x_1, \ldots, x_m\}$.

$$\mathbf{Q}\{\tilde{S}(x_1) \geq n_1, \ldots, \tilde{S}(x_m) \geq n_m\} \leq \mathbf{E}\left[I[\max_{\substack{y_i \in \Lambda_{x_i} \\ i:1,\ldots,m}} \sum_{i=1}^{m} S(x_i) \geq \sum_{i=1}^{m} n_i]\right] \tag{52}$$

with $I[\cdot]$ the indicator function.

$$\begin{aligned}(52) \quad &\leq \sum_{y_1,\ldots,y_m \in \mathbb{Z}^d} \mathbf{E}\left[\prod_{i=1}^{m} I[y_i \in \Lambda_{x_i}] I[\sum_{i=1}^{m} S(x_i) \geq \sum_{i=1}^{m} n_i]\right] \\ &\leq \sum_{y_1,\ldots,y_m \in \mathbb{Z}^d} \mathbf{E}\left[\prod_{i=1}^{m} G(x_i, y_i) \mathrm{e}^{-\lambda |x_i - y_i|} \exp[\delta \sum_{i=1}^{m} S(x_i)] \exp[-\delta \sum_{i=1}^{m} n_i]\right].\end{aligned} \tag{53}$$

Consider now the site percolation process in which we put $s(x) = +1$ on every odd lattice site $x \in \mathbb{Z}^d{}_{\text{odd}} \subset \mathbb{Z}^d$ and, with probability $p = \mathbf{E}[q_x]$ we put $s(x) = +1$ on the every even lattice site $x \in \mathbb{Z}^d{}_{\text{even}} \subset \mathbb{Z}^d$, with $\mathbb{Z}^d_{\text{even}} \cup \mathbb{Z}^d_{\text{odd}} = \mathbb{Z}^d$. Denote with $\langle \cdot \rangle_p$ the corresponding expectation value.
Note that $\{q_x, x \in \mathbb{Z}^d_{\text{even}}\}$ are independently distributed. Hence, due to the convexity of the exponential function and the FKG–inequality (53) is bounded by

$$\begin{aligned}(53) \quad &\leq \exp[-\delta \sum_{i=1}^{m} n_i] \\ &\times \sum_{y_1,\ldots,y_m \in \mathbb{Z}^d} \exp[\beta \sum_{i=1}^{m} |x_i - y_i|] \left\langle \prod_{i=1}^{m} I\{x_i \rightsquigarrow y_i\} \exp \delta \sum_{i=1}^{m} |C(x_i)| \right\rangle_p \\ &\leq K \exp[-\delta \sum_{i=1}^{m} n_i],\end{aligned} \tag{54}$$

with $I\{x \rightsquigarrow y\}$ the indicator function of the event that there is an open path from x to y and K a constant such that, for some $0 < \gamma = \gamma(\lambda) < \infty$,

$$K \geq \frac{1}{\gamma^m} \left\langle \mathrm{e}^{(\lambda+\gamma+\delta) \sum\limits_{i=1}^{m} |C(x_i)|} \right\rangle_p . \tag{55}$$

Because of the exponential distribution of the cluster size in the subcritical phase, (see Grimmett (1989)), K is finite when $p < p_0$ for some $p_0 = p_0(m, \delta, \gamma, \lambda)$. This proves $(m = 1)$ Proposition 1.
Combining (18) and (55) gives

$$(19) \leq mKC_1^{k_1+\ldots k_m} \prod_{i=1}^{m} \left[k_i! \sum_{n_i=0}^{\infty} n_i^{\frac{k_i}{\alpha}} \mathrm{e}^{-\delta n_i} \right] . \tag{56}$$

This finishes the proof of Theorem 2. ■

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